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# A toroidal compactification of the Fermi surface for the discrete Schrödinger operator

D. BÄTTIG

## 1. Introduction

Let  $\Lambda = a_1\mathbb{Z} \oplus a_2\mathbb{Z} \oplus a_3\mathbb{Z}$  be a lattice in  $\mathbb{R}^3$  and  $q$  a real valued square-integrable function on the torus  $\mathbb{R}^3/\Lambda$ . For each  $\xi = (\xi_1, \xi_2, \xi_3) \in S^1 \times S^1 \times S^1$  the self-adjoint boundary value problem, called the independent electron approximation of solid state physics (see [1]),

$$(-\Delta + q)\psi = \lambda\psi,$$

$$\psi(x + \gamma) = \xi_1^{\gamma_1} \xi_2^{\gamma_2} \xi_3^{\gamma_3} \psi(x) \quad \forall \gamma \in \Lambda,$$

has discrete spectrum, denoted by

$$E_1(\xi) \leq E_2(\xi) \leq E_3(\xi) \leq \dots$$

The (physical) Fermi surface for energy  $\lambda$  is the set

$$F_{\text{phys},\lambda}(q) := \{\xi \in S^1 \times S^1 \times S^1 \mid E_n(\xi) = \lambda \text{ for some } n \geq 1\}.$$

In [3] one defines the complex Fermi surface by

$$F_\lambda(q) := \{(\xi_1, \xi_2, \xi_3) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \mid \text{there exists a non-trivial function } \psi \text{ in } H_{\text{loc}}^2(\mathbb{R}^3) \text{ solving the above boundary value problem}\}.$$

Clearly  $F_\lambda(q)$  contains all points that can be reached by analytic continuation of  $F_{\text{phys},\lambda}(q)$ . Using regularized determinants (see [7]) it can be shown, that  $F_\lambda(q)$  is a complex analytic hypersurface in  $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ . In [3] it was shown that for potentials  $q(x)$  of the form  $p_1(x_1) + p_2(x_2) + p_3(x_3)$  or  $p_1(x_1) + p_2(x_2, x_3)$  the surface  $F_\lambda(q)$  is irreducible, i.e. in this case  $F_{\text{phys},\lambda}(q)$ , if it is a nonempty set of dimension two, determines  $F_\lambda(q)$  uniquely.

In this paper we consider a discrete version and show that for each (complex) potential  $q$ , the Fermi surface is always irreducible.

So let  $V : \mathbb{Z}^3 \rightarrow \mathbb{C}$  an arbitrary function periodic with respect to the lattice  $\Gamma$ . Furthermore let  $\Delta$  be the discrete Laplace operator defined by

$$(\Delta\psi)(m, n, p) = \psi(m-1, n, p) + \psi(m+1, n, p) + \psi(m, n-1, p) \\ + \psi(m, n+1, p) + \psi(m, n, p-1) + \psi(m, n, p+1)$$

for functions  $\psi : \mathbb{Z}^3 \rightarrow \mathbb{C}$ .

We are interested in the spectral problem

$$(-\Delta + V)\psi = \lambda\psi$$

with boundary conditions

$$\psi(m+a_1, n, p) = \xi_1\psi(m, n, p), \quad \psi(m, n+a_2, p) = \xi_2\psi(m, n, p), \\ \psi(m, n, p+a_3) = \xi_3\psi(m, n, p)$$

for functions  $\psi : \mathbb{Z}^3 \rightarrow \mathbb{C}$  and  $(\lambda, \xi_1, \xi_2, \xi_3) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ , and define as above the complex Fermi surface  $F_\lambda(V)$  for this discrete problem (see [4]). Furthermore we assume that  $a_1, a_2$  and  $a_3$  are relatively prime, positive natural numbers greater to two.

Due to the boundary conditions the spectral problem can be written in terms of the values  $\psi(m, n, p)$  with  $1 \leq m \leq a_1, 1 \leq n \leq a_2, 1 \leq p \leq a_3$ . The Fermi surface  $F_\lambda(V)$  is then given by the vanishing of the determinant of a certain  $a_1 a_2 a_3 \times a_1 a_2 a_3$ -matrix, or concrete, it is the zero-set of a polynomial  $P$  in the variables  $\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}$  and  $\xi_3, \xi_3^{-1}$ , where the coefficients depend on  $\lambda$ :

$$P = (-1)^{a_2 a_3 (a_1 - 1)} \xi_1^{a_2 a_3} + (-1)^{a_2 a_3 (a_1 - 1)} \xi_1^{-a_2 a_3} \\ + (-1)^{a_1 a_3 (a_2 - 1)} \xi_2^{a_1 a_3} + (-1)^{a_1 a_3 (a_2 - 1)} \xi_2^{-a_1 a_3} \\ + (-1)^{a_1 a_2 (a_3 - 1)} \xi_3^{a_1 a_2} + (-1)^{a_1 a_2 (a_3 - 1)} \xi_3^{-a_1 a_2} + \dots$$

lower order terms, i.e. an algebraic surface in  $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ .

For potentials  $V = C = \text{constant}$ ,  $F_\lambda(V)$  can be calculated explicitly, using discrete Fourier analysis. Namely let  $\mu_{a_i}$  be the multiplicative group of  $a_i$ -th root of unity. Then for  $\rho = (\rho_1, \rho_2, \rho_3) \in \mu_{a_1} \times \mu_{a_2} \times \mu_{a_3}$  consider the set

$$\tilde{F}_\lambda(V = C) = \bigcup_{\rho} \left\{ (z_1, z_2, z_3) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \mid \sum_{i=1}^3 ((\rho_i z_i)^{-1} + (\rho_i z_i)) = \lambda - C \right\}.$$

Now  $\mu_{a_1} \times \mu_{a_2} \times \mu_{a_3}$  acts on  $\tilde{F}_\lambda(V = C)$  by  $\rho \cdot z = (\rho_1 z_1, \rho_2 z_2, \rho_3 z_3)$ . Then one has  $F_\lambda(V) = \tilde{F}_\lambda(V = C) / \mu_{a_1} \times \mu_{a_2} \times \mu_{a_3}$ , and so  $F_\lambda(V)$  is irreducible.

It is well known, that the one-dimensional Bloch variety  $B(W)$ , defined by (see [4], [8])

$$B(W) = \{(\xi, \lambda) \in \mathbb{C}^* \times \mathbb{C} \mid \text{there exists a non-trivial function } \psi : \mathbb{Z} \rightarrow \mathbb{C} \text{ solving} \\ -[\psi(m-1) + \psi(m+1)] + W(m)\psi(m) = \lambda\psi(m), \psi(m+a) = \psi(m)\},$$

where  $W : \mathbb{Z} \rightarrow \mathbb{C}$  has period  $a$ , is a hyperelliptic curve of arithmetic genus  $a-1$ , which can be compactified by adding two smooth points.

In this paper we construct a compactification  $F_\lambda(V)_{\text{comp}}$  of  $F_\lambda(V)$ , such that the generic points of  $F_\lambda(V)_{\text{comp}}$  are smooth points of  $F_\lambda(V)_{\text{comp}}$ .

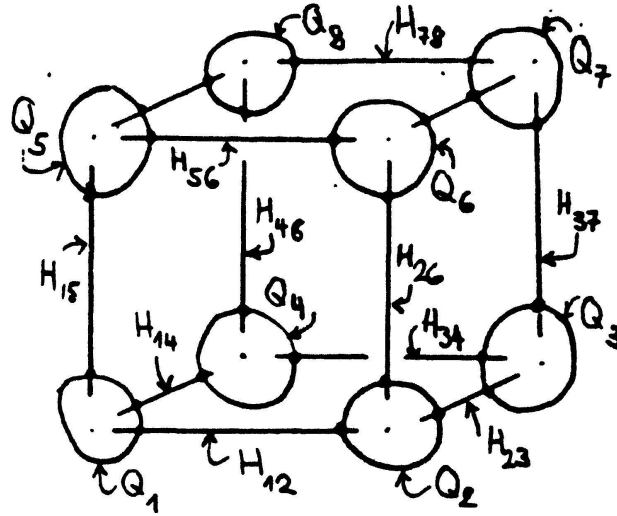
**THEOREM 1.**  $F_\lambda(V)_{\text{comp}} - F_\lambda(V)$  is the union of twenty algebraic curves due to twenty one-dimensional spectral problems:

- (i) eight rational curves  $Q_1, \dots, Q_8$  with  $(a_1-1)(a_2-1)(a_3-1) + \sum_{i \neq j} (a_i-1)(a_j-1)$  ordinary double points. These curves do not depend on the potential  $V$ .
- (ii) Twelve hyperelliptic curves  $H_{ij} : H_{12}, H_{34}, H_{56}, H_{78}$  (resp.  $H_{14}, H_{58}, H_{23}, H_{67}$ ; resp.  $H_{15}, H_{26}, H_{37}, H_{48}$ ) of arithmetic genus  $a_1-1$  (resp.  $a_2-1$ , resp.  $a_3-1$ ), each isomorphic to the one-dimensional Bloch variety  $B(W)$ , where  $W$  is the averaged potential

$$W(\cdot) = \frac{1}{a_2 a_3} \sum_{i=1}^{a_2} \sum_{k=1}^{a_3} V(\cdot, i, k)$$

$$\left( \text{resp. } \frac{1}{a_1 a_3} \sum_{i=1}^{a_1} \sum_{k=1}^{a_3} V(i, \cdot, k); \text{ resp. } \frac{1}{a_1 a_2} \sum_{i=1}^{a_1} \sum_{k=1}^{a_2} V(i, k, \cdot) \right).$$

- (iii)  $F_\lambda(V)_{\text{comp}}$  is smooth on all smooth points of  $F_\lambda(V)_{\text{comp}} - F_\lambda(V)$ .
- (iv) All the above curves intersect transversally, only on smooth points of  $F_\lambda(V)_{\text{comp}}$  and the intersection pattern is given by the following picture:





As an immediate consequence we get

**THEOREM 2.** *The Fermi surface  $F_\lambda(V)$  is irreducible.*

Naively one could try to compactify  $F_\lambda(V)$  by embedding  $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$  in  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  and closing the Fermi surface in there. This doesn't work, since the new points added to  $F_\lambda(V)$  are highly singular. Instead we construct (as in [2] and [8]) a compact three-dimensional torus embedding  $X_\Sigma$ , such that

$$F_\lambda(V) \subset \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \subset X_\Sigma.$$

A torus embedding  $X_\Sigma$  is a scheme such that algebraic torus  $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$  can be embedded in  $X_\Sigma$  in a way, that the action of the algebraic torus can be extended to the whole torus embedding  $X_\Sigma$ . Information and facts about torus embeddings can be found in [5], [6] and [9]. The closure of  $F_\lambda(V)$  in this space  $X_\Sigma$  (after resolution of certain singular points of  $X_\Sigma$ ) is the compactified Fermi surface  $F_\lambda(V)_{\text{comp}}$ .

Furthermore we not only construct the torus embedding  $X_\Sigma$ , but also an infinite-dimensional vector bundle  $Y$ , the vectorspace  $F$  of all functions  $\psi : \mathbb{Z} \rightarrow \mathbb{C}$  as fiber, on  $X_\Sigma$ . On  $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times F \subset Y$  we have four commuting operators:

$$-\Delta + V - \lambda 1, \quad S^{(a_1, 0, 0)} - \xi_1 1, \quad S^{(0, a_2, 0)} - \xi_2 1, \quad S^{(0, 0, a_3)} - \xi_3 1,$$

for  $(\xi_1, \xi_2, \xi_3) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$  (here  $S^{(\alpha, \beta, \gamma)}$  denotes the shift operator in direction  $(\alpha, \beta, \gamma)$ ). The Fermi surface is then the support of the bundle

$$\{(\xi_1, \xi_2, \xi_3, \psi) \in \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times F \mid \text{the above four operators have a common kernel } \psi\}.$$

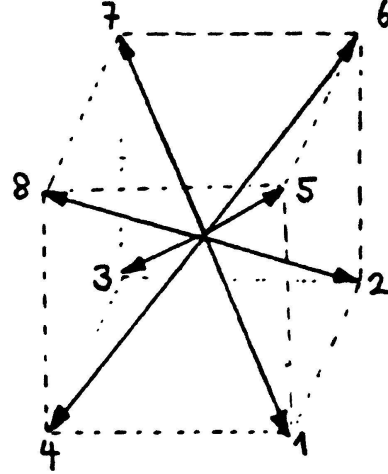
By extending this bundle to the whole  $X_\Sigma$  the rational and hyperelliptic curves mentioned in Theorem 1 will appear in a straightforward way.

Let us close by mentioning, that a similar construction was worked out in detail for the two-dimensional Bloch variety in [2].

## 2. The construction of the toroidal octahedron

In this chapter we construct the three-dimensional torus embedding, in which  $F_\lambda(V)$  will lie. Consider the eight vectors  $1, 2, \dots, 8$  in  $\mathbb{R}^3$  given by

$$\begin{aligned}
1 &:= (a_1, a_2, a_3), & 2 &:= (-a_1, a_2, a_3), \\
3 &:= (-a_1, -a_2, a_3), & 4 &:= (a_1, -a_2, -a_3), \\
5 &:= (a_1, a_2, -a_3), & 6 &:= (-a_1, a_2, -a_3), \\
7 &:= (-a_1, -a_2, -a_3), & 8 &:= (a_1, -a_2, -a_3).
\end{aligned}$$



We introduce the following notation:

$\sigma^i$  means the strongly convex polyhedral cone generated by the vector  $i$ .

So for example  $\sigma^{12} = \{t(a_1, a_2, a_3) + s(-a_1, a_2, a_3) \mid s, t \in \mathbb{R}_{\geq 0}\}$ , where  $\mathbb{R}_{\geq 0}$  denotes the non-negative real numbers.

We define the fan  $\Sigma$  to be the collection of the six three-dimensional cones  $\sigma^{1256}$ ,  $\sigma^{2367}$ ,  $\sigma^{3478}$ ,  $\sigma^{1458}$ ,  $\sigma^{5678}$  and  $\sigma^{1234}$  and all its faces. There are two-dimensional faces as  $\sigma^{12}$  or  $\sigma^{15}$ , one-dimensional faces as  $\sigma^1$ ,  $\sigma^2$  and one zero-dimensional face  $\sigma^0 = \{0\}$ . We call the torus embedding  $X_\Sigma$  associated to this fan toroidal octahedron. It is compact (see [5]). Explicitly  $X_\Sigma$  is given by a coordinate covering  $(X_\sigma)_{\sigma \in \Sigma}$ . The  $(X_\sigma)$ 's are (quasi)-affine varieties defined by

$$X_\sigma = \text{Spec } \mathbb{C}[\dots, \xi_1^{r_1} \xi_2^{r_2} \xi_3^{r_3}, \dots],$$

where  $\mathbb{C}[\dots, \xi_1^{r_1} \xi_2^{r_2} \xi_3^{r_3}, \dots]$  is the algebra generated by the polynomials  $\xi_1^{r_1} \xi_2^{r_2} \xi_3^{r_3}$  with  $(r_1, r_2, r_3)$  in  $\mathbb{Z}^3$  such that  $\langle (r_1, r_2, r_3), \sigma \rangle \geq 0$ .

If  $\sigma^\alpha$  and  $\sigma^\beta$  are two cones in  $\Sigma$ , then the charts  $X_{\sigma^\alpha}$  and  $X_{\sigma^\beta}$  are patched together along  $X_{\sigma^\alpha} \cap X_{\sigma^\beta}$ . So, for example  $X_{\sigma^{12}}$  and  $X_{\sigma^{13}}$  are glued together on  $X_{\sigma^{12}} \cap X_{\sigma^{13}} = X_{\sigma^1}$ .

Clearly we have  $X_{\sigma^0} = \text{Spec } \mathbb{C}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}, \xi_3, \xi_3^{-1}] = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ . So we embed the Fermi surface  $F_\lambda(V)$  by the inclusions

$$F_\lambda(V) \subset \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* = X_{\sigma^0} \subset X_\Sigma$$

in the toroidal octahedron.

In the following we analyze  $X_\Sigma$ .

Since the action of the algebraic torus  $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$  on itself can be extended to  $X_\Sigma$ , the toroidal octahedron is a disjoint union of  $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$  orbits. There is now a one-to-one correspondence between these orbits in  $X_\Sigma$  and the cones  $\sigma \in \Sigma$ ; so we can label an orbit by a cone  $\sigma : \mathbb{O}_\sigma$ . Furthermore we can organize this labeling such that (see [5]):

$$\mathbb{O}_\sigma \subset X_\Sigma, \quad \dim_{\mathbb{C}} \mathbb{O}_\sigma = 3 - \dim_{\mathbb{R}} \sigma,$$

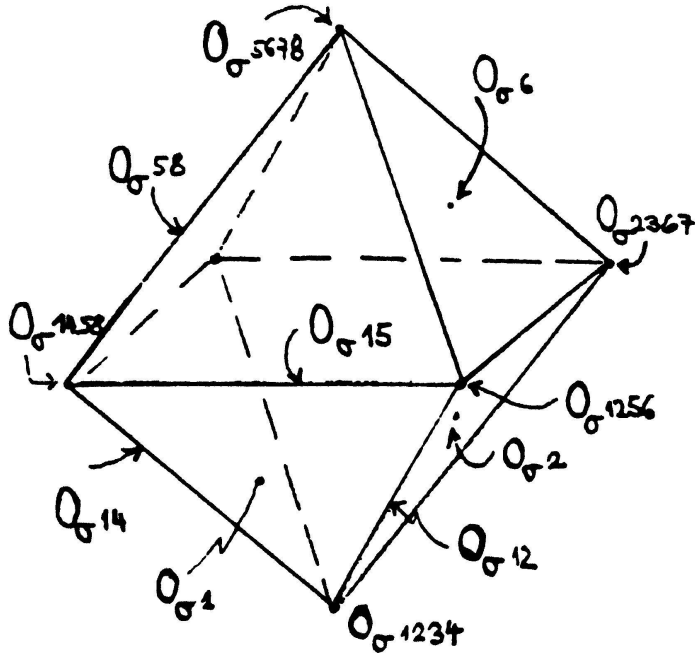
and  $\mathbb{O}_\sigma = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \cdot \lambda_b(0)$ , where  $\lambda_b(0)$  is the point

$$\lim_{t \rightarrow 0} X_\sigma(t) = \lim_{t \rightarrow 0} \text{Spec } \mathbb{C}[\dots, \xi_1^{r_1} \xi_2^{r_2} \xi_3^{r_3}, \dots] \Big|_{\text{setting } \xi_i = t^{b_i}},$$

where  $b = (b_1, b_2, b_3) \in \mathbb{Z}^3$  is a point in the interior of  $\sigma$ .

It is easy to draw a schematic picture of the toroidal octahedron  $X_\Sigma$  (compare with [2] and [7]):

The “interior” of this octahedron represents  $\mathbb{O}_{\sigma_0} = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ .



Using the symmetry of the fan  $\Sigma$  we can restrict to the orbits  $\mathbb{O}_{\sigma_0}$ ,  $\mathbb{O}_{\sigma_1}$ ,  $\mathbb{O}_{\sigma_{12}}$  and  $\mathbb{O}_{\sigma_{1234}}$ . Let  $y_0, z_0$  be integers with  $a_2 y_0 + a_3 z_0 = 1$ .

LEMMA 1. (i)  $X_{\sigma_0} = \text{Spec } \mathbb{C}[\xi_1, \xi_1^{-1}, \xi_2, \xi_2^{-1}, \xi_3, \xi_3^{-1}] = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$ .

(ii)  $X_{\sigma_1} = \text{Spec } \mathbb{C}[\xi_1^{-1}(\xi_2^{y_0} \xi_3^{z_0})^{a_1}, \xi_1(\xi_2^{y_0} \xi_3^{z_0})^{-a_1}, \xi_2^{y_0} \xi_3^{z_0}, \xi_2^{-a_3} \xi_3^{a_2}, \xi_2^{a_3} \xi_3^{-a_2}]$ , i.e.  $X_{\sigma_1}$  is isomorphic to  $\mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^*$  with local coordinates

$$u_1 := \xi_1^{-1}(\xi_2^{y_0} \xi_3^{z_0})^{a_1} \in \mathbb{C}^*, \quad v_1 := \xi_2^{y_0} \xi_3^{z_0} \in \mathbb{C} \quad \text{and} \quad w_1 := \xi_2^{-a_3} \xi_3^{a_2} \in \mathbb{C}^*.$$

Furthermore  $\mathbb{O}_{\sigma^1} = \{(u_1, w_1) \in \mathbb{C}^* \times \mathbb{C}^*, v_1 = 0\}$ .

(iii)  $X_{\sigma^{12}} = \text{Spec } \mathbb{C}[\xi_1^{-1}(\xi_2^{y_0}\xi_3^{z_0})^{a_1}, \xi_1(\xi_2^{y_0}\xi_3^{z_0})^{a_1}, \xi_2^{y_0}\xi_3^{z_0}, \xi_2^{-a_3}\xi_3^{a_2}, \xi_2^{a_3}\xi_3^{-a_2}]$  i.e.  $X_{\sigma^{12}}$  isomorphic to  $\text{Spec } \mathbb{C}[x, y, w, z, z^{-1}]/\langle xy = w^{2a_1}\rangle$ .  $\mathbb{O}_{\sigma^{12}} = \{z \in \mathbb{C}^*, x = y = w = 0\}$ , i.e. each point of  $\mathbb{O}_{\sigma^{12}}$  is singular of type  $A_{2a_1-1}$ .

*Proof*

(i) Trivial.

(ii) Clearly the three vectors  $(1, -a_1y_0, -a_1z_0), (0, y_0, z_0), (0, -a_3, a_2)$  form a  $\mathbb{Z}$ -basis of  $\mathbb{Z}^3$ . So each  $(r_1, r_2, r_3) \in \mathbb{Z}^3$  can be written as

$$(r_1, r_2, r_3) = r_1(1, -a_1y_0, -a_1z_0) + s(0, y_0, z_0) + t(0, -a_3, a_2) \quad (1)$$

with  $s, t \in \mathbb{Z}$ . Now  $\langle (r_1, r_2, r_3), \sigma^1 \rangle \geq 0$  exactly if  $a_1r_1 + a_2r_2 + a_3r_3 \geq 0$ . But  $s$  is equal to  $a_1r_1 + a_2r_2 + a_3r_3$  by (1). Therefore  $X_{\sigma^1}$  is given as stated in the lemma. Computing  $\mathbb{O}_{\sigma^1}$  is straightforward.

(iii) We have two  $\mathbb{Z}$ -bases of  $\mathbb{Z}^3$ ; first the three vectors  $(1, -a_1y_0, -a_1z_0), (0, y_0, z_0), (0, -a_3, a_2)$  and second the vectors  $(1, a_1y_0, a_1z_0), (0, y_0, z_0), (0, -a_3, a_2)$ . So for each  $(r_1, r_2, r_3) \in \mathbb{Z}^3$  we can write

$$(r_1, r_2, r_3) = r_1(1, -a_1y_0, -a_1z_0) + s(0, y_0, z_0) + t(0, -a_3, a_2), \quad (2)$$

$$(r_1, r_2, r_3) = r_1(1, a_1y_0, a_1z_0) + \tilde{s}(0, y_0, z_0) + \tilde{t}(0, -a_3, a_2), \quad (3)$$

with  $s, t, \tilde{s}, \tilde{t} \in \mathbb{Z}$ . Since  $\langle (r_1, r_2, r_3), \sigma^{12} \rangle \geq 0$  if and only if  $a_1r_1 + a_2r_2 + a_3r_3 \geq 0$  and  $-a_1r_1 + a_2r_2 + a_3r_3 \geq 0$  it follows with (2) and (3) that

$$a_1r_1 + a_2r_2 + a_3r_3 = s = 2a_1r_1 + \tilde{s} \geq 0,$$

$$a_1r_1 + a_2r_2 + a_3r_3 = \tilde{s} = -2a_1r_1 + s \geq 0.$$

Let first be  $r_1 \geq 0$ , then both inequalities are fulfilled exactly if  $\tilde{s} \geq 0$ . Secondly let  $r_1 \leq 0$ , then the necessary and sufficient condition is  $s \geq 0$ . This proves (iii) (again  $\mathbb{O}_{\sigma^{12}}$  is easy to calculate).

We do not need the chart  $X_{\sigma^{1234}}$  since we have:

**LEMMA 2.** *The closure  $\overline{F_\lambda(V)}$  of the Fermi surface  $F_\lambda(V)$  in  $X_\Sigma$  doesn't intersect the zero-dimensional orbits.*

*Proof.* It is enough to show that  $\mathbb{O}_{\sigma^{1234}} \cap \overline{F_\lambda(V)} = \emptyset$ . Since  $\dim_{\mathbb{C}} \mathbb{O}_{\sigma^{1234}} = 0$  the (singular) point  $\mathbb{O}_{\sigma^{1234}}$  has coordinates (in  $X_{\sigma^{1234}}$ )  $\xi_1^{r_1}\xi_2^{r_2}\xi_3^{r_3} = 0$  for all  $(r_1, r_2, r_3) \in \mathbb{Z}^3$  with  $\langle (r_1, r_2, r_3), \sigma^{1234} \rangle \geq 0$ .

Clearly  $\xi_3 \in \mathbb{C}$  is a coordinate of  $X_{\sigma^{1234}}$ , so the polynomial  $P$ , defining  $F_\lambda(V)$ , has a pol in  $\xi_3$  of order  $a_1 a_2$ . On the other hand since  $P = \sum_{ijk} a_{ijk}(\lambda) \xi_1^i \xi_2^j \xi_3^k$  with (due to the boundary conditions defining the Fermi surface)  $a_1|i| + a_2|j| + a_3|k| \leq a_1 a_2 a_3$  it follows that each summand  $\xi_1^i \xi_2^j \xi_3^k + a_1 a_2 \neq 1$  of the polynomial  $\xi_3^{a_1 a_2} P$  is a coordinate of  $X_{\sigma^{1234}}$ .

Therefore the closure of  $F_\lambda(V)$  in  $X_\Sigma$  lying in the chart  $X_{\sigma^{1234}}$  is given by the equation

$$\xi_3^{a_1 a_2} P = 0.$$

But  $\xi_3^{a_1 a_2} P|_{\mathcal{O}_{\sigma^{1234}}} = (-1)^{a_1 a_2 (a_3 - 1)} \neq 0$ .

Motivated from this lemma we are only are interested in the closure  $\overline{F_\lambda(V)}$  of  $F_\lambda(V)$  in

$$X_\Sigma^* = X_\Sigma - \{\text{union of the zero-dimensional orbits}\}.$$

### 3. The compactification

We consider the compactified Fermi surface as the solution of a spectral problem on a vectorbundle  $Y$  of infinite rank on  $X_\Sigma^*$ .

We define by  $F$  the infinite-dimensional vector space of all functions  $\psi : \mathbb{Z}^3 \rightarrow \mathbb{C}$ . The vectorbundle  $\pi : Y \rightarrow X_\Sigma^*$  will be trivial over each affine part  $X_\sigma$  of  $X_\Sigma^*$  ( $\sigma \in \Sigma$ ).

On  $Y|_{X_{\sigma^0}} = X_{\sigma^0} \times F = \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C}^* \times F$  we have four commuting operators

$$T - \lambda 1 := -\Delta + V - \lambda 1, \quad S^{(a_1, 0, 0)} - \xi_1 1, \quad S^{(0, a_2, 0)} - \xi_2 1, \quad S^{(0, 0, a_3)} - \xi_3 1,$$

for  $(\xi_1, \xi_2, \xi_3) \in X_{\sigma^0}$ .

**DEFINITION.** The (uncompactified) Fermi surface  $F_\lambda(V)$  is the support of the bundle

$$\{(\xi_1, \xi_2, \xi_3, y_0) \in X_{\sigma^0} \times F \mid \text{the above four operators have a common kernel } \psi_0\}.$$

By symmetry and lemma 2 (since we want the closure  $\overline{F_\lambda(V)}$  of  $F_\lambda(V)$  in  $X_\Sigma^*$  to coincide with the support of bundle on  $X_\Sigma^*$ ) it is enough to extend the vectorbundle  $Y|_{X_{\sigma^0}}$  on  $X_{\sigma^1}$  and  $X_{\sigma^{12}}$ . We give the transition functions, using lemma 1:

(i)  $X_{\sigma^1} = \text{Spec } \mathbb{C}[u_1, u_1^{-1}, v_1, w_1, w_1^{-1}]$ , i.e. the coordinates are  $(u_1, v_1, w_1) \in$

$\mathbb{C}^* \times \mathbb{C} \times \mathbb{C}^*$ . We now identify  $(\xi_1, \xi_2, \xi_3, \psi_0) \in X_{\sigma^0} \times F$  with  $(u_1, v_1, w_1, \psi_1) \in X_{\sigma^1} \times F$  on  $X_{\sigma^1} \cap X_{\sigma^0} = X_{\sigma^0}$  (or equivalently on  $v_1 \neq 0$ ) by

$$u_1 = \xi_1^{-1} (\xi_2^{y_0} \xi_3^{z_0})^{a_1}, \quad v_1 = \xi_2^{y_0} \xi_3^{z_0}, \quad w_1 = \xi_2^{-a_3} \xi_3^{a_2}$$

(this is the coordinate change on  $X_\Sigma$  from  $X_{\sigma^0}$  to  $X_{\sigma^1}$ ) and

$$\psi_0(m, n, p) = v_1^{m+n+p} \psi_1(m, n, p).$$

(ii) On  $X_{\sigma^{12}}$  we have coordinates  $(x, y, w, z) \in \mathbb{C}^3 \times \mathbb{C}^*$  with  $xy = w^{2a_1}$ . Identify  $(\xi_1, \xi_2, \xi_3, \psi_0) \in X_{\sigma^0} \times F$  with  $(x, y, w, z) \in X_{\sigma^{12}} \times F$  on  $X_{\sigma^{12}} \cap X_{\sigma^0}$  (i.e. on  $w \neq 0$ ) by

$$x = \xi_1^{-1} (\xi_2^{y_0} \xi_3^{z_0})^{a_1}, \quad y = \xi_1 (\xi_2^{y_0} \xi_3^{z_0})^{a_1}, \quad w = \xi_2^{y_0} \xi_3^{z_0}, \quad z = \xi_2^{-a_3} \xi_3^{a_2},$$

and

$$\psi_0(m, n, p) = w^{m+n+p} x^{-m/a_1} \psi_{12}(m, n, p).$$

Denote by  $\overline{F_\lambda(V)}$  the closure of  $F_\lambda(V)$  in  $X_\Sigma^*$ .

**PROPOSITION 1.**  $\overline{(F_\lambda(V) - F_\lambda(V))} \cap X_{\sigma^{12}}$  is the union of two rational curves  $Q_1$  and  $Q_2$  with the following properties;

- (i)  $Q_i$  ( $i = 1, 2$ ) has  $(a_1 - 1)(a_2 - 1)(a_3 - 1) + \sum_{i \neq j} (a_i - 1)(a_j - 1)$  ordinary double points.  $Q_i$  does not depend on the potential  $V$ .
- (ii)  $\overline{Q_1 \cap Q_2}$  is a point ( $=: P_{12}$ ).  $P_{12}$  lies on the singular orbit  $\mathbb{O}_{\sigma^{12}}$ .
- (iii)  $\overline{F_\lambda(V)}$  is smooth on all smooth points of  $Q_1 \cup Q_2 - \{P_{12}\}$ .

*Remark.* Since  $P_{12}$  is singular, we will resolve this point. The strict transformation of  $\overline{F_\lambda(V)}$  on the exceptional divisor is then one of the hyperelliptic curves mentioned in theorem 1 of the introduction.

**PROPOSITION 2.**  $Q_1$  is given on the chart  $X_{\sigma^1}$  as the support of the bundle  $(u_1, v_1, w_1, \psi_1) \in Y$  with

$$(S^{(-1,0,0)} + S^{(0,-1,0)} + S^{(0,0,-1)})\psi_1 = 0,$$

$$S^{(-a_1,0,0)}\psi_1 = u_1\psi_1, \quad S^{(0,-a_2a_3,a_2a_3)}\psi_1 = w_1\psi_1, \quad S^{(0,a_2y_0,a_3z_0)}\psi_1 = \psi_1$$

with  $v_1 = 0$ .

We first prove Proposition 2, then Proposition 1 will follow easily.

*Proof of proposition 2.* The spectral problem on  $X_{\sigma_0} \times F$

$$T\psi_0 = \lambda\psi_0, \quad S^{(a_1, 0, 0)}\psi_0 = \xi_1\psi_0, \quad S^{(0, a_2, 0)}\psi_0 = \xi_2\psi_0, \quad S^{(0, 0, a_3)}\psi_0 = \xi_3\psi_0$$

can be written alternatively as

$$\begin{aligned} TS^{(0, a_2 y_0, a_3 z_0)}\psi_0 &= \lambda \xi_2^{y_0} \xi_3^{z_0} \psi_0, & S^{(-a_1, 0, 0)}\psi_0 &= \xi_1^{-1} \psi_0, \\ S^{(0, a_2 y_0, a_3 z_0)}\psi_0 &= \xi_2^{y_0} \xi_3^{z_0} \psi_0, & S^{(0, -a_2 a_3, a_2 a_3)}\psi_0 &= \xi_2^{-a_3} \xi_3^{a_2} \psi_0, \end{aligned}$$

since the shift operators are invertible and the vectors  $(-a_1, 0, 0)$ ,  $(0, a_2 y_0, a_3 z_0)$ ,  $(0, -a_2 a_3, a_2 a_3)$  are also a basis for the lattice  $\Gamma$ .

By the construction of the vectorbundle  $Y$  these four equations transform to

$$\begin{aligned} &-(S^{(-1, a_2 y_0, a_3 z_0)}\psi_1 + v_1^2 S^{(1, a_2 y_0, a_3 z_0)}\psi_1 + S^{(0, a_2 y_0 - 1, a_3 z_0)}\psi_1 \\ &+ v_1^2 S^{(0, a_2 y_0 + 1, a_3 z_0)}\psi_1 + S^{(0, a_2 y_0, a_3 z_0 - 1)}\psi_1 + v_1^2 S^{(0, a_2 y_0, a_3 z_0 + 1)}\psi_1) \\ &+ v_1 V S^{(0, a_2 y_0, a_3 z_0)}\psi_1 = v_1 \lambda S^{(0, a_2 y_0, a_3 z_0)}\psi_1, \\ &S^{(-a_1, 0, 0)}\psi_1 = u_1 \psi_1, \quad S^{(0, a_2 y_0, a_3 z_0)}\psi_1 = \psi_1, \quad S^{(0, -a_2 a_3, a_2 a_3)}\psi_1 = w_1 \psi_1, \end{aligned}$$

on  $X_{\sigma_1} \times F$ .

But  $X_{\sigma_1} - X_{\sigma_0} = \{v_1 = 0\}$  and on the open set  $X_{\sigma_1} \cup X_{\sigma_0} = X_{\sigma_0}$  by the continuity of the transition function the spectral problems on  $Y|_{X_{\sigma_0}}$  and  $Y|_{X_{\sigma_1}}$  coincide.

Therefore  $(\overline{F_\lambda(V)} - F_\lambda(V)) \cap X_{\sigma_1}$  is the support of the following spectral problem

$$\begin{aligned} &(S^{(-1, a_2 y_0, a_3 z_0)} + S^{(0, a_2 y_0 - 1, a_3 z_0)} + S^{(0, a_2 y_0, a_3 z_0 - 1)})\psi_1 = 0, \\ &S^{(-a_1, 0, 0)}\psi_1 = u_1 \psi_1, \quad S^{(0, a_2 y_0, a_3 z_0)}\psi_1 = \psi_1, \quad S^{(0, -a_2 a_3, a_2 a_3)}\psi_1 = w_1 \psi_1, \end{aligned}$$

which leads immediately to proposition 2.

*Proof of proposition 1.* (i) Clearly  $Q_1$  does not depend on  $V$ . To calculate the genus of this curve, we consider a covering of  $Q_1$  with  $a_1 a_2 a_3$  sheets, as in [4].

Let  $\mu_{a_1}$  (resp.  $\mu_{a_2 a_3}$ ) be the multiplicative group of  $a_1$ th (resp.  $a_2 a_3$ th) root of unity and  $(z_1^{-a_1}, z_2^{-a_2 a_3}) := (u_1, w_1)$ . Then the functions  $e_\rho(z)(m, -n, n) = (\rho_1 z_1)^m (\rho_2 z_2)^n$  with  $\rho = (\rho_1, \rho_2) \in \mu_{a_1} \times \mu_{a_2 a_3}$  form a basis of the vectorspace of functions

$$\psi : \mathbb{Z}^2 \rightarrow \mathbb{C}, (m, -n, n) \rightarrow \psi(m, -n, n) \quad \text{with } S^{(-a_1, 0, 0)}\psi_1 = u_1 \psi_1,$$

$$S^{(0, -a_2 a_3, a_2 a_3)}\psi_1 = w_1 \psi_1.$$

The operator  $S^{(-1, a_2 y_0, a_3 z_0)} + S^{(0, a_2 y_0 - 1, a_3 z_0)} + S^{(0, a_2 y_0, a_3 z_0 - 1)}$  diagonalizes in this basis and considering the covering

$$c : \mathbb{C}^* \times \mathbb{C}^* \rightarrow \mathbb{C}^* \times \mathbb{C}^*, (z_1, z_2) \rightarrow (z_1^{-a_1}, z_2^{-a_2 a_3}) = (u_1, w_1)$$

$\tilde{Q}_1 = c^{-1}(Q_1)$  is given by

$$\bigcup_{\rho} \{(z_1, z_2) \in \mathbb{C}^* \times \mathbb{C}^* \mid (\rho_1 z_1)^{-1} + (\rho_2 z_2)^{a_2 y_0 - 1} + (\rho_2 z_2)^{a_2 y_0} = 0\}.$$

This means  $Q_1$  is rational. Let  $C_{\rho} = \{(\rho_1 z_1)^{-1} + (\rho_2 z_2)^{a_2 y_0 - 1} + (\rho_2 z_2)^{a_2 y_0} = 0\}$ . Perform the changes of coordinates

$$z_1 = y^{-a_2 y_0} x^{-a_3 z_0}, \quad z_2 = y x^{-1}$$

$C_{\rho}$  transforms to

$$\rho_1^{-1} x y + \rho_2^{-a_3 z_0} x + \rho_2^{a_2 y_0} y = 0.$$

Since  $a_2$  and  $a_3$  are relatively prime we have  $\mu_{a_2 a_3} = \mu_{a_2} \otimes \mu_{a_3}$  i.e. each  $\rho_2 \in \mu_{a_2 a_3}$  can be written as  $\tilde{\rho}_2 \tilde{\rho}_3^{-1}$  with  $\tilde{\rho}_2 \in \mu_{a_2}$ ,  $\tilde{\rho}_3 \in \mu_{a_3}$ . Therefore  $C_{\rho} = C_{(\rho_1, \tilde{\rho}_2, \tilde{\rho}_3)}$  is given by

$$\rho_1^{-1} x y + \tilde{\rho}_2^{-1} x + \tilde{\rho}_3^{-1} y = 0.$$

For  $\rho \in \mu_{a_1} \times \mu_{a_2} \times \mu_{a_3}$  the action on  $\tilde{Q}_1$  is given by  $\rho \cdot (x, y) = (\rho_1^{-1} \tilde{\rho}_2 x, \rho_1^{-1} \tilde{\rho}_3 y)$ , and we have  $\rho \cdot C_{\rho'} = C_{\rho \rho'}$ . Now by Bézout's Theorem  $C_1$  and  $C_{\rho}$  intersect transverse in  $\mathbb{C}^* \times \mathbb{C}^*$  in exactly one point, given by

$$x(\rho) = -\frac{\tilde{\rho}_3^{-1} - \tilde{\rho}_2^{-1}}{\rho_1^{-1} - \tilde{\rho}_2^{-1}}, \quad y(\rho) = -\frac{\tilde{\rho}_2^{-1} - \tilde{\rho}_3^{-1}}{\rho_1^{-1} - \tilde{\rho}_3^{-1}}$$

if  $\rho$  is not of the form  $(1, 1, \tilde{\rho}_3)$ ,  $(1, \tilde{\rho}_2, 1)$ ,  $(\rho_1, 1, 1)$ . In this second case we have  $C_1 \cap C_{\rho} \cap (\mathbb{C}^* \times \mathbb{C}^*) = \emptyset$ . To prove (i) it remains to show that

$$(x(\rho), y(\rho)) \neq (x(\rho'), y(\rho')) \quad \text{for } \rho \neq \rho',$$

because then  $Q_1$  has exactly

$$a_1 a_2 a_3 - 1 - (a_1 - 1) - (a_2 - 1) - (a_3 - 1)$$

ordinary double points.



Observe that  $\arg x(\rho) = \alpha = \text{angle in } \tilde{\rho}_2^{-1} \text{ of the triangle } \rho_1^{-1}, \tilde{\rho}_2^{-1}, \tilde{\rho}_3^{-1}$ . Put  $\rho_1^{-1} = e^{2\pi i m_1/a_1}$ ,  $\rho_1'^{-1} = e^{2\pi i m_1'/a_1}$ ,  $\tilde{\rho}_3^{-1} = e^{2\pi i m_3/a_3}$  and  $\tilde{\rho}_3'^{-1} = e^{2\pi i m_3'/a_3}$  with  $m_i, m_i' \in \{0, \dots, a_i - 1\}$ . One shows

$$\alpha = \pi \left| \frac{m_3}{a_3} - \frac{m_1}{a_1} \right|,$$

so if  $x(\rho) = x(\rho')$  then  $a_3(m_1 \pm m_1') = a_1(m_3 \pm m_3')$ . Since  $a_1$  and  $a_3$  are relatively prime it follows that

$$(m_1 \pm m_1', m_3 \pm m_3') \in \{(0, 0), \pm(a_1, a_3)\}.$$

In the first case we get either  $\rho_1 = \rho_1'$ ,  $\rho_3 = \tilde{\rho}_3'$  or  $\rho_1 = \rho_1'^{-1}$ ,  $\tilde{\rho}_3 = \tilde{\rho}_3'^{-1}$ . If  $\rho_1 = \rho_1'$ ,  $\tilde{\rho}_3 = \tilde{\rho}_3'$  using  $x(\rho) = x(\rho')$  we have  $\tilde{\rho}_2 = \tilde{\rho}_2'$ . If on the other hand  $\rho_1 = \rho_1'^{-1}$ ,  $\tilde{\rho}_3 = \tilde{\rho}_3'^{-1}$  by assuming  $\arg y(\rho) = \arg y(\rho')$  it follows  $\tilde{\rho}_2 = \tilde{\rho}_2'^{-1}$ , i.e.  $x(\rho)$  is real, so  $\alpha \in \{0, \pi\}$  and therefore  $\rho_1 = \tilde{\rho}_3 = 1$  which contradicts  $y(\rho) \in \mathbb{C}^*$ . The cases  $m_i \pm m_i' = \pm a_i$  are treated similarly.

(ii) The spectral problem on  $X_{\sigma 12} \times F$  is given in the coordinates  $(x, y, w, z)$  of  $X_{\sigma 12}$  by

$$\begin{aligned} TS^{(0, a_2 y_0, a_3 z_0)} \psi_0 &= \lambda w \psi_0, & S^{(-a_1, 0, 0)} \psi_0 &= x w^{-a_1} \psi_0, \\ S^{(0, a_2 y_0, a_3 z_0)} \psi_0 &= w \psi_0, & S^{(0, -a_2 a_3, a_2 a_3)} \psi_0 &= z \psi_0. \end{aligned}$$

By the construction of the vectorbundle the last three equations transform to

$$S^{(-a_1, 0, 0)} \psi_{12} = x w^{-a_1} \psi_{12}, \quad S^{(0, a_2 y_0, a_3 z_0)} \psi_{12}, \quad S^{(0, -a_2 a_3, a_2 a_3)} \psi_{12} = z \psi_{12}$$

The first equation gives (on  $x = y = w = 0$ ), using  $S^{(0, a_2 y_0, a_3 z_0)} \psi_{12} = \psi_{12}$ ,

$$(S^{(0, -1, 0)} + S^{(0, 0, -1)}) \psi_{12} = 0,$$

i.e.  $S^{(0, -1, 1)} \psi_{12} = -\psi_{12}$ .

It follows  $S^{(0, -a_2 a_3, a_2 a_3)} \psi_{12} = (-1)^{a_2 a_3} \psi_{12}$ , which leads to  $z = (-1)^{a_2 a_3}$ . This means  $\overline{F_\lambda(V)} \cap \mathbb{O}_{\sigma 12} = \text{one point } (P_{12})$  with coordinates  $x = y = w = 0$ ,  $z = (-1)^{a_2 a_3}$ .

(iii) On  $X_{\sigma 1}, \overline{F_\lambda(V)}$  is given as the zero set of polynomial  $P(u_1, u_1^{-1}, w_1, w_1^{-1}, v_1) = Q(u_1, u_1^{-1}, w_1, w_1^{-1}) + v_1 R(u_1, u_1^{-1}, w_1, w_1^{-1}, v_1)$ , where the zero set of  $Q$  describes  $Q_1$ . So  $\overline{F_\lambda(V)}$  is smooth on the smooth points of  $Q_1 \subset (X_{\sigma 1} \cap \{v_1 = 0\})$ .

Now we resolve the singular point  $P_{12}$  of type  $A_{2a_1-1}$ . Its coordinates are

$$x = y = w = 0, \quad z = (-1)^{a_2 a_3}.$$

Blowing up this point in  $\mathbb{C}^4$   $a_1$ -times, the exceptional divisor is the transverse union of  $a_1$  hyperplanes  $E_i$  ( $i = 1, \dots, a_1$ ), where the  $E_i$  is the exceptional divisor of the  $i$ th blowing up.

**PROPOSITION 3.** *The blowing up of  $\overline{F_\lambda(V)}$  at the point  $P_{12}$  intersects only the exceptional divisor  $E_{a_1}$ . The strict transform of  $\overline{F_\lambda(V)}$  (on  $E_{a_1}$ ) is a hyperelliptic curve of arithmetic genus  $a_1 - 1$ . The blowing up of  $\overline{F_\lambda(V)}$  is smooth on all smooth points of this curve. Furthermore the curve is determined by the following one-dimensional spectral problem*

$$\begin{aligned} S^{(-a_1, 0, 0)}\psi &= x_{a_1}\psi, & S^{(0, a_2 y_0, a_3 z_0)}\psi &= \psi, & S^{(0, -1, 1)}\psi &= -\psi, \\ & -\psi(m-1, n, p) - \psi(m+1, n, p) \\ & + \frac{1}{a_2 a_3} \left( \sum_{i=1}^{a_2} \sum_{j=1}^{a_3} V(m, i, j) \right) \psi(m, n, p) \\ & = \tilde{z}\psi(m, n, p) \end{aligned}$$

where the coordinates  $\tilde{z}, x_{a_1}$  are defined by resolving the point  $P_{12}$ :

$$\begin{aligned} w &= \mu, & x &= \mu^{a_1} x_{a_1}, & y &= \mu^{a_1} y_{a_1}, \\ (1 + (-1)^{a_2 a_3 - 1} z) &= a_2 a_3 (-1)^{a_2 y_0} \mu(\tilde{z} - \lambda) \end{aligned}$$

(here  $\mu = 0$  is the exceptional divisor  $E_{a_1}$ ).

Due to the shift operators  $S^{(0, a_2 y_0, a_3 z_0)}$  and  $S^{(0, -1, 1)}$  the curve on  $E_{a_1}$  is already determined by the values of  $\psi(m, n, p)$  on the line spanned by the vector  $(a_1, 0, 0)$ . Therefore Proposition 3 and Proposition 2 prove the theorems in the introduction.

*Proof.* We first calculate the strict transform of  $\overline{F_\lambda(V)}$  on  $E_{a_1}$ . Blowing up  $P_{12}$   $a_1$ -times, we get the coordinates

$$\begin{aligned} w &= \mu, & x &= \mu^{a_1} x_{a_1}, & y &= \mu^{a_1} y_{a_1}, \\ (1 + (-1)^{a_2 a_3 - 1} z) &= a_2 a_3 (-1)^{a_2 y_0} \mu(\tilde{z} - \lambda) \end{aligned}$$

Denote by  $U$  the chart generated by the coordinates  $(\mu, x_{a_1}, y_{a_1}, \tilde{z})$ , i.e.

$$U = \{(\mu, x_{a_1}, y_{a_1}, \tilde{z}) \in \mathbb{C} \times \mathbb{C}^* \times \mathbb{C}^* \times \mathbb{C} \mid x_{a_1} y_{a_1} = 1\}.$$

Now  $X_{\sigma_0} \cap U = X_{\sigma_0}$  and define the transition function for the vectorbundle  $Y$  by

$$\psi_0(m, n, p) = \mu^{n+p} \psi(m, n, p).$$

The spectral problem on  $X_{\sigma_0} \times F$  is given in the coordinates of  $U$  by

$$TS^{(0, a_2 y_0, a_3 z_0)} \psi_0 = \mu \lambda \psi_0 \quad (1)$$

$$S^{(-a_1, 0, 0)} \psi_0 = x_{a_1} \psi_0, \quad S^{(0, a_2 y_0, a_3 z_0)} \psi_0 = \mu \psi_0, \quad (2), (3)$$

$$\mu^{-1} \{1 + (-1)^{a_2 a_3 - 1} S^{(0, -a_2 a_3, a_2 a_3)}\} \psi_0 = a_2 a_3 (-1)^{a_2 y_0} (\tilde{z} - \lambda) \psi_0 \quad (4)$$

Using the transition function the equations (1), (2) and (3) transform to

$$\begin{aligned} & -S^{(0, a_2 y_0 - 1, a_3 z_0)} \psi - S^{(0, a_2 y_0, a_3 z_0 - 1)} \psi \\ & + \mu \{ -S^{(-1, a_2 y_0, a_3 z_0)} \psi - S^{(1, a_2 y_0, a_3 z_0)} \psi + VS^{(0, a_2 y_0, a_3 z_0)} \psi \} \\ & - \mu^2 \{ S^{(0, a_2 y_0 + 1, a_3 z_0)} \psi + S^{(0, a_2 y_0, a_3 z_0 + 1)} \psi \} = \mu \lambda \psi \end{aligned}$$

and

$$S^{(-a_1, 0, 0)} \psi = x_{a_1} \psi, \quad S^{(0, a_2 y_0, a_3 z_0)} \psi = \psi.$$

Therefore on  $E_{a_1} = \{\mu = 0\}$  we have

$$S^{(0, -1, 1)} \psi = -\psi, \quad S^{(-a_1, 0, 0)} \psi = x_{a_1} \psi, \quad S^{(0, a_2 y_0, a_3 z_0)} \psi = \psi.$$

To explore (4) observe that

$$1 + (-1)^{a_2 a_3 - 1} S^{(0, -a_2 a_3, a_2 a_3)} = \sum_{i=0}^{a_2 a_3 - 1} (-1)^i (S^{i(0, -1, 1)} + S^{(i+1)(0, -1, 1)})$$

On the other hand we have from (1)

$$\begin{aligned} & (S^{i(0, -1, 1)} + S^{(i+1)(0, -1, 1)}) \psi_0 = -S^{(-1, -i, i+1)} \psi_0 - S^{(1, -i, i+1)} \psi_0 \\ & - S^{(0, -i, i+2)} \psi_0 - S^{(0, -i+1, i+1)} \psi_0 + (V(m, n-i, p+i+1) - \lambda) S^{(0, -i, i+1)} \psi_0 \end{aligned}$$

Thus

$$\begin{aligned} \mu^{-1}\{1 + (-1)^{a_2 a_3 - 1} S^{(0, -a_2 a_3, a_2 a_3)}\} \psi_0 &= \mu^{-1} \sum_{i=0}^{a_2 a_3 - 1} (-1)^i \{ -\mu^{n+p+1} (S^{(-1, -i, i+1)} \psi \\ &+ S^{(1, -i, i+1)} \psi) - \mu^{n+p+2} (S^{(0, -i, i+2)} \psi + S^{(0, -i+1, i+1)} \psi) \\ &+ \mu^{n+p+1} (V(m, n-i, p+1+i) - \lambda) S^{(0, -i, i+1)} \psi \} \end{aligned}$$

So on  $\mu = 0$  (4) transforms to

$$\begin{aligned} \sum_{i=0}^{a_2 a_3 - 1} (-1)^i \{ -(S^{(-1, -i, i+1)} - S^{(1, -i, i+1)} + (V(m, n-i, p+1+i) \\ - \lambda) S^{(0, -i, i+1)} \psi \} = a_2 a_3 (-1)^{a_2 y_0} (\tilde{z} - \lambda) \psi. \end{aligned}$$

Since  $S^{(0, -1, 1)} \psi = -\psi$  we get

$$\begin{aligned} -a_2 a_3 S^{(-1, 0, 0)} \psi - a_2 a_3 S^{(1, 0, 0)} \psi - \lambda a_2 a_3 S^{(0, 0, 0)} \psi + \sum_{i=0}^{a_2 a_3 - 1} V(m, n-i, p+i) \psi \\ = a_2 a_3 (-1)^{a_2 y_0} (\tilde{z} - \lambda) S^{(0, 0, -1)} \psi. \end{aligned}$$

But  $S^{(0, 0, -1)} = S^{(0, -a_2 y_0, -a_3 z_0)} S^{-(0, -a_2 y_0, a_2 y_0)}$  and we have

$$-S^{(-1, 0, 0)} \psi - S^{(1, 0, 0)} \psi + \frac{1}{a_2 a_3} \sum_{i=0}^{a_2 a_3 - 1} V(m, n-i, p+i) \psi = \tilde{z} \psi.$$

Now  $a_2$  and  $a_3$  are relatively prime, therefore we get the desired spectral problem as in proposition 3.

Let now  $\pi_i$  be the  $i$ th blowing up of the point  $P_{12}$  and  $E_i$  the exceptional divisor. So we have

$$w = \mu, \quad x = \mu^i x_i, \quad y = \mu^i y_i, \quad (1 + (-1)^{a_2 a_3 - 1} z) = a_2 a_3 (-1)^{a_2 y_0} \mu (\tilde{z} - \lambda).$$

Let  $U_i = \{(\mu, x_i, y_i, \tilde{z}) \in \mathbb{C}^4 \mid x_i y_i = \mu^{2a_1 - 2i}\}$  be the new chart. On  $U_i \cap X_{\sigma_0} = X_{\sigma_0}$  define the transition function  $\psi_0(m, n, p) = \mu^{n+p} \psi_i(m, n, p)$ . The spectral problem on  $X_{\sigma_0} \times F$  is given by the equations (1), (3), (4) and

$$S^{(-a_1, 0, 0)} S^{(a_1 - i)(0, a_2 y_0, a_3 z_0)} \psi_0 = x_i \psi_0$$

$$S^{(a_1, 0, 0)} S^{(a_1 - i)(0, a_2 y_0, a_3 z_0)} \psi_0 = y_i \psi_0$$

The last two equations give on the exceptional divisor  $E_i = \{\mu = 0\}$   
 $x_i = y_i = 0$  for  $i \neq a_1$ , i.e.

$$\overline{\pi^{-1}(F_\lambda(V) - P_{12})} \cap E_i = (E_i)_{\text{singular}}$$

Denote by  $H_{12}$  the above hyperelliptic curve. Now  $F_\lambda(V)_{\text{comp}}$  is smooth on the smooth points of  $H_{12} - (H_{12} \cap Q_1 \cap Q_2)$  as in proposition 2. Observe that  $Q_1 \subset X_{\sigma_{12}}$  lies in the plane  $x = 0$ , so by the blowing-ups  $Q_1$  intersects  $H_{12}$  transversally at  $x_{a_1} = 0$  (and similarly  $Q_2$  intersects  $H_{12}$  at  $x_{a_1} = y_{a_1}^{-1} = \infty$ ), i.e. on (see the introduction) a smooth point of  $H_{12}$ . This proves proposition 3.

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