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Homology of maximal orders in central simple algebras

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§0. Introduction

Let R be a Dedekind domain whose quotient field K is a local or global field. Let L/K denote a finite separable extension and $\mathcal{O}$ the integral closure of R in L. It is known ([3]) that there are (non-canonical) isomorphisms of Hochschild homology groups

$$\mathrm{HH}_n^R(\mathcal{O}) \cong \mathrm{HH}_{n+2}^R(\mathcal{O}) \qquad \forall n \geq 1, \tag{0.1}$$

and moreover that $\mathrm{HH}_2^R(\mathcal{O}) = 0$. These two facts greatly facilitate the computation of the cyclic homology $\mathrm{HC}_*^R(\mathcal{O})$ ([3]).

This paper presents non-commutative analogues of the main results of [3]. In particular, L is to be replaced by a central simple algebra D over K, while $\mathcal{O}$ is taken to be a maximal R-order in D. Of course, $\mathcal{O}$ is not, in general, uniquely defined by R and D, but it turns out that the homology of $\mathcal{O}/R$ is independent of the choice of maximal order. We prove that (0.1) remains valid in the non-commutative context but that the *odd* Hochschild homology groups vanish.

The first section is devoted to generalities on Hochschild homology. We do not claim to prove any new result but only to establish notation and give self-contained proofs of various facts for which we could not find convenient references. The main results, the periodicity theorem and the vanishing theorem, are proved in §2 where R is a complete discrete valuation ring with perfect residue field. We construct an element in Hochschild *cohomology* such that Yoneda product gives the desired periodicity. The calculations needed to compute the low-dimensional Hochschild homology groups are laborious, but the resulting formulae are quite simple. The globalization is carried out in §3 and is completely standard. The vanishing of odd Hochschild homology causes the Connes sequence to split into small pieces, so it is

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easy to compute cyclic homology, in both the local and the global case, up to extension. This is explained in §4, where the extension problem is partially solved by an application of the universal coefficient theorem.

§1. Generalities

(1.1) Let R be a commutative ring and A an associative, unital flat R-algebra. Let $A^e = A \otimes_R A^{\mathrm{op}}$, where A^{op} is the opposite ring of A. Then A has a natural structure, A_l, of left A^e-module and a natural structure, A_r, of right A^e-module. Hochschild homology is defined as

$$\mathrm{HH}_n^R(A) := \mathrm{Tor}_n^{A^e}(A_r, A_l),$$

and Hochschild cohomology is defined

$$\mathrm{HH}_R^n(A) := \mathrm{Ext}_{A^e}^n(A_l, A_l).$$

The Yoneda pairing gives natural maps

$$\mathrm{HH}_R^m(A) \times \mathrm{HH}_R^n(A) \to \mathrm{HH}_R^{m+n}(A) \tag{1.1.1}$$

and

$$\mathrm{HH}_{m+n}^R(A) \times \mathrm{HH}_R^n(A) \to \mathrm{HH}_m^R(A). \tag{1.1.2}$$

(1.2) There is a natural resolution, known as the *standard resolution* ([1] IX §6), of A_l as A^e-module:

$$\cdots \xrightarrow{b'} A^{\otimes 3} \xrightarrow{b'} A^{\otimes 2} \xrightarrow{b'} A \longrightarrow 0 \tag{1.2.1}$$

where

$$b'(x_0 \otimes \cdots \otimes x_n) = \sum_{i=0}^{n-1} (-1)^i d_i(x_0 \otimes \cdots \otimes x_n)$$

and

$$d_i(x_0 \otimes \cdots \otimes x_n) = x_0 \otimes \cdots \otimes x_{i-1} \otimes x_i x_{i+1} \otimes x_{i+1} \otimes \cdots \otimes x_n.$$

The complex (1.2.1) is acyclic because

$$b'(1\otimes\alpha)=\alpha-1\otimes b'(\alpha).$$

Throughout this paper, A is always a projective R-module, so (1.2.1) is always a projective resolution. There are natural isomorphisms

$$A_r\otimes_{A^e}A^{\otimes n}\cong A^{\otimes n-1}$$

and

$$\mathrm{Hom}_{A^e}(A^{\otimes n},A)\cong\mathrm{Hom}_R(A^{\otimes n-2},A).$$

Therefore, Hochschild homology is the homology of the complex

$$\cdots\xrightarrow{b}A^{\otimes 3}\xrightarrow{b}A^{\otimes 2}\xrightarrow{b}A\longrightarrow 0$$

where

$$b(x_0\otimes\cdots\otimes x_n)=b'(x_0\otimes\cdots\otimes x_n)+(-1)^n x_n x_0\otimes\cdots\otimes x_{n-1}.$$

Similarly, HH* is the cohomology of the complex

$$0\to \mathrm{A}\to\mathrm{Hom}_R(A,A)\to\mathrm{Hom}_R(A^{\otimes 2},A)\to\mathrm{Hom}_R(A^{\otimes 3},A)\to\cdots,$$

where the boundary of the multi-R-linear function $f\colon A^{\times n}\to A$ is

$$\begin{aligned}(\partial f)(x_0,\ldots,x_n)=x_0f(x_1,\ldots,x_n)&-f(x_0x_1,x_2,\ldots,x_n)+f(x_0,x_1x_2,\ldots,x_n)\\ &-\cdots+(-1)^nf(x_0,\ldots,x_{n-1}x_n)\\ &-(-1)^nf(x_0,\ldots,x_{n-1})x_n.\end{aligned}$$

(1.3) Let $F\in\mathrm{Hom}_{A^e}(A^{\otimes n+2},A)$ represents an element of $\mathrm{HH}_R^n(A)$. We define F_0 to be the A^e-linear map sending

$$x_0\otimes\cdots\otimes x_{n+1}\mapsto F(x_0\otimes\cdots\otimes x_n\otimes 1)\otimes x_{n+1}.$$

We extend F_0 to a map of complexes

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & A^{\otimes n+4} & \longrightarrow & A^{\otimes n+3} & \longrightarrow & A^{\otimes n+2} & \longrightarrow & 0 \\
& \downarrow{\scriptstyle F_2} & & \downarrow{\scriptstyle F_1} & & \downarrow{\scriptstyle F_0} & & \\
\cdots \longrightarrow & A^{\otimes 4} & \longrightarrow & A^{\otimes 3} & \longrightarrow & A^{\otimes 2} & \longrightarrow & 0
\end{array}$$

where

$$F_m(x_0 \otimes \cdots \otimes x_{n+m+1}) = (-1)^{mn} F(x_0 \otimes \cdots \otimes x_n \otimes 1) \otimes x_{n+1} \otimes \cdots \otimes x_{n+m+1}.$$

This diagram commutes because the cocycle relation for HH^n implies

$$\begin{aligned}
F(b'(x_0 \otimes \cdots \otimes x_{n+1}) \otimes 1) &= F(b'(x_0 \otimes \cdots \otimes x_{n+1} \otimes 1) + (-1)^n x_0 \otimes \cdots \otimes x_{n+1}) \\
&= (-1)^n F(x_0 \otimes \cdots \otimes x_{n+1});
\end{aligned}$$

therefore,

$$\begin{aligned}
&F_{m-1}(b'(x_0 \otimes \cdots \otimes x_{n+1} \otimes y_1 \otimes \cdots \otimes y_m)) \\
&\quad = F_{m-1}(b'(x_0 \otimes \cdots \otimes x_{n+1}) \otimes y_1 \otimes \cdots \otimes y_m) \\
&\qquad - (-1)^n F_{m-1}(x_0 \otimes \cdots \otimes x_n \otimes b'(x_{n+1} \otimes y_0 \otimes \cdots \otimes y_m)) \\
&\quad = (-1)^{(m-1)n}(F(b'(x_0 \otimes \cdots \otimes x_{n+1} \otimes 1) \otimes y_1 \otimes \cdots \otimes y_m) \\
&\qquad - (-1)^n F(x_0 \otimes \cdots \otimes x_n \otimes 1) \otimes b'(y_0 \otimes \cdots \otimes y_m)) \\
&\quad = (-1)^{mn}(F(x_0 \otimes \cdots \otimes x_{n+1}) \otimes y_1 \otimes \cdots \otimes y_m \\
&\qquad - F(x_0 \otimes \cdots \otimes x_n \otimes 1) b'(x_{n+1} \otimes y_1 \otimes \cdots \otimes y_m)) \\
&\quad = b'(F_m(x_0 \otimes \cdots \otimes x_{n+1} \otimes y_1 \otimes \cdots \otimes y_m)).
\end{aligned}$$

We conclude that the pairing (1.1.2) is given by

$$(x_0 \otimes \cdots \otimes x_{m+n}) \times f \mapsto x_0 f(x_1, \ldots, x_n) \otimes x_{n+1} \otimes \cdots \otimes x_{m+n},$$

where

$$f(x_1, \ldots, x_n) := F(1 \otimes x_1 \otimes \cdots \otimes x_n \otimes 1).$$

(1.4) An element of $\mathrm{HH}^2_R(A)$ is called a *Hochschild extension* ([5] 25.C). It is equivalent to an R-split R-algebra extension of A by the square-zero ideal I,

$$0 \to I \to A_2 \to A \to 0,$$

such that I is a free A-module of rank 1. To obtain a cocycle representative, choose any R-linear section $c : A \to A_2$, and set

$$f(a, b) = c(a)c(b) - c(ab).$$

In particular, if V is a discrete valuation ring with uniformizer π and residue field k and W is a V-algebra, we have an extension

$$0 \longrightarrow W/\pi W \xrightarrow{\pi} W/\pi^2 W \longrightarrow W/\pi W \to 0$$

which gives rise to an element of $\mathrm{HH}^2_k(W/\pi W)$.

(1.5) Let A be an R-algebra and R' a commutative R-algebra, and let $A' = A \otimes_R R'$.
Then

$$A' \otimes_{R'} \cdots \otimes_{R'} A' \cong A \otimes_R R' \otimes_{R'} A \otimes_R \otimes R' \otimes_{R'} \otimes \cdots \otimes_R R'$$
$$\cong (A \otimes_R \cdots \otimes_R A) \otimes_R R',$$

so the universal coefficient spectral sequence says

$$E^2_{p,q} = \mathrm{Tor}^R_p(\mathrm{HH}^R_q(A), R') \Rightarrow \mathrm{HH}^{R'}_{p+1}(A).$$

When R' is flat over R,

$$\mathrm{HH}^{R'}_p(A') \cong \mathrm{HH}^R_p(A) \otimes_R R'. \tag{1.5.1}$$

When R is a discrete valuation ring with unifomizer π and residue field K,

$$\mathrm{HH}^{R/\pi^k R}_p(A/\pi^k A) \cong \mathrm{HH}^R_p(A) \otimes_R R/\pi^k R \oplus \mathrm{Tor}^R(\mathrm{HH}^R_{p-1}(A), R/\pi^k R).$$

PROPOSITION (1.6). *If A/R is an Azumaya algebra, then there exists an invertible R-module M such that*

$$\mathrm{HH}^R_p(A) = \begin{cases} M & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. By [2] 5.1, there exists an integer $n \in \mathbf{Z}^{\geq 1}$ and a faithfully flat étale R-algebra R' such that $A' = M_n(R')$. By Morita equivalence,

$$\mathrm{HH}_p^{R'}(A') = \begin{cases} R' & \text{if } p = 0, \\ 0 & \text{otherwise.} \end{cases}$$

By faithful flatness, $\mathrm{HH}_p^R(A) = 0$ for $p > 0$. By faithfully flat descent for line bundles, $\mathrm{HH}_0^R(A)$ is the module of sections of a line bundle over R, i.e., an invertible R-module.

COROLLARY (1.7). *If R is an integral domain with fraction field K and A is an R-order in a central simple algebra D/K, then* $\mathrm{HH}_p^R(A)$ *is a torsion module for $p > 0$.*

Proof. Setting $R' = K$, $A' = D$, and applying (1.5.1), the result is an immediate consequence of 1.6.

§2. Discrete valuation rings

(2.1) Let V be a complete discrete valuation ring with a uniformizer π, perfect residue field k, and fraction field K. Let k'/k be a cyclic extension of degree $n \geq 2$, and let σ denote a fixed generator of $\mathrm{Gal}(k'/k)$. There exists a unique discrete valuation ring V', finite and unramified over V, such that the residue field of V' is k' ([6] III Th. 2). If K' denotes its fraction field, there is a canonical isomorphism $\mathrm{Gal}(K'/K) \xrightarrow{\sim} \mathrm{Gal}(k'/k)$; we view σ indifferently as an automorphism of k' or of K'. By [7] IX Prop. 11, there exists a simple algebra D/K such that as K-vector space, $D \cong K'[x]/(x^n - \pi)$, and the multiplication is given by the rule

$$(k_1 x^{a_1})(k_2 x^{a_2}) = k_1 k_2^{\sigma^{a_1}} x^{a_1 + a_2}. \tag{2.1.1}$$

The reduced trace of an element $k \in K'$ is $\mathrm{Tr}_{K'|K}(k)$, while the trace of kx^a is zero for $1 \leq a < n$ ([7] IX §3 (8)). The (unique) maximal order $\mathcal{O}$ in D is the V-span of expressions vx^k, where $v \in V'$ and $k \geq 0$. There is a natural $\mathbf{Z}/n\mathbf{Z}$-grading, $\mathrm{Gr}^\bullet$ on $\mathcal{O}$ given by

$$\mathrm{Gr}^k(\mathcal{O}) = x^k V' = V' x^k, \qquad 0 \leq k < n.$$

PROPOSITION (2.2). *With notation as in* 2.1,

$$HH_0^V(\mathcal{O}) \cong V \oplus (k'/k).$$

Proof. Computing with the standard complex, we obtain

$$\mathrm{HH}_0^V(\mathcal{O}) \cong \mathcal{O}/[\mathcal{O}, \mathcal{O}].$$

By [6] III Prop. 12, we can choose $v_0 \in V'$ such that $V' = V[v_0]$; in particular, the (mod π) reduction of v_0 is a primitive element of the extension k'/k. For $1 \leq i < n$, $v_0^{\sigma^i} - v_0$ is therefore invertible, so the set of commutators

$$\{[v_0, vx^i] \mid v \in V'\} = \{v(v_0^{\sigma^i} - v_0)x^i \mid v \in V'\} = V'x^i = \mathrm{Gr}^i(\mathcal{O}). \tag{2.2.1}$$

Moreover,

$$\begin{aligned}\{[x, vx^{n-1}] \mid v \in V'\} &= \{\pi(v^\sigma - v) \mid v \in V'\}\\ &= \{x \in V' \mid \mathrm{Tr}_{K'/K}(x) = 0\}\pi \subset \mathrm{Gr}^0(\mathcal{O})\end{aligned} \tag{2.2.2}$$

by [6] X Prop. 1 and [6] XII Lemma 3. But every commutator lies in $x\mathcal{O}$ and has reduced trace 0, so every commutator is a sum of an element of (2.2.1) and an element of (2.2.2). Therefore,

$$\mathrm{HH}_0^V(\mathcal{O}) \cong \mathrm{Gr}^0(\mathcal{O})/\{x \in V' \mid \mathrm{Tr}_{K'/K}(x) = 0\}\pi.$$

As V'/V is unramified, the relative different is trivial, so the restriction of $\mathrm{Tr}_{K'/K}$ is a surjective map from V' to V. The proposition follows immediately.

PROPOSITION (2.3). *With notation as in* 2.1,

$$HH_1^V(\mathcal{O}) = 0.$$

Proof. We use the standard complex to compute Hochschild homology. For elements $\alpha, \beta \in \mathcal{O} \otimes_V \mathcal{O}$, we write $\alpha \sim \beta$ whenever $\alpha - \beta \in b(\mathcal{O}^{\otimes 3})$. Thus $\mathrm{HH}_1^V(\mathcal{O})$ consists of ker $(\mathcal{O} \otimes_V \mathcal{O} \to \mathcal{O})$ modulo the equivalence relation $\sim$. The relation

$$a \otimes xb \sim ax \otimes b + ba \otimes x \qquad \forall a, b \in \mathcal{O} \tag{2.3.1}$$

shows that every element of $\mathcal{O} \otimes_V \mathcal{O}$ is homologous to an element of $\mathcal{O} \otimes x + \mathcal{O} \otimes V'$.

The $\mathbf{Z}/n\mathbf{Z}$-grading $\mathrm{Gr}^\bullet$ on $\mathcal{O}$ extends to the standard complex for Hochschild homology and thus to a grading on Hochschild homology itself. The proposition is equivalent to the claim that $\mathrm{Gr}^k(\mathrm{HH}_1^V(\mathcal{O})) = 0$ for all k.

Suppose $k \neq 0$. Every $\alpha \in \mathrm{Gr}^k \ker(\mathcal{O} \otimes_V \mathcal{O} \to \mathcal{O})$ is homologous to an element $\alpha' \in \mathrm{Gr}^k \ker(\mathcal{O} \otimes_V \mathcal{O} \to \mathcal{O})$ such that

$$\alpha' \in vx^{k-1} \otimes x + V'x^k \otimes V', \qquad v \in V'.$$

As $v_0^{\sigma^k} - v_0$ is invertible, we can set

$$w = \frac{v}{v_0^{\sigma^k} - v_0},$$

so

$$b(wx^{k-1} \otimes v_0^{\sigma} \otimes x - wx^{k-1} \otimes x \otimes v_0) \in (wv_0^{\sigma^k} - v_0 w)x^{k-1} \otimes x + \mathcal{O} \otimes V'$$
$$= vx^{k-1} \otimes x + \mathcal{O} \otimes V'.$$

It follows that α' is homologous to

$$\alpha'' \in (V'x^k \otimes V') \cap \ker(\mathcal{O} \otimes \mathcal{O} \to \mathcal{O}).$$

As $V' = V[v_0]$, and

$$a \otimes v_0 b \sim av_0 \otimes b + ba \otimes v_0 \qquad \forall a, b \in \mathcal{O}. \tag{2.3.2}$$

Since $y \otimes 1 = b(y \otimes 1 \otimes 1)$ for all y, this means α'' is homologous to $v'x^k \otimes v_0 \in \ker(\mathcal{O} \otimes \mathcal{O} \to \mathcal{O})$ for some $v' \in V'$. Hence $v'x^k v_0 = v_0 v' x^k$, or

$$v'(v_0^{\sigma^k} - v_0)x^k = 0.$$

In other words, $v' = 0$, so $\alpha \sim 0$.

If $\alpha \in \mathrm{Gr}^0 \ker(\mathcal{O} \otimes_V \mathcal{O} \to \mathcal{O})$, we apply (2.3.1) to find $\alpha' \sim \alpha$ such that

$$\alpha' \in vx^{n-1} \otimes x + V' \otimes V',$$

for some $v \in V'$. As $b(\alpha') = 0$, $v^{\sigma} - v = 0$, so $v \in V$. But repeated application of (2.3.1) gives

$$\begin{aligned} 0 \sim w \otimes \pi &= w \otimes x^n \sim wx \otimes x^{n-1} + x^{n-1}x \\ &\sim wx^2 \otimes x^{n-2} + x^{n-2}wx \otimes x + x^{n-1}w \otimes x \\ &\sim \cdots \sim wx^{n-1} \otimes x + xwx^{n-2} \otimes x + \cdots + x^{n-1}w \otimes x \\ &= \mathrm{Tr}_{K'/K}(w)x^{n-1} \otimes x. \end{aligned}$$

As V' is unramified over V, $vx^{n-1} \otimes x \sim 0$. Every element in $V' \otimes V'$ is a boundary since by [6] III Prop. 14, $HH_1^V(V') = \Omega^1_{V'/V}$ is annihilated by the different of V'/V, which is the unit ideal.

KEY LEMMA (2.4). *Let X denote the kernel of $b' : \mathcal{O}^{\otimes 3} \to \mathcal{O} \otimes \mathcal{O}$. Then there exists a surjective map $\phi : X \to \mathcal{O}$ such that $M = \ker(\phi)$ is a projective left-$\mathcal{O}^e$-module.*

Proof. Let $\mathscr{A} = V'\{x\}$ denote the twisted polynomial ring in one variable x satisfying (2.1.1). Thus A is the quotient of $\mathscr{A}$ by the two-sided ideal $(x^n - \pi)$. By 1.4, the short exact sequence

$$0 \longrightarrow A \xrightarrow{x^n - \pi} \mathscr{A}/(x^n - \pi)^2 \mathscr{A} \longrightarrow A \longrightarrow 0,$$

defines an element of $HH_V^2(\mathcal{O})$ represented by the unique V-linear function satisfying

$$f(vx^a, wx^b) = \begin{cases} 0 & \text{if } a + b < n, \\ vw^{\sigma^a} x^{a+b-n} & \text{if } a + b \geq n, \end{cases} \tag{2.4.1}$$

for all $v, w \in V'$ and $a, b \in \{0, 1, \ldots, n-1\}$. This element gives (by 1.3) the vertical arrows of the commutative square

$$\begin{array}{ccc} \mathcal{O}^{\otimes 5} & \xrightarrow{b'} & \mathcal{O}^{\otimes 4} \\ \downarrow & & \downarrow \\ \mathcal{O}^{\otimes 3} & \xrightarrow{b'} & \mathcal{O}^{\otimes 2}. \end{array}$$

The induced map on b'-cokernels, $\phi : X \to \mathcal{O}$, is surjective because the map $\mathcal{O}^{\otimes 4} \to \mathcal{O}^{\otimes 2}$ is so; indeed $f(x, x^{n-1}) = 1$.

Every finite D-module is projective, so to prove $\ker(\phi)$ is projective, it suffices to prove the kernel of the map $\bar{\phi} : X/\pi X = \bar{X} \to \bar{\mathcal{O}} = \mathcal{O}/\pi\mathcal{O}$ is projective as a left-$\bar{\mathcal{O}}^e$-module. The advantage of working (mod π) is that $\bar{x}^n = 0$, so the grading of $\bar{\mathcal{O}}^{\otimes k}$ by total $\bar{x}$-degree is actually a $\mathbf{Z}$-grading rather than a $\mathbf{Z}/n\mathbf{Z}$-grading. Given a graded module $\bigoplus_{n \in N} M_n$ of finite k-vector spaces, there is an associated Poincaré polynomial $\Sigma_n \dim_l (M_n) t^n$. Thus, the polynomial associated to $\bar{\mathcal{O}}^{\otimes k}$ is $N(t)^k$, where

$$N(t) = n + nt + \cdots + nt^{n-1}.$$

The exactness of

$$\bar{\mathcal{O}}^{\otimes 3} \to \bar{\mathcal{O}}^{\otimes 2} \to \bar{\mathcal{O}} \to 0$$

implies that the Poincaré polynomial of $\bar{X}$ is

$$N(t)^3 - N(t)^2 + N(t).$$

Yoneda product with $f(x, y)$ has degree $-n$, so the polynomial of $\ker(\bar{\phi})$ is

$$\begin{aligned} &N(t)^3 - N(t)^2 + N(t) - N(t)t^n \\ &= N(t)^2\left(N(t) - 1 - \frac{t-1}{n}\right) \\ &= N(t)^2\left(nt^{n-1} + nt^{n-2} + \cdots + nt^2 + \frac{n^2-1}{n}t + \frac{n^2-n+1}{n}\right). \end{aligned}$$

As k'/k is Galois, there is an algebra isomorphism

$$k' \otimes_k k' \cong \underbrace{k' \oplus k' \oplus \cdots \oplus k'}_{n}.$$

If I is an irreducible direct summand of $k' \otimes_k k'$, we define

$$\bar{\mathcal{O}}^e_I = \bar{\mathcal{O}}^e \otimes_{k' \otimes k'} I.$$

This is a projective left ideal of $\bar{\mathcal{O}}^e$, and its Poincaré polynomial is $(1/n)N(t)^2$. More generally, if M is a graded $\bar{\mathcal{O}}^e$-module and $m \in M$ is an element of degree k such that $\bar{\mathcal{O}}^e m \cong \bar{\mathcal{O}}^e_I$, then the Poincaré polynomial of $\mathcal{O}^e m$ is $(1/n)t^k N(t)^2$. To prove the lemma, it suffices to prove that $\ker(\bar{\phi})$ is a direct sum of free modules and modules of the form $\bar{\mathcal{O}}^e_I$.

Given $v \in k'$ with monic minimal polynomial $\Sigma_i a_i v^i = 0$, we define

$$\alpha_v = \sum_p \sum_q a_{p+q+1} v^p \otimes v \otimes v^q.$$

Substituting $r = p + q$

$$b'(\alpha_v) = \sum_r a_{r+1}(v^{r+1} \otimes 1 - 1 \otimes v^{r+1}) = 0.$$

Next we define the k-linear operation

$$* : k'^{\otimes m} \times k[\bar{x}]^{\otimes m} \to \bar{\mathcal{O}}^{\otimes m}$$

such that

$$(v_1 \otimes \cdots \otimes v_m) * (\bar{x}^{a_1} \otimes \cdots \otimes \bar{x}^{a_m})$$
$$= v_1 \bar{x}^{a_1} \otimes v_2^{\sigma - a_1} \bar{x}^{a_2} \otimes v_3^{\sigma - a_1 - a_2} \bar{x}^{a_3} \otimes \cdots \otimes v_m^{\sigma - a_1 - \cdots - a_{m-1}}.$$

By construction,

$$d_i(\alpha * \delta) = d_i(\alpha) * d_i(\delta)$$

for all i, α, and δ. In particular, when $m = 3$,

$$b'(\alpha * \delta) = d_0(\alpha * \delta) - d_1(\alpha * \delta) = d_0(\alpha) * d_0(\delta) - d_1(\alpha) * d_1(\delta),$$

so if α and δ lie in $\ker(b')$, so does $\alpha * \delta$. Choose a basis $k_1, \ldots, k_n$ for k'/k, and let

$$p_{i,j} = \alpha_{k_j} * \left(1 \otimes \bar{x}^i \otimes 1 - \sum_{l=0}^{i-1} \bar{x}^{i-1-l} \otimes \bar{x} \otimes \bar{x}^l\right).$$

Then $p_{i,j} \in \ker(b')$ for all i and j. Next, let q_i denote generators of the irreducible $k' \otimes_k k'$-submodules of

$$\ker(k' \otimes_k k' \otimes_k k' \xrightarrow{b'} k' \otimes_k k') \subset \ker(\bar{\phi}). \tag{2.4.2}$$

Irreducible $k' \otimes_k k'$-modules are identified with k'-factors in $k' \otimes_k k'$ and are therefore of k-dimension n. It follows that the module (2.4.2) breaks up into $n^2 - n + 1$ irreducible pieces. Finally, let α_i denote generations of the irreducible $k' \otimes_k k'$-submodules of

$$\ker(k' \otimes_k k' \otimes_k k' \xrightarrow{m} k'), \tag{2.4.3}$$

where m denotes the multiplication map $a \otimes b \otimes c \mapsto abc$. The exact sequence of $k' \otimes k'$-modules

$$k'^{\otimes 3} \xrightarrow{b'} k'^{\otimes 2} \xrightarrow{b'} k' \longrightarrow 0$$

can be decomposed into a direct sum of $k' \otimes k'$-isotypic exact subcomplexes by tensoring with the irreducible summands of $k' \otimes k'$. In particular, as $d_0(\alpha_i)$ and $d_1(\alpha_i)$ lie in $\ker(b')$, we can find elements β_i and γ_i, each annihilated by the full maximal ideal annihilating α_i, such that

$$b(\beta_i) = d_0(\alpha_i); \qquad b(\gamma_i) = d_1(\alpha_i).$$

Setting

$$r_i = \alpha_i * (1 \otimes \bar{x} \otimes 1) - \beta_i * (\bar{x} \otimes 1 \otimes 1) + \gamma_i * (1 \otimes 1 \otimes \bar{x}),$$

we have

$$b'(r_i) = d_0(\alpha_i) * (\bar{x} \otimes 1) - d_1(\alpha_i) * (1 \otimes \bar{x})$$
$$- b'(\beta_i) * (\bar{x} \otimes 1) + b'(\gamma_i) * (1 \otimes \bar{x}) = 0.$$

As the k-dimension of (2.4.3) is $n^3 - n$, there are $n^2 - 1$ generators r_i.

We claim that the $\bar{\mathcal{O}}^e$-modules $\bar{\mathcal{O}}^e p_{i,j}$ are free, that the modules $\bar{\mathcal{O}}^e q_i$ and $\bar{\mathcal{O}}^e r_i$ are of the form $\bar{\mathcal{O}}^e_I$, and that

$$\ker(\bar{\phi}) = \left(\bigoplus_i \bar{\mathcal{O}}^e p_{i,j}\right) \oplus \left(\bigoplus_i \bar{\mathcal{O}}^e q_i\right) \oplus \left(\bigoplus_i \bar{\mathcal{O}}^e r_i\right). \tag{2.4.4}$$

The two sides have equal Poincaré polynomials, and $p_{i,j}$, q_i, $r_i \in \ker(\bar{\phi})$, so it suffices to prove that the factors on the right hand side of (2.4.4) are independent submodules of the left hand side. Left multiplication by an element of $\bar{\mathcal{O}}^e$ does not change the center coordinate of a decomposable tensor $a \otimes b \otimes c$. We can therefore write $\bar{\mathcal{O}}^{\otimes 3}$ as a sum of free $\bar{\mathcal{O}}^e$-modules:

$$\bar{\mathcal{O}}^{\otimes 3} = \bigoplus_{i=0}^{n-1} M_i, \qquad M_i = \bar{\mathcal{O}} \otimes k' \bar{x}^i \otimes \bar{\mathcal{O}}.$$

Each $p_{i,j}$ is the sum of terms of the form $\cdot \otimes k_j \bar{x}^i \otimes \cdot \in M_i$ and terms of lower central degree. Since the M_i-components are independent over $\bar{\mathcal{O}}^e$, and since all $q_i, r_i \in M_0 \oplus M_1$, the $p_{i,j}$ cannot appear in any non-trivial relation with the q_i and the r_i. Similarly, the r_i can be decomposed into their M_0 and M_1 coordinates. If

$$\sum_{i,j} (d_{i,j} \otimes e_{i,j})\alpha_i * (1 \otimes \bar{x} \otimes 1) = 0,$$

then

$$\sum_{i,j} (d_{i,j} \otimes e^{\sigma}_{i,j})\alpha_i = 0,$$

in which case

$$\sum_j (d_{i,j} \otimes e^{\sigma}_{i,j})\alpha_i = 0$$

for each i since the α_i generate independent submodules of $k'^{\otimes 3}$. As $\mathrm{Ann}(\alpha_i) = \mathrm{Ann}(\beta_i) = \mathrm{Ann}(\gamma_i)$,

$$\sum_j (d_{i,j} \otimes e^{\sigma}_{i,j})\beta_i = \sum_j (d_{i,j} \otimes e^{\sigma}_{i,j})\gamma_i = 0.$$

This implies

$$\left(\sum_j d_{i,j} \otimes e_{i,j}\right) r_i = 0. \tag{2.4.5}$$

Since the q_i have no M_1 component, in any relation between the q_i and the r_j, the q_i contribution is a sum of terms of the form (2.4.5). Finally, the $\bar{\mathcal{O}}^e q_i$ are linearly independent because $\bigoplus (k' \otimes k')q_i$ can be extended to a decomposition of $k'^{\otimes 3}$ into $k' \otimes k'$-irreducible components. Tensoring over $k' \otimes k'$ with $\bar{\mathcal{O}}^e$, we see that the $\bar{\mathcal{O}}^e q_i$ are independent factors of $\bar{\mathcal{O}}^{\otimes 3}$. The lemma follows.

THEOREM (2.5). *With notation as in* 2.1,

$$HH_i^V(\mathcal{O}) = \begin{cases} V \oplus (k'/k) & \text{if } i = 0, \\ k'/k & \text{if } i > 0 \text{ is even}, \\ 0 & \text{if } i \text{ is odd}. \end{cases} \tag{2.5.1}$$

Proof. The case $i = 0$ is Prop. 2.2. Let $X = \ker(\mathcal{O}^{\otimes 3} \to \mathcal{O}^{\otimes 2})$. The short exact sequence

$$0 \longrightarrow M \longrightarrow X \xrightarrow{\ \phi\ } \mathcal{O} \longrightarrow 0$$

gives rise to a long exact sequence

$$\cdots \to \mathrm{Tor}_n^{\mathcal{O}^e}(\mathcal{O}_r, M) \to \mathrm{Tor}_n^{\mathcal{O}^e}(\mathcal{O}_r, X) \to \mathrm{Tor}_n^{\mathcal{O}^e}(\mathcal{O}_r, \mathcal{O}_l) \to \mathrm{Tor}_{n-1}^{\mathcal{O}^e}(\mathcal{O}_r, M) \to \cdots. \tag{2.5.2}$$

As M is a projective left $\mathcal{O}^e$-module, for $n \geq 1$

$$\mathrm{Tor}_n^{\mathcal{O}^e}(\mathcal{O}_r, X) \hookrightarrow \mathrm{HH}_n^V(\mathcal{O}),$$

with equality for $n > 1$. The projective resolution

$$\cdots \to \mathcal{O}^{\otimes 5} \to \mathcal{O}^{\otimes 4} \to X \to 0$$

implies that

$$\mathrm{Tor}_n^{\mathcal{O}^e}(\mathcal{O}_r, X) \cong \mathrm{HH}_{n+2}^V(\mathcal{O})$$

for $n \geq 1$. By Prop. 2.3,

$$\mathrm{HH}_3^V(\mathcal{O}) \hookrightarrow \mathrm{HH}_1^V(\mathcal{O}) = 0,$$

and by (2.5.2),

$$\mathrm{HH}_{n+2}^V(\mathcal{O}) \xrightarrow{\sim} \mathrm{HH}_n^V(\mathcal{O}) \tag{2.5.3}$$

for $n > 1$. All that remains is the computation of $\mathrm{HH}_2^V(\mathcal{O})$.

To accomplish this, we note first that for all p, $\mathrm{HH}_p^V(\mathcal{O})$ is the homology of a complex of finitely generated V-modules, and therefore finitely generated. By (1.5.1) and Prop. 1.6,

$$\mathrm{HH}_p^V(\mathcal{O}) \otimes_V K \cong \mathrm{HH}_p^K(\mathcal{O} \otimes_V K) = 0,$$

for all $p > 0$. Therefore, $\mathrm{HH}_p^V(\mathcal{O})$ is a finitely generated torsion module. By (2.5.3), we may choose j such that π^j annihilates $\mathrm{HH}_p^V(\mathcal{O})$, for all $p > 0$. As $M/\pi^j M$ is a projective $\mathcal{O}^e/\pi^j\mathcal{O}^e$-module, we have

$$\mathrm{HH}_3^{V/\pi^j V}(\mathcal{O}/\pi^j\mathcal{O}) \hookrightarrow \mathrm{HH}_1^{V/\pi^j V}(\mathcal{O}/\pi^j\mathcal{O}).$$

The universal coefficient formula (1.5.2) implies

$$\mathrm{HH}_{2n+1}^{V/\pi^j V}(\mathcal{O}/\pi^j\mathcal{O}) = \begin{cases} \mathrm{Tor}\,(V/\pi^j V, \mathrm{HH}_0^V(\mathcal{O})) = k'/k & \text{if } n = 0, \\ \mathrm{Tor}\,(V/\pi^j V, \mathrm{HH}_2^V(\mathcal{O})) = \mathrm{HH}_2^V(\mathcal{O}) & \text{if } n > 0. \end{cases}$$

To prove the theorem, therefore, it suffices to prove that the inclusion (2.5.4) is an isomorphism. Since k'/k is π-torsion, it suffices to prove this for $j = 1$.

Let

$$\alpha = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} a_{i,j}\bar{x}^i \otimes b_{i,j}\bar{x}^j$$

represent a class in $\mathrm{HH}_1^k(\bar{\mathcal{O}})$. We can lift α to an element

$$\tilde{\alpha} = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \tilde{a}_{i,j}x^i \otimes \tilde{b}_{i,j}x^j \in \mathcal{O} \otimes \mathcal{O}.$$

As $b(\alpha) = 0$, $b(\tilde{\alpha})$ is divisible by $\pi = x^n$. Therefore,

$$b\left(\sum_{i+j<n} a_{i,j}\bar{x}^i \otimes b_{i,j}\bar{x}^j \right) = 0.$$

Without loss of generality, then, we may assume $a_{i,j} = b_{i,j} = 0$ when $i+j<n$. Setting

$$\beta = \sum_{i+j \geq n} a_{i,j}\bar{x}^i \otimes \bar{x}^{n-i} \otimes \bar{x}^i \otimes b_{i,j}\bar{x}^j \in \bar{\mathcal{O}}^{\otimes 4},$$

we have $b(\beta) = 0$, so β represents a class in $\mathrm{HH}_3^k(\bar{\mathcal{O}})$. The map $\mathrm{HH}_3^k(\bar{\mathcal{O}}) \to \mathrm{HH}_1^k(\bar{\mathcal{O}})$ is given by Yoneda product with the (mod π) reduction, $\bar{f}$, of the periodicity element (2.4.1). As

$$\bar{f}(\bar{x}^i, \bar{x}^{n-i}) = 1,$$

$\beta \mapsto \alpha$, which proves that (2.5.4) is surjective and therefore bijective.

§3. Dedekind domains

(3.1) Let R denote a Dedekind domain whose field of fractions, K is a global field, i.e., a finite extension of **Q** or a finitely generated extension of a finite field of transcendence degree 1. If M is a module over R and $\wp$ a maximal ideal of R, we write $M_\wp$ for the completion of the module M with respect to the ideal $\wp$. Let D

denote a central simple algebra over K. Then $[D:K]=n^2$ for some $n\in \mathbf{Z}^{\geq 1}$ ([7] IX Prop. 3 Cor. 3), and for all but a finite set of primes $\wp$,

$$D_\wp \cong M_n(K_\wp)$$

([7] XI Th. 1). In general $D_\wp \cong M_{f_\wp}(E_\wp)$, where $E_\wp/K_\wp$ is a division of dimension $e_\wp^2$ and $e_\wp f_\wp = n$ ([7] XI §1). The algebra D is said to *ramify* when $e_\wp > 1$, and we write $\mathrm{Ram}\,(D/R)$ for the set of ramifying primes.

(3.2) Let $\mathcal{O}$ be an order in D, i.e., a subring of D such that $\mathcal{O}_\wp$ is a compact open subgroup of $D_\wp$ for all maximal ideals $\wp$ of R. Then $\mathcal{O}_\wp$ is a maximal compact subring of $D_\wp$ for all but finitely many $\wp$ ([7] XI Th. 1). Every order is contained in a maximal order $\mathcal{O}'$ such that $\mathcal{O}'_\wp$ is a maximal compact subring of $D_\wp$ for all $\wp$; this is proved in the case of number fields K in [7] XI Prop. 4, but the argument also works for function fields. We treat only the case that $\mathcal{O}$ is a maximal order.

(3.3) The division algebras over a local field $K_\wp$ are classified by their invariants $\mathrm{inv}(D_\wp)\in \mathrm{Br}(K_\wp)\cong \mathbf{Q}/\mathbf{Z}$ ([7] XII Th. 1), where the dimension of the algebra is the square of the denominator of $\mathrm{inv}(D_\wp)$. Moreover, the algebra with invariant a/n, is generated over $K_\wp$ by the unramified extension $K'_\wp$ of $K_\wp$ of degree n and an element x satisfying (2.1.1), where σ is the a^{th} power of the Frobenius element in $\mathrm{Gal}\,(K'_\wp/K_\wp)$. The maximal order $\mathcal{O}_\wp$ of $D_\wp$ is the one described in 2.1. More generally, the maximal orders of $M_n(D_\wp)$ are all of the form $M_n(\mathcal{O}_\wp)$ ([7] X Th. 1).

THEOREM (3.4). *If A is a central simple algebra over K, and $\mathcal{O}$ is a maximal order of A with respect to R, then*

$$HH_i^R(\mathcal{O}) \cong \begin{cases} R \oplus \bigoplus_{\wp\in \mathrm{Ram}\,(A/R)} (R/\wp)^{e_\wp - 1} & \text{if } i=0,\\ \bigoplus_{\wp\in \mathrm{Ram}\,(A/R)} (R/\wp)^{e_\wp - 1} & \text{if } i>0 \text{ is even},\\ 0 & \text{if } i \text{ is odd}.\end{cases}$$

Proof. As $\mathcal{O}$ is a finite R-module, the Hochschild homology modules are finite over R. Assume $i>0$. Applying (1.5.1) for $R'=K$ and Prop. 1.6, we see that $\mathrm{HH}_i^R(\mathcal{O})$ is torsion. From the structure theorem for modules over Dedekind domains, we deduce

$$\mathrm{HH}_i^R(\mathcal{O}) \cong \bigoplus_\wp \mathrm{HH}_i^{R_\wp}(\mathcal{O}_\wp). \tag{3.4.1}$$

Now, $\mathcal{O}_{\wp}$ is maximal, so by Morita equivalence,

$$HH_i^{R_\wp}(\mathcal{O}_\wp) = HH_i^{R_\wp}(M_{f_\wp}) \cong HH_i^{R_\wp}(\mathcal{O}'_\wp),$$

where $\mathcal{O}'_\wp$ is the maximal order of a division algebra of degree $e_\wp^2$ over K. By Th. 2.5.,

$$HH_i^{R_\wp}(\mathcal{O}'_\wp) = \begin{cases} 0 & \text{if } i \text{ is odd,} \\ (R/\wp)^{e_\wp}/(R/\wp) & \text{if } i \text{ is even.} \end{cases}$$

In particular, the primes $\wp$ such that $e_\wp = 1$ contribute nothing to the sum (3.4.1), so they may be omitted from the summation. This completes the proof for $i > 0$.

For $i = 0$, we need to compute $\mathcal{O}/[\mathcal{O}, \mathcal{O}]$. We have already noted that commutators in a simple algebra have reduced trace zero. If $\mathcal{O}'_\wp$ is defined as above, we have already shown that the reduced trace map

$$\mathrm{Trd} : \mathcal{O}'_\wp \to R_\wp$$

is surjective. It follows that

$$\mathrm{Trd} : \mathcal{O}_\wp = M_{f_\wp}(\mathcal{O}_\wp) \to \mathcal{O}_\wp$$

is surjective. Since $\mathrm{Trd} : \mathcal{O} \to R$ is R-linear, its image is an ideal, and by the compatibility of trace with completion, we see that the image is in fact all of R. Therefore, R is a quotient of $HH_0^R(\mathcal{O})$, so there exists an R-splitting $HH_0^R(\mathcal{O}) \cong R \oplus M$. As $HH_0^K(A) = K$, M is a finite torsion module, so

$$M \cong \bigoplus_\wp HH_0^{R_\wp}(\mathcal{O}_\wp)/R_\wp.$$

The theorem follows from Prop. 2.2.

§4. Cyclic homology

(4.1) Let R be a ring and A an associative R-algebra. We define rotation operator $\rho : A \otimes_R \cdots \otimes_R A \to A \otimes_R \cdots \otimes_R A$ by setting

$$\rho(x_0 \otimes \cdots \otimes x_n) = x_n \otimes x_0 \otimes x_1 \otimes \cdots \otimes x_{n-1}.$$

The cyclic homology $\mathrm{HC}_r^R(A)$ is defined as the homology of $\mathrm{Tot}(X_{..})$, where $X_{..}$ is the Tsygan double complex ([4] §1):

$$\begin{array}{ccccccccccc}
 & & \vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
0 & \longleftarrow & A^{\otimes 4} & \xleftarrow{T} & A^{\otimes 4} & \xleftarrow{N} & A^{\otimes 4} & \xleftarrow{T} & A^{\otimes 4} & \xleftarrow{N} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
0 & \longleftarrow & A^{\otimes 3} & \xleftarrow{T} & A^{\otimes 3} & \xleftarrow{N} & A^{\otimes 3} & \xleftarrow{T} & A^{\otimes 3} & \xleftarrow{N} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
0 & \longleftarrow & A \otimes A & \xleftarrow{T} & A \otimes A & \xleftarrow{N} & A \otimes A & \xleftarrow{T} & A \otimes A & \xleftarrow{N} & \cdots \\
 & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
0 & \longleftarrow & A & \xleftarrow{0} & A & \xleftarrow{1} & A & \xleftarrow{0} & A & \xleftarrow{1} & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & & 0 & &
\end{array}$$

where

$$T = 1 - (-1)^{n+1}\rho; \qquad N = \frac{1 - (-\rho)^{n+1}}{1 - (-1)^{n+1}\rho}.$$

We use the first spectral sequence of this double complex, i.e., the one in which the vertical differential is applied first, known in this context as the Connes spectral sequence. As the b'-columns are exact, the E^1 term of this spectral sequence is of the form

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \ddots \\
\mathrm{HH}_4(A) & \leftarrow & 0 & \leftarrow & \mathrm{HH}_4(A) & \leftarrow & 0 & \leftarrow & \cdots \\
\mathrm{HH}_3(A) & \leftarrow & 0 & \leftarrow & \mathrm{HH}_3(A) & \leftarrow & 0 & \leftarrow & \cdots \\
\mathrm{HH}_2(A) & \leftarrow & 0 & \leftarrow & \mathrm{HH}_2(A) & \leftarrow & 0 & \leftarrow & \cdots \\
\mathrm{HH}_1(A) & \leftarrow & 0 & \leftarrow & \mathrm{HH}_1(A) & \leftarrow & 0 & \leftarrow & \cdots \\
\mathrm{HH}_0(A) & \leftarrow & 0 & \leftarrow & \mathrm{HH}_0(A) & \leftarrow & 0 & \leftarrow & \cdots
\end{array}$$

Thus the only non-zero differentials d_r for $r \geq 1$ are the even differentials.

(4.2) Let V, $\mathcal{O}$, k, and k' be defined as in §2. By (2.5.1), the Connes long exact sequence breaks up into short exact sequences

$$0 \to \mathrm{HH}^V_{2k}(\mathcal{O}) \to \mathrm{HC}^V_{2k}(\mathcal{O}) \to \mathrm{HH}^V_{2k-2}(\mathcal{O}) \to 0$$

and sequences $0 \to \mathrm{HC}^V_{2k+1}(\mathcal{O}) \to 0$. Thus,

$$\mathrm{HC}^V_r(\mathcal{O}) = 0 \qquad \forall r \text{ odd}, \tag{4.2.1}$$

while for r even, $\mathrm{HC}^V_r(\mathcal{O})$ has a filtration whose quotients are

$$V \oplus (k'/k), \underbrace{k'/k, \ldots, k'/k.}_{r/2} \tag{4.2.2}$$

We cannot fully settle the extension problem, but considerable light is cast on the question by the tensoring with $V/\pi V$.

(4.3) Consider the Connes spectral sequence for $\mathrm{HC}_*(\bar{\mathcal{O}}/k)$. We have already seen that

$$E^1_{p,q} = E^2_{p,q} = \begin{cases} \mathrm{HH}^k_q(\bar{\mathcal{O}}) & \text{for } p \text{ even}, \\ 0 & \text{for } p \text{ odd}. \end{cases}$$

The universal coefficient formula (1.5.2) implies

$$\mathrm{HH}^k_p(\bar{\mathcal{O}}) \cong \begin{cases} k' & \text{for } p = 0, \\ k'/k & \text{for } p > 0. \end{cases}$$

By horizontal periodicity, the differential

$$d_2 : E^2_{p,q} \to E^2_{p-2,q+1}$$

depends only on q and on the parity of p, and of course it is non-trivial only for p even. For $v \in k'$, let

$$\alpha_v = v\bar{x}^{n-1} \otimes \bar{x} \otimes \bar{x}^{n-1} \otimes \bar{x} \otimes \cdots \otimes \bar{x}^{n-1} \otimes \bar{x} \in \bar{\mathcal{O}}^{\otimes 2r},$$

where, as usual, $n = [k' : k]$. Then α_v represents an element of $E^2_{2,2r-1} = \mathrm{HH}^k_{2r-1}(\bar{\mathcal{O}})$. Its image under d_2 is represented by

$$T(-1 \otimes N(\alpha_v)) = \rho(1 \otimes N(\alpha_v)) - 1 \otimes N(\alpha_v).$$

The periodicity element $\bar{f}$ obtained by reducing (2.4.1) (mod π) defines a sequence of morphisms

$$\mathrm{HH}^k_{2r}(\bar{\mathcal{O}}) \xrightarrow{\sim} \mathrm{HH}^k_{2r-2}(\bar{\mathcal{O}}) \xrightarrow{\sim} \cdots \xrightarrow{\sim} \mathrm{HH}^k_2(\bar{\mathcal{O}}) \hookrightarrow \mathrm{HH}^k_0(\mathcal{O}) = k'.$$

Note that $\mathrm{HH}_2 \subset \mathrm{HH}_0$ consists of the set of trace 0 elements of k'. The composition of these morphisms is given by the map

$$F : \sum_i x_i^{(0)} \otimes \cdots \otimes x_i^{(2r)} \mapsto \sum_i a_i^{(0)} \bar{f}(a_i^{(1)}, a_i^{(2)}) \cdots \bar{f}(a_i^{(2r-1)}, a_i^{(2r)}).$$

In particular, any tensor monomial with $x_i^{(j)} \in k'$ for some $j > 0$, maps to zero. Thus

$$F(\rho(1 \otimes N(\alpha_v))) = 0.$$

From this we deduce that

$$\begin{aligned} F(T(-1 \otimes N(\alpha_v))) &= -F(1 \otimes N(\alpha_v)) \\ &= -F(1 \otimes \alpha_v) + F(1 \otimes \rho(\alpha_v)) - \cdots + F(1 \otimes \rho^{2r-1}(\alpha_v)) \\ &= -v + v^\sigma - v + v^\sigma - \cdots - v + v^\sigma = r(v^\sigma - v). \end{aligned}$$

As every trace 0 element of k' is of the form $v^\sigma - v$ (cf. 2.2), if r is invertible, d_2 maps $E^2_{2,2r-1}$ onto $E^2_{0,2r} = \mathrm{HH}^k_{2r}(\bar{\mathcal{O}})$. By the horizontal symmetry of the Tsygan complex, d_2 maps $E^2_{2p+2,2r-1}$ onto $E^2_{2p,2r}$. Thus, $E^3_{2p,q} = 0$ for $0 < q < \mathrm{char}(k)$ and for all q if $\mathrm{char}(k) = 0$. We conclude that if $r \in \mathrm{char}(k)$ or if $\mathrm{char}(k) = 0$,

$$\dim_k \mathrm{HC}_{2r}(\bar{\mathcal{O}}/k) \leq \dim_k E^2_{2r,0} = n. \tag{4.3.1}$$

PROPOSITION (4.4). *If char$(k) = 0$ or $r <$ char(k), then as a V-module,*

$$HC^V_{2r}(\mathcal{O}) \cong V \oplus (V/\pi^{r+1}V)^{n-1}. \tag{4.4.1}$$

Proof. Applying the universal coefficient theorem to the total Tsygan complex for $\mathcal{O}/V$, by (4.2.1),

$$HC_{2r}(\bar{\mathcal{O}}/k) \cong HC_{2r}^{V}(\mathcal{O}) \otimes_V k. \tag{4.4.2}$$

The structure theorem for finitely generated modules over discrete valuation rings says that as V-module,

$$HC_{2r}^{V}(\mathcal{O}) \cong V^a \oplus \bigoplus_{i=1}^{m} V/\pi^{b_i}V.$$

By (4.2.2), $a = 1$ and $\Sigma_i b_i = (n-1)(r+1)$. Moreover, $b_i \leq r+1$ because $V/\pi^{b_i}V$ has a filtration with $r+1$ quotients, each annihilated by π. By (4.3.1) and (4.4.2), $1+m \leq n$, but $HC_0^V(\mathcal{O}) \cong V \oplus k^{n-1}$ is a quotient of $HC_{2r}^V(\mathcal{O})$, so $m = n-1$. Together these facts imply (4.4.1).

PROPOSITION (4.5). *If R is a Dedekind domain whose field of fractions K is a global field and $\mathcal{O}$ is a maximal order of a central simple algebra D over K, then*

$$HC_r^R(\mathcal{O}) = \begin{cases} R \oplus M & \text{if } r \text{ is even,} \\ 0 & \text{if } r \text{ is odd,} \end{cases}$$

where M is a finite module of order

$$\prod_{\wp \in \mathrm{Ram}(D/R)} \|\wp\|^{k(e_\wp - 1)/2}.$$

If, moreover, $r/2$ is less than the residue characteristic of every prime in Ram (D/R), *then*

$$M \cong \bigoplus_{\wp \in \mathrm{Ram}(D/R)} (R/\wp^{k/2})^{e_\wp - 1}.$$

Proof. As cyclic homology groups are finitely generated, they are of the form $P \oplus M$, where P is projective and M is finite. The Chinese remainder theorem says that $M = \bigoplus_\wp M_\wp$. We apply the universal coefficient theorem to completions $R_\wp/R$. Equation (4.2.1) implies the vanishing of odd cyclic homology. By 4.2 and Th. 3.4, P has rank 1 and M has the given order. Prof. 4.4. gives M explicitly for $r/2$ less than for the residue characteristic of every $\wp \in$ Ram (D/R). To see that P is free we note that $HC_0^R(\mathcal{O}) = HH_0^R(\mathcal{O})$ contains R as a direct factor (Th. 3.4). As HC_0 is a factor of HC_r for even r, this implies that P is free.

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