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On planarity of graphs in 3-manifolds*

YING-QING WU

A graph Γ in a 3-manifold M is called *planar* if it is contained in an embedded 2-sphere in M . It is *abstractly planar* if it can be embedded into an abstract 2-sphere. In [3] Scharlemann and Thompson gave necessary and sufficient conditions for a graph Γ to be planar in S^3 (see Theorem 3 in section 3). The special case that Γ has a single vertex was proved by Gordon [1], while the generic case was shown [2] to be equivalent to: An abstractly planar graph Γ in S^3 is planar if and only if both $\Gamma - e$ and Γ/e are planar, where e is a noncycle edge of Γ . Fix an embedding of Γ in a 2-sphere F . We say that the embedding of Γ in S^3 is F -planar if it can be extended to an embedding of F into S^3 . It turns out that the above result is equivalent to: If both $\Gamma - e$ and Γ/e are F -planar, then Γ is also F -planar.

In this paper, we study the F -planarity of a graph Γ in a 3-manifold M , where F can be an arbitrary surface containing Γ , or more generally a 2-dimensional cell complex with Γ as 1-skeleton. An embedding of Γ in a 3-manifold M is called F -planar if it can be extended to an embedding of F in M . We are interested in the problem of whether the F -planarity of Γ is determined by that of $\Gamma - e$ and Γ/e . A statement parallel to the case of $F = S^2$ is not true in this general setting. However we will show it is true if Γ is a “regular” graph.

We first study the triviality of cycles. This can be considered a special case of the above problem, when the cell complex has only one 2-cell. A *cycle* of Γ is a subgraph C which is homeomorphic to a circle.

DEFINITION. Suppose Γ is embedded in a 3-manifold M . Then a cycle C of Γ is trivial (with respect to (M, Γ)), if it bounds a disk with interior disjoint from Γ .

In section 2 we prove a theorem about triviality of simple cycles. Note that if C is a cycle of Γ , and e is an edge intersecting C at most once, then C remains a cycle in both $\Gamma - e$ and Γ/e . Therefore it makes sense talking about the triviality of C with respect to $(M, \Gamma - e)$ and $(M/e, \Gamma/e)$.

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THEOREM 1. *Suppose Γ is a graph embedded in a 3-manifold M . Let C be a cycle in Γ , and let e be an edge of Γ with at most one end on C . If C is trivial with respect to both $(M, \Gamma - e)$ and $(M/e, \Gamma/e)$, then it is trivial with respect to (M, Γ) .*

A link L in S^3 is the unlink if each component of L is a trivial cycle. It turns out that this is also true for any abstractly planar graphs in a 3-manifold M :

THEOREM 2. *An abstractly planar graph Γ in M is planar if and only if all cycles of Γ are trivial.*

We will prove Theorem 2 in Section 3, and use these theorems to give an alternative proof of the Scharlemann–Thompson Theorem.

In Section 4, we study the F -planarity of graphs in arbitrary 3-manifolds M . Suppose Γ is a graph in a compact surface F . We assume that ∂F is either empty or a subgraph of Γ . An embedding of Γ into M is F -planar if it can be extended to an embedding of F into M . We call the closure of a component of $F - \Gamma$ a face of F . The graph Γ is called a *regular graph* in F if each face of F is a disk, and the intersection of any two faces is connected (or empty). Suppose e is an edge of Γ with at least one end in the interior of F . Then both $\Gamma - e$ and Γ/e can be considered as graphs in F in the natural way, so we can talk about the F -planarity of $\Gamma - e$ and Γ/e . The following theorem is proved in section 4.

THEOREM 5. *Suppose Γ is a regular graph on a surface F , and suppose Γ is embedded in a 3-manifold M . Let e be an edge of Γ with at least one end in $\text{Int } F$. If both Γ/e and $\Gamma - e$ are F -planar, then Γ is F -planar.*

The regularity condition on Γ is necessary. We will give an example of a graph Γ on a torus F that can be embedded into S^3 , so that both $\Gamma - e$ and Γ/e are F -planar, but Γ itself is not F -planar.

I would like to thank Marty Scharlemann for some helpful discussion on this topic, and to the referee for many useful comments.

1. Definitions and preliminaries

Given a graph Γ in a 3-manifold M , choose a regular neighborhood for each vertex and each edge of Γ , so that the disks $\partial N(v) \cap N(e)$ are mutually disjoint for all v and e . The union of all such neighborhoods forms a regular neighborhood $N(\Gamma)$ of Γ and we define the exterior of Γ to be $E(\Gamma) = M - \text{Int } N(\Gamma)$. For each vertex v , denote by $\delta(v)$ the punctured sphere $\partial N(v) - \bigcup \text{Int } N(e)$; similarly, for

each edge e , let $\delta(e)$ be the annulus $\partial N(e) - \bigcup \text{Int } N(v)$. Sometimes the graph may vary, in which case we use $\delta_\Gamma(e)$ and $\delta_\Gamma(v)$ to denote $\delta(e)$ and $\delta(v)$, respectively. If C is a cycle, or more generally a subgraph of Γ , we use $\delta(C)$ to denote the union of $\delta(t)$ with t ranges over all edges and vertices of C .

For an edge e in Γ , denote by $\Gamma - e$ the subgraph obtained from Γ by deleting the interior of the edge e . If e is not a loop, then Γ/e is a graph in M/e . Denote by \bar{e} the image of e in Γ/e . The quotient map $\pi : M \rightarrow M/e$ sends $N(\Gamma)$ to a regular neighborhood $N(\Gamma/e)$ of Γ/e in M/e , so it induces a homeomorphism $E(\Gamma) \cong E(\Gamma/e) = M/e - \text{Int } N(\Gamma/e)$. We identify $E(\Gamma)$ with $E(\Gamma/e)$ by this homeomorphism. Note that $\delta_{\Gamma/e}(\bar{e}) = \delta_\Gamma(v) \cup \delta_\Gamma(e) \cup \delta_\Gamma(v')$ if $\partial e = v \cup v'$.

If X is a subset of M , denote the number of components in X by $|X|$.

We define a *simple disk* to be a disk D in M which is bounded by a cycle of Γ , and has interior disjoint from Γ . Thus a cycle of Γ is a trivial cycle if and only if it bounds a simple disk.

Define a *normal structure* on $N(\Gamma)$ to be a set of line segments $\{l_x \mid x \in \partial N(\Gamma)\}$ as follows: For any vertex $v \in \Gamma$ and any $x \in \delta(v)$, let l_x be the straight line in $D^3 = N(v)$ connecting x to v . If e is not a loop, the closure of $N(e) - \bigcup \{l_x \mid x \in \bigcup \delta(v)\}$ has a product structure $e \times D^2$ such that for $x \in \partial e \times \partial D^2$, the l_x defined above is the line between x and a point in $\partial e \times 0$. Now for any $x \in p \times \partial D^2$ with $p \in e$, let l_x be the line connecting x to $p \times 0$. If e is a loop, $N(e) - \bigcup \{l_x \mid x \in \bigcup \delta(v)\}$ is homeomorphic to $e \times D^2 / \partial e \times 0$, so we can define l_x in the same way as above. For any $p \in e$, $p \times D^2$ is called a *meridian disk* of Γ (or e) at p , and $p \times \partial D^2$ is called a *meridian* of Γ .

Suppose P is a surface in $E(\Gamma)$. The normal extension D of P is the union of P and the lines l_x with $x \in P \cap \partial N(\Gamma)$. If P is a properly embedded disk in $E(\Gamma)$, and C is a cycle of Γ such that P intersects any meridian of C exactly once, and is disjoint from the other meridians of Γ , then D is a disc with $\partial D = C$. A surface S in M with ∂S in Γ is called in normal position if S is the normal extension of $S \cap E(\Gamma)$. The following lemma is useful in modifying disks to make their interiors disjoint.

LEMMA 1.1. *Suppose D_1, \dots, D_n are simple disks in M with mutually disjoint interiors. Suppose C is a trivial cycle, and $C \cap D_i$ is connected for all i . Then C bounds a simple disk D with interior disjoint from D_i for all i .*

Proof. By an isotopy we may assume D_1, \dots, D_n are in normal position. Let $P_i = D_i \cap E(\Gamma)$. Choose a simple disk D in normal position and bounded by C so that $P = D \cap E(\Gamma)$ is transverse to P_i , and $|P \cap (\bigcup P_i)|$ is minimal. Let A be the closure of $\text{Int } D \cap (\bigcup D_i)$. Since $A \cap N(\Gamma)$ consists of lines l_x with $x \in \partial P \cap (\bigcup P_i)$, we know that A is the union of some circles which may intersect Γ at one point, and

some arcs with different endpoints on Γ . These circles and arcs might intersect on Γ , but are otherwise disjoint. If A has some circles, choose a circle α which is innermost in some D_i , and let Δ and Δ_i be the disks it bounds in D and D_i respectively. Then $(D - \Delta) \cup \Delta_i$ can be rel ∂D isotoped into a disk D' with $|D' \cap (\bigcup P_i)| < |P \cap (\bigcup P_i)|$. If A has no circles but has some arcs, let β be an arc in A which is outermost in the sense that there is an arc γ in some $C \cap \partial D_i$, such that $\beta \cup \gamma$ bounds a disk Δ_i in D_i with $\text{Int } \Delta_i \cap D = \emptyset$. (This is possible because of the assumption that $C \cap \partial D_i$ is connected for all i). Let Δ be the disk in D with $\partial \Delta = \partial \Delta_i = \beta \cup \gamma$. Then a perturbation of $(D - \Delta) \cup \Delta_i$ produces a disk D' with $|D' \cap (\bigcup P_i)| < |P \cap (\bigcup P_i)|$. By the minimality of $|P \cap (\bigcup P_i)|$, neither case can happen. Therefore $A = \emptyset$. \square

In section 3 we will need some handle addition lemmas. Let F be a surface on the boundary of a 3-manifold M , and let J be a simple loop on F . Denote by $\tau(M, J)$ the manifold obtained from M by attaching a 2-handle along J , that is, $\tau(M, J) = M \cup (D^1 \times D^2)$, where $D^1 \times \partial D^2$ is identified with a regular neighborhood $N(J)$ of J in F . Denote by $\sigma(F, J)$ the surface $(F - N(J)) \cup (\partial D^1 \times D^2)$. We have the following generalized handle addition lemma.

LEMMA 1.2. *Suppose S is a surface on the boundary of a 3-manifold M . Let γ be a 1-manifold on S such that $S - \gamma$ is compressible, and let J be a circle in S disjoint from γ . If $\sigma(S, J)$ is compressible in $\tau(M, J)$ with D' a compressing disk, then $S - J$ has a compressing disk D such that $\partial D \cap \gamma \subset \partial D' \cap \gamma$.*

This was implied in the proof of [4, Thm 1]. It was shown that under the assumption we have $|\partial D \cap \gamma| \leq |\partial D' \cap \gamma|$, but the argument there has actually proved that $\partial D \cap \gamma \subset \partial D' \cap \gamma$.

2. Trivial cycles in a graph

Given a cycle C in $\Gamma \subset M$, and a noncycle edge e of Γ , if e does not have both endpoints on C , then C remains a cycle in $\Gamma - e$ and Γ/e . The following theorem shows that the triviality of C with respect to (M, Γ) is determined by that with respect to $(M, \Gamma - e)$ and $(M/e, \Gamma/e)$.

THEOREM 1. *Suppose Γ is a graph embedded in a 3-manifold M . Let C be a simple cycle in Γ , and let e be an edge of Γ with at most one end on C . If C is trivial with respect to both $(M, \Gamma - e)$ and $(M/e, \Gamma/e)$, then it is trivial with respect to (M, Γ) .*

Proof. The Theorem is simple when C is disjoint from e : Let $\pi : M \rightarrow M/e$ be the quotient map. By assumption C bounds a disk D in M/e with interior disjoint from Γ/e . Since e is disjoint from C , $\pi^{-1}(D)$ is a simple disk in M bounded by C .

Now we assume e has exactly one end on C . Since C is trivial with respect to $(M, \Gamma - e)$, there is a disk D in M such that $\partial D = C$, and $\text{Int } D \cap \Gamma = \text{Int } D \cap e$. Consider $E(\Gamma) = M - \text{Int } N(\Gamma)$. The surface $P = D \cap E(\Gamma)$ is a planar surface satisfying

(*1): ∂P consists of circles $\partial_0, \partial_1, \dots, \partial_n$, where $\partial_1, \dots, \partial_n$ are meridians of e on $\partial N(e)$, and ∂_0 is a curve on $\delta(C)$ intersecting each meridian of C at a single point.

Conversely, any planar surface P in $E(\Gamma)$ satisfying (*1) can be extended to a disk D in M such that $\partial D = C$ and $\text{Int } D \cap \Gamma = \text{Int } D \cap e$.

Now consider C as a cycle in Γ/e . Since C is trivial with respect to $(M/e, \Gamma/e)$, there is a disk D' in M/e bounded by C with $\text{Int } D'$ disjoint from Γ/e . The surface $Q = D' \cap E(\Gamma)$ is a disk satisfying

(*2): ∂Q is a curve on $\partial N(C \cup e)$, which intersects each meridian of C at a single point.

Conversely, any such disk Q can be extended to a disk D' in M/e with $\partial D' = C$ and $\text{Int } D' \cap (\Gamma/e) = \emptyset$.

We choose P and Q to satisfy (*1) and (*2), as well as the following general position and minimality conditions:

(*3): $n = |P \cap \delta(e)|$ is minimal, and $k = |Q \cap \delta(e)|$ is minimal.

(*4): P intersects Q transversely, and $|P \cap Q|$ is minimal subject to (*3).

(*5): $P \cap Q \cap \delta(e') = \emptyset$ for each edge e' in C .

(*5) is possible because by (*1) and (*2) each of $P \cap \delta(e')$ and $Q \cap \delta(e')$ is an essential arc in $\delta(e')$, so we can isotop Q to make them disjoint. Since k is minimal, $Q \cap \delta(e)$ consists of parallel essential arcs. So we may further assume

(*6): each component of $Q \cap \delta(e)$ intersects each ∂_j at a single point, $j = 1, \dots, n$.

If either $n = 0$ or $k = 0$, then an extension of P or Q is a disc D in M with $\partial D = C$ and $\text{Int } D \cap \Gamma = \emptyset$, so C is trivial with respect to (M, Γ) , as required. Hence we assume both n and k are positive. Label the components of ∂P so that, beginning with a point on $\partial N(C)$, an arc of $\partial Q \cap \delta(e)$ intersects $\partial_1, \partial_2, \dots, \partial_n$ successively.

A point of $\partial P \cap \partial Q$ is labeled i if it is a point on ∂_i . Thus any arc on $P \cap Q$ has a label on each of its end points.

LEMMA 2.1. *A component of $P \cap Q$ in P is an arc which is either essential or has both ends on ∂_0 .*

Proof. If $P \cap Q$ has some circle components, a 2-surgery of P along some disk in Q bounded by an innermost circle will reduce $|P \cap Q|$. Therefore $P \cap Q$ consists of arcs only.

If $P \cap Q$ has some arc which is inessential in P and has both ends on some ∂_j with $j \neq 0$, let α be an outermost one, so there is an arc β on ∂_j such that $\alpha \cup \beta$ bounds a disk Δ in P with interior disjoint from Q . A boundary compression of Q along Δ produces two disks, one of which satisfies (*2), but has less components of intersection with $\delta(e)$, contradicting the minimality of k . \square

LEMMA 2.2. *There is a label $i_0 > 0$ such that no arc of $P \cap Q$ has both ends labeled i_0 .*

Proof. Otherwise choose an α_i for each $i = 1, \dots, n$, with $\partial\alpha_i$ on ∂_i . Then the innermost such α_i will be an inessential arc on P . \square

Examine the order in which the indices appear on ∂Q . By (*6), if we delete all the 0 indices, the sequence is $1, 2, \dots, n, n, \dots, 2, 1$ repeated $k/2$ times. The 0 indices appear only possibly between two successive 1's.

LEMMA 2.3. *An arc α of $P \cap Q$ which is outermost in Q is of one of the following types.*

Type (i): α has both ends labeled 1 or both ends labeled n .

Type (ii): α has one end labeled 1 and the other labeled 0.

Proof. Note that if i, j are successive labels on ∂Q , then $|i - j| \leq 1$. Therefore if α is not of Type (i) or (ii), then the labels of α are either $\{0, 0\}$ or $\{i, i + 1\}$ for some $i > 0$. Let β be the arc on ∂Q so that $\alpha \cup \beta$ bounds a disk Δ in Q with interior disjoint from P .

Suppose α has label 0 on both endpoints. Then $\partial\alpha$ divides $\partial_0 \subset \partial P$ into two arcs ∂'_0 and ∂''_0 , one of which, say ∂''_0 , has the property that it intersects a meridian of C if and only if β does. So $\partial'_0 \cup \beta$ intersects any meridian of C at a single point. Let P_1 be the part of P bounded by $\partial'_0 \cup \alpha$. Then $P' = P_1 \cup \Delta$ satisfies (*1). Moreover, $|\partial P'| \leq |\partial P|$, and a perturbation of P' has less components of intersection with Q than P does. This is impossible by (*4).

Now suppose α has labels $\{i, i + 1\}$ for some $i > 0$. Then the normal extension of Δ is a disk Δ' in M such that $\partial\Delta' = \alpha' \cup \beta'$, where $\alpha' \subset D$, $\beta' \subset e$, and $\text{Int } \Delta' \cap \Gamma = \emptyset$. So we can isotope β' through Δ' to reduce $|D \cap e|$. This contradicts the minimality of n . \square

Note that the proof does not apply to the case when the labels of α are $\{0, 1\}$, since part of β' may be on C .

LEMMA 2.4. *There are at least two outermost edges α_1, α_2 of Type (ii).*

Proof. By Lemma 2.2, there is an index i_0 such that no arc in $P \cap Q$ has both ends labeled i_0 . Let A be the set of arcs in $P \cap Q$ with one end labeled i_0 . Let Δ be a disk in Q such that $\gamma = \partial\Delta - \partial Q$ is an arc in A , and Δ contains no other arcs in A . Note that there are at least two such Δ 's. So we need only to show that there is at least one type (ii) outermost edge in Δ .

Suppose there is no outermost arc of type (ii) in Δ . Then by Lemma 2.3, each outermost arc in Δ is of type (i), so the labels of the arc are either $\{1, 1\}$ or $\{n, n\}$. If there are two such outermost arcs, then the index i_0 appears between them, which is impossible by the definition of Δ . So there is only one outermost arc on Δ . This implies that the arcs of $P \cap Q$ are all parallel in Δ . It is now clear that every arc in Δ has the same index on both ends. Especially, both ends of γ are labeled i_0 , contradicting the choice of i_0 . \square

Now let Δ_1, Δ_2 be two disks in Q such that $\partial\Delta_i - \partial Q$ is an outermost arc of type (ii). Then the normal extension of Δ_i is a disk Δ'_i in M with $\partial\Delta'_i = \alpha_i \cup \beta_i \cup \gamma_i$, where α_i is an arc in D connecting a vertex v_i of C to the first intersection x of e with $\text{Int } D$, β_i is an arc on e connecting x to $v_0 = e \cap C$, and γ_i is an arc on C connecting v_0 to v_i . (γ_i may degenerate to a single point.) Since ∂Q intersects a meridian of C at a single point, the two arcs γ_1 and γ_2 cannot have an edge in common, and hence intersect only at v_0 . Thus $\Delta'_1 \cap \Delta'_2 = \beta_1 = \beta_2$, so $\Delta = \Delta'_1 \cup \Delta'_2$ is a disk in M . Let D_1 be the part of D bounded by $\partial\Delta$, and let D_2 be $(D - D_1) \cup \Delta$ pushed off $\beta_1 - v_0$. Then D_2 is a disk in M with $\partial D_2 = C$, and $|\text{Int } D_2 \cap e| \leq n - 1$. This contradicts the minimality of $n = |\text{Int } D \cap e|$. \square

3. Planar graphs in manifolds

In this section we will discuss the planarity of graphs in a 3-manifold. Suppose Γ is a graph embedded in M . An edge e of Γ is called a free edge if it is not a cycle, and one of its endpoints is not incident to any other edges. Clearly, if e is a free edge, then Γ is planar in M if and only if $\Gamma - e$ is planar. Therefore, without loss of generality we will always assume that Γ has no free edges.

We need the following definitions: A graph Γ in M is called *split* if there is a 2-sphere S in M which is disjoint from Γ , and separates M into M_1 and M_2 , such that both M_i contain part of Γ . It is called *decomposable* if there is a vertex $v \in \Gamma$ such that $\delta(v)$ has a compressing disk D in $E(\Gamma)$ which is separating. The following lemma and its proof is similar to that of [3, Lemma 1.3].

LEMMA 3.1. *Let Γ be a split or decomposable graph in a 3-manifold. If all proper subgraphs of Γ are planar, then Γ is planar.*

Proof. First assume Γ is split. Let S be a 2-sphere disjoint from Γ , separating M into M_1 and M_2 , such that $\Gamma_i = M_i \cap \Gamma$ are proper subgraphs of Γ . By assumption, there are 2-spheres $S_i \subset M$ such that $\Gamma_i \subset S_i$. By 2-surgery along disks bounded by innermost circles of $S_i \cap S$, we can delete all intersections of S_i with S , and get $S_i \subset M_i$. Tubing S_1 to S_2 gives a 2-sphere containing Γ .

Now suppose Γ is decomposable, and let D be a separating compressing disk of $\delta(v)$ in $E(\Gamma)$. It can be extended to a 2-sphere S in M so that $S \cap \Gamma = \{v\}$, and S separates M into M_1 and M_2 . Let $\Gamma_i = \Gamma \cap M_i$. Since Γ_i is planar, there is a 2-disk D_i in $E = M - \text{Int } N(v)$ which contains $\Gamma_i \cap E$. By surgery along disks bounded by innermost circles or outermost arcs of $D \cap D_i$ in D , we can assume $D_i \cap D = \emptyset$. Gluing a band on $\delta(v)$ to $D_1 \cup D_2$ produces a single disk containing $\Gamma \cap E$, which can be extended to a sphere in M containing Γ . \square

Define a cut point of Γ to be a vertex v such that $\Gamma - v$ has more components than Γ . Let $\{v_1, \dots, v_k\}$ be the cut points of Γ . Then there is a component X of $\Gamma - \{v_1, \dots, v_k\}$ which has the property that $\Gamma_1 = (\text{the closure of } X)$ contains at most one of these v_i ; for otherwise one can find a simple loop in Γ passing through some of the v_j 's, contradicting the definition of cutting points. This subgraph Γ_1 is connected, and has no cut point of its own. (It is possible that $\Gamma_1 = \Gamma$.)

Suppose Γ is abstractly planar. Embed Γ_1 into a 2-sphere S . Since Γ_1 is connected and has no cut points of its own, the closure of each components of $S - \Gamma_1$ is a disk. Let D_0, D_1, \dots, D_n be these disks. If Γ_1 contains a cut point v of Γ , choose D_0 to contain v . Let $D = S - \text{Int } D_0$.

Denote by Γ_1^c the closure of $\Gamma - \Gamma_1$. Then $\Gamma_1^c \cap \Gamma_1 = \{v\}$ or \emptyset , depending on whether Γ_1 contains a cut point v of Γ . Embed Γ_1^c into a disk D' so that $\partial D' \cap \Gamma_1^c = \{v\}$ or \emptyset accordingly. Glue D and D' together, we get an embedding of Γ into $S^2 = D \cup D'$. We fix this embedding.

Recall that D_1, \dots, D_n are the closures of the components of $D - \Gamma_1$.

LEMMA 3.2. *We can number the disks so that $B_k = D_1 \cup \dots \cup D_k$ is a disk for all k .*

Proof. If D_i is a disk such that $D_1 \cap D_i$ is not connected, then $D_1 \cup D_i$ is not simply connected, so there is a region Ω in D bounded by a boundary component of $D_1 \cup D_i$. Ω is the union of some D_j 's. Choose i so that Ω contains a minimal number of these disks. Since D_i is a disk, $\partial \Omega$ is not completely contained in ∂D_i , so there is a disk D_j in Ω which has an edge in common with D_1 . If $D_1 \cup D_j$ is not simply connected, then it bounds a region $\Omega' \subset \Omega$, contradicting the choice of D_i . Hence we can name this D_j as D_2 . Generally, if $B_k = D_1 \cup \dots \cup D_k$ is a disk, then by the same argument we can find D_{k+1} so that $B_k \cap D_{k+1}$ is an arc, and hence $B_k \cup D_{k+1}$ is a disk. The Lemma now follows by induction. \square

A link in S^3 is a trivial link if and only if all of its components are trivial. The following theorem shows that this is also true for graphs in 3-manifolds.

THEOREM 2. *An abstractly planar graph Γ in M is planar if and only if all cycles of Γ are trivial.*

Proof. We want to show that the inclusion $\Gamma \rightarrow M$ can be extended to an embedding of $\Gamma \cup B_n$ into M . This is done by induction. By assumption, ∂D_1 bounds a disk with interior disjoint from Γ , so we have an embedding of $\Gamma \cup D_1$ into M . Generally, suppose we have extended $\Gamma \rightarrow M$ to an embedding $i_k : \Gamma \cup B_k \rightarrow M$. By Lemma 3.2, B_k is a disk, and $B_k \cap D_{k+1}$ is an arc. Consider the graph $\Gamma' = \Gamma - \text{Int } B_k \subset M$. Then ∂B_k and ∂D_{k+1} are cycles in Γ' which are trivial with respect to (M, Γ') . So by Lemma 1.1, ∂D_{k+1} bounds a disk Δ_{k+1} which has interior disjoint from $\Gamma' \cup B_k$. Now we can define $i_{k+1} : \Gamma \cup D_1 \cup \dots \cup D_{k+1} \rightarrow M$ so that D_{k+1} is mapped to Δ_{k+1} . This completes the induction.

It follows that the image of B_n is an embedded disk Δ in M so that $\Delta \cap \Gamma = \Gamma_1$, and $\partial \Delta \subset \Gamma_1$. When $\Gamma = \Gamma_1$, this implies Γ is planar. When $\Gamma \neq \Gamma_1$, the set $\Gamma_1 \cup \Gamma_1^c$ is either empty or a cut point, which implies Γ is split or decomposable. By induction we may assume that all proper subgraphs of Γ are planar. The theorem now follows from Lemma 3.1. \square

As an application of the above theorems, we give an alternative proof of a theorem of Scharlemann and Thompson [3].

THEOREM 3. *A finite graph $\Gamma \subset S^3$ is planar if and only if*

- (a) Γ is abstractly planar;
- (b) every graph properly contained in Γ is planar;
- (c) $\pi_1(E(\Gamma))$ is a free group.

Proof. Since $\pi_1(E(\Gamma))$ is free, $E(\Gamma)$ is the connected sum of some handlebodies. If Γ is not connected, then it is split, and the theorem follows from Lemma 3.1. So we assume Γ is connected. When Γ has only one vertex, the theorem was proved in [1], so we assume Γ has some noncycle edge e . By induction on the number of edges in Γ , we may assume that Γ/e is planar for all such e .

According to Theorem 2, we need only to show that each cycle of Γ is trivial. Let C be a cycle in Γ . There are several cases.

CASE 1 (C does not contain all vertices of Γ). In this case there is some noncycle edge e which has at most one endpoint on C . Since both $\Gamma - e$ and Γ/e are planar, C is trivial with respect to both $(S^3, \Gamma - e)$ and $(S^3/e, \Gamma/e)$. By Theorem 1, C is also trivial with respect to (S^3, Γ) .

CASE 2 (Γ has some cycle edges). A cycle edge cannot contain all vertices of Γ because Γ has more than one vertex. By Case 1, a cycle edge is a trivial cycle, so it bounds a simple disk. It follows that Γ is decomposable, and the Theorem follows from Lemma 3.1.

In the remaining cases, all edges not in C are noncycle edges with both ends on C . Let e be such an edge. Its endpoints divide C into two arcs C_1 and C_2 .

CASE 3 (There is an edge e' which has one endpoint on each of $\text{Int } C_i$). Consider the cycle $C_i \cup e$. It is incident to just one endpoint of e' . By Case 1, $C_i \cup e$ bounds a disk D_i with interior disjoint from Γ . By Lemma 1.1, we can choose the D_i to have disjoint interiors. Thus $D = D_1 \cup D_2$ can be modified off e to become a simple disk bounded by C .

CASE 4 (No such edges e' as in Case 3 exist). Note that in this case \bar{e} , the image of e in Γ/e , is a cut point of Γ/e , and hence a decomposing point because Γ/e is planar. We want to apply Lemma 1.2 to our situation. To do this, let $M = E(\Gamma)$, and let $F = \partial N(C \cup e) - \text{Int } N(\Gamma)$. This is a punctured genus 2 surface, with one hole for each end of each edge which is not in $C \cup e$. Let e_1, \dots, e_k be the edges and v_1, \dots, v_k the vertices of C . Denote by m_i a meridian of e_i , and by J a meridian of e . Let $\gamma = m_1 \cup \dots \cup m_k$.

$F - \gamma$ is isotopic to $\delta(e) \cup \delta(v_1) \cup \dots \cup \delta(v_k) = \delta_{\Gamma/e}(\bar{e}) \cup (\bigcup \{\delta(v_i) \mid v_i \notin \partial e\})$. Since \bar{e} is a decomposing point of Γ/e , $\delta_{\Gamma/e}(\bar{e})$ is compressible in $E(\Gamma)$. So $F - \gamma$ is compressible.

Consider $\tau(E(\Gamma), J)$. This is the manifold obtained from $E(\Gamma)$ by attaching a 2-handle along a meridian of e , so it is actually the exterior of $\Gamma - e$. The surface $\sigma(F, J)$ is the punctured torus $\partial N(C) - \text{Int } N(\Gamma - e)$. Since $\Gamma - e$ is planar, C bounds a disk in M , which gives rise to a compressing disk D' of $\sigma(F, J)$ in $E(\Gamma - e)$, so that D' intersects each m_j at a single point. By Lemma 1.2, $F - J$ has a compressing disk D in $E(\Gamma)$ intersecting each m_j at most once. Since $F - J$ is a punctured torus, and m_j are meridians, if D is disjoint from some m_j , it is disjoint from all m_j , so it will be a compressing disk of some $\delta(v_i)$, which implies Γ is decomposable, and the Theorem follows. So we assume D intersects each m_j at one point. Then we can modify D so that ∂D intersects any meridian of C at a single point. The normal extension Δ of D is now a simple disk bounded by C . \square

4. F -planarity of graphs

Let F be a finite 2-dimensional cell complex with a connected graph Γ as its 1-skeleton. Γ is called a regular graph in F if the attaching map of each face (i.e.

2-cell) is a cycle in Γ , and the intersection of any two faces is connected. Suppose Γ is embedded in a 3-manifold M . Then Γ is called F -planar if it can be extended to an embedding of F in M . Suppose e is an edge of Γ which is not contained in the boundary of any faces of F . Then $\Gamma - e$ has $\Gamma - e$ as 1-skeleton, and F/e has Γ/e as 1-skeleton. To simplify notations, we call Γ/e (resp. $\Gamma - e$) F -planar if it is (F/e) -planar (resp. $(F - \text{Int } e)$ -planar). The following is a generalization of Theorem 1.

THEOREM 4. *Suppose F is a regular 2-complex with Γ as its 1-skeleton, and suppose Γ is embedded in a 3-manifold M . Let e be a noncycle edge of Γ such that both $\Gamma - e$ and Γ/e are F -planar. If e intersects each face of F at most at one of its endpoints, then Γ is F -planar.*

Proof. This follows from Theorem 1 and Lemma 1.1 by induction on the number of faces in F . \square

The most interesting case of F -planarity is when F is a surface. It was shown in [2] that Theorem 3 is equivalent to the following:

THEOREM 3'. *Let Γ be an abstractly planar graph in S^3 (or R^3). If Γ has a noncycle edge e such that both $\Gamma - e$ and Γ/e are planar, then Γ is planar.*

The following is a similar result for regular graphs in an arbitrary compact surface F . Suppose Γ is such a graph, and e is a noncycle edge which has at least one endpoint in the interior of F . Since $F/e \cong F$, both $\Gamma - e$ and Γ/e can be considered naturally as a graph in F .

THEOREM 5. *Suppose Γ is a regular graph on a surface F , and suppose Γ is embedded in a 3-manifold M . Let e be a noncycle edge of Γ with at least one end in $\text{Int } F$. If both Γ/e and $\Gamma - e$ are F -planar, then Γ is F -planar.*

Proof. We may assume that each end of e has valence at least 3, otherwise Γ is homeomorphic to Γ/e , and the planarity of Γ follows from that of Γ/e . Especially, an end of e in $\text{Int } F$ is incident to at least 3 faces of F .

Denote by D' , D'' the two disks incident to e . Consider the 2-complex $G = F - \text{Int } D' \cup \text{Int } D''$. First suppose e has both ends on some face D of G , then D contains $\partial D - e$ because $D \cap D'$ is connected. Similarly, D contains $\partial D'' - e$. By assumption ∂D is a cycle, so $\partial D = \partial(D' \cup D'')$. This is now a very special case: Γ has 3 edges and 2 vertices, and F is a 2-sphere. Since Γ/e is F -planar, Γ/e , and hence Γ , is contained in a 3-ball. Therefore the theorem follows from Theorem 3'.

Let D_1, \dots, D_n be the faces of G and consider G as a subset of M . Since D_i intersects e at most once, it remains a disk in M/e . By assumption Γ/e is F -planar in M/e , so $\partial D'/e$ bounds a disk Δ in M/e with $\text{Int } \Delta \cap \Gamma/e = \emptyset$. Since $\partial D' \cap D_i$ is connected, $\partial \Delta \cap D_i$ is connected for all $i = 1, \dots, n$. By Lemma 1.1 we can choose Δ so that $\text{Int } \Delta \cap D_i = \emptyset$ for $i = 1, \dots, n$. Let Q be the disk $\Delta \cap E(\Gamma/e) = \Delta \cap E(\Gamma)$ in $E(\Gamma)$. Q is disjoint from $\bigcup D_i$, and ∂Q intersects each meridian of $\partial D' - e$ at a single point. Let v be an end of e in $\text{Int } F$. Isotop Q so that $|\partial Q \cap \delta(e)|$ is minimal. Then $A = \partial Q \cap \delta(v)$ consists of arcs on the punctured sphere $\delta(v)$ which are all essential. As the circle ∂Q intersects a meridian of $\partial D' - e$ at a single point, there is an arc $\alpha \in A$ with exactly one end on the circle $J = \delta(v) \cap \delta(e)$, while all the other arcs in A have both ends on J . The arcs in A , being part of ∂Q , are disjoint from the disks D_1, \dots, D_n . Because F is a surface, and v is in $\text{Int } F$, these disks cut $\delta(v)$ into an annulus. It follows that all arcs in $A - \{\alpha\}$ are inessential, which is absurd unless α is the only arc in A . Therefore ∂Q intersects a meridian of e at a single point. The normal extension Δ' of Q is now a disk bounded by $\partial D'$, with interior disjoint from G . Similarly, there is a disk Δ'' bounded by $\partial D''$, such that $\text{Int } \Delta'' \cap G = \emptyset$, and by Lemma 1.1, it can be chosen so that $\Delta' \cap \Delta'' = e$. The surface $G \cup \Delta' \cup \Delta''$ is now an embedding of F in M . \square

The regularity condition in Theorem 4 is necessary. Consider the graph Γ on a torus F as shown in Figure 1. Embedding F into S^3 in the trivial way, we get a graph Γ_1 which is F -planar in S^3 . Let Γ_2 be the embedding of Γ in S^3 as shown in Figure 2, obtained from Γ_1 by interchanging a crossing in Figure 1. Let e be the edge shown in the figure. It is easy to see that $\Gamma_2 - e$ and Γ_2/e are isotopic to $\Gamma_1 - e$ and Γ_1/e respectively, so they are F -planar in S^3 . One can also isotop Γ_2 so that it lies on the trivial torus. But Γ_2 is not F -planar. To see this, one may need the following fact.

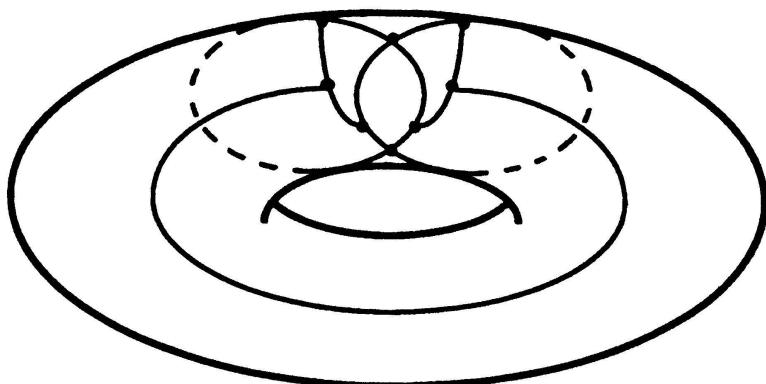


Figure 1

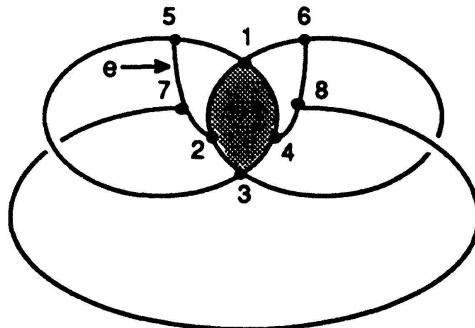


Figure 2

LEMMA 4.1. Suppose Γ is a graph in S^3 , and C is a trivial cycle with respect to (S^3, Γ) . If $\Gamma \cap E(C)$ is connected, then the simple disk D bounded by C is unique up to ambient isotopy fixing Γ .

Label the vertices of Γ_2 as in Figure 2. Denote by $C(i_1, \dots, i_k)$ the cycle successively passing through the vertices labeled i_1, \dots, i_k . Suppose Γ_2 is F -planar. Then $C(1, 2, 3, 4)$ and $C(1, 5, 3, 6)$ should bound simple disks with disjoint interiors. By Lemma 4.1, the disks are unique up to isotopy, so we can take the disk D bounded by $C(1, 2, 3, 4)$ to be the shaded region in Figure 2. Now $C(1, 5, 3, 6)$ cannot bound a disk with interior disjoint from $\Gamma_2 \cup D$, because it has linking number 1 with some curve in $\Gamma_2 \cup D - C(1, 5, 3, 6)$.

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