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## Inequivalent frame-spun knots with the same complement

ALEXANDER I. SUCIU<sup>1</sup>

### 1. Introduction

One of the basic questions of knot theory is: Is every  $n$ -knot determined by its complement? For  $n = 1$ , Gordon and Luecke [11] have recently given an affirmative answer to this question. For  $n \geq 2$ , there are at most two  $n$ -knots with the same complement [9], [4], [17], [15]. A knot which is determined by its complement is called reflexive. Knots that are spun [9], superspun [5], 2-twist-spun [10], [13], simple [18], stable [8], [22], and some others [21], [7], are known to be reflexive. Cappell and Shaneson [7] gave the first examples of knots which are not determined by their complements. Their method works for each  $n \geq 2$ , as long as certain integral, unimodular  $(n + 1) \times (n + 1)$  matrices exist; such matrices have been found only for  $n = 2, 3, 4$  and  $5$ . Shortly thereafter, Gordon [10] proved that odd-twist-spun  $n$ -knots with closed fiber covered by  $\mathbb{R}^{n+1}$  are non-reflexive. His method is known to yield examples only for  $n = 2$ . Other examples of 2-knots which are not determined by their complements were given in [20], [21], [13].

The main result of this paper is the following theorem.

**THEOREM 1.1.** *There exist non-reflexive  $n$ -knots for every  $n \equiv 3$  or  $4 \pmod{8}$ .*

We construct these  $n$ -knots by frame-spinning the 2-knots of Gordon. In doing so, we reprove Gordon's theorem under slightly more general conditions (Corollary 6.2), thus giving a new proof of the non-reflexivity of his 2-knots. The basic idea is to translate the question of reflexivity of the frame-spun knots into a question about homotopy groups of spheres, via a generalized Pontrjagin–Thom construction.

The process of frame-spinning was introduced by Roseman in [23]; it generalizes previous notions of spinning that go back to Artin. If  $K$  is an  $n$ -knot and  $M^k$  is a framed submanifold of  $S^{n+k}$ , with framing  $\varphi$ , one can spin  $K$  about  $M^k$  to get an  $(n + k)$ -knot  $\sigma_M^\varphi(K)$ . This is done by removing at each point of  $M^k \subset (S^{n+k+2}, S^{n+k})$  the transverse disk pair determined by the framing and gluing back the knotted disk pair determined by the  $n$ -knot.

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The question we investigate in this paper is: Is a frame-spun knot determined by its complement? Quite often, the answer is yes. Suppose  $M^k = S^k$ , standardly embedded in  $S^{n+k}$ , with framing given by a smooth map  $\varphi : S^k \rightarrow SO(n)$ . For  $k \geq 2$ , let  $\eta_k$  be a generator of  $\pi_{k+1}(S^k)$ . Given an  $n$ -knot  $K$ ,  $n \geq 2$ , we prove the following (Theorems 4.2 and 4.3): *If either  $K$  is reflexive, or  $[\varphi] \circ \eta_k$  is zero, then  $\sigma_{S^k}^\varphi(K)$  is reflexive.* This generalizes a result of Cappell [5].

In general though, the answer to the above question is no. For an arbitrary framed manifold  $(M^k, \varphi) \subset S^{n+k}$ , the Pontrjagin–Thom construction yields an element  $\alpha$  of  $\pi_{n+k}(S^n)$ . Suppose  $K$  is a fibered  $n$ -knot,  $n \geq 2$ , with aspherical closed fiber and odd order monodromy (such knots are known to exist only for  $n = 2$ ). We then prove the following (Theorem 6.3): *If the suspension of  $\alpha \circ \eta_{n+k}$  is non-zero, then  $\sigma_M^\varphi(K)$  is not reflexive.* For  $k \equiv 1$  or  $2 \pmod{8}$ , there are such  $\alpha$ 's in  $\pi_{k+2}(S^2)$ , by deep work of Mahowald [19]. This produces non-reflexive frame-spun  $(k+2)$ -knots by surjectivity of the Pontrjagin–Thom homomorphism.

Let us briefly sketch the proof of Theorem 6.3. In §5, we introduce the notion of spinning a closed manifold  $W^m$  about a framed manifold  $(M^k, \varphi)$ . This is done by removing at each point of  $M^k \subset S^{m+k}$  a transverse  $n$ -disk and gluing back a punctured copy of  $W^m$ . An essential feature of this construction is the existence of a ‘‘Pontrjagin–Thom’’ map,  $\sigma_M^\varphi(W) \rightarrow W$ , that may be used to differentiate among the various frame-spins of  $W$ . Now, as noticed by Roseman [23], the process of frame-spinning takes fibered knots to fibered knots. In our terminology, if  $K$  has closed fiber  $F^c$ , then  $\sigma_M^\varphi(K)$  has closed fiber the stabilized frame-spin of  $F^c$ . In case  $F^c$  is aspherical, we are able to distinguish between the closed fibers of two frame-spins of  $K$ , provided the two manifolds we spin about are not stably framed bordant (Theorem 5.2). In particular, if  $E(\alpha \circ \eta_{n+k}) \neq 0$ , the two  $S^1$ -spins of the closed fiber of  $\sigma_M^\varphi(K)$  are distinct. On the other hand, if  $K$  has odd order monodromy, so does  $\sigma_M^\varphi(K)$ , and therefore  $\sigma_M^\varphi(K)$  cannot be reflexive, for otherwise the two  $S^1$ -spins of its closed fiber would be equal.

In view of the above results, we venture the following

**CONJECTURE.** The knot  $\sigma_M^\varphi(K)$  is reflexive if and only if either  $K$  is reflexive, or  $\alpha \circ \eta_{n+k} = 0$ .

If the forward implication were true, one could produce examples of non-reflexive knots in the missing dimensions by frame-spinning the Cappell–Shaneson knots instead of Gordon’s knots.

I wish to thank J. Klein and M. Mahowald for valuable conversations. An early version of Theorem 5.2 dealt only with homology spheres. I am grateful to the referee for pointing out a gap in a subsequent generalization, and for suggesting the use of Lemma 2.1 to arrive at the right level of generality.

## 2. Knotted spheres

We start with some definitions and notation. All manifolds are to be compact, connected, oriented, and smooth; closed manifolds are those without boundary. Diffeomorphisms are denoted by  $\cong$ , homotopy equivalences by  $\simeq$ , reduced suspensions by  $\Sigma$ , and homotopy classes by  $[ \ ]$ .  $S^n$  is the  $n$ -sphere, and  $D^n$  the  $n$ -disk, with center 0.

An  $n$ -knot is a smooth submanifold  $K$  of  $S^{n+2}$  diffeomorphic to  $S^n$ . Two  $n$ -knots  $K$  and  $K'$  are *equivalent* ( $K \cong K'$ ) if there is a diffeomorphism of  $S^{n+2}$  taking  $K$  to  $K'$ .

Each knot  $K$  has a tubular neighborhood  $K \times D^2$ . The *exterior* of  $K$  is  $X(K) = S^{n+2} - K \times \text{int } D^2$ . It is a compact  $(n+2)$ -manifold, whose boundary is diffeomorphic to  $S^n \times S^1$ , and whose interior is diffeomorphic to the knot complement  $S^{n+2} - K$ . Equivalent knots have diffeomorphic complements, and thus, by uniqueness of tubular neighborhoods, diffeomorphic exteriors.

For  $n \geq 2$ , let the *Gluck twist*  $\tau_{n+1} : S^n \times S^1 \rightarrow S^n \times S^1$  be the involution given by  $\tau_{n+1}(x, t) = (\rho_{n+1}(t)(x), t)$ , where  $\rho_{n+1} : S^1 \rightarrow SO(n+1)$  is a smooth essential map. Consider the manifold  $\Sigma^{n+2} = X(K) \cup_{\tau_{n+1}} S^n \times D^2$ . It is easily seen to be a homotopy  $(n+2)$ -sphere. Thus  $\Sigma^{n+2}$  is homeomorphic to  $S^{n+2}$ . For  $n > 2$ , we may assume it is in fact diffeomorphic to  $S^{n+2}$ , by changing the smooth structure at a point if necessary. For  $n = 2$ , all the knots  $K$  we shall consider will have the property that  $\Sigma^4$  is diffeomorphic to  $S^4$ . The image of  $S^n \times \{0\}$  in  $S^{n+2}$  is a knot  $K^*$ , called the *Gluck reconstruction* of  $K$ .

By construction, the knot  $K^*$  has the same exterior as  $K$ . Gluck [9], Browder [4], Lashof and Shaneson [17], and Kato [15] showed that if  $K_0$  is another knot with  $X(K_0) \cong X(K)$ , then  $K_0$  is equivalent to  $K$  or  $K^*$ . Furthermore,  $K$  is equivalent to  $K^*$  if, and only if, there is a diffeomorphism of  $X(K)$  which restricts to  $\nu\tau_{n+1}$  on  $\partial X(K) = S^n \times S^1$ , where  $\nu$  belongs to the group generated by orientation reversals of the factors. In this case we say the knot  $K$  is *reflexive*.

An  $n$ -knot  $K$  is *fibred* if there is a smooth fibration  $\pi : X(K) \rightarrow S^1$  restricting on the boundary to  $pr_2 : S^n \times S^1 \rightarrow S^1$ . The inverse image of a point is a Seifert surface  $F^{n+1}$  for  $K$  called the *fiber*. The bundle is determined by the isotopy class of the *monodromy*, which is a diffeomorphism  $\theta$  of the fiber that restricts to the identity on the boundary  $S^n$ . For  $n > 1$ , the fiber depends on the choice of fibration; it is well-defined up to an  $s$ -cobordism. The *closed fiber* is the closed, smooth  $(n+1)$ -manifold  $F^c = F^{n+1} \cup D^{n+1}$ ; the *closed monodromy* is  $\theta^c = \theta \cup id$ . The closed fiber depends on the choice of boundary identification; it is well-defined up to connected sum with an exotic sphere.

A well-known way of creating fibred knots is by twist-spinning. If  $K$  is a knot in  $S^{n+2}$ , then the  $r$ -twist-spin of  $K$ ,  $K^{(r)}$ , is a fibred knot in  $S^{n+3}$ , with fiber the

punctured  $r$ -fold cyclic branched cover of  $(S^{n+2}, K)$  and monodromy the canonical branched covering transformation [27]. The Gluck reconstruction of  $K^{(r)}$  is a knot in a smooth  $S^{n+3}$  [10].

We conclude this section with a proposition about the equalizers of degree one maps from closed-up Seifert surfaces. For that, we need the following result of Jeff Smith, communicated to us by the referee.

**LEMMA 2.1.** *Let  $F$  be a Seifert surface for an  $n$ -knot, and  $i : S^n \rightarrow F$  be the inclusion of the boundary. Then  $\Sigma i$  is nullhomotopic.*

*Proof.* Let  $j : F \rightarrow F^c$  be the inclusion into the closed-up Seifert surface. We then have a cofiber sequence

$$S^n \xrightarrow{i} F \xrightarrow{j} F^c \xrightarrow{k} S^{n+1} \xrightarrow{\Sigma i} \Sigma F \xrightarrow{\Sigma j} \Sigma F^c \xrightarrow{\Sigma k} S^{n+2}$$

(see [25, p. 27]). The relative Pontrjagin–Thom collapse  $S^{n+2} \rightarrow \Sigma(F/\partial F) \simeq \Sigma F^c$  provides a section to  $\Sigma k$ . Thus  $\Sigma F^c \simeq \Sigma F \vee S^{n+2}$ , and we get a retract  $\Sigma F^c \rightarrow \Sigma F$  of  $\Sigma j$ . As  $\Sigma j \circ \Sigma i$  is nullhomotopic, it follows that  $\Sigma i$  is nullhomotopic.  $\square$

**PROPOSITION 2.2.** *Let  $F$  be a Seifert surface for an  $n$ -knot, and  $q : F^c \rightarrow S^{n+1}$  be a degree 1 map. Suppose  $f, g : S^{n+1} \rightarrow Z$  are two maps such that  $f \circ q \simeq g \circ q$ . Then  $f \simeq g$ .*

*Proof.* Since  $q$  has degree 1, it is homotopic to  $k$ , the cofiber of  $j$ . In a general cofiber sequence  $A \rightarrow B \xrightarrow{j} C \rightarrow \Sigma A \rightarrow \cdots$ , the group  $[\Sigma A, Z]$  acts transitively on the fibers of the function  $\gamma^* : [C, Z] \rightarrow [B, Z]$  (see [25, Proposition 2.48]). In our case, since  $\Sigma i \simeq *$ , the action of  $[\Sigma F, Z]$  on the fibers of  $q^*$  is trivial, and so  $q^*$  is injective.  $\square$

The proposition also holds for degree one maps  $q : \Sigma^m \rightarrow S^m$ , where  $\Sigma^m$  is an arbitrary homology  $m$ -sphere. For then  $q$  is an acyclic map, and we can quote Hausmann and Husemoller [12, Theorem 2.6]. In fact, the above proof closely follows theirs.

### 3. Framed manifolds

In this section we review some standard facts about framed manifolds and the Pontrjagin–Thom construction. More details can be found in [16], [3], [25].

Let  $M^k$  be a closed, smooth submanifold of  $S^{n+k}$ . A *framing*  $\varphi$  on  $M^k$  consists of a set of unit vectors  $\varphi_1(x), \dots, \varphi_n(x)$  varying smoothly with  $x \in M^k$  and

providing a basis for the normal space of  $M^k$  in  $S^{n+k}$  at  $x$ . Corresponding to the framing  $\varphi$  there is a uniquely defined trivialization  $M^k \times D^n$  of the unit normal bundle of  $M^k$  in  $S^{n+k}$ . The Pontrjagin–Thom construction yields a smooth map  $p(M, \varphi) : S^{n+k} \rightarrow S^n$ , sending  $S^{n+k} - M^k \times D^n$  to the lower hemisphere  $D_-^n$  and  $M^k \times D^n$  to the upper hemisphere  $D_+^n$ . The homotopy class of this map depends only on the framed bordism class of  $(M, \varphi)$ . The assignment  $(M, \varphi) \mapsto [p(M, \varphi)]$  establishes an isomorphism between the group of framed bordism classes of framed  $k$ -submanifolds of  $S^{n+k}$  and the homotopy group  $\pi_{n+k}(S^n)$ .

Given a fixed framing  $\varphi$  of  $M^k \subset S^{n+k}$ , another framing  $\psi$  determines a smooth map  $\hat{\psi} : M^k \rightarrow SO(n)$ . The trivialization  $M^k \times D^n$  corresponding to  $\psi$  depends up to isotopy only on the homotopy class  $[\hat{\psi}] \in [M^k, SO(n)]$ .

In the case where  $M^k = S^k$ , standardly embedded in  $S^{n+k}$ , there is a canonical choice of framing: the trivial framing  $1 = (e_{k+1}, \dots, e_{n+k})$ , where  $e_i$  is the  $i$ -th basis vector of  $\mathbb{R}^{n+k}$ . The framings of  $S^k$  then correspond to smooth maps  $\varphi : S^k \rightarrow SO(n)$ , and the isotopy classes of trivializations of the normal bundle to homotopy classes  $[\varphi] \in \pi_k(SO(n))$ . Moreover,  $[p(S^k, \varphi)] = J[\varphi]$ , where  $J : \pi_k(SO(n)) \rightarrow \pi_{n+k}(S^n)$  is the Hopf–Whitehead homomorphism.

The Freudenthal suspension homomorphism

$$E : \pi_{n+k}(S^n) \rightarrow \pi_{n+k+1}(S^{n+1}),$$

given by  $E[f] = [\Sigma f]$ , has the following geometric interpretation. Let  $\alpha \in \pi_{n+k}(S^n)$  be represented by a manifold  $M^k$  in  $S^{n+k}$  with framing  $\varphi = (\varphi_1, \dots, \varphi_n)$ . Then  $E\alpha$  is represented by the manifold  $M^k$  embedded in  $S^{n+k+1}$  with framing  $\varphi \oplus 1 = (\varphi_1, \dots, \varphi_n, e_{n+k+1})$ . In fact,  $\Sigma p(M, \varphi) = p(M, \varphi \oplus 1)$ .

Given  $\beta \in \pi_{n+k+l}(S^{n+k})$ , the composition map

$$\circ \beta : \pi_{n+k}(S^n) \rightarrow \pi_{n+k+l}(S^n)$$

can be interpreted as follows. Let  $\beta$  be represented by a manifold  $N^l$  in  $S^{n+k+l}$  with framing  $\psi$ , and let  $N^l \times D^{n+k}$  be the corresponding trivialization of the normal bundle. Let  $\alpha \in \pi_{n+k}(S^n)$  be represented by a manifold  $M^k \subset D^{n+k} \subset S^{n+k}$  with framing  $\varphi$  and trivialization  $M^k \times D^n$ . We get an embedding  $N^l \times M^k \times D^n \subset N^l \times D^{n+k} \subset S^{n+k+l}$ . The manifold  $N^l \times M^k$  with the respective framing  $\psi * \varphi$  represents  $\alpha \circ \beta \in \pi_{n+k+l}(S^n)$ . In fact,  $p(M, \varphi) \circ p(N, \psi) = p(N \times M, \psi * \varphi)$ .

#### 4. Frame-spun knots

We now describe the process, due to Roseman [23], of spinning an  $n$ -knot  $K$  about a framed submanifold  $(M^k, \varphi)$  of  $S^{n+k}$ . The resulting  $(n+k)$ -knot  $\sigma_M^{\varphi}(K)$  will be called the  $(M, \varphi)$ -spin of  $K$ .

Let  $M^k \times D^n$  be the trivialization of the unit normal bundle of  $M^k$  corresponding to  $\varphi$ . Let  $(D_-^{n+2}, D_-^n)$  be a standard disk pair embedded in  $(S^{n+2}, K)$ . Set  $(D_+^{n+2}, D_+^n) = (S^{n+2}, K) - (D_-^{n+2}, D_-^n)$ . Consider the unknot  $S^{n+k} = S^{n+k} \times \{0\} \subset S^{n+k+2} = S^{n+k} \times D^2 \cup D^{n+k+1} \times S^1$ . The knot  $\sigma_M^\varphi(K)$  consists of the  $(n+k)$ -sphere

$$(S^{n+k} - M^k \times \text{int } D^n) \cup_{M^k \times S^{n-1}} M^k \times D_+^n$$

embedded in the  $(n+k+2)$ -sphere

$$(S^{n+k+2} - M^k \times \text{int } (D^n \times D^2)) \cup_{M^k \times S^{n+1}} M^k \times D_+^{n+2}.$$

In other words, at each point of  $M^k \subset (S^{n+k+2}, S^{n+k})$ , we remove a transverse disk pair  $(D^n \times D^2, D^n)$  and glue back the knotted disk pair  $(D_+^{n+2}, D_+^n)$  to get  $\sigma_M^\varphi(K)$ . See Figure 1.

The disk  $D_+^n$  has exterior  $D_+^{n+2} - D_+^n \times \text{int } D^2$  diffeomorphic to  $X(K)$ , with boundary  $(D_+^n \cup D_-^n) \times S^1 \cong K \times S^1$ . Therefore, the exterior of the  $(M, \varphi)$ -spin of  $K$  is

$$X(\sigma_M^\varphi(K)) = (D^{n+k+1} - M^k \times \text{int } B^{n+1}) \times S^1 \cup_{M^k \times D_-^n \times S^1} M^k \times X(K),$$

where  $B^{n+1}$  is a standard disk in  $D^n \times D^2$  with boundary  $D^n \cup D_-^n$ .

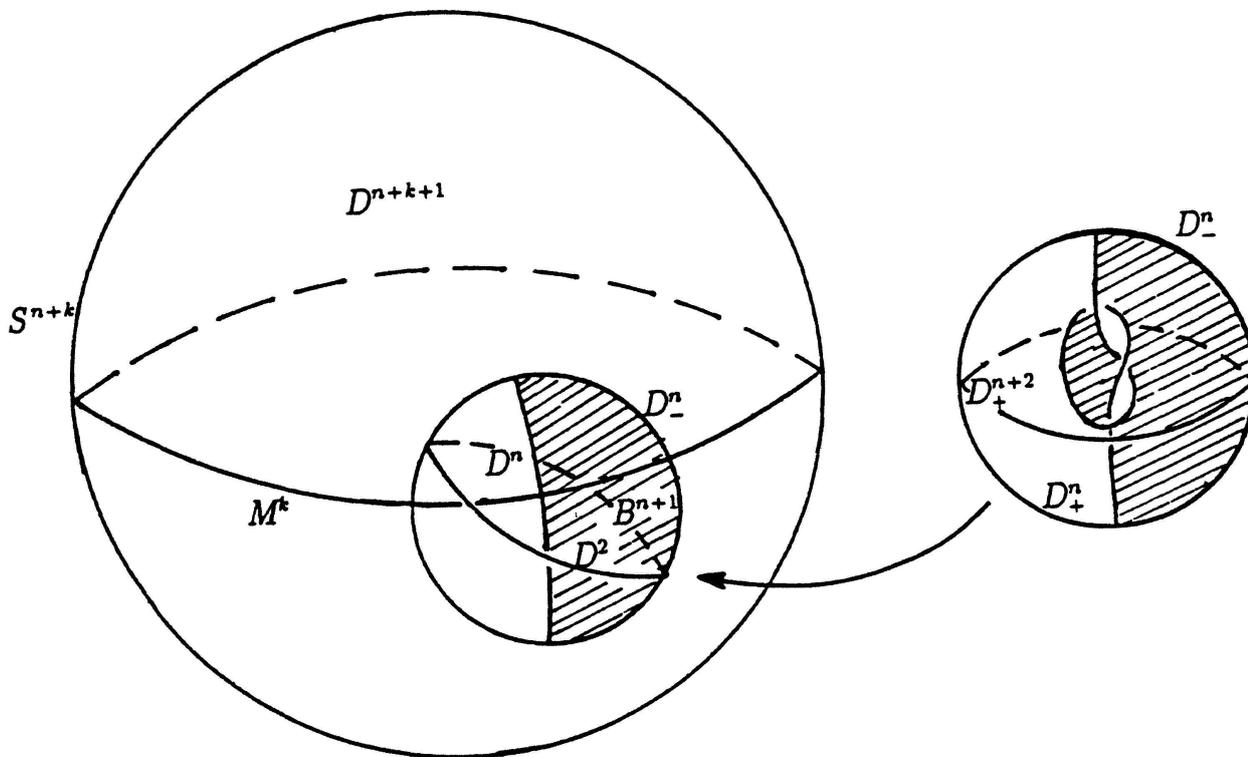


Figure 1

Some remarks on the construction are in order. First, notice that the manifold  $D^{n+k+1} - M^k \times \text{int } B^{n+1}$  is contractible. Therefore, by Van Kampen's theorem,  $\pi_1(X(\sigma_M^\varphi(K))) \cong \pi_1(X(K))$ . This means the group of the  $(M, \varphi)$ -spin of  $K$  is determined by that of  $K$  alone; it does not depend on the framed manifold  $(M, \varphi)$ . Also, the homotopy type of  $X(\sigma_M^\varphi(K))$  depends only on the homotopy type of  $X(K)$  and that of  $M^k$ ; it does not depend on the framing. In other words, for any two framings  $\varphi, \psi$  of  $M^k$ ,  $X(\sigma_M^\varphi(K)) \simeq X(\sigma_M^\psi(K))$ ; but, as we shall see in §5, there may be no homotopy equivalence preserving the boundaries.

Second, it should be noted that the  $(M, \varphi)$ -spin of  $K$  depends only on the isotopy class of the trivialization  $M^k \times D^n$  associated to the framing  $\varphi$ . If  $\psi$  is another framing of  $M^k$ , let  $\hat{\psi} : M^k \rightarrow SO(n)$  be the map it determines by comparison to  $\varphi$ . The exterior of  $\sigma_M^\psi(K)$  is obtained from that of  $\sigma_M^\varphi(K)$  by splitting along  $M^k \times D^n \times S^1$  and gluing back by the map  $(x, y, t) \mapsto (x, \hat{\psi}(x)(y), t)$ . Thus, if  $[\hat{\psi}] = 0$ , then  $\sigma_M^\psi(K)$  is equivalent to  $\sigma_M^\varphi(K)$ .

Finally, let us record the fact that in general a frame-spun knot depends on the given framed manifold, not just on the framed bordism class of that manifold. Indeed, if  $M_g^2$  is the surface of genus  $g$ , standardly embedded in  $S^3$ , and  $K$  is a non-trivial fibered classical knot, then  $M_g^2$  is framed null-bordant, yet  $\sigma_{M_g}(K) \not\cong \sigma_{M_h}(K)$  for  $g \neq h$ . In fact, the two frame-spun knots are fibered, with the fibers having non-isomorphic second homology groups (see [23] for the case  $K = \text{trefoil knot}$ , and [24] for the case  $g = 0, h = 1$ ).

The effect of iterated frame-spinning can be described as follows. Let  $(N^l, \psi)$  be a framed submanifold of  $S^{n+k+l}$ , with normal bundle  $N^l \times D^{n+k}$ . Consider the  $(N^l, \psi)$ -spin of the  $(M^k, \varphi)$ -spin of the knot  $K$ . It consists of the  $(n+k+l)$ -sphere

$$\begin{aligned} & (S^{n+k+l} - N^l \times \text{int } D^{n+k}) \\ & \cup_{N^l \times S^{n+k-1}} N^l \times [(D^{n+k} - M^k \times \text{int } D^n) \cup_{M^k \times S^{n-1}} M^k \times D_+^n] \\ & \cong (S^{n+k+l} - N^l \times M^k \times \text{int } D^n) \cup_{N^l \times M^k \times S^{n-1}} N^l \times M^k \times D_+^n, \end{aligned}$$

embedded in the  $(n+k+l+2)$ -sphere

$$\begin{aligned} & (S^{n+k+l+2} - N^l \times \text{int } (D^{n+k} \times D^2)) \cup_{N^l \times S^{n+k+1}} N^l \\ & \times [(D^{n+k+2} - M^k \times \text{int } (D^n \times D^2)) \cup_{M^k \times S^{n+1}} M^k \times D_+^{n+2}] \\ & \cong (S^{n+k+l+2} - N^l \times M^k \times \text{int } (D^n \times D^2)) \cup_{N^l \times M^k \times S^{n+1}} N^l \times M^k \times D_+^{n+2}. \end{aligned}$$

The framing of  $N^l \times M^k \subset S^{n+k+l}$  corresponding to the trivialization  $N^l \times M^k \times D^n$  obtained above is the product framing  $\psi * \varphi$ . Thus the resulting knot is the  $(N^l \times M^k, \psi * \varphi)$ -spin of  $K$ . We have proved

**PROPOSITION 4.1.** *The iterated frame-spun knot  $\sigma_N^\psi(\sigma_M^\varphi(K))$  is equivalent to  $\sigma_{N \times M}^{\psi * \varphi}(K)$ .  $\square$*

As mentioned in the introduction, we are primarily interested in the following question about frame-spun knots: Given a knot  $K$  and a framed manifold  $(M, \varphi)$ , is the knot  $\sigma_M^\varphi(K)$  determined by its complement? We conclude this section with two situations – one involving  $K$ , the other  $(M, \varphi)$  – where the answer is affirmative. We will come back to this question in §6 with a situation where the answer is negative.

Consider the case  $M^k = S^k$ , standardly embedded in  $S^{n+k}$ , with framing given by a smooth map  $\varphi : S^k \rightarrow SO(n)$ . The resulting frame-spun knots,  $\sigma_k^\varphi(K) = \sigma_{S^k}^\varphi(K)$ , first appeared in Hsiang and Sanderson [14]. When  $\varphi(x) = id$ , i.e. the framing is trivial, we get the superspin, or  $k$ -spin,  $\sigma_k(K)$ , of Cappell [5]. The exterior of the  $(k, \varphi)$ -spin of  $K$  is

$$X(\sigma_k^\varphi(K)) = D^{k+1} \times D_-^n \times S^1 \cup_{S^k \times D_-^n \times S^1} S^k \times X(K),$$

with gluing map  $(x, y, t) \mapsto (x, \varphi(x)(y), t)$ .

Now let  $K$  be an  $n$ -knot,  $n \geq 2$ . The following result establishes the relationship between  $(k, \varphi)$ -spinning and Gluck reconstruction.

**THEOREM 4.2.** *The knot  $\sigma_k^\varphi(K^*)$  is equivalent to  $\sigma_k^\varphi(K)^*$ . Thus, if  $K$  is reflexive,  $\sigma_k^\varphi(K)$  is also reflexive.*

*Proof.* Recall  $K^*$  is a knot in  $S^{n+2}$ , with exterior  $X(K)$ ; the ambient sphere is obtained by attaching  $S^n \times D^2$  to  $X(K)$  by the Gluck twist  $\tau_{n+1}(y, t) = (\rho_{n+1}(t)(y), t)$ . The  $(k, \varphi)$ -spin of  $K^*$  has exterior

$$X(\sigma_k^\varphi(K^*)) = D^{k+1} \times D_-^n \times S^1 \cup_{S^k \times D_-^n \times S^1} S^k \times X(K),$$

with gluing map  $(x, y, t) \mapsto (x, \varphi(x)(\rho_n(t)(y)), t)$ . There is a diffeomorphism  $X(\sigma_k^\varphi(K^*)) \rightarrow X(\sigma_k^\varphi(K))$  given by  $id \times \tau_n \cup id$ .

The ambient sphere  $S^{n+k+2}$  of  $\sigma_k^\varphi(K^*)$  is obtained by attaching  $S^{n+k} \times D^2$  to  $X(\sigma_k^\varphi(K))$  along  $D^{k+1} \times S^{n-1} \times S^1 \cup S^k \times D_+^n \times S^1 \cong S^{n+k} \times S^1$  by the map  $id \cup id \times \tau_n = \tau_{n+k+1}$ . It follows that  $\sigma_k^\varphi(K^*) \cong \sigma_k^\varphi(K)^*$ .  $\square$

A frame-spin of  $K$  may be reflexive even though  $K$  is not. Indeed, Gluck [9] showed that 1-spun knots are always reflexive. This was generalized to  $k$ -spun knots by Cappell [5]. The following theorem, based on Cappell's method, extends their results to certain  $(k, \varphi)$ -spun knots. First, some notation:  $\eta_2 = J[\rho_2]$  is the generator

of  $\pi_3(S^2) \cong \mathbb{Z}$  given by the Hopf map, and, for  $k > 2$ ,  $\eta_k = E^{k-2}\eta_2$  is the generator of  $\pi_{k+1}(S^k) = \mathbb{Z}_2$ . To keep things compact, we shall let  $\eta_1$  stand for  $\iota_1$ , the usual generator of  $\pi_1(S^1)$ .

**THEOREM 4.3.** *Let  $K$  be an  $n$ -knot and  $\varphi : S^k \rightarrow SO(n)$  a smooth map. If  $[\varphi] \circ \eta_k = 0$ , then  $\sigma_k^\varphi(K)$  is reflexive.*

*Proof.* Define a smooth map  $f_0 : D^{k+1} \times D_-^n \times S^1 \rightarrow D^{k+1} \times D_-^n \times S^1$  by  $f_0(x, y, t) = (\rho_{k+1}(t)(x), y, t)$ . Let  $\gamma : X(K) \rightarrow S^1$  be a smooth map which represents a generator of  $[X(K), S^1] \cong H^1(X(K); \mathbb{Z}) \cong \mathbb{Z}$  and which restricts on the boundary to  $pr_2 : S^n \times S^1 \rightarrow S^1$ . Then define a smooth map  $f_1 : S^k \times X(K) \rightarrow S^k \times X(K)$  by  $f_1(x, y) = (\rho_{k+1}(\gamma(y))(x), y)$ .

On  $S^k \times D_-^n \times S^1$ , both  $f_0$  and  $f_1$  restrict to  $(x, y, t) \mapsto (\rho_{k+1}(t)(x), y, t)$ . In order for this map to be compatible with the gluing determined by  $\varphi$  we must have

$$\varphi(\rho_{k+1}(t)(x)) = \varphi(x), \quad \text{for } x \in S^k, t \in S^1. \quad (*)$$

Let  $\tau_{k+1} : S^k \times S^1 \rightarrow S^k \times S^1$  be the Gluck twist and  $pr_1 : S^k \times S^1 \rightarrow S^k$  the projection map. Then (\*) is equivalent to  $\varphi \circ pr_1 \circ \tau_{k+1} = \varphi \circ pr_1$ .

If  $k = 1$ , then  $[\varphi] = 0$ , and we may assume, by homotoping  $\varphi$  if necessary, that (\*) holds. Otherwise, the only obstruction to a homotopy  $\varphi \circ pr_1 \circ \tau_{k+1} \simeq \varphi \circ pr_1$  is the class of the difference cocycle  $d(\varphi \circ pr_1 \circ \tau_{k+1}, \varphi \circ pr_1) \in H^{k+1}(S^k \times S^1; \pi_{k+1}(SO(n))) \cong \pi_{k+1}(SO(n))$ . By naturality, the obstruction equals  $[\varphi] \circ d(pr_1 \circ \tau_{k+1}, pr_1)$ . Since  $d(pr_1 \circ \tau_{k+1}, pr_1) = \eta_k$  (see [9]), the obstruction vanishes, and again we may assume that (\*) holds.

This permits us to glue the maps  $f_0$  and  $f_1$  to get a smooth map  $f : X(\sigma_k^\varphi(K)) \rightarrow X(\sigma_k^\varphi(K))$ . On the boundary  $S^{n+k} \times S^1$  the map  $f$  restricts to  $\tau_{n+k+1}$ . Thus  $\sigma_k^\varphi(K)$  is reflexive.  $\square$

As suggested in §1, the above theorems should generalize to arbitrary frame-spun knots. Namely, one should prove:

- (i)  $\sigma_M^\varphi(K^*) \cong \sigma_M^\varphi(K)^*$ .
- (ii) If  $[p(M, \varphi)] \circ \eta_{n+k} = 0$ , then  $\sigma_M^\varphi(K)$  is reflexive.

The difficulty one runs into is finding appropriate ‘‘Gluck twists’’ over  $(D^{n+k+1} - M^k \times \text{int } B^{n+1}) \times S^1$ .

## 5. Frame-spun fibers

In this section we introduce the notion of frame-spinning a closed manifold and use it to study the closed fiber of a frame-spun knot.

Let  $W^m$  be a closed, smooth  $m$ -manifold,  $m \geq 1$ . Let  $B^m$  be a fixed embedded disk in  $W^m$  and let  $W_0^m = W^m - \text{int } B^m$ . Let  $(M^k, \varphi)$  be a framed submanifold of  $S^{m+k}$ , with unit normal bundle  $M^k \times D^m$ . The  $(M, \varphi)$ -spin of  $W^m$  is the closed, smooth  $(m+k)$ -manifold

$$\sigma_M^\varphi(W^m) = (S^{m+k} - M^k \times \text{int } D^m) \cup_{M^k \times S^{m-1}} M^k \times W_0^m. \quad (\dagger)$$

That is to say, at each point of  $M^k \subset S^{m+k}$ , we remove a transverse disk  $D^m$  and glue back the punctured manifold  $W_0^m$ . Notice the frame-spin of  $S^m$  is just  $S^{m+k}$ .

If  $M^k = S^k$ , with framing  $\varphi : S^k \rightarrow SO(n)$ , the resulting frame-spun manifold is  $\sigma_k^\varphi(W^m) = D^{k+1} \times S^{m-1} \cup_{S^k \times S^{m-1}} S^k \times W_0^m$ , with gluing map  $(x, y) \mapsto (x, \varphi(x)(y))$ . In case the framing is trivial, we get the  $k$ -spin,  $\sigma_k(W^m)$ , of Cappell [5]. In case  $k=1, m \geq 3$ , there are two possible  $S^1$ -spins,  $\sigma_1(W^m)$  and  $\sigma'_1(W^m)$ , corresponding to the framings 1 and  $\rho_m$  (Plotnick [20]). The two pieces of  $\sigma'_1(W^m)$  get glued along  $S^1 \times S^{m-1}$  by the Gluck twist. Thus, if the Gluck twist extends to a diffeomorphism of  $S^1 \times W_0^m$  (for example, if  $W^m$  admits a smooth  $S^1$ -action with codimension 2 fixed-point set), then  $\sigma_1(W^m)$  is diffeomorphic to  $\sigma'_1(W^m)$ .

Frame-spinning behaves nicely with respect to fundamental groups. If  $m \geq 3$ , then  $\pi_1(S^{m+k} - M^k \times \text{int } D^m) = 0$ ,  $\pi_1(W_0^m) \cong \pi_1(W^m)$ , and so, by Van Kampen's theorem,  $\pi_1(\sigma_M^\varphi(W^m)) \cong \pi_1(W^m)$ .

The Pontrjagin–Thom construction can be extended to frame-spun manifolds. Indeed, the decomposition  $(\dagger)$  yields a smooth map

$$p(W, M, \varphi) : \sigma_M^\varphi(W^m) \rightarrow W^m$$

that sends  $S^{m+k} - M^k \times \text{int } D^m$  to  $B^m$  and  $M^k \times W_0^m$  to  $W_0^m$ . Clearly,  $p(S^m, M, \varphi)$  is just  $p(M, \varphi)$ . Moreover,  $p(W, M, \varphi) \circ p(\sigma_M^\varphi(W), N, \psi) = p(W, N \times M, \psi * \varphi)$ .

The frame-spinning construction enjoys the following naturality properties. Let  $V^m$  be another manifold with a fixed embedded disk. Let  $f : W^m \rightarrow V^m$  be a degree 1 smooth map preserving the chosen disks. Define the  $(M, \varphi)$ -spin of  $f$  to be the (degree 1) smooth map

$$\sigma_M^\varphi(f) : \sigma_M^\varphi(W^m) \rightarrow \sigma_M^\varphi(V^m)$$

obtained by piecing together the maps  $id_{S^{m+k} - M^k \times \text{int } D^m}$  and  $id_{M^k \times f|_{W_0^m}}$ . Then  $f \circ p(W, M, \varphi) = p(V, M, \varphi) \circ \sigma_M^\varphi(f)$ . Moreover, if  $g : V^m \rightarrow U^m$  is another degree 1 smooth map preserving base disks, then  $\sigma_M^\varphi(g \circ f) = \sigma_M^\varphi(g) \circ \sigma_M^\varphi(f)$ .

Having defined the process of frame-spinning a knot, respectively a manifold, we now relate the two notions in case the knot we start with is fibered. Unlike twist-spinning, the process of spinning doesn't create essentially new fibrations. But it does the next best thing. As recognized by Andrews and Sumners [2],  $k$ -spinning

takes fibered knots to fibered knots. This was generalized to frame-spinning by Roseman [23, Lemma 1]. Let us identify the frame-spun fiber and monodromy in our terminology.

Let  $K$  be a fibered  $n$ -knot, with fibration of the exterior  $\pi : X(K) \rightarrow S^1$ , fiber  $F$ , and monodromy  $\theta$ . Denote by  $F^c$  the closed fiber of  $K$  (so that  $F_0^c = F$ ). Let  $M^k$  be a submanifold  $S^{n+k}$  with framing  $\varphi$ . The exterior of the  $(M, \varphi)$ -spin of  $K$  admits a corresponding frame-spun fibration  $pr_2 \circ \pi \circ pr_2 : (D^{n+k+1} - M^k \times \text{int } B^{n+1}) \times S^1 \cup_{M^k \times D_-^n \times S^1} M^k \times X(K) \rightarrow S^1$ . Its fiber is

$$F(\sigma_M^\varphi(K)) = (D^{n+k+1} - M^k \times \text{int } B^{n+1}) \cup_{M^k \times D_-^n} M^k \times F,$$

and its monodromy is  $id \cup id \times \theta$ . The closed fiber of  $\sigma_M^\varphi(K)$  is

$$\begin{aligned} F^c(\sigma_M^\varphi(K)) &= F(\sigma_M^\varphi(K)) \cup_{S^{n+k}} D_1^{n+k+1} \\ &\cong [(D^{n+k+1} - M^k \times \text{int } B^{n+1}) \\ &\quad \cup_{S^{n+k} - M^k \times \text{int } D^n} (D_1^{n+k+1} - M^k \times \text{int } B_1^{n+1})] \cup_{M^k \times S^n} M^k \times F \\ &= (S^{n+k+1} - M^k \times \text{int } D^{n+1}) \cup_{M^k \times S^n} M^k \times F, \end{aligned}$$

where  $D^{n+1} = B^{n+1} \cup_{D^n} B_1^{n+1}$  (see Figure 2). The trivialization  $M^k \times D^{n+1}$  of the normal bundle of  $M^k$  in  $S^{n+k+1}$  corresponds to the stabilized framing  $\varphi \oplus 1$ . We thus have proved

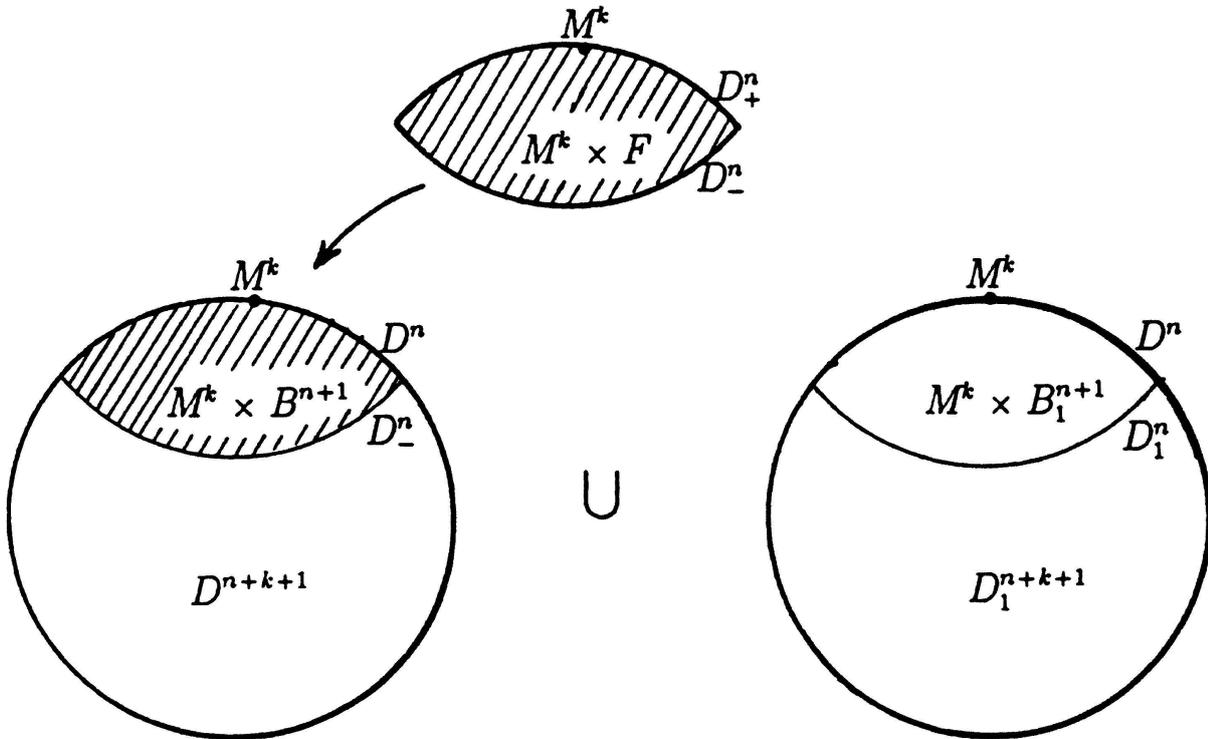


Figure 2

**PROPOSITION 5.1.** *If  $K$  is a fibered knot, then  $\sigma_M^\varphi(K)$  is also fibered, with closed fiber  $\sigma_M^{\varphi \oplus 1}(F^c)$  and closed monodromy  $\sigma_M^{\varphi \oplus 1}(\theta^c)$ .  $\square$*

We now address the following question: Given a closed  $m$ -manifold,  $W^m$ , and two framed  $k$ -submanifolds of  $S^{m+k}$ ,  $(M^k, \varphi)$  and  $(N^k, \psi)$ , are the corresponding frame-spins,  $\sigma_M^\varphi(W^m)$  and  $\sigma_N^\psi(W^m)$ , homotopy equivalent? As the case  $W^m = S^m$  illustrates, the answer may be yes. But in general, the expected answer is no. If  $M^k \not\cong N^k$ , one can often distinguish between the two frame-spins by means of their homology or the homology of their universal covers. For example, if  $W^3 \not\cong S^3$ , then  $\sigma_{S^k}(W^3) \not\cong \sigma_{S^l \times S^{k-l}}(W^3)$  (see [24] for a proof and generalizations). If  $M^k \cong N^k$ , the difference between the two frame-spins of  $W^m$  is more subtle. The next theorem shows that we still may tell them apart, provided  $W^m$  is the closed fiber of a knot, its universal cover is contractible, and  $(M^k, \varphi \oplus 1)$  is not framed cobordant to  $(N^k, \psi \oplus 1)$ .

**THEOREM 5.2.** *Let  $K$  be a fibered  $n$ -knot,  $n \geq 2$ , with aspherical closed fiber. Let  $(M^k, \varphi)$  and  $(N^k, \psi)$  be two framed  $k$ -submanifolds of  $S^{n+k}$  such that  $E[p(M, \varphi)] \neq E[p(N, \psi)]$ . Then  $F^c(\sigma_M^\varphi(K)) \not\cong F^c(\sigma_N^\psi(K))$ .*

*Proof.* Suppose  $F^c(\sigma_M^\varphi(K)) \cong F^c(\sigma_N^\psi(K))$ . Let  $F^c$  be the closed fiber of  $K$ . By Proposition 5.1, there is a homotopy equivalence  $g : \sigma_M^{\varphi \oplus 1}(F^c) \rightarrow \sigma_N^{\psi \oplus 1}(F^c)$ . Let  $g_*$  be the induced automorphism on  $\pi = \pi_1(F^c)$ . Since  $F^c$  is a  $K(\pi, 1)$ ,  $g_*$  extends to a homotopy equivalence  $h : F^c \rightarrow F^c$ . Moreover,  $h \circ p(F^c, M, \varphi \oplus 1) \cong p(F^c, N, \psi \oplus 1) \circ g$ , again by asphericity of  $F^c$ .

Now let  $q : F^c \rightarrow S^{n+1}$  be the map sending  $F^{n+1}$  to  $D_+^{n+1}$  and  $B^{n+1}$  to  $D_-^{n+1}$ . Changing the orientation of  $S^{n+1}$  if necessary, we see that  $q$  has degree 1. Hence there is a homotopy equivalence  $\bar{h} : S^{n+1} \rightarrow S^{n+1}$  such that  $\bar{h} \circ q = q \circ h$ . The maps  $\sigma_M^{\varphi \oplus 1}(q)$  and  $\sigma_N^{\psi \oplus 1}(q)$  also have degree 1, so there is a homotopy equivalence  $\bar{g} : S^{n+k+1} \rightarrow S^{n+k+1}$  such that  $\bar{g} \circ \sigma_M^{\varphi \oplus 1}(q) = \sigma_N^{\psi \oplus 1}(q) \circ g$ .

We thus have the diagram

$$\begin{array}{ccccc}
 & & \sigma_N^{\psi \oplus 1}(F^c) & \xrightarrow{p(F^c, N, \psi \oplus 1)} & F^c \\
 & \nearrow g & \downarrow \sigma_N^{\psi \oplus 1}(q) & & \nearrow h \\
 & & S^{n+k+1} & \xrightarrow{p(N, \psi \oplus 1)} & S^{n+1} \\
 \sigma_M^{\varphi \oplus 1}(F^c) & \xrightarrow{p(F^c, M, \varphi \oplus 1)} & F^c & & \downarrow q \\
 \downarrow \sigma_M^{\varphi \oplus 1}(q) & \nearrow \bar{g} & \downarrow q & \nearrow \bar{h} & \\
 S^{n+k+1} & \xrightarrow{p(M, \varphi \oplus 1)} & S^{n+1} & & 
 \end{array}$$

with the top and side squares commuting up to homotopy. Hence  $p(M, \varphi \oplus 1) \circ \sigma_M^{\varphi \oplus 1}(q) \simeq p(N, \psi \oplus 1) \circ \sigma_M^{\varphi \oplus 1}(q)$ . Since  $\sigma_M^{\varphi \oplus 1}(F^c)$  is the closed fiber of  $\sigma_M^{\varphi}(K)$ , and  $\sigma_M^{\varphi \oplus 1}(q)$  has degree 1, Proposition 2.2 implies  $p(M, \varphi \oplus 1) \simeq p(N, \psi \oplus 1)$ . This is a contradiction, and we are done.  $\square$

**REMARK.** The knot exteriors  $X(\sigma_M^{\varphi}(K))$  and  $X(\sigma_N^{\psi}(K))$  are not homotopy equivalent (rel. boundary). This follows from the preceding theorem by a standard argument: Suppose there is a homotopy equivalence (rel.  $\partial$ ) of the knot exteriors. It lifts to a homotopy equivalence (rel.  $\partial$ ) of the infinite cyclic covers  $F(\sigma_M^{\varphi}(K)) \times \mathbb{R} \simeq F(\sigma_N^{\psi}(K)) \times \mathbb{R}$ . This yields a homotopy equivalence  $F(\sigma_M^{\varphi}(K)) \rightarrow F(\sigma_N^{\psi}(K))$ , which is the identity on the boundary  $S^{n+k}$ , and thus extends to a homotopy equivalence of the closed fibers. For example, if  $K$  is a 2-knot with aspherical closed fiber (see [10], [13] for such knots), and  $\varphi : S^1 \rightarrow SO(2)$  has odd degree, then  $X(\sigma_1^{\varphi}(K)) \not\simeq X(\sigma_1(K))$  (rel.  $\partial$ ). Or, if  $K$  is a Cappell–Shaneson 3-knot with closed fiber the 4-torus [7], and  $\varphi : S^k \rightarrow SO(3)$  satisfies  $J[\varphi] \neq 0$ , then  $X(\sigma_k^{\varphi}(K)) \not\simeq X(\sigma_k(K))$  (rel.  $\partial$ ).

**COROLLARY 5.3.** *Let  $K$  be a fibered  $n$ -knot,  $n \geq 2$ , with aspherical closed fiber  $F^c$ . Then  $\sigma_1(F^c) \not\simeq \sigma'_1(F^c)$ .*  $\square$

**REMARK.** For 3-dimensional manifolds, more is true. With some additional work, we can show that given *any* aspherical  $W^3$ , the two  $S^1$ -spins of  $W^3$  are homotopically distinct. This result was first proved by Plotnick [20, Theorem 3.1], using intersection forms on universal covers. He also showed [20, Theorem 5.1] that there is no “special” homotopy equivalence between the two spins of  $W^3$ , provided not all summands of  $W^3$  are  $S^2 \times S^1$  of  $\Sigma^3/\pi$ , where  $\Sigma^3$  is a homotopy 3-sphere,  $\pi$  is a finite group acting freely on  $\Sigma^3$ , and all Sylow subgroups of  $\pi$  are cyclic. We can sharpen this last result in some cases. For example,  $\sigma_1(\Sigma^3/I^*) \not\simeq \sigma'_1(\Sigma^3/I^*)$ , where  $I^*$  is the binary icosahedral group.

## 6. Non-reflexive knots

We now return to the problem of reflexivity of knots, more specifically, of frame-spun fibered knots. Under certain assumptions on the fibering and on the framing, these knots will prove to be non-reflexive. We start with the following necessary condition for reflexivity. The idea of the proof is similar to that of [10, Proposition 4.2] and [20, Theorem 6.2].

**PROPOSITION 6.1.** *Let  $K$  be a fibered  $n$ -knot,  $n \geq 2$ , with odd order monodromy. If  $K$  is reflexive, then  $\sigma_1(F^c(K)) \cong \sigma'_1(F^c(K))$ .*

*Proof.* Let  $\theta$  be the monodromy of  $K$ , and  $r$  its order. Since  $K \cong K^*$ , there is a diffeomorphism  $f$  of the exterior  $F \times_{\theta} S^1$  which restricts on the boundary to  $v\tau_{n+1}$ , where  $\tau_{n+1}$  is the Gluck twist and  $v$  is a composite of orientation reversals of the factors of  $S^n \times S^1$ . Lift  $f$  to a diffeomorphism  $\tilde{f}$  of the  $r$ -fold cover  $F \times S^1$ . Since  $r$  is odd,  $\tilde{f}$  restricts on the boundary to  $v\tau_{n+1}$ . It is now a simple matter to extend  $\tilde{f}$  to a diffeomorphism  $\sigma_1(F^c) \rightarrow \sigma'_1(F^c)$ .  $\square$

This proposition, together with Corollary 5.3, implies

**COROLLARY 6.2.** *Let  $K$  be a fibered  $n$ -knot,  $n \geq 2$ , with aspherical closed fiber and odd order monodromy. Then  $K$  is not reflexive.*  $\square$

This result was first proved by Gordon [10] under the extra assumptions that  $K$  be a twist-spun knot and the universal cover of  $F^c(K)$  be  $\mathbb{R}^{n+1}$ . He used it to produce examples of non-reflexive 2-knots as follows. Let  $p, q, r$  be integers greater than 1, with  $p$  and  $q$  coprime,  $r$  odd, and  $1/p + 1/q + 1/r \leq 1$ . Denote by  $K_{p,q}$  the  $(p, q)$ -torus knot in  $S^3$ . The  $r$ -twist-spin  $K_{p,q}^{(r)}$  is a knot in  $S^4$  with closed fiber the aspherical Brieskorn 3-manifold  $\Sigma(p, q, r)$  and monodromy of order  $r$ . Therefore  $K_{p,q}^{(r)}$  is not reflexive. (In fact, according to Hillman and Plotnick [13], no  $r$ -twist-spin of a non-trivial prime, simple classical knot with  $r > 2$  is reflexive).

For  $n > 2$  this method doesn't work, as there are no known examples of aspherical  $(n+1)$ -manifolds that are cyclic branched covers of a knotted pair  $(S^{n+1}, S^{n-1})$ . Consequently, a stronger result is required in order to produce high-dimensional non-reflexive knots.

**THEOREM 6.3.** *Let  $K$  be a fibered  $n$ -knot,  $n \geq 2$ , with aspherical closed fiber and odd order monodromy. If  $E[p(M^k, \varphi)] \circ \eta_{n+k+1} \neq 0$ , then  $\sigma_M^{\varphi}(K)$  is not reflexive.*

*Proof.* Consider  $F^c(\sigma_M^{\varphi}(K))$ , the closed fiber of  $\sigma_M^{\varphi}(K)$ . By Propositions 5.1 and 4.1, its two  $S^1$ -spins are

$$\sigma_1(F^c(\sigma_M^{\varphi}(K))) \cong F^c(\sigma_{S^1}^{1* \varphi} M(K)) \quad \text{and} \quad \sigma'_1(F^c(\sigma_M^{\varphi}(K))) \cong F^c(\sigma_{S^1}^{\rho* \varphi} M(K)).$$

As  $[p(S^1 \times M, 1 * \varphi)] = 0$  and  $[p(S^1 \times M, \rho_{n+k} * \varphi)] = [p(M, \varphi)] \circ \eta_{n+k}$ , Theorem 5.2 implies  $\sigma_1(F^c(\sigma_M^{\varphi}(K))) \neq \sigma'_1(F^c(\sigma_M^{\varphi}(K)))$ .

Let  $\theta$  be the monodromy of  $K$ . Since  $\theta$  has odd order, the monodromy  $id \cup id \times \theta$  of  $\sigma_M^{\varphi}(K)$  also has odd order. It follows from Proposition 6.1 that  $\sigma_M^{\varphi}(K)$  is not reflexive.  $\square$

We now can prove Theorem 1.1. Let  $K$  be a fibered 2-knot with aspherical closed fiber and odd order monodromy, e.g. one of the twist-spun knots mentioned

above. To find a non-reflexive  $n$ -knot,  $n \geq 3$ , it is enough to find an element  $\alpha \in \pi_n(S^2)$  such that

$$E\alpha \circ \eta_{n+1} \neq 0.$$

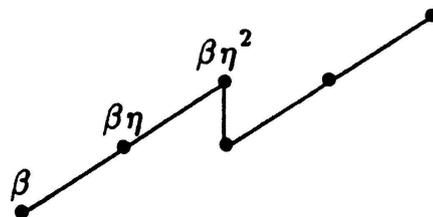
For if  $(M^{n-2}, \varphi)$  is a framed submanifold of  $S^n$  such that  $[p(M^{n-2}, \varphi)] = \alpha$ , then  $\sigma_M^\varphi(K)$  is a knot in  $S^{n+2}$  which, by Theorem 6.3, is not reflexive.

We will show that such elements  $\alpha$  of  $\pi_n(S^2)$  exist, provided that  $n \equiv 3$  or  $n \equiv 4 \pmod{8}$ . A search through Toda's book [26] produces the following table:

$n$	$\alpha$	$E\alpha \circ \eta_{n+1}$
3	$j\eta_2, j$ odd	$\eta_3\eta_4$
4	$\eta_2\eta_3$	$\eta_3\eta_4\eta_5 = 2\nu'$
11	$\eta_2\epsilon_3$	$\eta_3\epsilon_4\eta_{12} = 2\epsilon'$
12	$\eta_2\mu_3$	$\eta_3\mu_4\eta_{13} = 2\mu'$
19	$\eta_2\mu_3\sigma_{12}$	$\eta_3\mu_4\sigma_{13}\eta_{20} = 2\mu'\sigma_{14}$
20	$\eta_2\bar{\mu}_3$	$\eta_3\bar{\mu}_4\eta_{21} = 2\bar{\mu}'$

The classes  $\epsilon_3, \mu_3, \bar{\mu}_3$  are certain Toda brackets defined in [26],  $\sigma_8$  is the generator of  $\pi_{15}(S^8)$  given by the Hopf map, and  $\sigma_k = E^{k-8}\sigma_8$ . It is readily seen that the elements in the right-hand column are all non-zero (they have order exactly 2). This proves our claim for  $n \leq 20$ .

For higher values of  $n$ , we must appeal to deeper results in homotopy theory. Let  $\alpha = \eta_2\beta^{(n)}$ , where  $\beta^{(n)} \in \pi_n(S^3)$  is defined inductively by Adams periodicity [1]:  $\beta^{(11)} = \epsilon_3$ ,  $\beta^{(12)} = \mu_3$ , and  $\beta^{(n)}$  is the Toda bracket  $\{\beta^{(n-8)}, 2i_{n-8}, 8\sigma_{n-8}\}$ . At the level of the  $E_2$ -term of the unstable Adams spectral sequence (mod 2) for  $S^3$ , the elements  $\beta^{(n)}$  appear at the beginning of two periodic families of "lightning flashes" [19, p. 107]:



It follows from a fundamental theorem of Mahowald that  $\beta^{(n)}\eta_n\eta_{n+1}$  is essential; it is detected by the composite of the bo-Hurewicz map with the Snaith map

$\pi_{n-1}(\Omega^3 S^3) \rightarrow \pi_{n-1}(Q\mathbb{R}P^2) \rightarrow \pi_{n-1}(\Omega^\infty(\mathbb{R}P^2 \wedge \text{bo}))$  [19, Theorem 1.5]. An elementary computation using [26, Proposition 3.2] and the injectivity of  $E : \pi_{n+2}(S^3) \rightarrow \pi_{n+3}(S^4)$  shows  $\beta^{(n)}\eta_n\eta_{n+1} = \eta_3 \circ E\beta^{(n)} \circ \eta_{n+1}$ . Hence  $E\alpha \circ \eta_{n+1} \neq 0$ . This finishes the proof of Theorem 1.1.  $\square$

For each  $n \equiv 3$  or  $4 \pmod{8}$ , there exist infinitely many distinct non-reflexive  $n$ -knots. We can show this two ways. First, we may choose infinitely many triples  $(p, q, r)$  as in the paragraph following Corollary 6.2 so that the manifolds  $\Sigma(p, q, r)$  have pairwise non-isomorphic fundamental groups. Thus, if  $(M^{n-2}, \varphi)$  is as in the proof of Theorem 1.1, the  $n$ -knots  $\sigma_M^\varphi(K_{p,q}^{(r)})$  are non-reflexive and have distinct groups. Second, we may fix a triple  $(p, q, r)$ , with  $r$  not coprime to  $p$ ; then the manifold  $\Sigma(p, q, r)$  is not a homology 3-sphere. For  $n > 3$  and  $i > 0$ , let  $M_i^{n-2} = M^{n-2} \#_1^i S^1 \times S^{n-3}$ ; it is a framed submanifold of  $S^n$ , with framing  $\varphi_i$  equal to  $\varphi$  on the first factor and the trivial framing on the other factors. The knots  $\sigma_{M_i}^{\varphi_i}(K_{p,q}^{(r)})$ ,  $i = 1, 2, \dots$ , are non-reflexive, have isomorphic groups, but are pairwise non-equivalent: they can be distinguished by the homology of their fibers.

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