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Autor: Lickorish, W.B.R.

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Calculations with the Temperley–Lieb algebra

W. B. R. LICKORISH

1. Introduction

An entirely elementary proof of the existence of the $SU(2)_q$ invariants of Witten [15], for closed 3-manifolds and for links in 3-manifolds, was given by the author in [9] and [10]. The foundation of that proof was just the use of the Kauffman bracket-polynomial (that gives the almost trivial approach to the Jones polynomial) and its formalisation as the Temperley–Lieb algebra. A 3-manifold invariant is associated with each primitive $4r^{\text{th}}$ root of unity; a formula for the invariant involves coefficients that depend on r and the chosen root. The analysis in [9] showed these coefficients exist, as solutions to some linear equations, but it gave no way, and indeed little hope, of writing down a reasonable formula for them. That defect is removed by this paper, so that an existence proof *and* a satisfactory formula for these 3-manifold invariants can now be derived from the Kauffman bracket approach. The formula is, as expected, essentially the same as that derived from quantum groups; a precise relationship will be given. The quantum group approach was pioneered by Reshetikhin and Turaev [11] and explored further by Kirby and Melvin [5]. In fact in what follows an analogue is found, using the Temperley–Lieb algebra, of the basic procedure (of the quantum group approach) of allocating irreducible representations of $SU(2)_q$ to the components of a link or of a tangle. The techniques here explained are also used to give short proofs of the symmetry principle of Kirby and Melvin and of formulae for the bracket polynomial of (diagrams of) Hopf links and of certain cablings.

The much appreciated hospitality of the Mathematical Sciences Research Institute (Berkeley, California, U.S.A.) produced the conception and much of the writing of this paper.

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2. Linear skein theory

Linear skein theory is useful rather than profound. A précis will recall the idea and establish some notation; more discussion is in [8], [9], [2] and [13]. Throughout, A is a fixed complex number that will eventually sometimes be required to be a primitive $4r^{\text{th}}$ root of unity. Consider link-diagrams of arcs and closed curves in a square for which the boundary of the arcs is m standard points on the left-hand side of the square and m on the right-hand side. The m^{th} Temperley–Lieb algebra V_m consists of formal linear sums, over $\mathbb{C}$, of such diagrams quotiented by

(i) planar isotopy fixed on the boundary of the square;
(ii) $X \cup (\text{closed component with no crossing}) = (-A^{-2} - A^2)X$;
(iii) $\times = A \asymp + A^{-1})($.

Here X is any diagram, and (iii) refers to three diagrams identical except where shown. Juxtaposition of diagrams induces the product in V_m which, as an algebra, is generated by the elements $1_m, e_1, e_2, \ldots, e_{m-1}$ shown in Figure 1, where the convention is used that a non-negative integer i beside a curve indicates the presence of i copies of that curve, all parallel in the plane. These generators satisfy the following relations

$$1_m x = x = x 1_m \qquad \text{for all } x \in V_m,$$

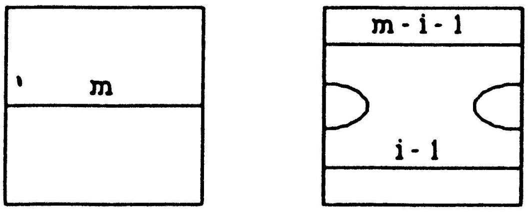
$$e_i^2 = (-A^{-2} - A^2)e_i,$$

$$e_i e_j = e_j e_i \qquad \text{if } |i-j| \geq 2,$$

$$e_i e_{i\pm 1} e_i = e_i \qquad \text{provided } e_{i\pm 1} \text{ is defined.}$$

V_0 is the 1-dimensional vector space spanned by the empty diagram 1_0; it will thus be identified with $\mathbb{C}$, 1_0 corresponding to $1 \in \mathbb{C}$. A link-diagram X of *closed* curves in the interior of a square (or in $\mathbb{R}^2$) thus represents an element of V_0 and hence of $\mathbb{C}$. By definition, that element is $\langle X \rangle$, the Kauffman bracket of X.

The same manoeuvres can be executed with any surface (with or without specified boundary points) in place of the square. Let $\mathfrak{A}$ be the complex vector



1_m e_i

Figure 1

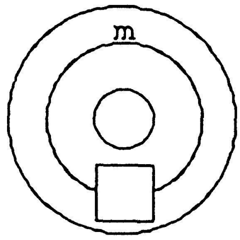


Figure 2

space of link-diagrams of closed curves in an *annulus* modulo relations the same as before, except that (ii) now refers only to components with no crossing that are *nul-homotopic* in the annulus. The process of forming one annulus from two, by identifying together one boundary component from each annulus, induces a product structure on $\mathfrak{A}$. Note that in $\mathfrak{A}$, and also in V_m, regularly isotopic diagrams (that is diagrams related by a sequence of Reidemeister moves of types II and III) represent the same elements. Thus $\mathfrak{A}$ becomes a commutative algebra. Let $\alpha \in \mathfrak{A}$ be represented by the diagram consisting of just one embedded curve encircling the annulus. Then α^m is represented by m parallel curves encircling the annulus and $\mathfrak{A}$ is the polynomial algebra $\mathbb{C}[\alpha]$. The operation of inserting a diagram that represents an element of V_m into the small square in the annulus of Figure 2 (where the sides of the square are connected by m parallel arcs around the annulus) induces a well-defined linear map $\Theta_m : V_m \to \mathfrak{A}$ and $\Theta_m(1_m) = \alpha^m$.

Suppose that D is a planar diagram of an ordered n-component link. Neighbourhoods of these components may be taken to be n annuli immersed in the plane with over and under crossing information preserved from the crossings of D. Consider the operation of taking n link diagrams in n standard annuli, inserting them in the immersed annuli, obeying the over and under crossing instructions in the obvious way, and then evaluating the Kauffman bracket. This operation induces a well defined multi-linear map

$$\Phi_D : \mathfrak{A} \times \mathfrak{A} \times \cdots \times \mathfrak{A} \to \mathbb{C}.$$

(A choice of orientation on the curves is not needed for this.) Figure 3 shows some link diagrams that will be used in conjunction with this idea.

Recall from [9] that a planar link-diagram D is said to *represent* a framed link L if D is a diagram for L in the usual sense and the framing of each component of L is the writhe (that is, the sum of the ± 1 signs of the crossings) of the sub-diagram of that component. The main result of [9] that describes the Witten 3-manifold invariants (for $SU(2)_q$) can, in this notation, be restated as follows.

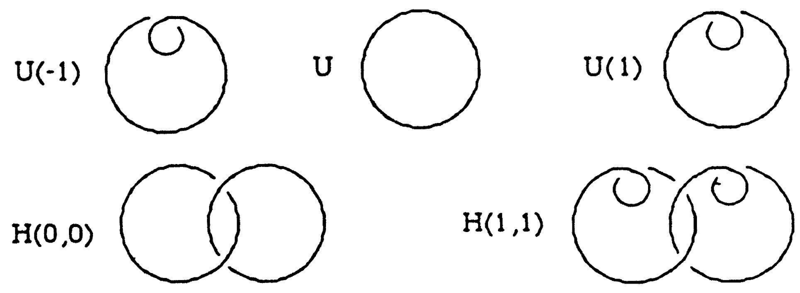


Figure 3

THEOREM 1. *Let A be a primitive $4r^{th}$ root of unity. Corresponding to A there is a unique element $\mathbf{a} \in \mathfrak{A}$ that is in the span of $\{\alpha^0, \alpha, \alpha^2, \ldots, \alpha^{r-2}\}$ such that*

$$\Phi_{H(1,1)}(\mathbf{a}, \ ) = \Phi_U(\ )$$

as maps from $\mathfrak{A}$ to $\mathbb{C}$. Suppose the 3-manifold M is obtained by surgery on the n-component framed link L that is represented by a planar diagram D, and let σ and ν be the signature and nullity of the linking matrix of L (with the framings on the diagonal). Then the complex number

$$(\Phi_{U(-1)}(\mathbf{a}))^{(\sigma+\nu-n)/2}\Phi_D(\mathbf{a}, \mathbf{a}, \ldots, \mathbf{a})$$

is an invariant of M.

(In [9], $\mathbf{a}$ was written as $\Sigma_{i=0}^{r-2} \lambda_i \alpha^i$, a linear sum of base elements of $\mathfrak{A}$, and thus $\Phi_D(\mathbf{a}, \mathbf{a}, \ldots, \mathbf{a})$ appeared as $\Sigma_{c \in C(n,r)} \lambda_{c(1)}\lambda_{c(2)}, \ldots, \lambda_{c(n)}\langle c * D\rangle$, where $C(n, r)$ was the set of all functions $\{1, 2, \ldots, n\} \to \{0, 1, \ldots, r-2\}$ and $c * D$ was the c-induced paralleling of D.) This paper is devoted to finding an explicit expression for $\mathbf{a}$ as a linear sum of base elements of $\mathfrak{A}$; it transpires that, at least in the first instance, the base $\{\alpha^i\}$ is *not* the base to use.

3. Chebyshev polynomials

In what follows, a Chebyshev polynomial $S_n(x)$ will occur in a fairly natural way. Although only elementary facts are required concerning these classical polynomials it will help to list them here. Proofs are but exercises for the reader; together with much other information they can be found in [12].

For an integer n, $n \geq 0$, the n^{th} *Chebyshev polynomial* of the second kind (renormalised) is the polynomial $S_n(x)$ defined inductively by $S_0(x) = 1$, $S_1(x) = x$ and

$$S_n(x) = xS_{n-1}(x) - S_{n-2}(x).$$

Note that $S_n(x)$ is a monic polynomial of degree n with integer coefficients, and $S_n(x) = (-1)^n S_n(-x)$. The notation is motivated by the equation

$$S_n(2\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$$

which follows from

$$(t - t^{-1})S_n(t + t^{-1}) = t^{n+1} - t^{-n-1}.$$

The actual expansion of $S_n(x)$ is given by

$$S_n(x) = \sum_{\substack{k \\ 0 \le 2k \le n}} (-1)^k \binom{n-k}{k} x^{n-2k}.$$

Sometimes this will be abbreviated to

$$S_n(x) = \sum_{k \ge 0} s_{n,k} x^k,$$

where $s_{n,k} = 0$ if $k > n$ and $s_{n,n} = 1$. The zeros of the polynomial are apparent from the factorisation

$$S_n(x) = \prod_{k=1}^{n} \left(x - 2\cos\frac{k\pi}{n+1}\right).$$

$S_n(x)$ is also seen to be the determinant of the matrix (with zeros off the three diagonals)

$$\begin{pmatrix} x & 1 & & & & \\ 1 & x & 1 & & & \\ & 1 & x & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & x & 1 \\ & & & & 1 & x \end{pmatrix}.$$

To whet the appetite, if $\mathfrak{U}^n$ is the n-dimensional irreducible representation of $\mathbf{sl}_2\mathbb{C}$ then the Clebsch–Gordan formula states that, in the Grothendieck representation ring, the irreducible representations satisfy the Chebyshev recurrence formula

$$\mathfrak{U}^{n+1} = \mathfrak{U}^2 \otimes \mathfrak{U}^n - \mathfrak{U}^{n-1}$$

with $\mathfrak{U}^1 = \mathbb{C}$ so that $\mathfrak{U}^{n+1} = S_n(\mathfrak{U}^2)$.

4. A base for $\mathfrak{A}$

In [9] attention was drawn to a sequence of elements $f_0, f_1, \ldots, f_{m-1}$ in the Temperley–Lieb algebra V_m. These elements were discovered by Jones [3] and formulated by Wenzl [14]. Define Δ_n to be $S_n(-A^{-2} - A^2)$, so that

$$\Delta_n = (-1)^n (A^{2(n+1)} - A^{-2(n+1)})(A^2 - A^{-2})^{-1}.$$

Then f_i is defined by $f_0 = 1_m$ and

$$f_i = f_{i-1} - (\Delta_{i-1}/\Delta_i) f_{i-1} e_i f_{i-1};$$

there is here a proviso that A be chosen so that $\Delta_1 \Delta_2 \cdots \Delta_i \neq 0$, otherwise f_i is not defined. Then short inductive arguments (see [14] or [9]) show that

$$f_i^2 = f_i,$$

$$f_i e_j = 0 = e_j f_i \qquad \text{if } j \leq i,$$

$$f_i e_j = e_j f_i \qquad \text{if } i + 2 \leq j.$$

Now, so that m may be allowed to vary, let $f^{(m)}$ be the final element (i.e. f_{m-1}) of this sequence of elements in V_m. Then $(f^{(m)})^2 = f^{(m)}$, and $f^{(m)} e_i = 0 = e_i f^{(m)}$ for all $e_i \in V_m$. Further (see [14] or [9]) $(1_m - f^{(m)})$ belongs to, and is an identity for, the subalgebra of V_m generated by $\{e_1, e_2, \ldots, e_{m-1}\}$. The fact that this subalgebra was known to be semi-simple, and hence to have an identity, led Jones to the discovery of $f^{(m)}$.

The Markov trace on V_m is the linear map $\operatorname{tr} : V_m \to \mathbb{C}$ induced by the operation of taking a link-diagram in a square representing an element of V_m, joining the m points on the left side to those on the right side by disjoint arcs in the plane outside the square, and then evaluating the Kauffman bracket. In the above terminology, $\operatorname{tr} x = \Phi_U(\Theta_m(x))$. It is shown in [14] and [9] and verified in the corollary to the next theorem that

$$\operatorname{tr}(f^{(m)}) = \Delta_m.$$

Now define $\phi_n \in \mathfrak{A}$ by $\phi_n = S_n(\alpha)$, the n^{th} Chebyshev polynomial evaluated in the algebra $\mathfrak{A}$ on the generator α. Then $\phi_n = \Sigma_{k \geq 0} s_{n,k}\alpha^k$, where $s_{n,k} = 0$ if $k > n$ and $s_{n,n} = 1$. Thus $\{\phi_0, \phi_1, \ldots, \phi_n\}$ and $\{\alpha^0, \alpha, \alpha^2, \ldots, \alpha^n\}$ are both bases for the same subspace of $\mathfrak{A}$; they are related by a lower triangular matrix with ones on the diagonal which is hence invertible. The base $\{\phi_0, \phi_1, \phi_2, \ldots\}$ will turn out to be distinctly easier for the purposes of calculation.

THEOREM 2. *Suppose that* $\Delta_1 \Delta_2 \cdots \Delta_{m-1} \neq 0$ (*so that* $f^{(m)}$ *is defined*). *Then*

$$\phi_m = \Theta_m(f^{(m)}).$$

Proof. First note that for al $x, y \in V_m$, $\Theta_m(xy) = \Theta_m(yx)$. This follows from the trick of pushing a diagram to the right of x all the way around the annulus until it is on the left of x. Now, working entirely with V_m, the definitions give that

$$\Theta_m(f^{(m)}) = \Theta_m(f_{m-1}) = \Theta_m(f_{m-2}) - (\Delta_{m-2}/\Delta_{m-1})\Theta_m(f_{m-2}e_{m-1}f_{m-2}).$$

However

$$\Theta_m(f_{m-2}) = \alpha\Theta_{m-1}(f^{(m-1)})$$

and

$$\Theta_m(f_{m-2}e_{m-1}f_{m-2}) = \Theta_m(e_{m-1}(f_{m-2})^2) = \Theta_m(e_{m-1}f_{m-2}).$$

But

$$\begin{aligned}\Theta_m(e_{m-1}f_{m-2}) &= \Theta_m(e_{m-1}f_{m-3}) - (\Delta_{m-3}/\Delta_{m-2})\Theta_m(e_{m-1}f_{m-3}e_{m-2}f_{m-3}) \\ &= (-A^{-2} - A^2)\Theta_{m-2}(f^{(m-2)}) \\ &\quad -(\Delta_{m-3}/\Delta_{m-2})\Theta_m(e_{m-1}e_{m-2}f_{m-3}) \\ &= ((-A^{-2} - A^2) - (\Delta_{m-3}/\Delta_{m-2}))\Theta_{m-2}(f^{(m-2)}).\end{aligned}$$

The very last step should be clear from Figure 4. By definition $\Delta_n = S_n(-A^{-2} - A^2)$ so

$$(\Delta_{m-2}/\Delta_{m-1})((-A^{-2} - A^2) - (\Delta_{m-3}/\Delta_{m-2})) = 1.$$

Thus $\Theta_m(f^{(m)}) = \alpha\Theta_{m-1}(f^{(m-1)}) - \Theta_{m-2}(f^{(m-2)})$. Of course, $\Theta_0(f^{(0)}) = \alpha^0$ and $\Theta_1(f^{(1)}) = \alpha$, so the proof is complete.

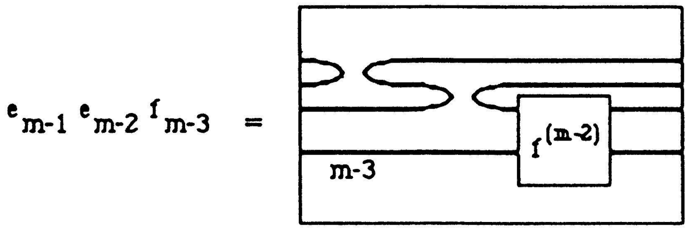


Figure 4

COROLLARY. $\operatorname{tr} f^{(m)} = \Phi_U(\phi_m) = \Delta_m$.

Proof. Note that $\Phi_U(\alpha^n) = (-A^{-2} - A^2)^n$, so that the theorem implies that

$$\operatorname{tr} f^{(m)} = \Phi_U(\Theta_m(f^{(m)})) = \Phi_U(\phi_m) = \Phi_U(S_m(\alpha)) = S_m(-A^{-2} - A^2) = \Delta_m.$$

5. Special elements of V_m

The element $f^{(m)}$ of V_m interacts with the standard generators of V_m in a marvelously easy manner: $f^{(m)}1_m = f^{(m)}$ and $f^{(m)}e_i = 0 = e_i f^{(m)}$. As a vector space V_m has a natural base consisting of all diagrams in the square (with the given $2m$ boundary points) that have no crossing and no closed loop. For each $x \in V_m$ let $1_m^*(x)$ be the coefficient of 1_m in the expansion of x as a linear sum of these base elements. Of course, every base element other than 1_m is a non-empty product of the e_i's. Thus the following result is clear.

LEMMA 3. *If* $x \in V_m$, $f^{(m)}x = xf^{(m)} = (1_m^*(x))f^{(m)}$.

This means that $f^{(m)}x$ is a 'scalar multiple' of $f^{(m)}$ for any x in V_m. That might be compared with the fact that, in the language of the quantum group approach, if the relevant R-matrix-induced functor is applied to a $1-1$ tangle whose strings are labelled with irreducible representations, then, by Schur's Lemma, there results an endomorphism, of the module assigned to the through-string, that is a scalar multiple of the identity. Likewise, use of $\phi_m \in \mathfrak{A}$ is an analogue of 'colouring' a closed string with the $(m+1)$-dimensional irreducible representation of $\mathbf{sl}_2\mathbb{C}$. Theorem 2 links this with the use of $f^{(m)}$.

Expressing an element x of V_m as a linear sum of elements of the natural base is, in general, not at all easy. However finding $1_m^*(x)$ is a much easier task; in two important instances it was carried out in [10] and [9] thus giving a proof of the next lemma. Only simple manipulations of the relations that define V_m were used.

Figure 5

LEMMA 4. *Let z_m and $(x_m)^i$ be the elements of V_m corresponding to the diagrams in Figure 5, then*

$$1_m^*(z_m) = (-1)^m A^{m^2+2m},$$

$$1_m^*((x_m)^i) = (-A^{2(m+1)} - A^{-2(m+1)})^i.$$

With the aid of this result calculations can begin. Recall the diagrams of Figure 3.

LEMMA 5

$$(A^2 - A^{-2})\Phi_{H(1,1)}(\phi_i, \phi_j) = A^{i^2+2i+j^2+2j}(A^{2(i+1)(j+1)} - A^{-2(i+1)(j+1)}).$$

Proof. Suppose that the (non-zero) complex number A is chosen so that $\Delta_1\Delta_2 \cdots \Delta_{\max\{i,j\}-1} \neq 0$ so $f^{(i)}$ and $f^{(j)}$ are defined. Then

$$\begin{aligned}\Phi_{H(1,1)}(\phi_i, \phi_j) &= \Phi_{H(1,1)}(\Theta_i f^{(i)}, \Theta_j f^{(j)}) = \Phi_{H(0,0)}(\Theta_i(z_i f^{(i)}, \Theta_j(z_j f^{(j)})) \\ &= (-1)^{i+j} A^{i^2+2i+j^2+2j}\Phi_{H(0,0)}(\phi_i, \phi_j)\end{aligned}$$

using Lemmas 3 and 4. Further use of these lemmas, and use of the corollary to Theorem 2, shows that

$$\begin{aligned}\Phi_{H(0,0)}(\phi_i, \phi_j) &= \Phi_{H(0,0)}(\Theta_i f^{(i)}, S_j(\alpha)) = \sum_{k\geq 0} s_{j,k}\Phi_U(\Theta_i(x_i^k f^{(i)})) \\ &= S_j(-A^{2(i+1)} - A^{-2(i+1)})\Phi_U(\Theta_i f^{(i)}) \\ &= S_j(-A^{2(i+1)} - A^{-2(i+1)})S_i(-A^{-2} - A^2).\end{aligned}$$

The result now follows by using the results on Chebyshev polynomials, and, being an identity between Laurent polynomials that has now been shown to be true for all but finitely many values of A, it is true for all A.

6. An explicit form of the 3-manifold invariant

In this section A is taken to be a primitive $4r^{\text{th}}$ root of unity as required in Theorem 1. That theorem asserts that the formula

$$\Phi_{H(1,1)}(\mathbf{a},\ ) = \Phi_U(\ )$$

defines a unique element $\mathbf{a} \in \mathfrak{A}$ in

$$\text{Span}\{\alpha^0, \alpha, \alpha^2, \ldots, \alpha^{r-2}\} = \text{Span}\{\phi_0, \phi_1, \ldots, \phi_{r-2}\}.$$

This element can now be expressed in terms of $\{\phi_0, \phi_1, \ldots, \phi_{r-2}\}$:

LEMMA 6. $\mathbf{a} = \Sigma_{k=0}^{r-2} \mu_k \phi_k$ *where*

$$\mu_k = \left(\sum_{n=1}^{4r} A^{n^2}\right)^{-1} 2A^{r^2+3}(-1)^k(A^{2(k+1)} - A^{-2(k+1)}).$$

Proof. Certainly $\mathbf{a} = \Sigma_{k=0}^{r-2} \mu_k \phi_k$ for some $\mu_k \in \mathbb{C}$. By the defining formula for $\mathbf{a}$,

$$\sum_{k=0}^{r-2} \mu_k \Phi_{H(1,1)}(\phi_k, \phi_j) = \Phi_U(\phi_j)$$

for all j. Thus, by Lemma 5 and the corollary to Theorem 2,

$$\sum_{k=0}^{r-2} \mu_k A^{(k+1)^2+(j+1)^2-2}(A^{2(k+1)(j+1)} - A^{-2(k+1)(j+1)})$$

$$=(-1)^j(A^{2(j+1)} - A^{-2(j+1)}).$$

However, in a similar calculation, Kirby and Melvin (see [5] Lemma 5.1) use elementary manipulations to show that, for any integers j and l,

$$\sum_{k=0}^{r-2} (A^{2l(k+1)} - A^{-2l(k+1)})(A^{2(k+1)(j+1)} - A^{-2(k+1)(j+1)})A^{(k+1)^2+(j+1)^2+l^2}$$

$$= -(A^{2l(j+1)} - A^{-2l(j+1)})\frac{1}{2}\left(\sum_{k=1}^{4r} A^{k^2}\right).$$

The result then follows using the substitution $l = r + 1$, because $A^{2r} = -1$ implies

$$A^{2(r+1)(k+1)} - A^{-2(r+1)(k+1)} = (-1)^{k+1}(A^{2(k+1)} - A^{-2(k+1)}).$$

Note that the proof of this lemma has proved (by actually finding it) that there is an $\mathbf{a}$ such that, as maps $\mathfrak{A} \to \mathbb{C}$, $\Phi_{H(1,1)}(\mathbf{a}, \ ) = \Phi_U(\ )$. The proof was by verifying the equality on the base $\{\phi_i\}$; only results from this paper (including the quoted Lemma 4 and the very elementary quote from [5] in the proof of Lemma 6) have been used. The existence of such an $\mathbf{a}$ is the only difficult fact that is needed in deducing the existence of the 3-manifold invariant from Kirby's surgery theorem. The only use made of the fact that A is a root of unity is in the proof of Lemma 6.

A tangible form for the theorem requires a calculation of $\Phi_{U(-1)}(\mathbf{a})$ and this is now performed.

LEMMA 7. $\Phi_{U(-1)}(\mathbf{a}) = (-1)^{r+1}A^6(\bar{G}/G)$ *where* G *is the Gauss sum defined by* $G(A) = (\Sigma_{k=1}^{4r} A^{k^2})$.

Proof. The link-diagram $U(1)$ is the diagram $U(-1)$ with its crossing switched. Letting the map

$$\Phi_{H(1,1)}(\mathbf{a}, \ ) = \Phi_U(\ )$$

operate on ϕ_0 (the empty diagram in the annulus) shows that $\Phi_{U(1)}(\mathbf{a}) = 1$. So, letting $\mathbf{a} = \Sigma_{k=0}^{r-2} \mu_k \phi_k$, where the μ_k are determined in Lemma 6,

$$\sum_{k=0}^{r-2} \mu_k \Phi_{U(1)}(\phi_k) = 1.$$

Complex conjugation, and the recollection that, when $|A| = 1$, the conjugate of $\langle D \rangle$ is $\langle D\text{-reflected} \rangle$, shows that

$$\sum_{k=0}^{r-2} \bar{\mu}_k \Phi_{U(-1)}(\phi_k) = 1.$$

But, from Lemma 6 and using $A^{2r} = -1$,

$$\frac{\bar{\mu}_k}{\mu_k} = (-1)^{r+1} A^{-6} \frac{G}{\bar{G}}$$

and this is independent of k. Thus

$$(-1)^{r+1}A^{-6}\frac{G}{\bar{G}}\Phi_{U(-1)}(\mathbf{a}) = \frac{\bar{\mu}_k}{\mu_k}\sum_{k=0}^{r-2}\mu_k\Phi_{U(-1)}(\phi_k) = 1.$$

The following reformulation follows at once from these calculations.

THEOREM 1 (restated). *Let A be a primitive $4r^{th}$ root of unity. Suppose the 3-manifold M is obtained by surgery on the n-component framed link L that is represented by a planar diagram D, and let σ and ν be the signature and nullity of the linking matrix of L (with the framings on the diagonal). Then the complex number*

$$\left((-1)^{r+1}A^{6}\frac{\bar{G}}{G}\right)^{(\sigma+\nu-n)/2}\Phi_D(\mathbf{a}, \mathbf{a}, \ldots, \mathbf{a})$$

is an invariant of M, where $G(A) = (\Sigma_{k=1}^{4r} A^{k^2})$ and $\mathbf{a} \in \mathfrak{A}$ is given by

$$\mathbf{a} = 2G^{-1}A^{r^2+3}\sum_{k=0}^{r-2}(-1)^k(A^{2(k+1)} - A^{-2(k+1)})\phi_k.$$

If desired, coordinates can be changed to those corresponding to the base $\{\alpha^i\}$ of $\mathfrak{A}$ in the following manner. Let $\nu_k = (-1)^k(A^{2(k+1)} - A^{-2(k+1)})$. Then, the series expansion for the Chebyshev polynomial gives

$$\sum_{k=0}^{r-2}\nu_k\phi_k = \sum_{k=0}^{r-2}\nu_k S_k(\alpha^k) = \sum_{k=0}^{r-2}\sum_{\substack{i \\ 0 \le 2i \le k}}\nu_k(-1)^i\binom{k-i}{i}\alpha^{k-2i}.$$

Writing $k - 2i = c$, this becomes

$$\sum_{c=0}^{r-2}\sum_{\substack{i \\ 0 \le 2i \le r-2-c}}\nu_{c+2i}(-1)^i\binom{c+i}{i}\alpha^c.$$

Hence $\mathbf{a} = \Sigma_{c=0}^{r-2}\lambda_c\alpha^c$ where

$$\lambda_c = 2G^{-1}A^{r^2+3}\sum_{\substack{i \\ 0 \le 2i \le r-2-c}}(-1)^{c+i}\binom{c+i}{i}(A^{2(c+2i+1)} - A^{-2(c+2i+1)}).$$

Thus, in the notation of [9], the 3-manifold invariant becomes the product of

$$\sum_{c \in C(n,r)}\left(\prod_{j=1}^{n}\left(\sum_{\substack{i \\ 0 \le 2i \le r-2-c(j)}}(-1)^{c(j)+i}\binom{c(j)+1}{i}\right.\right.$$

$$\times\, (A^{2(c(j)+2i+1)} - A^{-2(c(j)+2i+1)})\Big)\langle c * D\rangle\Big)$$

with

$$\left((-1)^{r+1}A^{6}\frac{\bar{G}}{G}\right)^{(\sigma+v-n)/2}(2G^{-1}A^{r^2+3})^{n}.$$

Gauss sums are a well understood part of elementary number theory. In particular, when $A = e^{i\pi/(2r)}$, $G(A) = 2\sqrt{2r}\, e^{i\pi/4}$; a good account is given in [6].

If t is a primitive $4r^{\text{th}}$ root of unity then so is $-t$. To align the above formulae with those of Kirby and Melvin [5], derived from the quantum group approach of Reshetikhin and Turaev, it is necessary to make the substitution $A = -t$. It seems desirable to record the way the invariant of this paper attunes with that of [5] in order to prevent a profusion of such invariants all known imprecisely to be manifestations of the same thing. Unfortunately the invariant of [5] is there discussed in detail only when $t = e^{i\pi/(2r)}$.

PROPOSITION 8. *For any closed oriented* 3*-manifold* M, *and any integer* $r \geq 3$, *the invariant described in Theorem* 1 *when* $A = -e^{i\pi/(2r)}$ *is equal to*

$$e^{i\pi(6-3r)v/(4r)}\tau_r(M)$$

where $\tau_r(M)$ *is the invariant of* [5] *and* v *is the first Betti number of* M (*and hence* v *is still the nullity of the previously discussed matrix*).

Proof. The checking of this proposition involves just a little more than a merging of notations. First note that, for a $4r^{\text{th}}$ root of unity A, it is easy to check that $A^{r^2}G(-A) = G(A)$. In the notation of the proof of Lemma 7, the substitution $A = -t$ gives

$$\frac{\mu_k}{\bar{\mu}_k} = -t^6\frac{\overline{G(t)}}{G(t)}.$$

Further, the formula for λ_c becomes

$$G(t)\lambda_c = -2t^3(t^2 - t^{-2}(-1)^c\{c\}$$

where

$$\{c\} = \sum_{\substack{i\\ 0\leq 2i\leq r-2-c}} (-1)^i\binom{c+i}{i}[c+2i+1],$$

[ ] being defined by $[k](t^2 - t^{-2}) = (t^{2k} - t^{-2k})$. (In [5] "$\{c\}$" is written "$\langle c\rangle$".) Thus, when $A = -t$, the invariant of Theorem 1 is the product of

$$\left(\frac{-t^6\overline{G(t)}}{G(t)}\right)^{(\sigma+\nu-n)/2}\left(\frac{-2t^3(t^2-t^{-2})}{G(t)}\right)^n$$

and

$$\sum_{c\in C(n,r)} (-1)^{\sum_{j=1}^n c(j)}\{c(1)\}\{c(2)\}\cdots\{c(n)\}\langle c*D\rangle_{A=-t}.$$

Now suppose that X is any link diagram with $\#X$ components. If $w(X)$ is the sum of the signs of the crossings of X with respect to some choice of orientation, $i^{w(X)}$ is independent of that choice ($i^2 = -1$). Then, for any t, the Kauffman bracket polynomial satisfies

$$\langle X\rangle_{A=-t} = (-1)^{\#X}(-i)^{w(X)}\langle X\rangle_{A=it}.$$

The invariant of link diagrams $J_X(t)$ that emerges from quantum groups is shown in [5] to be given by $J_X(t) = (-i)^{w(X)}\langle X\rangle_{A=it}$. It seems that this formula comes naturally from that representation theory approach. This means that the final summation above is

$$\sum_{c\in C(n,r)} \{c(1)\}\{c(2)\}\cdots\{c(n)\}J_{c*D}(t).$$

If now one makes the substitution $t = e^{i\pi/(2r)}$ so that $G(t) = 2\sqrt{2r}\,e^{i\pi/4}$ then

$$\frac{-t^6\overline{G(t)}}{G(t)} = e^{i\pi(6-3r)/(2r)},$$

$$\frac{-2t^3(t^2-t^{-2})}{G(t)} = \sqrt{\frac{2}{r}}\sin\frac{\pi}{r}\,e^{i\pi(6-3r)/(4r)}.$$

The invariant of Theorem 1, when $A = -e^{i\pi/(2r)}$, then becomes

$$e^{i\pi(6-3r)(\sigma+\nu)/(4r)}\left(\sqrt{\frac{2}{r}}\sin\frac{\pi}{r}\right)^n \sum_{c\in C(n,r)} \{c(1)\}\{c(2)\}\cdots\{c(n)\}J_{c*D}(t)$$

and this is $e^{i\pi(6-3r)\nu/(4r)}\tau_r(M)$ according to Theorem 4.17 of [5].

7. Further formulae

The Temperley–Lieb algebra methods of this paper can be used to confirm quickly other results first obtained by means of quantum groups; it is hoped that such confirmation will give additional credence to both approaches. The proposition that follows is a version of the Symmetry Principle of [5]. In [5] this principle was a crucial part of the whole theory. That is not the case here, but it is also valuable in simplifying the calculation of the invariants of specific 3-manifolds (see [5]).

PROPOSITION 9. *Let D be a planar link diagram, with $D_1, D_2, \ldots, D_n$ being the subdiagrams of the link's components. Let A be a primitive $4r^{th}$ root of unity and suppose $k \le r-2$. Let $w(D_1)$ be the writhe of D_1. Then for any non-negative integers $i(2), i(3), \ldots, i(n)$,*

$$\Phi_D(\phi_k, \alpha^{i(2)}, \alpha^{i(3)}, \ldots, \alpha^{i(n)})$$
$$= (-1)^{\Lambda + r}((-1)^{k+r+1}A^{-r^2})^{w(D_1)}\Phi_D(\phi_{r-2-k}, \alpha^{i(2)}, \alpha^{i(3)}, \ldots, \alpha^{i(n)})$$

where $\Lambda = \Sigma_{j=2}^{n} i(j)\Lambda_2(D_1, D_j)$, with Λ_2 denoting the linking number modulo two.

Proof. First note that it is sufficient to check this formula in the case when D_1 is a diagram of the unknot. This is because crossings of D_1 can be switched using Theorem 1. To be more precise, let D and D' be the diagrams shown in Figure 6 (where X represents an arbitrary diagram). By Theorem 1,

$$\Phi_D(\phi_k, \alpha^{i(2)}, \alpha^{i(3)}, \ldots, \alpha^{i(n)}) = \Phi_{D'}(\phi_k, \alpha^{i(2)}, \alpha^{i(3)}, \ldots, \alpha^{i(n)}, \mathbf{a}).$$

Now

$$((-1)^{k+r+1}A^{-r^2})^{w(D_1)} = ((-1)^{k+r+1}A^{-r^2})^{w(D_1')}$$

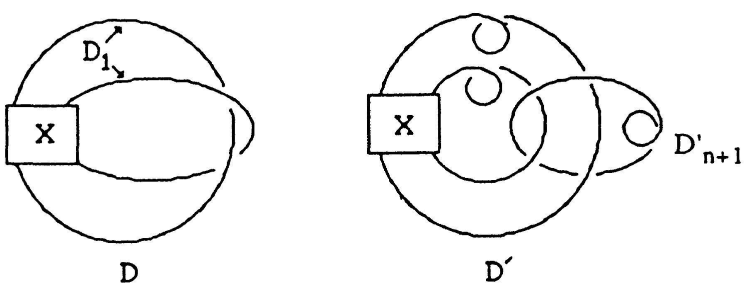


Figure 6

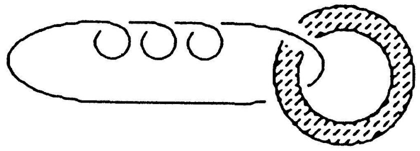


Figure 7

because $w(D_1')$ and $w(D_1)$ are equal modulo four, and $A^{4r}=1$. Further, $\Lambda_2(D'_{n+1}, D'_1)=0$.

Now suppose that D_1 is indeed a diagram of the unknot. All the terms of the formula to be verified are invariant under regular isotopy, so it may be assumed that D_1 is a closed curve with no self crossing except for $w(D_1)$ kinks, as in Figure 7, and that the remaining components are a diagram in the annulus of Figure 7. If D^* denotes D with the kinks in D_1 removed then, by Theorem 2 and Lemmas 3 and 4,

$$((-1)^k A^{k^2+2k})^{w(D_1)}\Phi_{D^*}(\phi_k, \alpha^{i(2)}, \alpha^{i(3)}, \ldots, \alpha^{i(n)}) = \Phi_D(\phi_k, \alpha^{i(2)}, \alpha^{i(3)}, \ldots, \alpha^{i(n)})$$

But $(-1)^k A^{k^2+2k} = (-1)^{r-2-k}A^{(r-2-k)^2+2(r-2-k)}(-1)^{k+r+1}A^{-r^2}$, and this means that it is just required to check the formula for D^*.

Thus it may be assumed that D_1 has no self crossing, and the remaining components form a link diagram in the annulus of Figure 7. Hence (recalling the diagram $H(0,0)$ of Figure 3) it is sufficient to prove that

$$\Phi_{H(0,0)}(\phi_k, \beta) = (-1)^{b+r}\Phi_{H(0,0)}(\phi_{r-2-k}, \beta)$$

where the element β of $\mathfrak{A}$ is represented by a diagram that represents b in the modulo two first homology of the annulus (that is, β encircles the annulus b times modulo two). However, β can be expressed as a linear sum of the α^i using the defining relations of $\mathfrak{A}$, and each such usage produces diagrams with the *same* value of b. Thus it remains to check that

$$\Phi_{H(0,0)}(\phi_k, \alpha^i) = (-1)^{i+r}\Phi_{H(0,0)}(\phi_{r-2-k}, \alpha^i),$$

and this follows from the fact that, by Theorem 2 and Lemmas 3 and 4,

$$\Phi_{H(0,0)}(\phi_k, \alpha^i) = (-A^{2(k+1)} - A^{-2(k+1)})^i(-1)^k(A^{2(k+1)} - A^{-2(k+1)}) \times (A^2 - A^{-2})^{-1}$$

The proof is complete.

In the preceding discussion, two bases, $\{\alpha^0, \alpha, \alpha^2, \ldots\}$ and $\{\phi_0, \phi_1, \phi_2, \ldots\}$, have been used for the linear skein of the annulus, namely the vector space $\mathfrak{A}$. The first seems geometrically natural whilst use of the second facilitates calculations. By definition, the Chebyshev polynomials express the second base in terms of the first, but in performing calculations concerning parallels of link diagrams, it may well be desirable to express the first base in terms of the second. Such an expression follows easily from the next simple (and presumably 'well known') lemma. First, however, extend the definition of the Chebyshev polynomial $S_n(x)$ to allow n to be negative by just continuing to use (in the 'negative direction') the recurrence $S_n(x) = xS_{n-1}(x) - S_{n-2}(x)$ with initial conditions $S_0(x) = 1$, $S_1(x) = x$ as before. Then $S_{-n}(x) = -S_{n-2}(x)$ and, from this, properties of $S_n(x)$ for $n < 0$ can easily be deduced. For example, the identity

$$(t - t^{-1})S_n(t + t^{-1}) = t^{n+1} - t^{-n-1}$$

is true for all integers n.

LEMMA 10. *For any non-negative integer n,*

$$x^{n+1} = \sum_{r=0}^{n} \binom{n}{r} S_{n-2r+1}(x).$$

Proof. Use induction. The formula is true when $n = 0$ so assume it for a given n. Then

$$\begin{aligned} x^{n+2} &= \sum_{r=0}^{n} \binom{n}{r} (S_{n-2r+2}(x) + S_{n-2r}(x)) \\ &= S_{-n}(x) + \sum_{r=1}^{n} \left(\binom{n}{r} + \binom{n}{r-1}\right) S_{n-2r+2}(x) + S_{n+2}(x) \\ &= \sum_{r=0}^{n+1} \binom{n+1}{r} S_{n-2r+2}(x). \end{aligned}$$

Of course, in the algebra $\mathfrak{A}$, the definition $\phi_n = S_n(\alpha)$ should now be extended to negative integers n. The identity $\phi_{-n} = -\phi_{n-2}$ enables an easy verification to be made of the formula that describes how $\Phi_D(\phi_n, \ldots,)$ changes, for any link diagram D with components $D_1, D_2, \ldots$, if a kink is removed in D_1 (namely, for *all* n, 'multiply by $(-1)^n A^{n^2+2n}$'). Also, if D_2 simply links D_1 with just two crossings, D_2 having no other crossing, and if D' is D with D_2 removed, then it can similarly be checked that

$$\Phi_D(\phi_n, \alpha^i, \ldots,) = (-A^{2(n+1)} - A^{-2(n+1)})^i \Phi_{D'}(\phi_n, \ldots,)$$

for *all* integers n.

PROPOSITION 11. (i) *In the algebra* $\mathfrak{A}$, *for* $n \geq 0$,

$$\alpha^{n+1} = \sum_{r=0}^{n} \binom{n}{r} \phi_{n-2r+1}.$$

(ii) *The bracket polynomial of the* $(n+1)$*-parallel of the diagram* $U(1)$ *of Figure* 3, *which is a diagram of the* $(n+1)$*-component positive Hopf link, is*

$$(A^2 - A^{-2})^{-1} \sum_{r=0}^{n} \binom{n}{r} A^{(n-2r+2)^2-1}(A^{2(n-2r+2)} - A^{-2(n-2r+2)}).$$

Alternatively this can be expressed as

$$A^2(A^2 - A^{-2})^{-1} \sum_{r=0}^{n-1} \binom{n-1}{r} A^{(n-2r+1)^2}(A^{4(n-2r+1)} - A^{-4(n-2r+1)}).$$

(This final formula has already been proved, using other methods, by Kirby and Melvin.)

Proof. The first formula follows at once from Lemma 10. The bracket polynomial of the $(n+1)$-parallel of the diagram $U(1)$ is $\Phi_{U(1)}(\alpha^{n+1})$. By (i) this is

$$\sum_{r=0}^{n} \binom{n}{r} \Phi_{U(1)}(\phi_{n-2r+1})$$

and, removing the kink of $U(1)$ by the above mentioned formula, and using the Corollary to Theorem 2 (which is still valid for negative m), this becomes the first formula of (ii). However,

$$\Phi_{U(1)}(\alpha^{n+1}) = \Phi_{H(1,1)}(\alpha^n, \alpha).$$

Using (i) this becomes

$$\sum_{r=0}^{n-1} \binom{n-1}{r} \Phi_{H(1,1)}(\phi_{n-2r}, \alpha).$$

Removing the two kinks as described above, and then removing the second component, this is

$$-A^3\sum_{r=0}^{n-1}\binom{n-1}{r}(-1)^n A^{(n-2r+1)^2-1}(-A^{2(n-2r+1)}-A^{-2(n-2r+1)})\Phi_U(\phi_{n-2r}).$$

The Corollary to Theorem 2 then produces the required formula.

This last proposition facilitates the calculation of the Kauffman bracket of a parallel of a link by re-expressing the problem in terms of the more amenable $\{\phi_n\}$-basis of $\mathfrak{A}$. The analogous calculation for when a component is replaced by a single curve running around it n times will now be treated in a similar way.

PROPOSITION 12. *Let y_n, $n \geq 1$, be the element of the Temperley–Lieb algebra V_n represented by the first diagram of Figure* 8. *Then*

$$\Theta_n(y_n) = A^{n-3}(A^2\phi_n + A^{-2}\phi_{-n}).$$

Proof. Figure 8 represents an identity in V_n. Applying Θ_n to it gives

$$\Theta_n(y_n) = A\alpha\Theta_{n-1}(y_{n-1}) - A^2\Theta_{n-2}(y_{n-2}).$$

Hence, letting $u_n = A^{3-n}\Theta_n(y_n)$,

$$u_n = \alpha u_{n-1} - u_{n-2}.$$

This recurrence relation for u_n is indeed that for the Chebyshev polynomials (with α the indeterminate). Attention must be given to the initial conditions which are

$$u_0 = A^{-2} + A^2, \qquad u_1 = A^2\alpha.$$

It is easy to check that the required solution is

$$u_n = A^2 S_n(\alpha) - A^{-2}S_{n-2}(\alpha);$$

this gives the required formula.

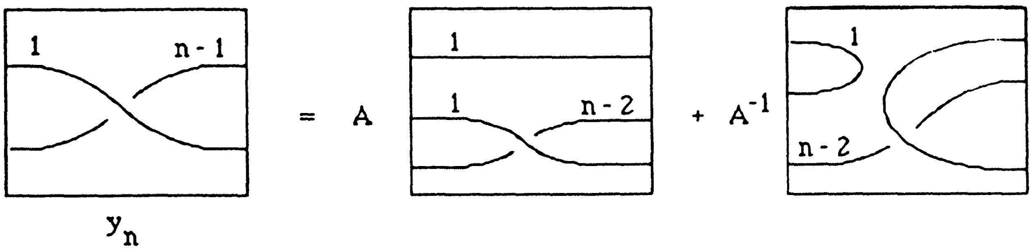


Figure 8

COROLLARY. *The Kauffman bracket polynomial of the diagram of the* $(n, kn+1)$*-torus knot that is the closure of* $y_n(z_n)^k$ *is*

$$(-1)^{nk+n}A^{n^2k+n-3}(A^2-A^{-2})^{-1}\{A^{2(nk+n+2)}-A^{2(nk-n)} \\ +A^{-2(nk+n)}-A^{-2(nk-n+2)}\}.$$

Proof. The required polynomial is $\Phi_U(\Theta_n(y_n(z_n)^k)$ and use of the proposition shows this to be

$$A^{n-3}\Phi_U(A^2((-1)^nA^{n^2+2n})^k\phi_n + A^{-2}((-1)^nA^{n^2-2n})^k\phi_{-n}),$$

and this is

$$A^{n-3+n^2k}(-1)^{n(k+1)}\{A^{2(nk+1)}(A^{2(n+1)}-A^{-2(n+1)}) \\ +A^{2(-nk-1)}(A^{2(-n+1)}-A^{-2(-n+1)})\}(A^2-A^{-2})^{-1}$$

which is the required formula.

Formulae for the Jones polynomial of torus knots are, of course, already known, so this corollary is only intended as illustration of how the proposition might be used. The result does agree with Jones' formula for his polynomial (in the indeterminate $\sqrt{t}$) of a torus knot at least up to choice of $\pm\sqrt{t}$.

Finally a very brief remark on invariants of framed links in 3-manifolds is in order. Suppose a framed link L in S^3, with components $L_1, L_2, \ldots, L_m$, is represented by a diagram D. If $n < m$ let M be the 3-manifold obtained by surgery along $L_1, L_2, \ldots, L_n$. The remaining components $L_{n+1}, L_{n+2}, \ldots, L_m$ form a framed link in M. An invariant of this link is the $(m-n)$*-multilinear form* on $\mathfrak{A}$ given by

$$(\Phi_{U(-1)}(\mathbf{a}))^{(\sigma+\nu-n)/2}\Phi_D(\mathbf{a}, \mathbf{a}, \ldots, \mathbf{a}, \ , \ , \ldots, \ )$$

where Φ_D acts on n copies of $\mathbf{a}$ and $m-n$ blanks, $\mathbf{a}$ is the element defined in Theorem 1 and σ and ν are the signature and nullity of the linking matrix of $L_1, L_2, \ldots, L_n$.

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Department of Pure Mathematics
16 Mill Lane
Cambridge, CB2 1SB

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