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## On planarity of graphs in 3-manifolds\*

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A graph  $\Gamma$  in a 3-manifold  $M$  is called *planar* if it is contained in an embedded 2-sphere in  $M$ . It is *abstractly planar* if it can be embedded into an abstract 2-sphere. In [3] Scharlemann and Thompson gave necessary and sufficient conditions for a graph  $\Gamma$  to be planar in  $S^3$  (see Theorem 3 in section 3). The special case that  $\Gamma$  has a single vertex was proved by Gordon [1], while the generic case was shown [2] to be equivalent to: An abstractly planar graph  $\Gamma$  in  $S^3$  is planar if and only if both  $\Gamma - e$  and  $\Gamma/e$  are planar, where  $e$  is a noncycle edge of  $\Gamma$ . Fix an embedding of  $\Gamma$  in a 2-sphere  $F$ . We say that the embedding of  $\Gamma$  in  $S^3$  is *F-planar* if it can be extended to an embedding of  $F$  into  $S^3$ . It turns out that the above result is equivalent to: If both  $\Gamma - e$  and  $\Gamma/e$  are *F-planar*, then  $\Gamma$  is also *F-planar*.

In this paper, we study the *F-planarity* of a graph  $\Gamma$  in a 3-manifold  $M$ , where  $F$  can be an arbitrary surface containing  $\Gamma$ , or more generally a 2-dimensional cell complex with  $\Gamma$  as 1-skeleton. An embedding of  $\Gamma$  in a 3-manifold  $M$  is called *F-planar* if it can be extended to an embedding of  $F$  in  $M$ . We are interested in the problem of whether the *F-planarity* of  $\Gamma$  is determined by that of  $\Gamma - e$  and  $\Gamma/e$ . A statement parallel to the case of  $F = S^2$  is not true in this general setting. However we will show it is true if  $\Gamma$  is a “regular” graph.

We first study the triviality of cycles. This can be considered a special case of the above problem, when the cell complex has only one 2-cell. A *cycle* of  $\Gamma$  is a subgraph  $C$  which is homeomorphic to a circle.

**DEFINITION.** Suppose  $\Gamma$  is embedded in a 3-manifold  $M$ . Then a cycle  $C$  of  $\Gamma$  is trivial (with respect to  $(M, \Gamma)$ ), if it bounds a disk with interior disjoint from  $\Gamma$ .

In section 2 we prove a theorem about triviality of simple cycles. Note that if  $C$  is a cycle of  $\Gamma$ , and  $e$  is an edge intersecting  $C$  at most once, then  $C$  remains a cycle in both  $\Gamma - e$  and  $\Gamma/e$ . Therefore it makes sense talking about the triviality of  $C$  with respect to  $(M, \Gamma - e)$  and  $(M/e, \Gamma/e)$ .

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**THEOREM 1.** *Suppose  $\Gamma$  is a graph embedded in a 3-manifold  $M$ . Let  $C$  be a cycle in  $\Gamma$ , and let  $e$  be an edge of  $\Gamma$  with at most one end on  $C$ . If  $C$  is trivial with respect to both  $(M, \Gamma - e)$  and  $(M/e, \Gamma/e)$ , then it is trivial with respect to  $(M, \Gamma)$ .*

A link  $L$  in  $S^3$  is the unlink if each component of  $L$  is a trivial cycle. It turns out that this is also true for any abstractly planar graphs in a 3-manifold  $M$ :

**THEOREM 2.** *An abstractly planar graph  $\Gamma$  in  $M$  is planar if and only if all cycles of  $\Gamma$  are trivial.*

We will prove Theorem 2 in Section 3, and use these theorems to give an alternative proof of the Scharlemann–Thompson Theorem.

In Section 4, we study the  $F$ -planarity of graphs in arbitrary 3-manifolds  $M$ . Suppose  $\Gamma$  is a graph in a compact surface  $F$ . We assume that  $\partial F$  is either empty or a subgraph of  $\Gamma$ . An embedding of  $\Gamma$  into  $M$  is  $F$ -planar if it can be extended to an embedding of  $F$  into  $M$ . We call the closure of a component of  $F - \Gamma$  a face of  $F$ . The graph  $\Gamma$  is called a *regular graph* in  $F$  if each face of  $F$  is a disk, and the intersection of any two faces is connected (or empty). Suppose  $e$  is an edge of  $\Gamma$  with at least one end in the interior of  $F$ . Then both  $\Gamma - e$  and  $\Gamma/e$  can be considered as graphs in  $F$  in the natural way, so we can talk about the  $F$ -planarity of  $\Gamma - e$  and  $\Gamma/e$ . The following theorem is proved in section 4.

**THEOREM 5.** *Suppose  $\Gamma$  is a regular graph on a surface  $F$ , and suppose  $\Gamma$  is embedded in a 3-manifold  $M$ . Let  $e$  be an edge of  $\Gamma$  with at least one end in  $\text{Int } F$ . If both  $\Gamma/e$  and  $\Gamma - e$  are  $F$ -planar, then  $\Gamma$  is  $F$ -planar.*

The regularity condition on  $\Gamma$  is necessary. We will give an example of a graph  $\Gamma$  on a torus  $F$  that can be embedded into  $S^3$ , so that both  $\Gamma - e$  and  $\Gamma/e$  are  $F$ -planar, but  $\Gamma$  itself is not  $F$ -planar.

I would like to thank Marty Scharlemann for some helpful discussion on this topic, and to the referee for many useful comments.

## 1. Definitions and preliminaries

Given a graph  $\Gamma$  in a 3-manifold  $M$ , choose a regular neighborhood for each vertex and each edge of  $\Gamma$ , so that the disks  $\partial N(v) \cap N(e)$  are mutually disjoint for all  $v$  and  $e$ . The union of all such neighborhoods forms a regular neighborhood  $N(\Gamma)$  of  $\Gamma$  and we define the exterior of  $\Gamma$  to be  $E(\Gamma) = M - \text{Int } N(\Gamma)$ . For each vertex  $v$ , denote by  $\delta(v)$  the punctured sphere  $\partial N(v) - \bigcup \text{Int } N(e)$ ; similarly, for

each edge  $e$ , let  $\delta(e)$  be the annulus  $\partial N(e) - \bigcup \text{Int } N(v)$ . Sometimes the graph may vary, in which case we use  $\delta_r(e)$  and  $\delta_r(v)$  to denote  $\delta(e)$  and  $\delta(v)$ , respectively. If  $C$  is a cycle, or more generally a subgraph of  $\Gamma$ , we use  $\delta(C)$  to denote the union of  $\delta(t)$  with  $t$  ranges over all edges and vertices of  $C$ .

For an edge  $e$  in  $\Gamma$ , denote by  $\Gamma - e$  the subgraph obtained from  $\Gamma$  by deleting the interior of the edge  $e$ . If  $e$  is not a loop, then  $\Gamma/e$  is a graph in  $M/e$ . Denote by  $\bar{e}$  the image of  $e$  in  $\Gamma/e$ . The quotient map  $\pi : M \rightarrow M/e$  sends  $N(\Gamma)$  to a regular neighborhood  $N(\Gamma/e)$  of  $\Gamma/e$  in  $M/e$ , so it induces a homeomorphism  $E(\Gamma) \cong E(\Gamma/e) = M/e - \text{Int } N(\Gamma/e)$ . We identify  $E(\Gamma)$  with  $E(\Gamma/e)$  by this homeomorphism. Note that  $\delta_{\Gamma/e}(\bar{e}) = \delta_r(v) \cup \delta_r(e) \cup \delta_r(v')$  if  $\partial e = v \cup v'$ .

If  $X$  is a subset of  $M$ , denote the number of components in  $X$  by  $|X|$ .

We define a *simple disk* to be a disk  $D$  in  $M$  which is bounded by a cycle of  $\Gamma$ , and has interior disjoint from  $\Gamma$ . Thus a cycle of  $\Gamma$  is a trivial cycle if and only if it bounds a simple disk.

Define a *normal structure* on  $N(\Gamma)$  to be a set of line segments  $\{l_x \mid x \in \partial N(\Gamma)\}$  as follows: For any vertex  $v \in \Gamma$  and any  $x \in \delta(v)$ , let  $l_x$  be the straight line in  $D^3 = N(v)$  connecting  $x$  to  $v$ . If  $e$  is not a loop, the closure of  $N(e) - \bigcup \{l_x \mid x \in \bigcup \delta(v)\}$  has a product structure  $e \times D^2$  such that for  $x \in \partial e \times \partial D^2$ , the  $l_x$  defined above is the line between  $x$  and a point in  $\partial e \times 0$ . Now for any  $x \in p \times \partial D^2$  with  $p \in e$ , let  $l_x$  be the line connecting  $x$  to  $p \times 0$ . If  $e$  is a loop,  $N(e) - \bigcup \{l_x \mid x \in \bigcup \delta(v)\}$  is homeomorphic to  $e \times D^2 / \partial e \times 0$ , so we can define  $l_x$  in the same way as above. For any  $p \in e$ ,  $p \times D^2$  is called a *meridian disk* of  $\Gamma$  (or  $e$ ) at  $p$ , and  $p \times \partial D^2$  is called a *meridian* of  $\Gamma$ .

Suppose  $P$  is a surface in  $E(\Gamma)$ . The normal extension  $D$  of  $P$  is the union of  $P$  and the lines  $l_x$  with  $x \in P \cap \partial N(\Gamma)$ . If  $P$  is a properly embedded disk in  $E(\Gamma)$ , and  $C$  is a cycle of  $\Gamma$  such that  $P$  intersects any meridian of  $C$  exactly once, and is disjoint from the other meridians of  $\Gamma$ , then  $D$  is a disc with  $\partial D = C$ . A surface  $S$  in  $M$  with  $\partial S$  in  $\Gamma$  is called in normal position if  $S$  is the normal extension of  $S \cap E(\Gamma)$ . The following lemma is useful in modifying disks to make their interiors disjoint.

**LEMMA 1.1.** *Suppose  $D_1, \dots, D_n$  are simple disks in  $M$  with mutually disjoint interiors. Suppose  $C$  is a trivial cycle, and  $C \cap D_i$  is connected for all  $i$ . Then  $C$  bounds a simple disk  $D$  with interior disjoint from  $D_i$  for all  $i$ .*

*Proof.* By an isotopy we may assume  $D_1, \dots, D_n$  are in normal position. Let  $P_i = D_i \cap E(\Gamma)$ . Choose a simple disk  $D$  in normal position and bounded by  $C$  so that  $P = D \cap E(\Gamma)$  is transverse to  $P_i$ , and  $|P \cap (\bigcup P_i)|$  is minimal. Let  $A$  be the closure of  $\text{Int } D \cap (\bigcup D_i)$ . Since  $A \cap N(\Gamma)$  consists of lines  $l_x$  with  $x \in \partial P \cap (\bigcup P_i)$ , we know that  $A$  is the union of some circles which may intersect  $\Gamma$  at one point, and

some arcs with different endpoints on  $\Gamma$ . These circles and arcs might intersect on  $\Gamma$ , but are otherwise disjoint. If  $A$  has some circles, choose a circle  $\alpha$  which is innermost in some  $D_i$ , and let  $\Delta$  and  $\Delta_i$  be the disks it bounds in  $D$  and  $D_i$  respectively. Then  $(D - \Delta) \cup \Delta_i$  can be rel  $\partial D$  isotoped into a disk  $D'$  with  $|D' \cap (\bigcup P_i)| < |P \cap (\bigcup P_i)|$ . If  $A$  has no circles but has some arcs, let  $\beta$  be an arc in  $A$  which is outermost in the sense that there is an arc  $\gamma$  in some  $C \cap \partial D_i$ , such that  $\beta \cup \gamma$  bounds a disk  $\Delta_i$  in  $D_i$  with  $\text{Int } \Delta_i \cap D = \emptyset$ . (This is possible because of the assumption that  $C \cap \partial D_i$  is connected for all  $i$ ). Let  $\Delta$  be the disk in  $D$  with  $\partial \Delta = \partial \Delta_i = \beta \cup \gamma$ . Then a perturbation of  $(D - \Delta) \cup \Delta_i$  produces a disk  $D'$  with  $|D' \cap (\bigcup P_i)| < |P \cap (\bigcup P_i)|$ . By the minimality of  $|P \cap (\bigcup P_i)|$ , neither case can happen. Therefore  $A = \emptyset$ .  $\square$

In section 3 we will need some handle addition lemmas. Let  $F$  be a surface on the boundary of a 3-manifold  $M$ , and let  $J$  be a simple loop on  $F$ . Denote by  $\tau(M, J)$  the manifold obtained from  $M$  by attaching a 2-handle along  $J$ , that is,  $\tau(M, J) = M \cup (D^1 \times D^2)$ , where  $D^1 \times \partial D^2$  is identified with a regular neighborhood  $N(J)$  of  $J$  in  $F$ . Denote by  $\sigma(F, J)$  the surface  $(F - N(J)) \cup (\partial D^1 \times D^2)$ . We have the following generalized handle addition lemma.

**LEMMA 1.2.** *Suppose  $S$  is a surface on the boundary of a 3-manifold  $M$ . Let  $\gamma$  be a 1-manifold on  $S$  such that  $S - \gamma$  is compressible, and let  $J$  be a circle in  $S$  disjoint from  $\gamma$ . If  $\sigma(S, J)$  is compressible in  $\tau(M, J)$  with  $D'$  a compressing disk, then  $S - J$  has a compressing disk  $D$  such that  $\partial D \cap \gamma \subset \partial D' \cap \gamma$ .*

This was implied in the proof of [4, Thm 1]. It was shown that under the assumption we have  $|\partial D \cap \gamma| \leq |\partial D' \cap \gamma|$ , but the argument there has actually proved that  $\partial D \cap \gamma \subset \partial D' \cap \gamma$ .

## 2. Trivial cycles in a graph

Given a cycle  $C$  in  $\Gamma \subset M$ , and a noncycle edge  $e$  of  $\Gamma$ , if  $e$  does not have both endpoints on  $C$ , then  $C$  remains a cycle in  $\Gamma - e$  and  $\Gamma/e$ . The following theorem shows that the triviality of  $C$  with respect to  $(M, \Gamma)$  is determined by that with respect to  $(M, \Gamma - e)$  and  $(M/e, \Gamma/e)$ .

**THEOREM 1.** *Suppose  $\Gamma$  is a graph embedded in a 3-manifold  $M$ . Let  $C$  be a simple cycle in  $\Gamma$ , and let  $e$  be an edge of  $\Gamma$  with at most one end on  $C$ . If  $C$  is trivial with respect to both  $(M, \Gamma - e)$  and  $(M/e, \Gamma/e)$ , then it is trivial with respect to  $(M, \Gamma)$ .*

*Proof.* The Theorem is simple when  $C$  is disjoint from  $e$ : Let  $\pi : M \rightarrow M/e$  be the quotient map. By assumption  $C$  bounds a disk  $D$  in  $M/e$  with interior disjoint from  $\Gamma/e$ . Since  $e$  is disjoint from  $C$ ,  $\pi^{-1}(D)$  is a simple disk in  $M$  bounded by  $C$ .

Now we assume  $e$  has exactly one end on  $C$ . Since  $C$  is trivial with respect to  $(M, \Gamma - e)$ , there is a disk  $D$  in  $M$  such that  $\partial D = C$ , and  $\text{Int } D \cap \Gamma = \text{Int } D \cap e$ . Consider  $E(\Gamma) = M - \text{Int } N(\Gamma)$ . The surface  $P = D \cap E(\Gamma)$  is a planar surface satisfying

(\*1):  $\partial P$  consists of circles  $\partial_0, \partial_1, \dots, \partial_n$ , where  $\partial_1, \dots, \partial_n$  are meridians of  $e$  on  $\partial N(e)$ , and  $\partial_0$  is a curve on  $\delta(C)$  intersecting each meridian of  $C$  at a single point.

Conversely, any planar surface  $P$  in  $E(\Gamma)$  satisfying (\*1) can be extended to a disk  $D$  in  $M$  such that  $\partial D = C$  and  $\text{Int } D \cap \Gamma = \text{Int } D \cap e$ .

Now consider  $C$  as a cycle in  $\Gamma/e$ . Since  $C$  is trivial with respect to  $(M/e, \Gamma/e)$ , there is a disk  $D'$  in  $M/e$  bounded by  $C$  with  $\text{Int } D'$  disjoint from  $\Gamma/e$ . The surface  $Q = D' \cap E(\Gamma)$  is a disk satisfying

(\*2):  $\partial Q$  is a curve on  $\partial N(C \cup e)$ , which intersects each meridian of  $C$  at a single point.

Conversely, any such disk  $Q$  can be extended to a disk  $D'$  in  $M/e$  with  $\partial D' = C$  and  $\text{Int } D' \cap (\Gamma/e) = \emptyset$ .

We choose  $P$  and  $Q$  to satisfy (\*1) and (\*2), as well as the following general position and minimality conditions:

(\*3):  $n = |P \cap \delta(e)|$  is minimal, and  $k = |Q \cap \delta(e)|$  is minimal.

(\*4):  $P$  intersects  $Q$  transversely, and  $|P \cap Q|$  is minimal subject to (\*3).

(\*5):  $P \cap Q \cap \delta(e') = \emptyset$  for each edge  $e'$  in  $C$ .

(\*5) is possible because by (\*1) and (\*2) each of  $P \cap \delta(e')$  and  $Q \cap \delta(e')$  is an essential arc in  $\delta(e')$ , so we can isotop  $Q$  to make them disjoint. Since  $k$  is minimal,  $Q \cap \delta(e)$  consists of parallel essential arcs. So we may further assume

(\*6): each component of  $Q \cap \delta(e)$  intersects each  $\partial_j$  at a single point,  $j = 1, \dots, n$ .

If either  $n = 0$  or  $k = 0$ , then an extension of  $P$  or  $Q$  is a disc  $D$  in  $M$  with  $\partial D = C$  and  $\text{Int } D \cap \Gamma = \emptyset$ , so  $C$  is trivial with respect to  $(M, \Gamma)$ , as required. Hence we assume both  $n$  and  $k$  are positive. Label the components of  $\partial P$  so that, beginning with a point on  $\partial N(C)$ , an arc of  $\partial Q \cap \delta(e)$  intersects  $\partial_1, \partial_2, \dots, \partial_n$  successively.

A point of  $\partial P \cap \partial Q$  is labeled  $i$  if it is a point on  $\partial_i$ . Thus any arc on  $P \cap Q$  has a label on each of its end points.

**LEMMA 2.1.** *A component of  $P \cap Q$  in  $P$  is an arc which is either essential or has both ends on  $\partial_0$ .*

*Proof.* If  $P \cap Q$  has some circle components, a 2-surgery of  $P$  along some disk in  $Q$  bounded by an innermost circle will reduce  $|P \cap Q|$ . Therefore  $P \cap Q$  consists of arcs only.

If  $P \cap Q$  has some arc which is inessential in  $P$  and has both ends on some  $\partial_j$  with  $j \neq 0$ , let  $\alpha$  be an outermost one, so there is an arc  $\beta$  on  $\partial_j$  such that  $\alpha \cup \beta$  bounds a disk  $\Delta$  in  $P$  with interior disjoint from  $Q$ . A boundary compression of  $Q$  along  $\Delta$  produces two disks, one of which satisfies (\*2), but has less components of intersection with  $\delta(e)$ , contradicting the minimality of  $k$ .  $\square$

LEMMA 2.2. *There is a label  $i_0 > 0$  such that no arc of  $P \cap Q$  has both ends labeled  $i_0$ .*

*Proof.* Otherwise choose an  $\alpha_i$  for each  $i = 1, \dots, n$ , with  $\partial\alpha_i$  on  $\partial_i$ . Then the innermost such  $\alpha_i$  will be an inessential arc on  $P$ .  $\square$

Examine the order in which the indices appear on  $\partial Q$ . By (\*6), if we delete all the 0 indices, the sequence is  $1, 2, \dots, n, n, \dots, 2, 1$  repeated  $k/2$  times. The 0 indices appear only possibly between two successive 1's.

LEMMA 2.3. *An arc  $\alpha$  of  $P \cap Q$  which is outermost in  $Q$  is of one of the following types.*

Type (i):  $\alpha$  has both ends labeled 1 or both ends labeled  $n$ .

Type (ii):  $\alpha$  has one end labeled 1 and the other labeled 0.

*Proof.* Note that if  $i, j$  are successive labels on  $\partial Q$ , then  $|i - j| \leq 1$ . Therefore if  $\alpha$  is not of Type (i) or (ii), then the labels of  $\alpha$  are either  $\{0, 0\}$  or  $\{i, i + 1\}$  for some  $i > 0$ . Let  $\beta$  be the arc on  $\partial Q$  so that  $\alpha \cup \beta$  bounds a disk  $\Delta$  in  $Q$  with interior disjoint from  $P$ .

Suppose  $\alpha$  has label 0 on both endpoints. Then  $\partial\alpha$  divides  $\partial_0 \subset \partial P$  into two arcs  $\partial'_0$  and  $\partial''_0$ , one of which, say  $\partial''_0$ , has the property that it intersects a meridian of  $C$  if and only if  $\beta$  does. So  $\partial'_0 \cup \beta$  intersects any meridian of  $C$  at a single point. Let  $P_1$  be the part of  $P$  bounded by  $\partial'_0 \cup \alpha$ . Then  $P' = P_1 \cup \Delta$  satisfies (\*1). Moreover,  $|\partial P'| \leq |\partial P|$ , and a perturbation of  $P'$  has less components of intersection with  $Q$  than  $P$  does. This is impossible by (\*4).

Now suppose  $\alpha$  has labels  $\{i, i + 1\}$  for some  $i > 0$ . Then the normal extension of  $\Delta$  is a disk  $\Delta'$  in  $M$  such that  $\partial\Delta' = \alpha' \cup \beta'$ , where  $\alpha' \subset D$ ,  $\beta' \subset e$ , and  $\text{Int } \Delta' \cap \Gamma = \emptyset$ . So we can isotop  $\beta'$  through  $\Delta'$  to reduce  $|D \cap e|$ . This contradicts the minimality of  $n$ .  $\square$

Note that the proof does not apply to the case when the labels of  $\alpha$  are  $\{0, 1\}$ , since part of  $\beta'$  may be on  $C$ .

LEMMA 2.4. *There are at least two outermost edges  $\alpha_1, \alpha_2$  of Type (ii).*

*Proof.* By Lemma 2.2, there is an index  $i_0$  such that no arc in  $P \cap Q$  has both ends labeled  $i_0$ . Let  $A$  be the set of arcs in  $P \cap Q$  with one end labeled  $i_0$ . Let  $\Delta$  be a disk in  $Q$  such that  $\gamma = \partial\Delta - \partial Q$  is an arc in  $A$ , and  $\Delta$  contains no other arcs in  $A$ . Note that there are at least two such  $\Delta$ 's. So we need only to show that there is at least one type (ii) outermost edge in  $\Delta$ .

Suppose there is no outermost arc of type (ii) in  $\Delta$ . Then by Lemma 2.3, each outermost arc in  $\Delta$  is of type (i), so the labels of the arc are either  $\{1, 1\}$  or  $\{n, n\}$ . If there are two such outermost arcs, then the index  $i_0$  appears between them, which is impossible by the definition of  $\Delta$ . So there is only one outermost arc on  $\Delta$ . This implies that the arcs of  $P \cap Q$  are all parallel in  $\Delta$ . It is now clear that every arc in  $\Delta$  has the same index on both ends. Especially, both ends of  $\gamma$  are labeled  $i_0$ , contradicting the choice of  $i_0$ .  $\square$

Now let  $\Delta_1, \Delta_2$  be two disks in  $Q$  such that  $\partial\Delta_i - \partial Q$  is an outermost arc of type (ii). Then the normal extension of  $\Delta_i$  is a disk  $\Delta'_i$  in  $M$  with  $\partial\Delta'_i = \alpha_i \cup \beta_i \cup \gamma_i$ , where  $\alpha_i$  is an arc in  $D$  connecting a vertex  $v_i$  of  $C$  to the first intersection  $x$  of  $e$  with  $\text{Int } D$ ,  $\beta_i$  is an arc on  $e$  connecting  $x$  to  $v_0 = e \cap C$ , and  $\gamma_i$  is an arc on  $C$  connecting  $v_0$  to  $v_i$ . ( $\gamma_i$  may degenerate to a single point.) Since  $\partial Q$  intersects a meridian of  $C$  at a single point, the two arcs  $\gamma_1$  and  $\gamma_2$  cannot have an edge in common, and hence intersect only at  $v_0$ . Thus  $\Delta'_1 \cap \Delta'_2 = \beta_1 = \beta_2$ , so  $\Delta = \Delta'_1 \cup \Delta'_2$  is a disk in  $M$ . Let  $D_1$  be the part of  $D$  bounded by  $\partial\Delta$ , and let  $D_2$  be  $(D - D_1) \cup \Delta$  pushed off  $\beta_1 - v_0$ . Then  $D_2$  is a disk in  $M$  with  $\partial D_2 = C$ , and  $|\text{Int } D_2 \cap e| \leq n - 1$ . This contradicts the minimality of  $n = |\text{Int } D \cap e|$ .  $\square$

### 3. Planar graphs in manifolds

In this section we will discuss the planarity of graphs in a 3-manifold. Suppose  $\Gamma$  is a graph embedded in  $M$ . An edge  $e$  of  $\Gamma$  is called a free edge if it is not a cycle, and one of its endpoints is not incident to any other edges. Clearly, if  $e$  is a free edge, then  $\Gamma$  is planar in  $M$  if and only if  $\Gamma - e$  is planar. Therefore, without loss of generality we will always assume that  $\Gamma$  has no free edges.

We need the following definitions: A graph  $\Gamma$  in  $M$  is called *split* if there is a 2-sphere  $S$  in  $M$  which is disjoint from  $\Gamma$ , and separates  $M$  into  $M_1$  and  $M_2$ , such that both  $M_i$  contain part of  $\Gamma$ . It is called *decomposable* if there is a vertex  $v \in \Gamma$  such that  $\delta(v)$  has a compressing disk  $D$  in  $E(\Gamma)$  which is separating. The following lemma and its proof is similar to that of [3, Lemma 1.3].

LEMMA 3.1. *Let  $\Gamma$  be a split or decomposable graph in a 3-manifold. If all proper subgraphs of  $\Gamma$  are planar, then  $\Gamma$  is planar.*

*Proof.* First assume  $\Gamma$  is split. Let  $S$  be a 2-sphere disjoint from  $\Gamma$ , separating  $M$  into  $M_1$  and  $M_2$ , such that  $\Gamma_i = M_i \cap \Gamma$  are proper subgraphs of  $\Gamma$ . By assumption, there are 2-spheres  $S_i \subset M$  such that  $\Gamma_i \subset S_i$ . By 2-surgery along disks bounded by innermost circles of  $S_i \cap S$ , we can delete all intersections of  $S_i$  with  $S$ , and get  $S_i \subset M_i$ . Tubing  $S_1$  to  $S_2$  gives a 2-sphere containing  $\Gamma$ .

Now suppose  $\Gamma$  is decomposable, and let  $D$  be a separating compressing disk of  $\delta(v)$  in  $E(\Gamma)$ . It can be extended to a 2-sphere  $S$  in  $M$  so that  $S \cap \Gamma = \{v\}$ , and  $S$  separates  $M$  into  $M_1$  and  $M_2$ . Let  $\Gamma_i = \Gamma \cap M_i$ . Since  $\Gamma_i$  is planar, there is a 2-disk  $D_i$  in  $E = M - \text{Int } N(v)$  which contains  $\Gamma_i \cap E$ . By surgery along disks bounded by innermost circles or outermost arcs of  $D \cap D_i$  in  $D$ , we can assume  $D_i \cap D = \emptyset$ . Gluing a band on  $\delta(v)$  to  $D_1 \cup D_2$  produces a single disk containing  $\Gamma \cap E$ , which can be extended to a sphere in  $M$  containing  $\Gamma$ .  $\square$

Define a cut point of  $\Gamma$  to be a vertex  $v$  such that  $\Gamma - v$  has more components than  $\Gamma$ . Let  $\{v_1, \dots, v_k\}$  be the cut points of  $\Gamma$ . Then there is a component  $X$  of  $\Gamma - \{v_1, \dots, v_k\}$  which has the property that  $\Gamma_1 = (\text{the closure of } X)$  contains at most one of these  $v_i$ ; for otherwise one can find a simple loop in  $\Gamma$  passing through some of the  $v_j$ 's, contradicting the definition of cutting points. This subgraph  $\Gamma_1$  is connected, and has no cut point of its own. (It is possible that  $\Gamma_1 = \Gamma$ .)

Suppose  $\Gamma$  is abstractly planar. Embed  $\Gamma_1$  into a 2-sphere  $S$ . Since  $\Gamma_1$  is connected and has no cut points of its own, the closure of each components of  $S - \Gamma_1$  is a disk. Let  $D_0, D_1, \dots, D_n$  be these disks. If  $\Gamma_1$  contains a cut point  $v$  of  $\Gamma$ , choose  $D_0$  to contain  $v$ . Let  $D = S - \text{Int } D_0$ .

Denote by  $\Gamma_1^c$  the closure of  $\Gamma - \Gamma_1$ . Then  $\Gamma_1^c \cap \Gamma_1 = \{v\}$  or  $\emptyset$ , depending on whether  $\Gamma_1$  contains a cut point  $v$  of  $\Gamma$ . Embed  $\Gamma_1^c$  into a disk  $D'$  so that  $\partial D' \cap \Gamma_1^c = \{v\}$  or  $\emptyset$  accordingly. Glue  $D$  and  $D'$  together, we get an embedding of  $\Gamma$  into  $S^2 = D \cup D'$ . We fix this embedding.

Recall that  $D_1, \dots, D_n$  are the closures of the components of  $D - \Gamma_1$ .

**LEMMA 3.2.** *We can number the disks so that  $B_k = D_1 \cup \dots \cup D_k$  is a disk for all  $k$ .*

*Proof.* If  $D_i$  is a disk such that  $D_1 \cap D_i$  is not connected, then  $D_1 \cup D_i$  is not simply connected, so there is a region  $\Omega$  in  $D$  bounded by a boundary component of  $D_1 \cup D_i$ .  $\Omega$  is the union of some  $D_j$ 's. Choose  $i$  so that  $\Omega$  contains a minimal number of these disks. Since  $D_i$  is a disk,  $\partial\Omega$  is not completely contained in  $\partial D_i$ , so there is a disk  $D_j$  in  $\Omega$  which has an edge in common with  $D_1$ . If  $D_1 \cup D_j$  is not simply connected, then it bounds a region  $\Omega' \subset \Omega$ , contradicting the choice of  $D_i$ . Hence we can name this  $D_j$  as  $D_2$ . Generally, if  $B_k = D_1 \cup \dots \cup D_k$  is a disk, then by the same argument we can find  $D_{k+1}$  so that  $B_k \cap D_{k+1}$  is an arc, and hence  $B_k \cup D_{k+1}$  is a disk. The Lemma now follows by induction.  $\square$

A link in  $S^3$  is a trivial link if and only if all of its components are trivial. The following theorem shows that this is also true for graphs in 3-manifolds.

**THEOREM 2.** *An abstractly planar graph  $\Gamma$  in  $M$  is planar if and only if all cycles of  $\Gamma$  are trivial.*

*Proof.* We want to show that the inclusion  $\Gamma \rightarrow M$  can be extended to an embedding of  $\Gamma \cup B_n$  into  $M$ . This is done by induction. By assumption,  $\partial D_1$  bounds a disk with interior disjoint from  $\Gamma$ , so we have an embedding of  $\Gamma \cup D_1$  into  $M$ . Generally, suppose we have extended  $\Gamma \rightarrow M$  to an embedding  $i_k : \Gamma \cup B_k \rightarrow M$ . By Lemma 3.2,  $B_k$  is a disk, and  $B_k \cap D_{k+1}$  is an arc. Consider the graph  $\Gamma' = \Gamma - \text{Int } B_k \subset M$ . Then  $\partial B_k$  and  $\partial D_{k+1}$  are cycles in  $\Gamma'$  which are trivial with respect to  $(M, \Gamma')$ . So by Lemma 1.1,  $\partial D_{k+1}$  bounds a disk  $\Delta_{k+1}$  which has interior disjoint from  $\Gamma' \cup B_k$ . Now we can define  $i_{k+1} : \Gamma \cup D_1 \cup \cdots \cup D_{k+1} \rightarrow M$  so that  $D_{k+1}$  is mapped to  $\Delta_{k+1}$ . This completes the induction.

It follows that the image of  $B_n$  is an embedded disk  $\Delta$  in  $M$  so that  $\Delta \cap \Gamma = \Gamma_1$ , and  $\partial \Delta \subset \Gamma_1$ . When  $\Gamma = \Gamma_1$  this implies  $\Gamma$  is planar. When  $\Gamma \neq \Gamma_1$ , the set  $\Gamma_1 \cup \Gamma_1^c$  is either empty or a cut point, which implies  $\Gamma$  is split or decomposable. By induction we may assume that all proper subgraphs of  $\Gamma$  are planar. The theorem now follows from Lemma 3.1.  $\square$

As an application of the above theorems, we give an alternative proof of a theorem of Scharlemann and Thompson [3].

**THEOREM 3.** *A finite graph  $\Gamma \subset S^3$  is planar if and only if*

- (a)  $\Gamma$  is abstractly planar;
- (b) every graph properly contained in  $\Gamma$  is planar;
- (c)  $\pi_1(E(\Gamma))$  is a free group.

*Proof.* Since  $\pi_1(E(\Gamma))$  is free,  $E(\Gamma)$  is the connected sum of some handlebodies. If  $\Gamma$  is not connected, then it is split, and the theorem follows from Lemma 3.1. So we assume  $\Gamma$  is connected. When  $\Gamma$  has only one vertex, the theorem was proved in [1], so we assume  $\Gamma$  has some noncycle edge  $e$ . By induction on the number of edges in  $\Gamma$ , we may assume that  $\Gamma/e$  is planar for all such  $e$ .

According to Theorem 2, we need only to show that each cycle of  $\Gamma$  is trivial. Let  $C$  be a cycle in  $\Gamma$ . There are several cases.

**CASE 1** ( $C$  does not contain all vertices of  $\Gamma$ ). In this case there is some noncycle edge  $e$  which has at most one endpoint on  $C$ . Since both  $\Gamma - e$  and  $\Gamma/e$  are planar,  $C$  is trivial with respect to both  $(S^3, \Gamma - e)$  and  $(S^3/e, \Gamma/e)$ . By Theorem 1,  $C$  is also trivial with respect to  $(S^3, \Gamma)$ .

CASE 2 ( $\Gamma$  has some cycle edges). A cycle edge cannot contain all vertices of  $\Gamma$  because  $\Gamma$  has more than one vertex. By Case 1, a cycle edge is a trivial cycle, so it bounds a simple disk. It follows that  $\Gamma$  is decomposable, and the Theorem follows from Lemma 3.1.

In the remaining cases, all edges not in  $C$  are noncycle edges with both ends on  $C$ . Let  $e$  be such an edge. Its endpoints divide  $C$  into two arcs  $C_1$  and  $C_2$ .

CASE 3 (There is an edge  $e'$  which has one endpoint on each of  $\text{Int } C_i$ ). Consider the cycle  $C_i \cup e$ . It is incident to just one endpoint of  $e'$ . By Case 1,  $C_i \cup e$  bounds a disk  $D_i$  with interior disjoint from  $\Gamma$ . By Lemma 1.1, we can choose the  $D_i$  to have disjoint interiors. Thus  $D = D_1 \cup D_2$  can be modified off  $e$  to become a simple disk bounded by  $C$ .

CASE 4 (No such edges  $e'$  as in Case 3 exist). Note that in this case  $\bar{e}$ , the image of  $e$  in  $\Gamma/e$ , is a cut point of  $\Gamma/e$ , and hence a decomposing point because  $\Gamma/e$  is planar. We want to apply Lemma 1.2 to our situation. To do this, let  $M = E(\Gamma)$ , and let  $F = \partial N(C \cup e) - \text{Int } N(\Gamma)$ . This is a punctured genus 2 surface, with one hole for each end of each edge which is not in  $C \cup e$ . Let  $e_1, \dots, e_k$  be the edges and  $v_1, \dots, v_k$  the vertices of  $C$ . Denote by  $m_i$  a meridian of  $e_i$ , and by  $J$  a meridian of  $e$ . Let  $\gamma = m_1 \cup \dots \cup m_k$ .

$F - \gamma$  is isotopic to  $\delta(e) \cup \delta(v_1) \cup \dots \cup \delta(v_k) = \delta_{\Gamma/e}(\bar{e}) \cup (\bigcup \{\delta(v_i) \mid v_i \notin \partial e\})$ . Since  $\bar{e}$  is a decomposing point of  $\Gamma/e$ ,  $\delta_{\Gamma/e}(\bar{e})$  is compressible in  $E(\Gamma)$ . So  $F - \gamma$  is compressible.

Consider  $\tau(E(\Gamma), J)$ . This is the manifold obtained from  $E(\Gamma)$  by attaching a 2-handle along a meridian of  $e$ , so it is actually the exterior of  $\Gamma - e$ . The surface  $\sigma(F, J)$  is the punctured torus  $\partial N(C) - \text{Int } N(\Gamma - e)$ . Since  $\Gamma - e$  is planar,  $C$  bounds a disk in  $M$ , which gives rise to a compressing disk  $D'$  of  $\sigma(F, J)$  in  $E(\Gamma - e)$ , so that  $D'$  intersects each  $m_j$  at a single point. By Lemma 1.2,  $F - J$  has a compressing disk  $D$  in  $E(\Gamma)$  intersecting each  $m_j$  at most once. Since  $F - J$  is a punctured torus, and  $m_j$  are meridians, if  $D$  is disjoint from some  $m_j$ , it is disjoint from all  $m_j$ , so it will be a compressing disk of some  $\delta(v_i)$ , which implies  $\Gamma$  is decomposable, and the Theorem follows. So we assume  $D$  intersects each  $m_j$  at one point. Then we can modify  $D$  so that  $\partial D$  intersects any meridian of  $C$  at a single point. The normal extension  $\Delta$  of  $D$  is now a simple disk bounded by  $C$ .  $\square$

#### 4. $F$ -planarity of graphs

Let  $F$  be a finite 2-dimensional cell complex with a connected graph  $\Gamma$  as its 1-skeleton.  $\Gamma$  is called a regular graph in  $F$  if the attaching map of each face (i.e.

2-cell) is a cycle in  $\Gamma$ , and the intersection of any two faces is connected. Suppose  $\Gamma$  is embedded in a 3-manifold  $M$ . Then  $\Gamma$  is called  $F$ -planar if it can be extended to an embedding of  $F$  in  $M$ . Suppose  $e$  is an edge of  $\Gamma$  which is not contained in the boundary of any faces of  $F$ . Then  $f - \text{Int } e$  has  $\Gamma - e$  as 1-skeleton, and  $F/e$  has  $\Gamma/e$  as 1-skeleton. To simplify notations, we call  $\Gamma/e$  (resp.  $\Gamma - e$ )  $F$ -planar if it is  $(F/e)$ -planar (resp.  $(F - \text{Int } e)$ -planar). The following is a generalization of Theorem 1.

**THEOREM 4.** *Suppose  $F$  is a regular 2-complex with  $\Gamma$  as its 1-skeleton, and suppose  $\Gamma$  is embedded in a 3-manifold  $M$ . Let  $e$  be a noncycle edge of  $\Gamma$  such that both  $\Gamma - e$  and  $\Gamma/e$  are  $F$ -planar. If  $e$  intersects each face of  $F$  at most at one of its endpoints, then  $G$  is  $F$ -planar.*

*Proof.* This follows from Theorem 1 and Lemma 1.1 by induction on the number of faces in  $F$ .  $\square$

The most interesting case of  $F$ -planarity is when  $F$  is a surface. It was shown in [2] that Theorem 3 is equivalent to the following:

**THEOREM 3'.** *Let  $\Gamma$  be an abstractly planar graph in  $S^3$  (or  $R^3$ ). If  $\Gamma$  has a noncycle edge  $e$  such that both  $\Gamma - e$  and  $\Gamma/e$  are planar, then  $\Gamma$  is planar.*

The following is a similar result for regular graphs in an arbitrary compact surface  $F$ . Suppose  $\Gamma$  is such a graph, and  $e$  is a noncycle edge which has at least one endpoint in the interior of  $F$ . Since  $F/e \cong F$ , both  $\Gamma - e$  and  $\Gamma/e$  can be considered naturally as a graph in  $F$ .

**THEOREM 5.** *Suppose  $\Gamma$  is a regular graph on a surface  $F$ , and suppose  $\Gamma$  is embedded in a 3-manifold  $M$ . Let  $e$  be a noncycle edge of  $\Gamma$  with at least one end in  $\text{Int } F$ . If both  $\Gamma/e$  and  $\Gamma - e$  are  $F$ -planar, then  $\Gamma$  is  $F$ -planar.*

*Proof.* We may assume that each end of  $e$  has valence at least 3, otherwise  $\Gamma$  is homeomorphic to  $\Gamma/e$ , and the planarity of  $\Gamma$  follows from that of  $\Gamma/e$ . Especially, an end of  $e$  in  $\text{Int } F$  is incident to at least 3 faces of  $F$ .

Denote by  $D'$ ,  $D''$  the two disks incident to  $e$ . Consider the 2-complex  $G = F - \text{Int } D' \cup \text{Int } D''$ . First suppose  $e$  has both ends on some face  $D$  of  $G$ , then  $D$  contains  $\partial D - e$  because  $D \cap D'$  is connected. Similarly,  $D$  contains  $\partial D'' - e$ . By assumption  $\partial D$  is a cycle, so  $\partial D = \partial(D' \cup D'')$ . This is now a very special case:  $\Gamma$  has 3 edges and 2 vertices, and  $F$  is a 2-sphere. Since  $\Gamma/e$  is  $F$ -planar,  $\Gamma/e$ , and hence  $\Gamma$ , is contained in a 3-ball. Therefore the theorem follows from Theorem 3'.

Let  $D_1, \dots, D_n$  be the faces of  $G$  and consider  $G$  as a subset of  $M$ . Since  $D_i$  intersects  $e$  at most once, it remains a disk in  $M/e$ . By assumption  $\Gamma/e$  is  $F$ -planar in  $M/e$ , so  $\partial D'/e$  bounds a disk  $\Delta$  in  $M/e$  with  $\text{Int } \Delta \cap \Gamma/e = \emptyset$ . Since  $\partial D' \cap D_i$  is connected,  $\partial \Delta \cap D_i$  is connected for all  $i = 1, \dots, n$ . By Lemma 1.1 we can choose  $\Delta$  so that  $\text{Int } \Delta \cap D_i = \emptyset$  for  $i = 1, \dots, n$ . Let  $Q$  be the disk  $\Delta \cap E(\Gamma/e) = \Delta \cap E(\Gamma)$  in  $E(\Gamma)$ .  $Q$  is disjoint from  $\bigcup D_i$ , and  $\partial Q$  intersects each meridian of  $\partial D' - e$  at a single point. Let  $v$  be an end of  $e$  in  $\text{Int } F$ . Isotop  $Q$  so that  $|\partial Q \cap \delta(e)|$  is minimal. Then  $A = \partial Q \cap \delta(v)$  consists of arcs on the punctured sphere  $\delta(v)$  which are all essential. As the circle  $\partial Q$  intersects a meridian of  $\partial D' - e$  at a single point, there is an arc  $\alpha \in A$  with exactly one end on the circle  $J = \delta(v) \cap \delta(e)$ , while all the other arcs in  $A$  have both ends on  $J$ . The arcs in  $A$ , being part of  $\partial Q$ , are disjoint from the disks  $D_1, \dots, D_n$ . Because  $F$  is a surface, and  $v$  is in  $\text{Int } F$ , these disks cut  $\delta(v)$  into an annulus. It follows that all arcs in  $A - \{\alpha\}$  are inessential, which is absurd unless  $\alpha$  is the only arc in  $A$ . Therefore  $\partial Q$  intersects a meridian of  $e$  at a single point. The normal extension  $\Delta'$  of  $Q$  is now a disk bounded by  $\partial D'$ , with interior disjoint from  $G$ . Similarly, there is a disk  $\Delta''$  bounded by  $\partial D''$ , such that  $\text{Int } \Delta'' \cap G = \emptyset$ , and by Lemma 1.1, it can be chosen so that  $\Delta' \cap \Delta'' = e$ . The surface  $G \cup \Delta' \cup \Delta''$  is now an embedding of  $F$  in  $M$ .  $\square$

The regularity condition in Theorem 4 is necessary. Consider the graph  $\Gamma$  on a torus  $F$  as shown in Figure 1. Embedding  $F$  into  $S^3$  in the trivial way, we get a graph  $\Gamma_1$  which is  $F$ -planar in  $S^3$ . Let  $\Gamma_2$  be the embedding of  $\Gamma$  in  $S^3$  as shown in Figure 2, obtained from  $\Gamma_1$  by interchanging a crossing in Figure 1. Let  $e$  be the edge shown in the figure. It is easy to see that  $\Gamma_2 - e$  and  $\Gamma_2/e$  are isotopic to  $\Gamma_1 - e$  and  $\Gamma_1/e$  respectively, so they are  $F$ -planar in  $S^3$ . One can also isotop  $\Gamma_2$  so that it lies on the trivial torus. But  $\Gamma_2$  is not  $F$ -planar. To see this, one may need the following fact.

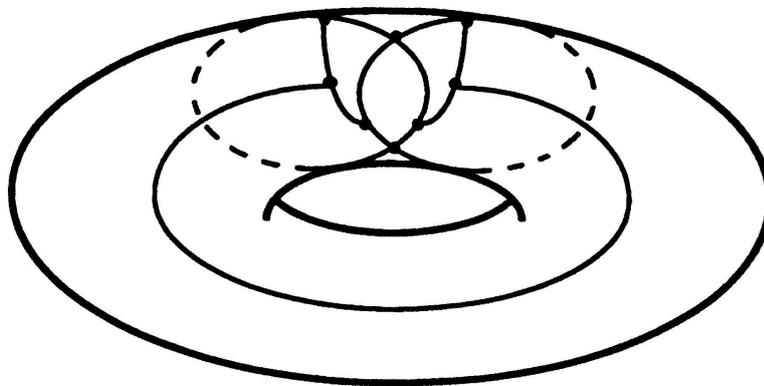


Figure 1

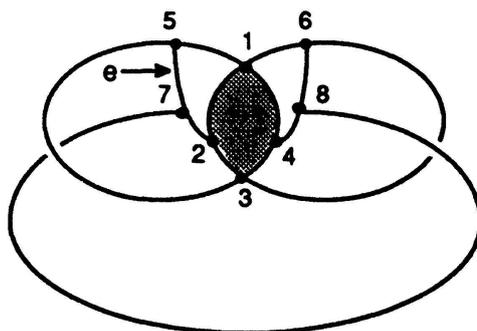


Figure 2

LEMMA 4.1. *Suppose  $\Gamma$  is a graph in  $S^3$ , and  $C$  is a trivial cycle with respect to  $(S^3, \Gamma)$ . If  $\Gamma \cap E(C)$  is connected, then the simple disk  $D$  bounded by  $C$  is unique up to ambient isotopy fixing  $\Gamma$ .*

Label the vertices of  $\Gamma_2$  as in Figure 2. Denote by  $C(i_1, \dots, i_k)$  the cycle successively passing through the vertices labeled  $i_1, \dots, i_k$ . Suppose  $\Gamma_2$  is  $F$ -planar. Then  $C(1, 2, 3, 4)$  and  $C(1, 5, 3, 6)$  should bound simple disks with disjoint interiors. By Lemma 4.1, the disks are unique up to isotopy, so we can take the disk  $D$  bounded by  $C(1, 2, 3, 4)$  to be the shaded region in Figure 2. Now  $C(1, 5, 3, 6)$  cannot bound a disk with interior disjoint from  $\Gamma_2 \cup D$ , because it has linking number 1 with some curve in  $\Gamma_2 \cup D - C(1, 5, 3, 6)$ .

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