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Inequivalent frame-spun knots with the same complement

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1. Introduction

One of the basic questions of knot theory is: Is every n-knot determined by its complement? For $n = 1$, Gordon and Luecke [11] have recently given an affirmative answer to this question. For $n \geq 2$, there are at most two n-knots with the same complement [9], [4], [17], [15]. A knot which is determined by its complement is called reflexive. Knots that are spun [9], superspun [5], 2-twist-spun [10], [13], simple [18], stable [8], [22], and some others [21], [7], are known to be reflexive. Cappell and Shaneson [7] gave the first examples of knots which are not determined by their complements. Their method works for each $n \geq 2$, as long as certain integral, unimodular $(n+1) \times (n+1)$ matrices exist; such matrices have been found only for $n = 2$, 3, 4 and 5. Shortly thereafter, Gordon [10] proved that odd-twist-spun n-knots with closed fiber covered by $\mathbb{R}^{n+1}$ are non-reflexive. His method is known to yield examples only for $n = 2$. Other examples of 2-knots which are not determined by their complements were given in [20], [21], [13].

The main result of this paper is the following theorem.

THEOREM 1.1. *There exist non-reflexive n-knots for every $n \equiv 3$ or 4 (mod 8).*

We construct these n-knots by frame-spinning the 2-knots of Gordon. In doing so, we reprove Gordon's theorem under slightly more general conditions (Corollary 6.2), thus giving a new proof of the non-reflexivity of his 2-knots. The basic idea is to translate the question of reflexivity of the frame-spun knots into a question about homotopy groups of spheres, via a generalized Pontrjagin–Thom construction.

The process of frame-spinning was introduced by Roseman in [23]; it generalizes previous notions of spinning that go back to Artin. If K is an n-knot and M^k is a framed submanifold of S^{n+k}, with framing φ, one can spin K about M^k to get an $(n+k)$-knot $\sigma_M^{\varphi}(K)$. This is done by removing at each point of $M^k \subset (S^{n+k+2}, S^{n+k})$ the transverse disk pair determined by the framing and gluing back the knotted disk pair determined by the n-knot.

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The question we investigate in this paper is: Is a frame-spun knot determined by its complement? Quite often, the answer is yes. Suppose $M^k = S^k$, standardly embedded in S^{n+k}, with framing given by a smooth map $\varphi : S^k \to SO(n)$. For $k \geq 2$, let η_k be a generator of $\pi_{k+1}(S^k)$. Given an n-knot K, $n \geq 2$, we prove the following (Theorems 4.2 and 4.3): *If either K is reflexive, or $[\varphi] \circ \eta_k$ is zero, then $\sigma^{\varphi}_{S^k}(K)$ is reflexive.* This generalizes a result of Cappell [5].

In general though, the answer to the above question is no. For an arbitrary framed manifold $(M^k, \varphi) \subset S^{n+k}$, the Pontrjagin–Thom construction yields an element α of $\pi_{n+k}(S^n)$. Suppose K is a fibered n-knot, $n \geq 2$, with aspherical closed fiber and odd order monodromy (such knots are known to exist only for $n = 2$). We then prove the following (Theorem 6.3): *If the suspension of $\alpha \circ \eta_{n+k}$ is non-zero, then $\sigma^{\varphi}_M(K)$ is not reflexive.* For $k \equiv 1$ or $2 \pmod 8$, there are such α's in $\pi_{k+2}(S^2)$, by deep work of Mahowald [19]. This produces non-reflexive frame-spun $(k+2)$-knots by surjectivity of the Pontrjagin–Thom homomorphism.

Let us briefly sketch the proof of Theorem 6.3. In §5, we introduce the notion of spinning a closed manifold W^m about a framed manifold (M^k, φ). This is done by removing at each point of $M^k \subset S^{m+k}$ a transverse n-disk and gluing back a punctured copy of W^m. An essential feature of this construction is the existence of a "Pontrjagin–Thom" map, $\sigma^{\varphi}_M(W) \to W$, that may be used to differentiate among the various frame-spins of W. Now, as noticed by Roseman [23], the process of frame-spinning takes fibered knots to fibered knots. In our terminology, if K has closed fiber F^c, then $\sigma^{\varphi}_M(K)$ has closed fiber the stabilized frame-spin of F^c. In case F^c is aspherical, we are able to distinguish between the closed fibers of two frame-spins of K, provided the two manifolds we spin about are not stably framed bordant (Theorem 5.2). In particular, if $E(\alpha \circ \eta_{n+k}) \neq 0$, the two S^1-spins of the closed fiber of $\sigma^{\varphi}_M(K)$ are distinct. On the other hand, if K has odd order monodromy, so does $\sigma^{\varphi}_M(K)$, and therefore $\sigma^{\varphi}_M(K)$ cannot be reflexive, for otherwise the two S^1-spins of its closed fiber would be equal.

In view of the above results, we venture the following

CONJECTURE. The knot $\sigma^{\varphi}_M(K)$ is reflexive if and only if either K is reflexive, or $\alpha \circ \eta_{n+k} = 0$.

If the forward implication were true, one could produce examples of non-reflexive knots in the missing dimensions by frame-spinning the Cappel–Shaneson knots instead of Gordon's knots.

I wish to thank J. Klein and M. Mahowald for valuable conversations. An early version of Theorem 5.2 dealt only with homology spheres. I am grateful to the referee for pointing out a gap in a subsequent generalization, and for suggesting the use of Lemma 2.1 to arrive at the right level of generality.

2. Knotted spheres

We start with some definitions and notation. All manifolds are to be compact, connected, oriented, and smooth; closed manifolds are those without boundary. Diffeomorphisms are denoted by $\cong$, homotopy equivalences by $\simeq$, reduced suspensions by Σ, and homotopy classes by [ ]. S^n is the n-sphere, and D^n the n-disk, with center 0.

An *n-knot* is a smooth submanifold K of S^{n+2} diffeomorphic to S^n. Two n-knots K and K' are *equivalent* ($K \cong K'$) if there is a diffeomorphism of S^{n+2} taking K to K'.

Each knot K has a tubular neighborhood $K \times D^2$. The *exterior* of K is $X(K) = S^{n+2} - K \times \operatorname{int} D^2$. It is a compact $(n+2)$-manifold, whose boundary is diffeomorphic to $S^n \times S^1$, and whose interior is diffeomorphic to the knot complement $S^{n+2} - K$. Equivalent knots have diffeomorphic complements, and thus, by uniqueness of tubular neighborhoods, diffeomorphic exteriors.

For $n \geq 2$, let the *Gluck twist* $\tau_{n+1} : S^n \times S^1 \to S^n \times S^1$ be the involution given by $\tau_{n+1}(x, t) = (\rho_{n+1}(t)(x), t)$, where $\rho_{n+1} : S^1 \to SO(n+1)$ is a smooth essential map. Consider the manifold $\Sigma^{n+2} = X(K) \cup_{\tau_{n+1}} S^n \times D^2$. It is easily seen to be a homotopy $(n+2)$-sphere. Thus Σ^{n+2} is homeomorphic to S^{n+2}. For $n > 2$, we may assume it is in fact diffeomorphic to S^{n+2}, by changing the smooth structure at a point if necessary. For $n = 2$, all the knots K we shall consider will have the property that Σ^4 is diffeomorphic to S^4. The image of $S^n \times \{0\}$ in S^{n+2} is a knot K^*, called the *Gluck reconstruction* of K.

By construction, the knot K^* has the same exterior as K. Gluck [9], Browder [4], Lashof and Shaneson [17], and Kato [15] showed that if K_0 is another knot with $X(K_0) \cong X(K)$, then K_0 is equivalent to K or K^*. Furthermore, K is equivalent to K^* if, and only if, there is a diffeomorphism of $X(K)$ which restricts to $\upsilon\tau_{n+1}$ on $\partial X(K) = S^n \times S^1$, where υ belongs to the group generated by orientation reversals of the factors. In this case we say the knot K is *reflexive*.

An n-knot K is *fibered* if there is a smooth fibration $\pi : X(K) \to S^1$ restricting on the boundary to $pr_2 : S^n \times S^1 \to S^1$. The inverse image of a point is a Seifert surface F^{n+1} for K called the *fiber*. The bundle is determined by the isotopy class of the *monodromy*, which is a diffeomorphism θ of the fiber that restricts to the identity on the boundary S^n. For $n > 1$, the fiber depends on the choice of fibration; it is well-defined up to an s-cobordism. The *closed fiber* is the closed, smooth $(n+1)$-manifold $F^c = F^{n+1} \cup D^{n+1}$; the *closed monodromy* is $\theta^c = \theta \cup id$. The closed fiber depends on the choice of boundary identification; it is well-defined up to connected sum with an exotic sphere.

A well-known way of creating fibered knots is by twist-spinning. If K is a knot in S^{n+2}, then the r-twist-spin of K, $K^{(r)}$, is a fibered knot in S^{n+3}, with fiber the punctured r-fold cyclic branched cover of (S^{n+2}, K) and monodromy the canonical branched covering transformation [27]. The Gluck reconstruction of $K^{(r)}$ is a knot in a smooth S^{n+3} [10].

We conclude this section with a proposition about the equalizers of degree one maps from closed-up Seifert surfaces. For that, we need the following result of Jeff Smith, communicated to us by the referee.

LEMMA 2.1. *Let F be a Seifert surface for an n-knot, and $i : S^n \to F$ be the inclusion of the boundary. Then $\Sigma\, i$ is nullhomotopic.*

Proof. Let $j : F \to F^c$ be the inclusion into the closed-up Seifert surface. We then have a cofiber sequence

$$S^n \xrightarrow{i} F \xrightarrow{j} F^c \xrightarrow{k} S^{n+1} \xrightarrow{\Sigma i} \Sigma F \xrightarrow{\Sigma j} \Sigma F^c \xrightarrow{\Sigma k} S^{n+2}$$

(see [25, p. 27]). The relative Pontrjagin–Thom collapse $S^{n+2} \to \Sigma (F/\partial F) \simeq \Sigma F^c$ provides a section to Σk. Thus $\Sigma F^c \simeq \Sigma F \vee S^{n+2}$, and we get a retract $\Sigma F^c \to \Sigma F$ of Σj. As $\Sigma j \circ \Sigma i$ is nullhomotopic, it follows that Σi is nullhomotopic. □

PROPOSITION 2.2. *Let F be a Seifert surface for an n-knot, and $q : F^c \to S^{n+1}$ be a degree 1 map. Suppose $f, g : S^{n+1} \to Z$ are two maps such that $f \circ q \simeq g \circ q$. Then $f \simeq g$.*

Proof. Since q has degree 1, it is homotopic to k, the cofiber of j. In a general cofiber sequence $A \to B \xrightarrow{\gamma} C \to \Sigma A \to \cdots$, the group $[\Sigma A, Z]$ acts transitively on the fibers of the function $\gamma^* : [C, Z] \to [B, Z]$ (see [25, Proposition 2.48]). In our case, since $\Sigma i \simeq *$, the action of $[\Sigma F, Z]$ on the fibers of q^* is trivial, and so q^* is injective. □

The proposition also holds for degree one maps $q : \Sigma^m \to S^m$, where Σ^m is an arbitrary homology m-sphere. For then q is an acyclic map, and we can quote Hausmann and Husemoller [12, Theorem 2.6]. In fact, the above proof closely follows theirs.

3. Framed manifolds

In this section we review some standard facts about framed manifolds and the Pontrjagin–Thom construction. More details can be found in [16], [3], [25].

Let M^k be a closed, smooth submanifold of S^{n+k}. A *framing* φ on M^k consists of a set of unit vectors $\varphi_1(x), \ldots, \varphi_n(x)$ varying smoothly with $x \in M^k$ and providing a basis for the normal space of M^k in S^{n+k} at x. Corresponding to the framing φ there is a uniquely defined trivialization $M^k \times D^n$ of the unit normal bundle of M^k in S^{n+k}. The Pontrjagin–Thom construction yields a smooth map $p(M, \varphi) : S^{n+k} \to S^n$, sending $S^{n+k} - M^k \times D^n$ to the lower hemisphere D^n_- and $M^k \times D^n$ to the upper hemisphere D^n_+. The homotopy class of this map depends only on the framed bordism class of (M, φ). The assignment $(M, \varphi) \mapsto [p(M, \varphi)]$ establishes an isomorphism between the group of framed bordism classes of framed k-submanifolds of S^{n+k} and the homotopy group $\pi_{n+k}(S^n)$.

Given a fixed framing φ of $M^k \subset S^{n+k}$, another framing ψ determines a smooth map $\hat{\psi} : M^k \to SO(n)$. The trivialization $M^k \times D^n$ corresponding to ψ depends up to isotopy only on the homotopy class $[\hat{\psi}] \in [M^k, SO(n)]$.

In the case where $M^k = S^k$, standardly embedded in S^{n+k}, there is a canonical choice of framing: the trivial framing $1 = (e_{k+1}, \ldots, e_{n+k})$, where e_i is the i-th basis vector of $\mathbb{R}^{n+k}$. The framings of S^k then correspond to smooth maps $\varphi : S^k \to SO(n)$, and the isotopy classes of trivializations of the normal bundle to homotopy classes $[\varphi] \in \pi_k(SO(n))$. Moreover, $[p(S^k, \varphi)] = J[\varphi]$, where $J : \pi_k(SO(n)) \to \pi_{n+k}(S^n)$ is the Hopf–Whitehead homomorphism.

The Freudenthal suspension homomorphism

$$E : \pi_{n+k}(S^n) \to \pi_{n+k+1}(S^{n+1}),$$

given by $E[f] = [\Sigma f]$, has the following geometric interpretation. Let $\alpha \in \pi_{n+k}(S^n)$ be represented by a manifold M^k in S^{n+k} with framing $\varphi = (\varphi_1, \ldots, \varphi_n)$. Then $E\alpha$ is represented by the manifold M^k embedded in S^{n+k+1} with framing $\varphi \oplus 1 = (\varphi_1, \ldots, \varphi_n, e_{n+k+1})$. In fact, $\Sigma\, p(M, \varphi) = p(M, \varphi \oplus 1)$.

Given $\beta \in \pi_{n+k+l}(S^{n+k})$, the composition map

$$\circ\, \beta : \pi_{n+k}(S^n) \to \pi_{n+k+l}(S^n)$$

can be interpreted as follows. Let β be represented by a manifold N^l in S^{n+k+l} with framing ψ, and let $N^l \times D^{n+k}$ be the corresponding trivialization of the normal bundle. Let $\alpha \in \pi_{n+k}(S^n)$ be represented by a manifold $M^k \subset D^{n+k} \subset S^{n+k}$ with framing φ and trivialization $M^k \times D^n$. We get an embedding $N^l \times M^k \times D^n \subset N^l \times D^{n+k} \subset S^{n+k+l}$. The manifold $N^l \times M^k$ with the respective framing $\psi * \varphi$ represents $\alpha \circ \beta \in \pi_{n+k+l}(S^n)$. In fact, $p(M, \varphi) \circ p(N, \psi) = p(N \times M, \psi * \varphi)$.

4. Frame-spun knots

We now describe the process, due to Roseman [23], of spinning an n-knot K about a framed submanifold (M^k, φ) of S^{n+k}. The resulting $(n+k)$-knot $\sigma^{\varphi}_M(K)$ will be called the *(M, φ)-spin* of K.

Let $M^k \times D^n$ be the trivialization of the unit normal bundle of M^k corresponding to φ. Let (D_-^{n+2}, D_-^n) be a standard disk pair embedded in (S^{n+2}, K). Set $(D_+^{n+2}, D_+^n) = (S^{n+2}, K) - (D_-^{n+2}, D_-^n)$. Consider the unknot $S^{n+k} = S^{n+k} \times \{0\} \subset S^{n+k+2} = S^{n+k} \times D^2 \cup D^{n+k+1} \times S^1$. The knot $\sigma_M^\varphi(K)$ consists of the $(n+k)$-sphere

$$(S^{n+k} - M^k \times \operatorname{int} D^n) \cup_{M^k \times S^{n-1}} M^k \times D_+^n$$

embedded in the $(n+k+2)$-sphere

$$(S^{n+k+2} - M^k \times \operatorname{int}(D^n \times D^2)) \cup_{M^k \times S^{n+1}} M^k \times D_+^{n+2}.$$

In other words, at each point of $M^k \subset (S^{n+k+2}, S^{n+k})$, we remove a transverse disk pair $(D^n \times D^2, D^n)$ and glue back the knotted disk pair (D_+^{n+2}, D_+^n) to get $\sigma_M^\varphi(K)$. See Figure 1.

The disk D_+^n has exterior $D_+^{n+2} - D_+^n \times \operatorname{int} D^2$ diffeomorphic to $X(K)$, with boundary $(D_+^n \cup D_-^n) \times S^1 \cong K \times S^1$. Therefore, the exterior of the (M, φ)-spin of K is

$$X(\sigma_M^\varphi(K)) = (D^{n+k+1} - M^k \times \operatorname{int} B^{n+1}) \times S^1 \cup_{M^k \times D_-^n \times S^1} M^k \times X(K),$$

where B^{n+1} is a standard disk in $D^n \times D^2$ with boundary $D^n \cup D_-^n$.

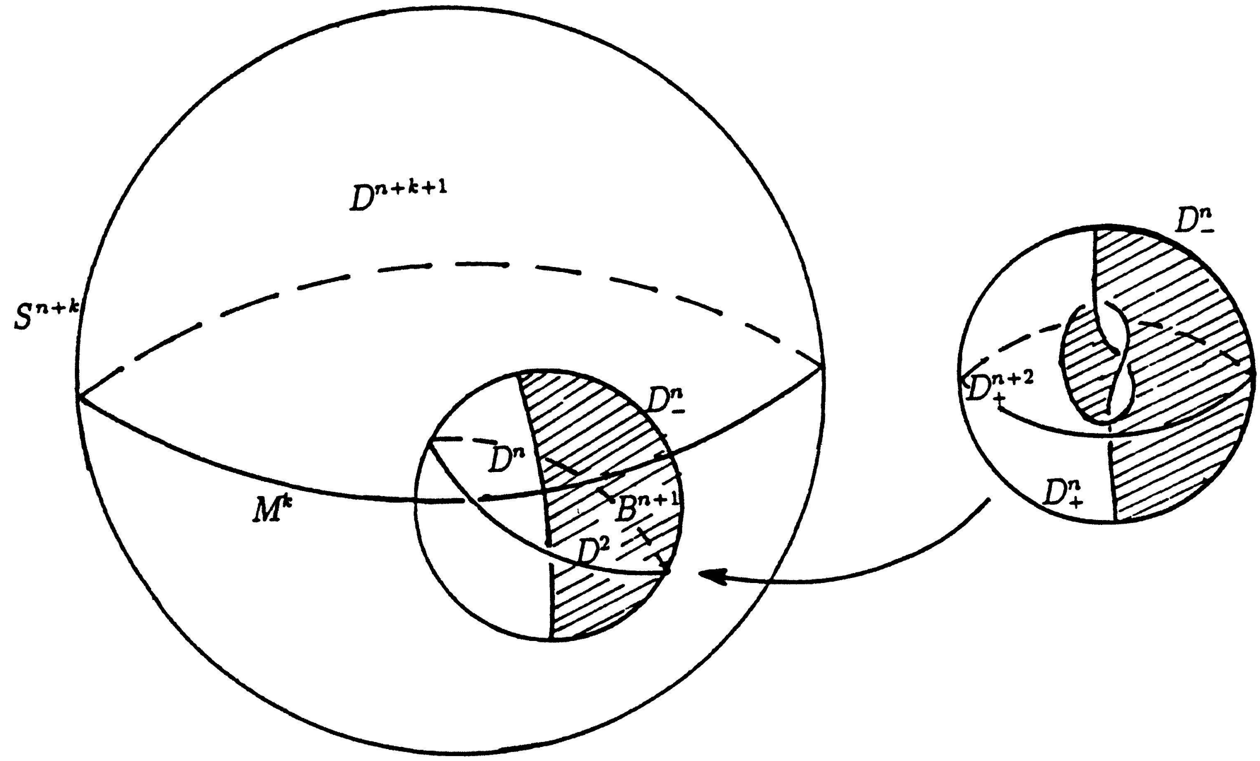


Figure 1

Some remarks on the construction are in order. First, notice that the manifold $D^{n+k+1} - M^k \times \operatorname{int} B^{n+1}$ is contractible. Therefore, by Van Kampen's theorem, $\pi_1(X(\sigma^{\varphi}_M(K))) \cong \pi_1(X(K))$. This means the group of the (M, φ)-spin of K is determined by that of K alone; it does not depend on the framed manifold (M, φ). Also, the homotopy type of $X(\sigma^{\varphi}_M(K))$ depends only on the homotopy type of $X(K)$ and that of M^k; it does not depend on the framing. In other words, for any two framings φ, ψ of M^k, $X(\sigma^{\varphi}_M(K)) \simeq X(\sigma^{\psi}_M(K))$; but, as we shall see in §5, there may be no homotopy equivalence preserving the boundaries.

Second, it should be noted that the (M, φ)-spin of K depends only on the isotopy class of the trivialization $M^k \times D^n$ associated to the framing φ. If ψ is another framing of M^k, let $\hat{\psi} : M^k \to SO(n)$ be the map it determines by comparison to φ. The exterior of $\sigma^{\psi}_M(K)$ is obtained from that of $\sigma^{\varphi}_M(K)$ by splitting along $M^k \times D^n_- \times S^1$ and gluing back by the map $(x, y, t) \mapsto (x, \hat{\psi}(x)(y), t)$. Thus, if $[\hat{\psi}] = 0$, then $\sigma^{\psi}_M(K)$ is equivalent to $\sigma^{\varphi}_M(K)$.

Finally, let us record the fact that in general a frame-spun knot depends on the given framed manifold, not just on the framed bordism class of that manifold. Indeed, if M^2_g is the surface of genus g, standardly embedded in S^3, and K is a non-trivial fibered classical knot, then M^2_g is framed null-bordant, yet $\sigma_{M_g}(K) \not\cong \sigma_{M_h}(K)$ for $g \neq h$. In fact, the two frame-spun knots are fibered, with the fibers having non-isomorphic second homology groups (see [23] for the case $K =$ trefoil knot, and [24] for the case $g = 0, h = 1$).

The effect of iterated frame-spinning can be described as follows. Let (N^l, ψ) be a framed submanifold of S^{n+k+l}, with normal bundle $N^l \times D^{n+k}$. Consider the (N^l, ψ)-spin of the (M^k, φ)-spin of the knot K. It consists of the $(n + k + l)$-sphere

$$\begin{aligned}
&(S^{n+k+l} - N^l \times \operatorname{int} D^{n+k}) \\
&\qquad \cup_{N^l \times S^{n+k-1}} N^l \times [(D^{n+k} - M^k \times \operatorname{int} D^n) \cup_{M^k \times S^{n-1}} M^k \times D^n_+] \\
&\qquad \cong (S^{n+k+l} - N^l \times M^k \times \operatorname{int} D^n) \cup_{N^l \times M^k \times S^{n-1}} N^l \times M^k \times D^n_+,
\end{aligned}$$

embedded in the $(n + k + l + 2)$-sphere

$$\begin{aligned}
&(S^{n+k+l+2} - N^l \times \operatorname{int} (D^{n+k} \times D^2)) \cup_{N^l \times S^{n+k+1}} N^l \\
&\qquad \times [(D^{n+k+2} - M^k \times \operatorname{int} (D^n \times D^2)) \cup_{M^k \times S^{n+1}} M^k \times D^{n+2}_+] \\
&\qquad \cong (S^{n+k+l+2} - N^l \times M^k \times \operatorname{int} (D^n \times D^2)) \cup_{N^l \times M^k \times S^{n+1}} N^l \times M^k \times D^{n+2}_+.
\end{aligned}$$

The framing of $N^l \times M^k \subset S^{n+k+l}$ corresponding to the trivialization $N^l \times M^k \times D^n$ obtained above is the product framing $\psi * \varphi$. Thus the resulting knot is the $(N^l \times M^k, \psi * \varphi)$-spin of K. We have proved

PROPOSITION 4.1. *The iterated frame-spun knot* $\sigma_N^\psi(\sigma_M^\varphi(K))$ *is equivalent to* $\sigma_{N\times M}^{\psi*\varphi}(K)$. □

As mentioned in the introduction, we are primarily interested in the following question about frame-spun knots: Given a knot K and a framed manifold (M, φ), is the knot $\sigma_M^\varphi(K)$ determined by its complement? We conclude this section with two situations – one involving K, the other (M, φ) – where the answer is affirmative. We will come back to this question in §6 with a situation where the answer is negative.

Consider the case $M^k = S^k$, standardly embedded in S^{n+k}, with framing given by a smooth map $\varphi : S^k \to SO(n)$. The resulting frame-spun knots, $\sigma_k^\varphi(K) = \sigma_{S^k}^\varphi(K)$, first appeared in Hsiang and Sanderson [14]. When $\varphi(x) = id$, i.e. the framing is trivial, we get the superspin, or k-spin, $\sigma_k(K)$, of Cappell [5]. The exterior of the (k, φ)-spin of K is

$$X(\sigma_k^\varphi(K)) = D^{k+1} \times D_-^n \times S^1 \cup_{S^k \times D_-^n \times S^1} S^k \times X(K),$$

with gluing map $(x, y, t) \mapsto (x, \varphi(x)(y), t)$.

Now let K be an n-knot, $n \geq 2$. The following result establishes the relationship between (k, φ)-spinning and Gluck reconstruction.

THEOREM 4.2. *The knot* $\sigma_k^\varphi(K^*)$ *is equivalent to* $\sigma_k^\varphi(K)^*$. *Thus, if K is reflexive,* $\sigma_k^\varphi(K)$ *is also reflexive.*

Proof. Recall K^* is a knot in S^{n+2}, with exterior $X(K)$; the ambient sphere is obtained by attaching $S^n \times D^2$ to $X(K)$ by the Gluck twist $\tau_{n+1}(y, t) = (\rho_{n+1}(t)(y), t)$. The (k, φ)-spin of K^* has exterior

$$X(\sigma_k^\varphi(K^*)) = D^{k+1} \times D_-^n \times S^1 \cup_{S^k \times D_-^n \times S^1} S^k \times X(K),$$

with gluing map $(x, y, t) \mapsto (x, \varphi(x)(\rho_n(t)(y)), t)$. There is a diffeomorphism $X(\sigma_k^\varphi(K^*)) \to X(\sigma_k^\varphi(K))$ given by $id \times \tau_n \cup id$.

The ambient sphere S^{n+k+2} of $\sigma_k^\varphi(K^*)$ is obtained by attaching $S^{n+k} \times D^2$ to $X(\sigma_k^\varphi(K))$ along $D^{k+1} \times S^{n-1} \times S^1 \cup S^k \times D_+^n \times S^1 \cong S^{n+k} \times S^1$ by the map $id \cup id \times \tau_n = \tau_{n+k+1}$. It follows that $\sigma_k^\varphi(K^*) \cong \sigma_k^\varphi(K)^*$. □

A frame-spin of K may be reflexive even though K is not. Indeed, Gluck [9] showed that 1-spun knots are always reflexive. This was generalized to k-spun knots by Cappell [5]. The following theorem, based on Cappell's method, extends their results to certain (k, φ)-spun knots. First, some notation: $\eta_2 = J[\rho_2]$ is the generator of $\pi_3(S^2) \cong \mathbb{Z}$ given by the Hopf map, and, for $k > 2$, $\eta_k = E^{k-2}\eta_2$ is the generator of $\pi_{k+1}(S^k) = \mathbb{Z}_2$. To keep things compact, we shall let η_1 stand for ι_1, the usual generator of $\pi_1(S^1)$.

THEOREM 4.3. *Let K be an n-knot and $\varphi : S^k \to SO(n)$ a smooth map. If $[\varphi] \circ \eta_k = 0$, then $\sigma_k^\varphi(K)$ is reflexive.*

Proof. Define a smooth map $f_0 : D^{k+1} \times D^n_- \times S^1 \to D^{k+1} \times D^n_- \times S^1$ by $f_0(x, y, t) = (\rho_{k+1}(t)(x), y, t)$. Let $\gamma : X(K) \to S^1$ be a smooth map which represents a generator of $[X(K), S^1] \cong H^1(X(K); \mathbb{Z}) \cong \mathbb{Z}$ and which restricts on the boundary to $pr_2 : S^n \times S^1 \to S^1$. Then define a smooth map $f_1 : S^k \times X(K) \to S^k \times X(K)$ by $f_1(x, y) = (\rho_{k+1}(\gamma(y))(x), y)$.

On $S^k \times D^n_- \times S^1$, both f_0 and f_1 restrict to $(x, y, t) \mapsto (\rho_{k+1}(t)(x)), y, t)$. In order for this map to be compatible with the gluing determined by φ we must have

$$\varphi(\rho_{k+1}(t)(x)) = \varphi(x), \qquad \text{for } x \in S^k, t \in S^1. \tag{$*$}$$

Let $\tau_{k+1} : S^k \times S^1 \to S^k \times S^1$ be the Gluck twist and $pr_1 : S^k \times S^1 \to S^k$ the projection map. Then $(*)$ is equivalent to $\varphi \circ pr_1 \circ \tau_{k+1} = \varphi \circ pr_1$.

If $k = 1$, then $[\varphi] = 0$, and we may assume, by homotoping φ if necessary, that $(*)$ holds. Otherwise, the only obstruction to a homotopy $\varphi \circ pr_1 \circ \tau_{k+1} \simeq \varphi \circ pr_1$ is the class of the difference cocycle $d(\varphi \circ pr_1 \circ \tau_{k+1}, \varphi \circ pr_1) \in H^{k+1}(S^k \times S^1; \pi_{k+1}(SO(n))) \cong \pi_{k+1}(SO(n))$. By naturality, the obstruction equals $[\varphi] \circ d(pr_1 \circ \tau_{k+1}, pr_1)$. Since $d(pr_1 \circ \tau_{k+1}, pr_1) = \eta_k$ (see [9]), the obstruction vanishes, and again we may assume that $(*)$ holds.

This permits us to glue the maps f_0 and f_1 to get a smooth map $f : X(\sigma_k^\varphi(K)) \to X(\sigma_k^\varphi(K))$. On the boundary $S^{n+k} \times S^1$ the map f restricts to τ_{n+k+1}. Thus $\sigma_k^\varphi(K)$ is reflexive. □

As suggested in §1, the above theorems should generalize to arbitrary frame-spun knots. Namely, one should prove:

(i) $\sigma_M^\varphi(K^*) \cong \sigma_M^\varphi(K)^*$.
(ii) If $[p(M, \varphi)] \circ \eta_{n+k} = 0$, then $\sigma_M^\varphi(K)$ is reflexive.

The difficulty one runs into is finding appropriate "Gluck twists" over $(D^{n+k+1} - M^k \times \operatorname{int} B^{n+1}) \times S^1$.

5. Frame-spun fibers

In this section we introduce the notion of frame-spinning a closed manifold and use it to study the closed fiber of a frame-spun knot.

Let W^m be a closed, smooth m-manifold, $m \geq 1$. Let B^m be a fixed embedded disk in W^m and let $W_0^m = W^m - \operatorname{int} B^m$. Let (M^k, φ) be a framed submanifold of S^{m+k}, with unit normal bundle $M^k \times D^m$. The (M, φ)-*spin* of W^m is the closed, smooth $(m+k)$-manifold

$$\sigma_M^\varphi(W^m) = (S^{m+k} - M^k \times \operatorname{int} D^m) \cup_{M^k \times S^{m-1}} M^k \times W_0^m. \qquad (\dagger)$$

That is to say, at each point of $M^k \subset S^{m+k}$, we remove a transverse disk D^m and glue back the punctured manifold W_0^m. Notice the frame-spin of S^m is just S^{m+k}.

If $M^k = S^k$, with framing $\varphi : S^k \to SO(n)$, the resulting frame-spun manifold is $\sigma_k^\varphi(W^m) = D^{k+1} \times S^{m-1} \cup_{S^k \times S^{m-1}} S^k \times W_0^m$, with gluing map $(x, y) \mapsto (x, \varphi(x)(y))$. In case the framing is trivial, we get the k-spin, $\sigma_k(W^m)$, of Cappell [5]. In case $k = 1, m \geq 3$, there are two possible S^1-spins, $\sigma_1(W^m)$ and $\sigma_1'(W^m)$, corresponding to the framings 1 and ρ_m (Plotnick [20]). The two pieces of $\sigma_1'(W^m)$ get glued along $S^1 \times S^{m-1}$ by the Gluck twist. Thus, if the Gluck twist extends to a diffeomorphism of $S^1 \times W_0^m$ (for example, if W^m admits a smooth S^1-action with codimension 2 fixed-point set), then $\sigma_1(W^m)$ is diffeomorphic to $\sigma_1'(W^m)$.

Frame-spinning behaves nicely with respect to fundamental groups. If $m \geq 3$, then $\pi_1(S^{m+k} - M^k \times \operatorname{int} D^m) = 0$, $\pi_1(W_0^m) \cong \pi_1(W^m)$, and so, by Van Kampen's theorem, $\pi_1(\sigma_M^\varphi(W^m)) \cong \pi_1(W^m)$.

The Pontrjagin–Thom construction can be extended to frame-spun manifolds. Indeed, the decomposition $(\dagger)$ yields a smooth map

$$p(W, M, \varphi) : \sigma_M^\varphi(W^m) \to W^m$$

that sends $S^{m+k} - M^k \times \operatorname{int} D^m$ to B^m and $M^k \times W_0^m$ to W_0^m. Clearly, $p(S^m, M, \varphi)$ is just $p(M, \varphi)$. Moreover, $p(W, M, \varphi) \circ p(\sigma_M^\varphi(W), N, \psi) = p(W, N \times M, \psi * \varphi)$.

The frame-spinning construction enjoys the following naturality properties. Let V^m be another manifold with a fixed embedded disk. Let $f : W^m \to V^m$ be a degree 1 smooth map preserving the chosen disks. Define the (M, φ)-*spin* of f to be the (degree 1) smooth map

$$\sigma_M^\varphi(f) : \sigma_M^\varphi(W^m) \to \sigma_M^\varphi(V^m)$$

obtained by piecing together the maps $id_{S^{m+k} - M^k \times \operatorname{int} D^m}$ and $id_{M^k} \times f|_{W_0^m}$. Then $f \circ p(W, M, \varphi) = p(V, M, \varphi) \circ \sigma_M^\varphi(f)$. Moreover, if $g : V^m \to U^m$ is another degree 1 smooth map preserving base disks, then $\sigma_M^\varphi(g \circ f) = \sigma_M^\varphi(g) \circ \sigma_M^\varphi(f)$.

Having defined the process of frame-spinning a knot, respectively a manifold, we now relate the two notions in case the knot we start with is fibered. Unlike twist-spinning, the process of spinning doesn't create essentially new fibrations. But it does the next best thing. As recognized by Andrews and Sumners [2], k-spinning takes fibered knots to fibered knots. This was generalized to frame-spinning by Roseman [23, Lemma 1]. Let us identify the frame-spun fiber and monodromy in our terminology.

Let K be a fibered n-knot, with fibration of the exterior $\pi : X(K) \to S^1$, fiber F, and monodromy θ. Denote by F^c the closed fiber of K (so that $F_0^c = F$). Let M^k be a submanifold S^{n+k} with framing φ. The exterior of the (M, φ)-spin of K admits a corresponding frame-spun fibration $pr_2 \cup \pi \circ pr_2 : (D^{n+k+1} - M^k \times \text{int } B^{n+1}) \times S^1 \cup_{M^k \times D^n_- \times S^1} M^k \times X(K) \to S^1$. Its fiber is

$$F(\sigma_M^\varphi(K)) = (D^{n+k+1} - M^k \times \text{int } B^{n+1}) \cup_{M^k \times D^n_-} M^k \times F,$$

and its monodromy is $id \cup id \times \theta$. The closed fiber of $\sigma_M^\varphi(K)$ is

$$\begin{aligned} F^c(\sigma_M^\varphi(K)) &= F(\sigma_M^\varphi(K)) \cup_{S^{n+k}} D_1^{n+k+1} \\ &\cong [(D^{n+k+1} - M^k \times \text{int } B^{n+1}) \\ &\qquad \cup_{S^{n+k} - M^k \times \text{int } D^n} (D_1^{n+k+1} - M^k \times \text{int } B_1^{n+1})] \cup_{M^k \times S^n} M^k \times F \\ &= (S^{n+k+1} - M^k \times \text{int } D^{n+1}) \cup_{M^k \times S^n} M^k \times F, \end{aligned}$$

where $D^{n+1} = B^{n+1} \cup_{D^n} B_1^{n+1}$ (see Figure 2). The trivialization $M^k \times D^{n+1}$ of the normal bundle of M^k in S^{n+k+1} corresponds to the stabilized framing $\varphi \oplus 1$. We thus have proved

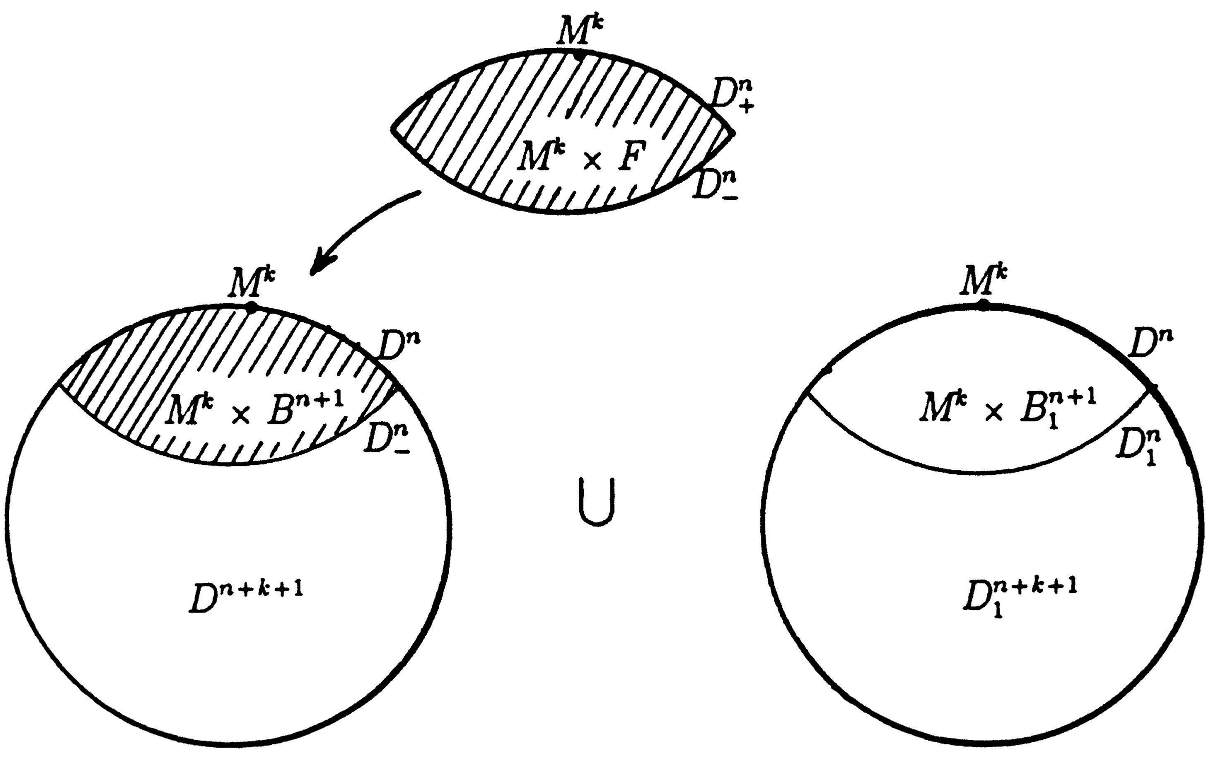


Figure 2

PROPOSITION 5.1. *If K is a fibered knot, then $\sigma_M^\varphi(K)$ is also fibered, with closed fiber $\sigma_M^{\varphi\oplus 1}(F^c)$ and closed monodromy $\sigma_M^{\varphi\oplus 1}(\theta^c)$.* □

We now address the following question: Given a closed m-manifold, W^m, and two framed k-submanifolds of S^{m+k}, (M^k, φ) and (N^k, ψ), are the corresponding frame-spins, $\sigma_M^\varphi(W^m)$ and $\sigma_N^\psi(W^m)$, homotopy equivalent? As the case $W^m = S^m$ illustrates, the answer may be yes. But in general, the expected answer is no. If $M^k \not\simeq N^k$, one can often distinguish between the two frame-spins by means of their homology or the homology of their universal covers. For example, if $W^3 \not\simeq S^3$, then $\sigma_{S^k}(W^3) \not\simeq \sigma_{S^l \times S^{k-l}}(W^3)$ (see [24] for a proof and generalizations). If $M^k \simeq N^k$, the difference between the two frame-spins of W^m is more subtle. The next theorem shows that we still may tell them apart, provided W^m is the closed fiber of a knot, its universal cover is contractible, and $(M^k, \varphi \oplus 1)$ is not framed cobordant to $(N^k, \psi \oplus 1)$.

THEOREM 5.2. *Let K be a fibered n-knot, $n \geq 2$, with aspherical closed fiber. Let (M^k, φ) and (N^k, ψ) be two framed k-submanifolds of S^{n+k} such that $E[p(M, \varphi)] \neq E[p(N, \psi)]$. Then $F^c(\sigma_M^\varphi(K)) \not\simeq F^c(\sigma_N^\psi(K))$.*

Proof. Suppose $F^c(\sigma_M^\varphi(K)) \simeq F^c(\sigma_N^\psi(K))$. Let F^c be the closed fiber of K. By Proposition 5.1, there is a homotopy equivalence $g : \sigma_M^{\varphi\oplus 1}(F^c) \to \sigma_N^{\psi\oplus 1}(F^c)$. Let g_* be the induced automorphism on $\pi = \pi_1(F^c)$. Since F^c is a $K(\pi, 1)$, g_* extends to a homotopy equivalence $h : F^c \to F^c$. Moreover, $h \circ p(F^c, M, \varphi \oplus 1) \simeq p(F^c, N, \psi \oplus 1) \circ g$, again by asphericity of F^c.

Now let $q : F^c \to S^{n+1}$ be the map sending F^{n+1} to D_+^{n+1} and B^{n+1} to D_-^{n+1}. Changing the orientation of S^{n+1} if necessary, we see that q has degree 1. Hence there is a homotopy equivalence $\bar{h} : S^{n+1} \to S^{n+1}$ such that $\bar{h} \circ q = q \circ h$. The maps $\sigma_M^{\varphi\oplus 1}(q)$ and $\sigma_N^{\psi\oplus 1}(q)$ also have degree 1, so there is a homotopy equivalence $\bar{g} : S^{n+k+1} \to S^{n+k+1}$ such that $\bar{g} \circ \sigma_M^{\varphi\oplus 1}(q) = \sigma_N^{\psi\oplus 1}(q) \circ g$.

We thus have the diagram

$$\begin{tikzcd}
 & \sigma_N^{\psi\oplus 1}(F^c) \arrow[rr, "p(F^c, N, \psi\oplus 1)"] \arrow[dd, "\sigma_N^{\psi\oplus 1}(q)"] & & F^c \arrow[dd, "q"] \\
\sigma_M^{\varphi\oplus 1}(F^c) \arrow[ur, "g"] \arrow[rr, "p(F^c, M, \varphi\oplus 1)"] \arrow[dd, "\sigma_M^{\varphi\oplus 1}(q)"'] & & F^c \arrow[ur, "h"] \arrow[dd, "q"] & \\
 & S^{n+k+1} \arrow[rr, "p(N, \psi\oplus 1)"] & & S^{n+1} \\
S^{n+k+1} \arrow[ur, "\bar{g}"] \arrow[rr, "p(M, \varphi\oplus 1)"'] & & S^{n+1} \arrow[ur, "\bar{h}"'] &
\end{tikzcd}$$

with the top and side squares commuting up to homotopy. Hence $p(M, \varphi \oplus 1) \circ \sigma_M^{\varphi \oplus 1}(q) \simeq p(N, \psi \oplus 1) \circ \sigma_M^{\varphi \oplus 1}(q)$. Since $\sigma_M^{\varphi \oplus 1}(F^c)$ is the closed fiber of $\sigma_M^{\varphi}(K)$, and $\sigma_M^{\varphi \oplus 1}(q)$ has degree 1, Proposition 2.2 implies $p(M, \varphi \oplus 1) \simeq p(N, \psi \oplus 1)$. This is a contradiction, and we are done. □

REMARK. The knot exteriors $X(\sigma_M^{\varphi}(K))$ and $X(\sigma_N^{\psi}(K))$ are not homotopy equivalent (rel. boundary). This follows from the preceding theorem by a standard argument: Suppose there is a homotopy equivalence (rel. ∂) of the knot exteriors. It lifts to a homotopy equivalence (rel. ∂) of the infinite cyclic covers $F(\sigma_M^{\varphi}(K)) \times \mathbb{R} \simeq F(\sigma_N^{\psi}(K)) \times \mathbb{R}$. This yields a homotopy equivalence $F(\sigma_M^{\varphi}(K)) \to F(\sigma_N^{\psi}(K))$, which is the identity on the boundary S^{n+k}, and thus extends to a homotopy equivalence of the closed fibers. For example, if K is a 2-knot with aspherical closed fiber (see [10], [13] for such knots), and $\varphi : S^1 \to SO(2)$ has odd degree, then $X(\sigma_1^{\varphi}(K)) \not\simeq X(\sigma_1(K))$ (rel. ∂). Or, if K is a Cappell–Shaneson 3-knot with closed fiber the 4-torus [7], and $\varphi : S^k \to SO(3)$ satisfies $J[\varphi] \neq 0$, then $X(\sigma_k^{\varphi}(K)) \not\simeq X(\sigma_k(K))$ (rel. ∂).

COROLLARY 5.3. *Let K be a fibered n-knot, $n \geq 2$, with aspherical closed fiber F^c. Then $\sigma_1(F^c) \not\simeq \sigma_1'(F^c)$.* □

REMARK. For 3-dimensional manifolds, more is true. With some additional work, we can show that given *any* aspherical W^3, the two S^1-spins of W^3 are homotopically distinct. This result was first proved by Plotnick [20, Theorem 3.1], using intersection forms on universal covers. He also showed [20, Theorem 5.1] that there is no "special" homotopy equivalence between the two spins of W^3, provided not all summands of W^3 are $S^2 \times S^1$ of Σ^3/π, where Σ^3 is a homotopy 3-sphere, π is a finite group acting freely on Σ^3, and all Sylow subgroups of π are cyclic. We can sharpen this last result in some cases. For example, $\sigma_1(\Sigma^3/I^*) \not\simeq \sigma_1'(\Sigma^3/I^*)$, where I^* is the binary icosahedral group.

6. Non-reflexive knots

We now return to the problem of reflexivity of knots, more specifically, of frame-spun fibered knots. Under certain assumptions on the fibering and on the framing, these knots will prove to be non-reflexive. We start with the following necessary condition for reflexivity. The idea of the proof is similar to that of [10, Proposition 4.2] and [20, Theorem 6.2].

PROPOSITION 6.1. *Let K be a fibered n-knot, $n \geq 2$, with odd order monodromy. If K is reflexive, then $\sigma_1(F^c(K)) \cong \sigma_1'(F^c(K))$.*

Proof. Let θ be the monodromy of K, and r its order. Since $K \cong K^*$, there is a diffeomorphism f of the exterior $F \times_\theta S^1$ which restricts on the boundary to $\upsilon\tau_{n+1}$, where τ_{n+1} is the Gluck twist and υ is a composite of orientation reversals of the factors of $S^n \times S^1$. Lift f to a diffeomorphism $\tilde{f}$ of the r-fold cover $F \times S^1$. Since r is odd, $\tilde{f}$ restricts on the boundary to $\upsilon\tau_{n+1}$. It is now a simple matter to extend $\tilde{f}$ to a diffeomorphism $\sigma_1(F^c) \to \sigma'_1(F^c)$. $\square$

This proposition, together with Corollary 5.3, implies

COROLLARY 6.2. *Let K be a fibered n-knot, $n \geq 2$, with aspherical closed fiber and odd order monodromy. Then K is not reflexive.* $\square$

This result was first proved by Gordon [10] under the extra assumptions that K be a twist-spun knot and the universal cover of $F^c(K)$ be $\mathbb{R}^{n+1}$. He used it to produce examples of non-reflexive 2-knots as follows. Let p, q, r be integers greater than 1, with p and q coprime, r odd, and $1/p + 1/q + 1/r \leq 1$. Denote by $K_{p,q}$ the (p, q)-torus knot in S^3. The r-twist-spin $K_{p,q}^{(r)}$ is a knot in S^4 with closed fiber the aspherical Brieskorn 3-manifold $\Sigma(p, q, r)$ and monodromy of order r. Therefore $K_{p,q}^{(r)}$ is not reflexive. (In fact, according to Hillman and Plotnick [13], *no* r-twist-spin of a non-trivial prime, simple classical knot with $r > 2$ is reflexive).

For $n > 2$ this method doesn't work, as there are no known examples of aspherical $(n+1)$-manifolds that are cyclic branched covers of a knotted pair (S^{n+1}, S^{n-1}). Consequently, a stronger result is required in order to produce high-dimensional non-reflexive knots.

THEOREM 6.3. *Let K be a fibered n-knot, $n \geq 2$, with aspherical closed fiber and odd order monodromy. If $E[p(M^k, \varphi)] \circ \eta_{n+k+1} \neq 0$, then $\sigma_M^\varphi(K)$ is not reflexive.*

Proof. Consider $F^c(\sigma_M^\varphi(K))$, the closed fiber of $\sigma_M^\varphi(K)$. By Propositions 5.1 and 4.1, its two S^1-spins are

$$\sigma_1(F^c(\sigma_M^\varphi(K)) \cong F^c(\sigma_{S^1 \times M}^{1 * \varphi}(K)) \quad \text{and} \quad \sigma'_1(F^c(\sigma_M^\varphi(K)) \cong F^c(\sigma_{S^1 \times M}^{\rho * \varphi}(K)).$$

As $[p(S^1 \times M, 1 * \varphi] = 0$ and $[p(S^1 \times M, \rho_{n+k} * \varphi)] = [p(M, \varphi)] \circ \eta_{n+k}$, Theorem 5.2 implies $\sigma_1(F^c(\sigma_M^\varphi(K)) \not\cong \sigma'_1(F^c(\sigma_M^\varphi(K))$.

Let θ be the monodromy of K. Since θ has odd order, the monodromy $id \cup id \times \theta$ of $\sigma_M^\varphi(K)$ also has odd order. It follows from Proposition 6.1 that $\sigma_M^\varphi(K)$ is not reflexive. $\square$

We now can prove Theorem 1.1. Let K be a fibered 2-knot with aspherical closed fiber and odd order monodromy, *e.g.* one of the twist-spun knots mentioned above. To find a non-reflexive n-knot, $n \geq 3$, it is enough to find an element $\alpha \in \pi_n(S^2)$ such that

$$E\alpha \circ \eta_{n+1} \neq 0.$$

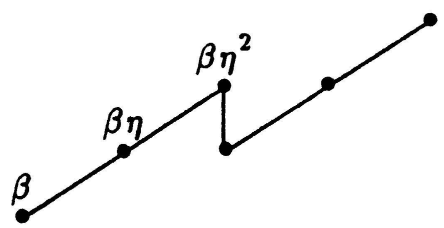
For if (M^{n-2}, φ) is a framed submanifold of S^n such that $[p(M^{n-2}, \varphi)] = \alpha$, then $\sigma_M^\varphi(K)$ is a knot in S^{n+2} which, by Theorem 6.3, is not reflexive.

We will show that such elements α of $\pi_n(S^2)$ exist, provided that $n \equiv 3$ or $n \equiv 4$ (mod 8). A search through Toda's book [26] produces the following table:

n	α	$E\alpha \circ \eta_{n+1}$
3	$j\eta_2, j$ odd	$\eta_3\eta_4$
4	$\eta_2\eta_3$	$\eta_3\eta_4\eta_5 = 2\nu'$
11	$\eta_2\epsilon_3$	$\eta_3\epsilon_4\eta_{12} = 2\epsilon'$
12	$\eta_2\mu_3$	$\eta_3\mu_4\eta_{13} = 2\mu'$
19	$\eta_2\mu_3\sigma_{12}$	$\eta_3\mu_4\sigma_{13}\eta_{20} = 2\mu'\sigma_{14}$
20	$\eta_2\bar{\mu}_3$	$\eta_3\bar{\mu}_4\eta_{21} = 2\bar{\mu}'$

The classes ϵ_3, μ_3, $\bar{\mu}_3$ are certain Toda brackets defined in [26], σ_8 is the generator of $\pi_{15}(S^8)$ given by the Hopf map, and $\sigma_k = E^{k-8}\sigma_8$. It is readily seen that the elements in the right-hand column are all non-zero (they have order exactly 2). This proves our claim for $n \leq 20$.

For higher values of n, we must appeal to deeper results in homotopy theory. Let $\alpha = \eta_2\beta^{(n)}$, where $\beta^{(n)} \in \pi_n(S^3)$ is defined inductively by Adams periodicity [1]: $\beta^{(11)} = \epsilon_3$, $\beta^{(12)} = \mu_3$, and $\beta^{(n)}$ is the Toda bracket $\{\beta^{(n-8)}, 2\iota_{n-8}, 8\sigma_{n-8}\}$. At the level of the E_2-term of the unstable Adams spectral sequence (mod 2) for S^3, the elements $\beta^{(n)}$ appear at the beginning of two periodic families of "lightning flashes" [19, p. 107]:



It follows from a fundamental theorem of Mahowald that $\beta^{(n)}\eta_n\eta_{n+1}$ is essential; it is detected by the composite of the bo-Hurewicz map with the Snaith map

$\pi_{n-1}(\Omega^3 S^3) \to \pi_{n-1}(Q\mathbb{RP}^2) \to \pi_{n-1}(\Omega^\infty(\mathbb{RP}^2 \wedge \mathrm{bo}))$ [19, Theorem 1.5]. An elementary computation using [26, Proposition 3.2] and the injectivity of $E: \pi_{n+2}(S^3) \to \pi_{n+3}(S^4)$ shows $\beta^{(n)}\eta_n\eta_{n+1} = \eta_3 \circ E\beta^{(n)} \circ \eta_{n+1}$. Hence $E\alpha \circ \eta_{n+1} \neq 0$. This finishes the proof of Theorem 1.1. □

For each $n \equiv 3$ or $4 \pmod 8$, there exist infinitely many distinct non-reflexive n-knots. We can show this two ways. First, we may choose infinitely many triples (p, q, r) as in the paragraph following Corollary 6.2 so that the manifolds $\Sigma(p, q, r)$ have pairwise non-isomorphic fundamental groups. Thus, if (M^{n-2}, φ) is as in the proof of Theorem 1.1, the n-knots $\sigma_M^\varphi(K_{p,q}^{(r)})$ are non-reflexive and have distinct groups. Second, we may fix a triple (p, q, r), with r not coprime to p; then the manifold $\Sigma(p, q, r)$ is not a homology 3-sphere. For $n > 3$ and $i > 0$, let $M_i^{n-2} = M^{n-2} \#_1^i S^1 \times S^{n-3}$; it is a framed submanifold of S^n, with framing φ_i equal to φ on the first factor and the trivial framing on the other factors. The knots $\sigma_{M_i}^{\varphi_i}(K_{p,q}^{(r)})$, $i = 1, 2, \ldots$, are non-reflexive, have isomorphic groups, but are pairwise non-equivalent: they can be distinguished by the homology of their fibers.

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