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An asymptotic formula for the eta invariants of hyperbolic 3-manifolds

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Let M be an oriented complete finite-volume hyperbolic 3-manifold with one cusp. Suppose that an oriented basis $\mathbf{l}, \mathbf{m}$, for the first homology at the cusp has been chosen, so that we can speak of $M(p, q)$, the result of (p, q)-Dehn filling the cusp – that is, we replace the cusp by a solid torus in which the class $p\mathbf{m} + q\mathbf{l}$ is null-homologous. Thurston's hyperbolic Dehn surgery theorem (see [T], [NZ]) tells us that $M(p, q)$ has a hyperbolic structure for $p^2 + q^2$ sufficiently large. In [Y] T. Yoshida proves a formula for the eta invariant $\eta(M(p, q))$ in terms of Thurston's analytic Dehn surgery parameter $u(p, q)$ and additional structure on M (various frame fields). For M equal to the figure-eight knot complement he gives a simpler and more explicit version of the formula which does not invoke the extra structure. The purpose of this note is to show that a formula of this simpler type can be derived in general from Yoshida's result. Our basic result is that:

THEOREM 1. *Suppose that the basis* $\mathbf{m}, \mathbf{l}$ *at the cusp is chosen so that* $\mathbf{l}$ *is a "longitude", that is, it is null-homologous in* M. *Then, with ingredients to be described below, for* $p^2 + q^2$ *sufficiently large*:

$$\frac{1}{\pi^2} \operatorname{Vol}(M(p, q)) + 3i\eta(M(p, q)) = f(u(p, q)) - \frac{1}{2\pi} \lambda(\gamma(p, q)) - iI(p, q).$$

$f(u)$ is, up to an imaginary constant, the complex analytic function (of the analytic Dehn surgery parameter u) which arose in the main results of [Y] and [NZ]. $I(p, q)$ is an integer depending only on p and q which we describe explicitly below. Finally, $\gamma(p, q)$ is the geodesic core of the Dehn filling and $\lambda(\gamma(p, q))$ is its "complex length" $\lambda(\gamma) = \text{length}(\gamma) + i \text{ torsion}(\gamma)$. We must, however, be careful about the branch of torsion (γ) that we use here, since torsion (γ) is only well defined modulo 2π. For $p^2 + q^2$ large, some branch of torsion (γ) is close to $2\pi q'/p$, where q' satisfies

$$0 \le \frac{q'}{p} < 1 \quad \text{and} \quad qq' \equiv -1 \pmod{|p|},$$

and this is the branch of torsion (γ) which we choose. In fact (cf. [Mh] and [NZ]), an appropriate branch of torsion $(\gamma) - 2\pi q'/p$ is the restriction of a real analytic function of u which vanishes at $u = 0$, namely (notation as in Sect. 4 and [NZ])

$$\text{torsion}\,(\gamma) - 2\pi \frac{q'}{p} = \arg \frac{v}{p} = \frac{2\pi}{pq} - \arg \frac{u}{q}.$$

With q' as above, $I(p, q)$ can be given by the formulae

$$I(p, q) = \frac{1}{p}(3\,\mathrm{def}\,(p; q, 1) + q - q') \qquad \text{if } p > 0,$$

$$I(p, q) = I(-p, -q) \qquad \text{if } p < 0,$$

where

$$\mathrm{def}\,(p; q, 1) = -\sum_{k=1}^{p-1} \cot \frac{k\pi}{p} \cot \frac{kq\pi}{p}$$

is the Hizebruch defect (H], [H-Z]).

Alternatively, if $p/q > 0$ is given by a continued fraction

$$\frac{p}{q} = b_1 - \cfrac{1}{b_2 - \cfrac{1}{\ddots - \cfrac{1}{b_k}}}$$

with $b_i \geq 2$ for $i \geq 2$, then

$$I(p, q) = -1 + \sum_{i=1}^{k} (3 - b_i),$$

while if $p/q < 0$ then $I(p, q)$ can be computed from:

$$I(p, q) = -1 - I(p, -q) \qquad \text{if } p > 1,$$

$$I(1, q) = q.$$

The formula for $I(p,q)$ in terms of a continued fraction is equivalent to the recurrence relations

$$I(bq-p,q) = I(q,p) + 3 - b \qquad \text{if } p, q, bq-p > 0,$$

$$I(p,1) = 2-p.$$

These and other relations follow easily from the properties of the Hirzebruch defect discussed in [H] and [H-Z]. For example

$$I(p-bq, -q) = I(q,p) - 3 - b \qquad \text{if } p, q, p-bq > 0,$$

$$I(p, bp+q) = b + 2 - I(q,p) \qquad \text{if } p, q > 0.$$

The latter gives the fastest computation of $I(p,q)$ in practice.

We prove Theorem 1 in Section 1 and generalize it to several cusps in Section 2 (Theorem 2). In Section 3 we deduce that the eta invariants of Dehn fillings of a given hyperbolic M are dense in $\mathbf{R}$.

It is worth remarking that in [N] a simplicial formula for $f(u)$ is given in terms of the Rogers dilogarithm function $R(z)$, analogous to Yoshida's formula in the special case of the figure-eight knot complement. In Section 4 we give explicit formulae for the complements of the figure-eight knot (Yoshida's case) and the Whitehead link.

The formulae of Theorems 1 and 2 are presumably valid throughout hyperbolic Dehn surgery space, rather than just for p^2+q^2 large, but out proof, which uses nothing but equation (1.1) below and some general considerations, does not show this.

A note on orientations. We chose $(\mathbf{l}, \mathbf{m})$ to be an oriented basis for the first homology of the cusp torus with its inherited complex structure. This is because if $\mathbf{l}$ and $\mathbf{m}$ are a standard longitude and meridian of a hyperbolic knot complement then they have this orientation. In the discussion of Dehn surgery on the figure-eight knot complement in [T] the opposite orientation convention was used, due to the cusp torus being viewed from inside the manifold rather than outside. This non-standard convention was used also in [NZ] and [Y] ($\mathbf{m}$ is drawn with standard orientation in Fig. 18 of [NZ], but this was inconsistent with the text). Thus (p,q)-Dehn surgery in those discussions would be $(p,-q)$-Dehn surgery in the convention which we follow here, affecting some signs in some formulae.

Acknowledgements

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1. Proof

Our starting point is a formula involving the Chern–Simons invariant CS $(M(p, q))$ which was conjectured in [NZ] and proven in [Y]:

$$\frac{1}{\pi^2}\operatorname{Vol}(M(p,q)) + 2i\operatorname{CS}(M(p,q)) = f(u(p,q)) - \frac{1}{2\pi}\lambda(p,q) \pmod{i\mathbf{Z}}. \quad (1.1)$$

For any closed Riemannian $(4k-1)$-manifold N, it is known [APS] that

$$3\eta(N) \equiv 2\operatorname{CS}(N) \pmod{\mathbf{Z}}, \quad (1.2)$$

so (1.1) can be rewritten:

$$\frac{1}{\pi^2}\operatorname{Vol}(M(p,q)) + 3i\eta(M(p,q)) = f(u(p,q)) - \frac{1}{2\pi}\lambda(\gamma(p,q)) - iI(p,q;M) \quad (1.3)$$

where $I(p, q; M)$ is an integer that depends on p, q, and M.

We shall prove Theorem 1 in two steps:

Step 1: For $p^2 + q^2$ large, $I(p, q; M) = I(p, q) + C(M)$, for some integer valued function $I(p, q)$ of p and q and some function $C(M)$;

Step 2: the function $I(p, q)$ is as described in the Introduction.

This will prove Theorem 1, since for fixed M the constant $C(M)$ can be absorbed into the analytic part $f(u)$.

Step 1. It suffices to show that $I(p, q; M)$ has the form

$$I(p,q;M) = I(p,q) + C(M) + o(1) \qquad \text{as } p^2 + q^2 \to \infty, \quad (1.4)$$

since the fact that $I(p, q; M) - I(p, q)$ is integral forces the $o(1)$ term to vanish for $p^2 + q^2$ large. The imaginary part of (1.3) can be written:

$$\begin{aligned} -I(p, q; M) &= 3\eta(M(p, q)) + \frac{1}{2\pi} \text{torsion}\,(\gamma(p, q)) + g(u(p, q)) \\ &= 3\eta(M(p, q)) + \frac{1}{2\pi} \text{torsion}\,(\gamma(p, q)) \\ &\quad + g(0) + o(1) \qquad \text{as } p^2 + q^2 \to \infty, \end{aligned} \tag{1.5}$$

where $g(u)$ is the real analytic function equal to the imaginary part of $-f$. As mentioned in the Introduction, by [Mh] or [NZ] we have

$$\text{torsion}\,(\gamma(p, q)) = 2\pi \frac{q'}{p} + o(1), \tag{1.6}$$

so

$$-I(p, q; M) = 3\eta(M(p, q)) + \frac{q'}{p} + g(0) + o(1) \qquad \text{as } p^2 + q^2 \to \infty. \tag{1.7}$$

We need to analyze $\eta(M(p, q))$.

Recall from [T] that $M_{[0,\epsilon)}$ is the "thin part" of M consisting of all points which lie on some essential curve of M of length less than ϵ. If ϵ is sufficiently small ($\epsilon < 0.1$ suffices) then one of the components of $M_{[0,\epsilon)}$ is a neighborhood N of the cusp. The boundary of $M - N$ is a flat torus. If this flat torus has area δ, then it is not hard to see that $\epsilon < 1.1\sqrt{\delta}$. Thus, by specifying a small value of δ we determine an ϵ which gives a neighborhood N as above. We fix such a value of δ and denote $M_0 = M - N$. Also, let N_1 be the smaller neighborhood determined by boundary area $\delta/2$ and put $M_1 = M - N_1$, so $M_0 \subset M_1$. Denote $K = M_1 - \text{int}\,(M_0)$.

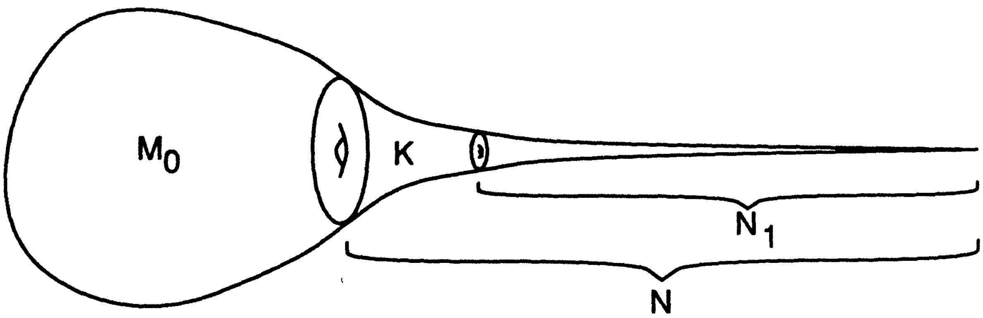

Figure 1

For $p^2 + q^2$ sufficiently large, $M(p, q)$ has a hyperbolic structure almost isometric to the union of M_1 and a suitably metrized solid torus $T(p, q; M)$. Thus on $M(p, q) \times I$ we can find a metric which is almost a product metric, and which has $M(p, q)$ and $M_1 \cup T(p, q; M)$ as its two boundary components, and has product metric in a collar neighborhood of each boundary component. Denote $M(p, q) \times I$ with this metric by Y_0.

Let T be a Riemannian solid torus with boundary isometric to $\partial(M_1)$ and with a neighborhood of the boundary isometric to K, as indicated in the very schematic Fig. 2. The "longitudinal" homology class $\mathbf{l}$ in K should represent 0 in T.

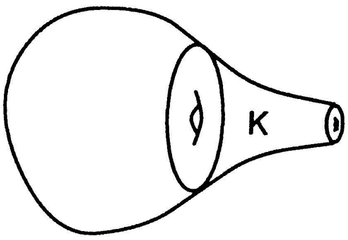

Figure 2

$T \cup T(p, q; M)$ is the lens space $L(p, q)$ with a Riemannian metric on it. Put the product metric on $L(p, q) \times I$. Let Y_1 be the result of pasting Y_0 to $L(p, q) \times I$ along the copy of $T(p, q; M)$ in the boundary of each.

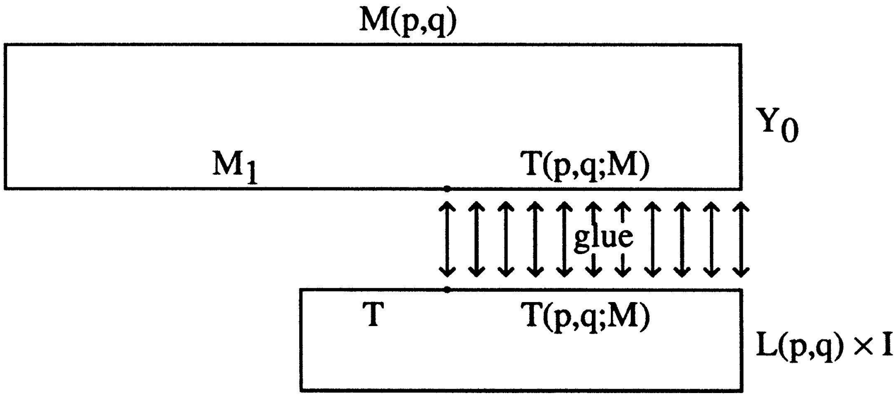

Figure 3

Let $Y(p, q)$ be Y_1 modified in a small neighborhood U of the boundary component $M_1 \cup T$ to make that boundary component smooth with product metric in a neighborhood of it. This modification should be done independently of p and q. Call the resulting Riemannian boundary component M'.

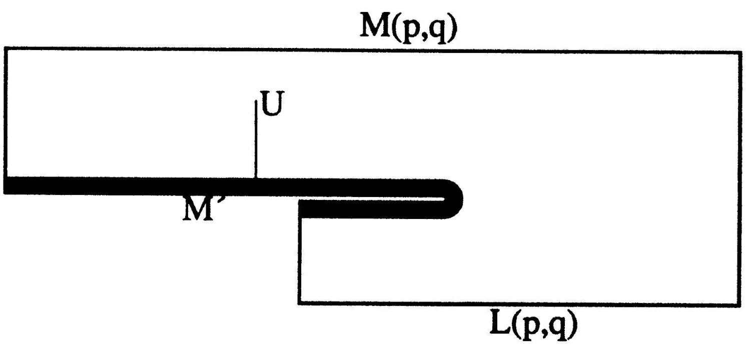

Figure 4

If p_1 is the first Pontryagin form on $Y(p, q)$, then the definition of the η-invariant says that

$$\eta(\partial Y(p, q)) = \int_{Y(p,q)} \frac{p_1}{3} - \operatorname{sign} Y(p, q). \tag{1.8}$$

This can be re-written

$$\eta(M(p, q)) = \eta(L(p, q)) - \eta(M') + \int_{Y(p,q)} \frac{p_1}{3} - \operatorname{sign} Y(p, q). \tag{1.9}$$

Since the metric on $Y(p, q)$ is almost a product outside the neighborhood U, p_1 almost vanishes outside U, so

$$\int_{Y(p,q)} \frac{p_1}{3} = \int_U \frac{p_1}{3} + o(1). \tag{1.10}$$

Moreover, it is easy to compute (e.g. using Wall's formula [W]) that $\operatorname{sign} Y(p, q) = 0$, so (1.9) becomes

$$\eta(M(p, q)) = \eta(L(p, q)) + C_0(M) + o(1), \tag{1.11}$$

where $C_0(M)$ is independent of p and q. Thus formula (1.7) becomes

$$-I(p, q; M) = 3\eta(L(p, q)) + \frac{q'}{p} + C_1(M) + o(1), \tag{1.12}$$

with $C_1(M) = 3C_0(M) + g(0)$.

For fixed p and q, the metric on $L(p, q)$ may be taken to depend only on the original shape τ of the cusp torus of M (and to depend continuously on this τ). To complete the proof of (1.4) we must show that $3\eta(L(p, q))$ has the form $-I(p, q) - (q'/p) + C_2(\tau) + o(1)$. In [Mh; Section 3.1] it is shown, in a context similar to ours, that

$$\begin{aligned} 3\eta(L(p, q)) &= 2\,\mathrm{CS}\,(L(p, q)) - I(p, q; \tau) \\ &= \int_F Q - \frac{1}{2\pi}(\text{torsion}\,(\gamma(p, q)) - \text{torsion}\,(\gamma)) - I(p, q; \tau), \end{aligned} \tag{1.13}$$

where $I(p, q; \tau)$ is some integer and F is an orthonormal frame field on $L(p, q)$

which is singular at the (p, q) core and at the core γ of the torus T. Moreover, the frame field F hardly depends on p and q for p^2+q^2 large, so, by (1.6), (1.13) can be written

$$3\eta(L(p, q)) = C_2(\tau) + o(1) - \frac{q'}{p} - I(p, q; \tau). \tag{1.14}$$

Up to $o(1)$, equation (1.14) equates the integer valued function $I(p, q; \tau)$ of τ with a continuous function of τ, so $I(p, q; \tau)$ is independent of τ, completing the proof of Step 1.

Step 2. We must show that, up to a constant, the function $I(p, q)$ can be given by the formulae claimed in the Introduction. Yoshida's calculation (in [Y, Theorem 3 and subsequent discussion]; recall that his q is the negative of ours) for the figure-eight knot complement gives

$$3\eta(M(p, q)) = -\frac{1}{p}(3 \operatorname{def}(p; q, 1) + q) + o(1), \tag{1.15}$$

so the result follows by inserting this in equation (1.7).

2. Several cusps

In Theorem 1 we chose the basis element $\mathbf{l}$ to be a "longitude." This was used to see that sign $Y(p, q) = 0$ in the step from equation (1.9) to equation (1.11). Without this choice of $\mathbf{l}$, the formula of Theorem 1 must be corrected by $3i$ sign $Y(p, q)$:

$$\begin{aligned}\frac{1}{\pi^2}&\operatorname{Vol}(M(p, q)) + 3i\eta(M(p, q))\\ &= f(u(p, q)) - \frac{1}{2\pi}\lambda(\gamma(p, q)) - iI(p, q) - 3i \operatorname{sign} Y(p, q).\end{aligned} \tag{2.1}$$

The analytic function $f(u)$ in this formulation differs from the one in Theorem 1 by a constant which depends on the choice of basis. The ingredient sign $Y(p, q)$ is 0 if any two of longitude, $\mathbf{l}, p\mathbf{m}+q\mathbf{l}$ are linearly dependent, and otherwise we have longitude $= a\mathbf{l} + b(p\mathbf{m}+q\mathbf{l})$ for some $a, b \in \mathbf{Q}$ and then sign $Y(p, q) = \operatorname{sign} abp$.

If M is a complete finite-volume hyperbolic 3-manifold with several cusps, then there is, in general, no natural choice of "longitudes" at the cusps, so the analog of

Theorem 1 must be stated in the form corresponding to equation (2.1). Suppose M has h cusps and a basis of homology $\mathbf{l}_j$, $\mathbf{m}_j$ has been chosen at the j-th cusp for each j. Let $(\mathbf{p}, \mathbf{q}) = (p_1, q_1; \dots; p_h, q_h)$ and let $M(\mathbf{p}, \mathbf{q})$ denote the result of Dehn filling all the cusps of M to kill the class $p_j\mathbf{m}_j + q_j\mathbf{l}_j$ at the j-th cusp for each j. Essentially the same proof as in Section 1 shows:

THEOREM 2. *If each $p_j^2 + q_j^2$ is sufficiently large then*

$$\left(\frac{1}{\pi^2}\operatorname{Vol} + 3i\eta\right)(M(\mathbf{p}, \mathbf{q}))$$
$$= f(\mathbf{u}(\mathbf{p}, \mathbf{q})) - 3i \operatorname{sign} Y(\mathbf{p}, \mathbf{q}) - \sum_{j=1}^{h}\left(\frac{1}{2\pi}\lambda(\gamma_j(\mathbf{p}, \mathbf{q})) + iI(p_j, q_j)\right).$$

Here $Y(\mathbf{p}, \mathbf{q})$ is the result of pasting $L(p_j, q_j) \times I$ to $M(\mathbf{p}, \mathbf{q}) \times I$ for each j, as in Section 1. Its signature sign $Y(\mathbf{p}, \mathbf{q})$ can be computed by Wall's formula [W] and depends only on $\operatorname{Ker}(H_1(\text{cusps}) \to H_1(M))$ and the elements $\mathbf{l}_1, \dots, \mathbf{l}_h$, $p_1\mathbf{l}_1 + q_1\mathbf{m}_1, \dots, p_h\mathbf{l}_h + q_h\mathbf{m}_h$, of $H_1(\text{cusps})$.

On the other hand, Theorem 2 can be derived as a formal consequence of Theorem 1. This implies that the Wall non-additivity term sign $Y(\mathbf{p}, \mathbf{q})$ can be computed in terms of the function $I(p, q)$. Such formulae are known, see [M] and [M–S].

An amusing consequence of our result, probably of little use, is that if one knows $\eta(M(\mathbf{p}, \mathbf{q}))$ for all sufficiently large $p_j^2 + q_j^2$, then one can compute $\operatorname{Ker}(H_1(\text{cusps}) \to H_1(M))$. We leave the proof of this to the reader.

3. Density of eta

Let M be a hyperbolic 3-manifold with one cusp.

THEOREM 3. *$\eta(M(p, q))$ takes on a dense set of values in $\mathbf{R}$ as $p^2 + q^2 \to \infty$.*

Proof. By (1.7) and our first formula for $I(p, q)$,

$$\eta(M(p, q)) = G(p, q) - \frac{1}{p}\operatorname{def}(p; q, 1) - \frac{q}{3p}, \tag{3.1}$$

where $G(p, q)$ approaches a constant value G as $p^2 + q^2 \to \infty$.

That $I(p, q)$ is an integer (see also [H–Z], Section 5.1, formula (19)) implies

$$\frac{1}{p}\operatorname{def}(p; q, 1) + \frac{q}{3p} = \frac{q'}{3p} \pmod{\tfrac{1}{3}\mathbf{Z}} \tag{3.2}$$

where q' satisfies $0 \le q'/p < 1$ and $qq' \equiv -1 \pmod{|p|}$. It follows that by judicious choice of q and p we can achieve a dense set of values for $(1/p)\operatorname{def}(p; q, 1) + q/(3p)$ in the circle $\mathbf{R}/(\frac{1}{3}\mathbf{Z})$, and in fact we can do this while assuming that $p^2 + q^2$ is large (compare with Section 5.3 of [Mh]). This easily proves density for η in $\mathbf{R}/(\frac{1}{3}\mathbf{Z})$, but to get density in $\mathbf{R}$ we need a little more work.

By the reciprocity formula for Dedekind sums (see [H–Z], Section 5.1, Theorem 1) we have the formula (equivalent to the last recursion formula for $I(p, q)$ in the Introduction)

$$\frac{1}{p}\operatorname{def}(p; q, 1) + \frac{q}{3p} = -\frac{1}{q}\operatorname{def}(q; p, 1) - \frac{p}{3q} - \frac{1}{3pq} + 1. \tag{3.3}$$

Of course, we also have that

$$\operatorname{def}(p; q, 1) = \operatorname{def}(p; q + tp, 1) \qquad \text{for all } t \in \mathbf{Z}. \tag{3.4}$$

To prove the Theorem, for given $\epsilon > 0$ and real b we must find (p_0, q_0) with $p_0^2 + q_0^2$ large so that $|\eta(M(p_0, q_0)) - b)| < \epsilon$.

Since $G(p, q) \to G$ as $p^2 + q^2 \to \infty$, we can find $N \in \mathbf{Z}_+$ such that for $p^2 + q^2 > N$ we have $|G - G(p, q)| < \epsilon/6$ and $1/|pq| < \epsilon/6$ and $M(p, q)$ is hyperbolic.

Choose positive integers p and q with $p^2 + q^2 > N$ so that

$$\frac{q'}{3p} + (b - G) = \frac{u}{3} + \delta,$$

with $|\delta| < \epsilon/4$ and $u \in \mathbf{Z}$. Note that, by (3.2),

$$\frac{q'}{3p} = \frac{q}{3p} + \frac{1}{p}\operatorname{def}(p; q, 1) - \frac{v}{3}$$

with $v \in \mathbf{Z}$. Let $t = u + v$. Then by (3.1) and the lat two formulae,

$$\begin{aligned}\left|\eta(M(p,q)) + \frac{t}{3} - b\right| &= \left|G(p,q) - \frac{1}{p}\operatorname{def}(p;q,1) - \frac{q}{3p} + \frac{u+v}{3} - b\right| \\ &= \left|G(p,q) - \frac{q'}{3p} + \frac{u}{3} - b\right| \\ &\leq |G(p,q) - G| + \left|G - \frac{q'}{3p} + \frac{u}{3} - b\right| \\ &< \frac{\epsilon}{6} + |\delta| < \frac{\epsilon}{2}.\end{aligned}$$

Suppose t is negative. Then by (3.1) and (3.4)

$$\begin{aligned}&\left|\eta(M(p,q)) + \frac{t}{3} - \eta(M(p,q-tp))\right| \\ &= \left|\left(G(p,q) - \frac{1}{p}\operatorname{def}(p;q,1) - \frac{q}{3p} + \frac{t}{3}\right)\right. \\ &\quad \left. - \left(G(p,q-tp) - \frac{1}{p}\operatorname{def}(p;q-tp,1) - \frac{q-tp}{3p}\right)\right| \\ &= |G(p,q) - G(p,q-tp| \leq |G(p,q) - G| + |G - G(p,q-tp)| \\ &< \frac{\epsilon}{6} + \frac{\epsilon}{6} < \frac{\epsilon}{2}.\end{aligned}$$

Thus

$$|\eta(M(p,q-tp)) - b| < \epsilon,$$

so in this case we take $p_0 = p$ and $q_0 = q - tp$.

If t is positive then $p^2 + (q - tp)^2$ might not be greater than N, so we must modify the above argument. Formulae (3.1) and (3.3) are used at the first step.

$$\begin{aligned}
&\left|(\eta(M(p, q)) + \frac{t}{3} - \eta(M(p + tq, q))\right| \\
&= \left|\left(G(p, q) + \frac{1}{q} \operatorname{def}(q; p, 1) + \frac{p}{3q} + \frac{1}{3pq} - 1 + \frac{t}{3}\right)\right. \\
&\quad \left. - \left(G(p + tq, q) + \frac{1}{q} \operatorname{def}(q; p + tq, 1) + \frac{p + tq}{3q} + \frac{1}{3(p + tq)q} - 1\right)\right| \\
&= \left|G(p, q) - G(p + tq, q) + \frac{1}{3pq} - \frac{1}{3(p + tq)q}\right| \\
&\leq |G(p, q) - G| + |G - G(p + tq, q)| + \left|\frac{1}{3pq} - \frac{1}{3(p + tq)q}\right| \\
&\leq \frac{\epsilon}{6} + \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{2}.
\end{aligned}$$

Thus

$$|\eta(M(p + tq, q)) - b| < \epsilon,$$

so in this case we take $p_0 = p + tq$ and $q_0 = q$, completing the proof.

4. Examples

The figure-eight knot complement. The standard ideal triangulation of the figure-eight knot complement m (see e.g., [T] and [NZ]) has two ideal 3-simplices with parameters z and w satisfying the consistency relation

$$\log z + \log (1 - z) + \log w + \log (1 - w) = 0. \tag{4.1}$$

(Here log will always denote the standard branch of natural log on the complex plane split along $(-\infty, 0]$.) The usual analytic Dehn surgery parameter is

$$u = \log w + \log (1 - z). \tag{4.2}$$

However, z and w (constrained by equation (4.1)) are analytic functions of u, so we shall write the analytic part of the equation of Theorem 1 as a function of z and w. The real Dehn surgery parameter (p, q) is determined from z and w by the equation

$$pu + qv = 2\pi i \quad \text{with} \quad v = -\log z^2(1-z)^2. \tag{4.3}$$

(The sign of v is opposite to that used in [T], [NZ], and [Y], to conform with standard orientation conventions.) Let $R(z)$ denote the Roger's dilogarithm

$$\begin{aligned} R(z) &= \frac{1}{2}\log z \log(1-z) + \mathrm{Li}_2(z) \\ &= \frac{1}{2}\log z \log(1-z) - \int_0^z \log(1-t)\,d\log t. \end{aligned} \tag{4.4}$$

The ingredients in Theorem 1 in this case are

$$f(u) = \frac{1}{\pi^2 i}\left(R(z) + R(w) - \frac{\pi^2}{6}\right), \tag{4.5}$$

$$\lambda(\gamma(p, q)) = \frac{q'}{p}\,2\pi i + \frac{1}{p}\,v, \tag{4.6}$$

so the formula of Theorem 1 becomes

$$\begin{aligned} &\frac{1}{\pi^2}\mathrm{Vol}\,(M(p, q)) + 3i\eta(M(p, q)) \\ &= \frac{1}{\pi^2 i}\left(R(z) + R(w) - \frac{\pi^2}{6}\right) - \frac{q'}{p}\,i + \frac{1}{2\pi p}\log z^2(1-z)^2 - iI(p, q) \\ &= \frac{1}{\pi^2 i}\left(R(z) + R(w) - \frac{\pi^2}{6}\right) + \frac{1}{2\pi p}\log z^2(1-z)^2 - \frac{i}{p}(3\,\mathrm{def}\,(p; q, 1) + q). \end{aligned} \tag{4.7}$$

Indeed, the imaginary part of this equation is Theorem 3 of Yoshida [Y] (with sign of q reversed – see above), while the real part reduces easily to the usual simplicial formula for volume (see [T], [NZ], [N]),. Note that in this case Yoshida shows that the formula is valid for all coprime p and q (other than the ones that do not give a hyperbolic structure: $(p, \pm 1)$ with $|p| \le 4$, and $(\pm 1, 0)$).

The complement of the Whitehead link. In [T] Thurston describes how to obtain the complement W of the Whitehead link (Fig. 5) by identifying faces of an ideal octahedron in pairs. The identification matches face A with A', B with B', etc., in Fig. 6, so as to respect the labelling of the edges.

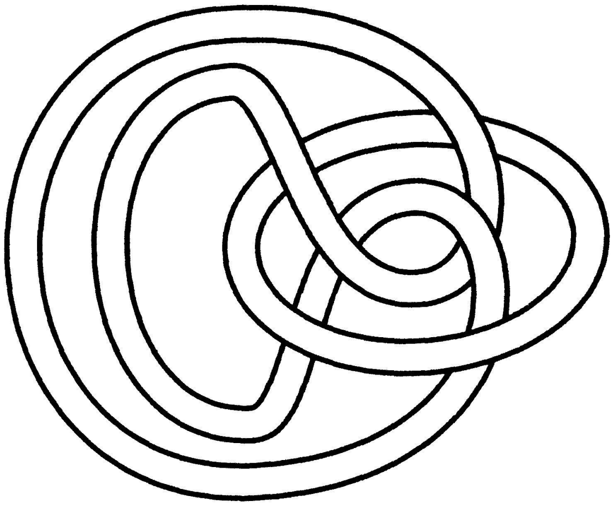

Figure 5

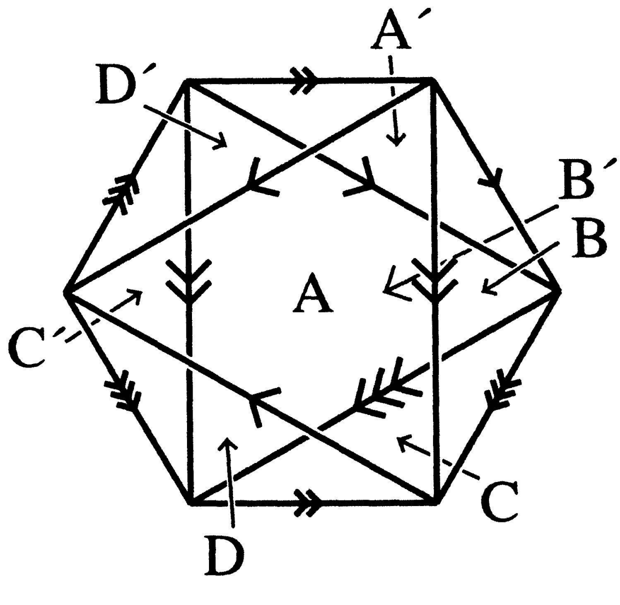

Figure 6

By subdividing the octahedron (Fig. 7), we obtain an ideal triangulation of W with four simplices.

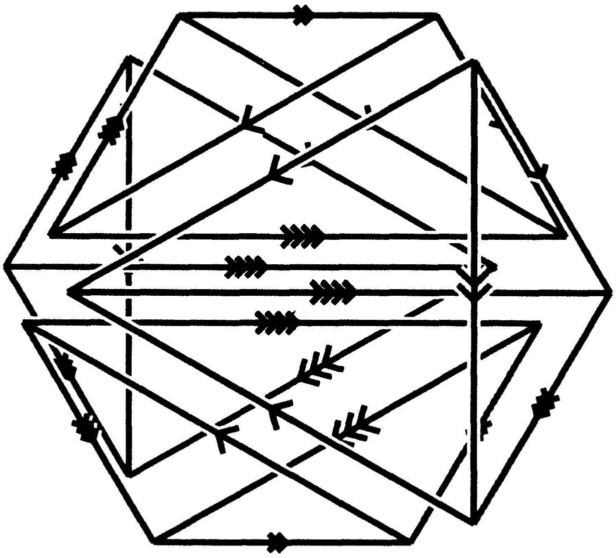

Figure 7

By cutting off the ends of W one obtains a compact manifold-with-boundary $\bar{W}$ which can be obtained by identifying truncated tetrahedra as in Fig. 8. The two boundary tori of $\bar{W}$ are triangulated as in Fig. 9, where the vertices are labelled according to the edges of the triangulation of W. Careful inspection shows that the standard topological meridian and longitude of each component of the Whitehead link are as indicated in Fig. 10, where we have also included labels for the complex parameters of the four tetrahedra.

We can read off from Fig. 10 the consistency relations at the four edges:

$$\log w'' + \log z' + \log x'' + \log w' + \log x'' + \log y' + \log w'' + \log x' = 2\pi i,$$

$$\log w + \log x + \log y + \log z = 2\pi i,$$

$$\log z'' + \log w' + \log y'' + \log z' + \log y'' + \log x' + \log z'' + \log y' = 2\pi i,$$

$$\log w + \log x + \log z + \log y = 2\pi i.$$

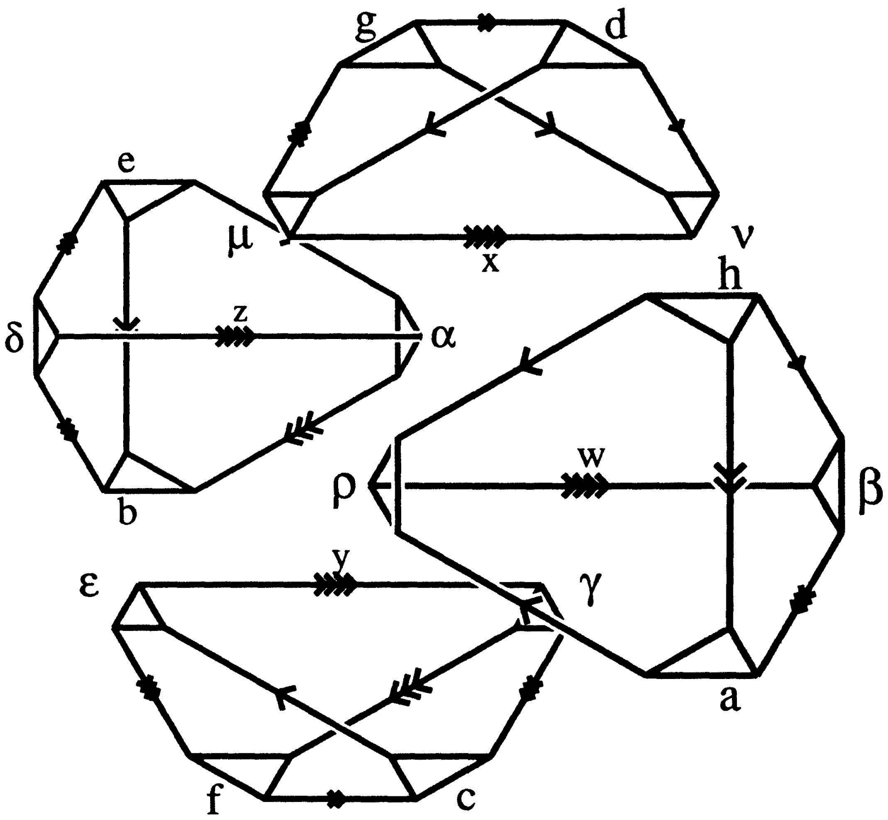

Figure 8

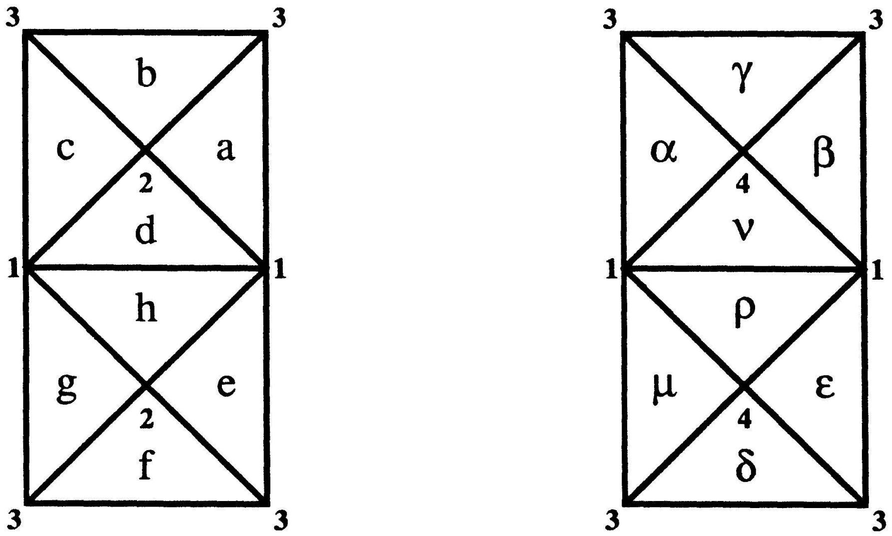

Figure 9

Since $x' = (x-1)/x$ and $x'' = 1/(1-x)$ and similarly for w, y, z, these simplify to the two relations:

$$\begin{aligned} &\log w + \log x + \log y + \log z = 2\pi i, \\ &\log(1-w) + \log(1-x) - \log(1-y) - \log(1-z) = 0. \end{aligned} \tag{4.8}$$

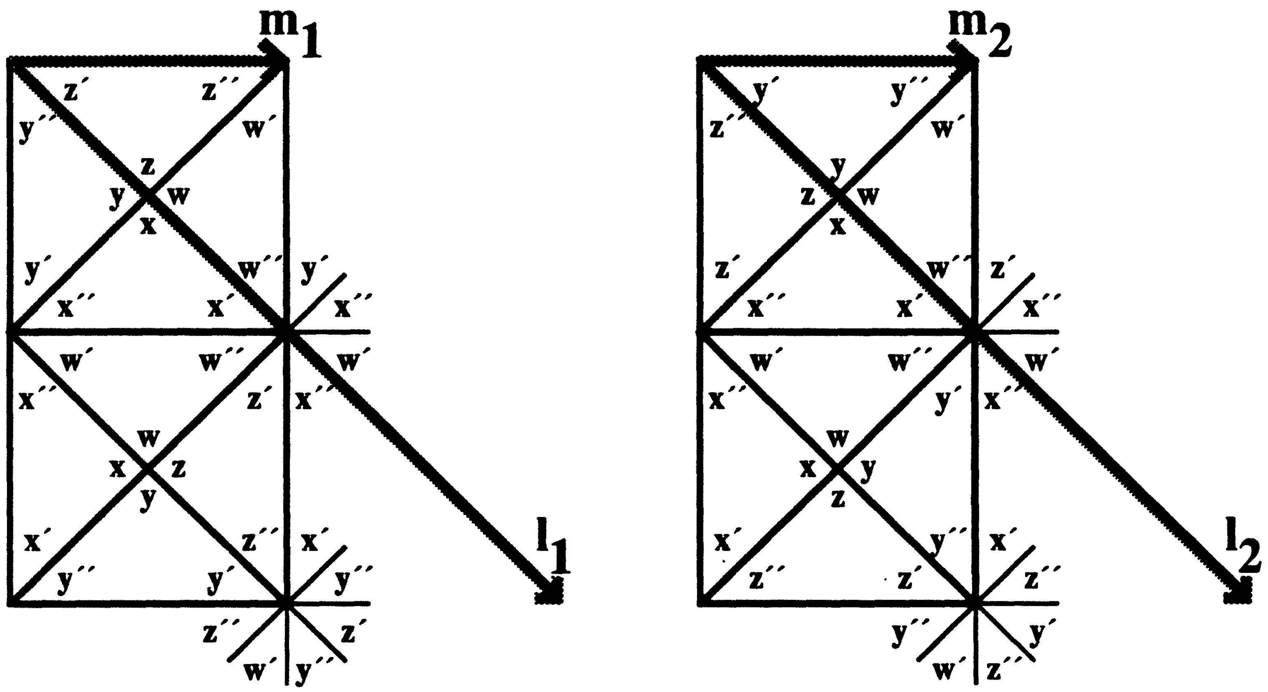

Figure 10

Similarly, we can read off the parameters u_1, v_1, u_2, v_2, which describe the holonomy of the meridians and longitudes $\mathbf{m}_1$, $\mathbf{l}_1$, $\mathbf{m}_2$, $\mathbf{l}_2$, at the two cusps:

$$\begin{aligned}
u_1 &= \log y'' + \log z' + \log z'' + \log w' - \pi i,\\
v_1 &= \log y'' + \log y + \log x + \log x' + \log w'' + \log z' + \log x''\\
&\quad + \log x + \log y + \log y' + \log z'' + \log w' - 4\pi i,\\
u_2 &= \log z'' + \log y' + \log y'' + \log w' - \pi i,\\
v_2 &= \log z'' + \log z + \log x + \log x' + \log w'' + \log y' + \log x''\\
&\quad + \log x + \log z + \log z' + \log y'' + \log w' - 4\pi i.
\end{aligned}$$

Expressed in terms of x, y, z, w and simplified using (4.8) this gives:

$$\begin{aligned}
u_1 &= \log(w-1) + \log x + \log y - \log(y-1) - \pi i,\\
v_1 &= 2\log x + 2\log y - 2\pi i,\\
u_2 &= \log(w-1) + \log x + \log z - \log(z-1) - \pi i,\\
v_2 &= 2\log x + 2\log z - 2\pi i.
\end{aligned} \tag{4.9}$$

(u_1, u_2) can be taken as the analytic Dehn surgery parameter, in which case w, x, y, z, constrained by equations (4.8), become analytic functions of this parameter, but we shall give the formula of Theorem 1 in terms of the tetrahedral parameters w, x, y, and z. The real Dehn surgery parameters (p_1, q_1) and (p_2, q_2) are determined by the equations

$$\begin{aligned} p_1 u_1 + q_1 v_1 &= 2\pi i, \\ p_2 u_2 + q_2 v_2 &= 2\pi i. \end{aligned} \tag{4.10}$$

The term $-3i$ sign $Y(\mathbf{p}, \mathbf{q})$ of Theorem 2 vanishes in this case (this happens for any link complement whose components have pairwise linking numbers zero if one chooses the $\mathbf{m}_j$ and $\mathbf{l}_j$ to be topological meridans and longitudes for the link components). The analytic function f of Theorem 2 in this case is

$$f(\mathbf{u}) = \frac{1}{\pi^2 i}\left(R(w) + R(x) + R\left(\frac{1}{1-y}\right) + R\left(\frac{1}{1-z}\right)\right) + 2i, \tag{4.11}$$

and the complex lengths of geodesics are given by

$$\lambda(\gamma_j(\mathbf{p}, \mathbf{q})) = \frac{q_j'}{p_j} 2\pi i + \frac{1}{p_j} v_j, \qquad j = 1, 2. \tag{4.12}$$

Indeed, up to an imaginary constant (4.11) follows from [N] (it suffices to check that the real part of the formula of Theorem 2 gives volume correctly). The value $2i$ of the imaginary constant was determined numerically by noting that $W(1, 1; p, q)$ is (p, q)-Dehn surgery on the figure-eight knot complement, so $\eta W(1, 1; p, q) = -\eta W(1, 1, p, -q)$. Using known values of the Chern–Simons invariant, e.g., for the Whitehead link complement itself or for the figure-eight knot complement, it was easy to see that the constant had to be a multiple of $i/2$, so the numerical experiments did not have to be accurate.

In fact the formula has been programmed to over 50 digit accuracy. In most cases the eta invariant appears to be irrational (however, it is not *proved* irrational in any example – the same holds for volume). In many cases one expects rational eta invariant for geometric reasons (some cover has an orientation reversing self-homeomorphism), and the computation bears this out. As a sample computation: for $N = W(3, -2\colon 6, -1)$ the formula gives

$$\begin{aligned} &\mathrm{Vol}\,(N) + 3i\pi^2\eta(N) \\ &\qquad \approx 1.01494160640965362502120255427452028594168930753029979 + 0i. \end{aligned}$$

Note that its volume is that of a regular ideal tetrahedron – half that of the figure-eight knot complement; N is presumably arithmetic over the field $\mathbf{Q}(\sqrt{-3})$. Jeff Weeks has informed us that this manifolds admits an orientation-reversing diffeomorphism, explaining the computed vanishing of eta.

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