Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 67 (1992)

Artikel: On the minimal surfaces of Riemann.
Autor: Toubiana, Eric
DOI: https://doi.org/10.5169/seals-51110

Comment. Math. Helvetici 67 (1992) 546–570 0010–2571/92/040546–25$1.50 + 0.20/0

On the minimal surfaces of Riemann

ERIC TOUBIANA

§1. Introduction

Let γ_i, $i = 1, 2$ be plane Jordan curves in horizontal planes P_i, $i = 1, 2$, $P_1 \neq P_2$, we know that under conditions on γ_i, for example γ_1 not too far from γ_2, γ_1 and γ_2 bound a least area minimal annulus M between P_1 and P_2. Meeks and White [7] were able to prove that when the γ_i are convex there are at most two minimal annuli bounded by $\gamma_1 \cup \gamma_2$. Assuming that the γ_i are convex, Shiffman [10] proved that if M is a minimal annulus bounded by $\gamma_1 \cup \gamma_2$ then for each horizontal plane P between P_1 and P_2, the intersection $P \cap M$ is again a convex Jordan curve, furthermore if γ_1 and γ_2 are circles, then $P \cap M$ is also a circle.

In view of this last result it is natural to ask what happens when two straight lines replace the Jordan curves. Namely, let D_i, $i = 1, 2$, be straight lines in horizontal planes P_i, $i = 1, 2$, $P_1 \neq P_2$. Let us assume that D_1 makes an angle θ with D_2, $\theta \in [0, \pi]$. Now let M be a minimal annulus between P_1 and P_2 bounded by $D_1 \cup D_2$. If P is a horizontal plane between P_1 and P_2, what can we say about the intersection $P \cap M$?

For example, if $\theta = 0$ i.e. D_1 and D_2 are parallel, Riemann [9] has constructed a minimal embedded annulus S between P_1 and P_2 bounded by $D_1 \cup D_2$. Furthermore the intersection with any horizontal plane is a circle, see [3] for a detailed description of Riemann's examples.

In the same paper [3], Hoffman, Karcher and Rosenberg proved that if D_1 and D_2 are parallel, i.e. $\theta = 0$, then the only minimal properly embedded annulus between P_1 and P_2, bounded by $D_1 \cup D_2$ is precisely Riemann's example.

Here we shall prove that the case $\theta \neq 0$ does not occur, namely we show the following.

THEOREM 1. *Let D_i, $i = 1, 2$ be straight lines in horizontal planes P_i, $i = 1, 2$, $P_1 \neq P_2$, let $\theta \in [0, \pi]$ be the angle that D_1 makes with D_2. Let us assume there exists a minimal properly embedded annulus M between P_1 and P_2 bounded by $D_1 \cup D_2$.*

Then necessarily $\theta = 0$ and M is Riemann's example S.

Let us assume now $\theta = 0$, i.e. D_1 *and* D_2 *parallel. We can generalize Riemann's examples to yield a family of minimal surfaces* S_k *with the following properties*:

(1) $S_0 = S$ *where* S *is Riemann's example.*

(2) *For every integer* $k, k \geq 0$, S_k *is a minimal immersed annulus between* P_1 *and* P_2, *bounded by* $D_1 \cup D_2$ *such that after reflection about the lines* $D_1, D_2, \ldots$ *we get a complete minimal surface in* R^3, *which we call again* S_k. *Furthermore* S_k *is invariant under the translation* $X \to X + 2u$ *where* u *is the vector orthogonal to* D_1 *translating* D_1 *to* D_2. *Also each end of* S_k *in* R^3 *is a flat horizontal end,* i.e. *an end asymptotic to a horizontal plane, and the projection of every end over any horizontal plane is a* $(4k + 1)$ *to* 1 *map.*

The following gives the Weierstrass representation of this family.

THEOREM 2. *Let* T *be a rectangular torus,* i.e. $T^2 = C/\Gamma$ *where* $\Gamma = \{2\omega_1 p + 2\omega_3 q, p, q \in \mathbf{Z}, \omega_1 \in \mathbf{R}_*^+, \omega_3 \in i\mathbf{R}_*^+\}$.

Then for every $k, k \in \mathbf{N}$ *the following data* (g_k, η_k) *is the Weierstrass representation of the surface* S_k *described above.*

$$g_k = \lambda_k [P(z) - P(\omega_2)]^{2k+1},$$

$$\eta_k = \frac{dz}{g_k(z)} = \frac{dz}{\lambda_k [P(z) - P(\omega_2)]^{2k+1}},$$

where $\omega_2 = \omega_1 + \omega_3$, $z \in T^2 - \{0, \omega_2\}$, $\lambda_k = i\sqrt{(-2/P''(\omega_2))^{2k+1}}$ *and* P *is the* P-*function of Weierstrass.*

We prove Theorems 1 and 2 in §2 and §3 respectively.

§2. Proof of Theorem 1

We are going to use the Weierstrass representation for a minimal surface, see Lawson [5] p. 113, and the reflection principle, namely if a minimal surface in $\mathbf{R}^3$ has a piece γ of a straight line in its boundary then we can extend minimally this surface along γ by the reflection about the line defined by γ, see [5] p. 82.

Let us take the notation of Theorem 1. By the reflection principle we can extend M to a complete properly embedded minimal surface in $\mathbf{R}^3$, let us call M again this surface. As the composition of the two reflections about D_1 and D_2 is a screw-motion $S_{2\theta}$, i.e. the 2θ-rotation with respect to the x_3-axis composed with a vertical translation, we get that M is globally invariant by the screw-motion $S_{2\theta}$:

$$S_{2\theta}(M) = M.$$

Then we get a quotient surface $M/_{S_{2\theta}}$ in $\mathbf{R}^3/_{S_{2\theta}}$. By construction this surface is topologically a two-punctured torus properly and minimally embedded in $\mathbf{R}^3/_{S_{2\theta}}$. By a theorem of Meeks and Rosenberg [6] we deduce that $M/_{S_{2\theta}}$ has finite total curvature. From this we get that each end of M is conformally a punctured disc, see [5] p. 130. At last we deduce that the ends of M are parallel, flat, embedded and also the Gauss map g of M has order two at each end (this last claim comes from the fact that if E_1 is the end of M passing through the straight line D_1 then, by construction, $E_1 - P_1$ has two connected components). Subsequently we assume that D_1 is the x_1-axis in the plane $P_1 = \{x_3 = 0\}$.

LEMMA 1. *The extended complete minimal surface M in $\mathbf{R}^3$ is parameterized by $C = C - (\Gamma \cup (z_0 + \Gamma))$ where*:

$$\begin{cases} \Gamma = \{2\omega_1 \cdot p + 2\pi i \cdot q, p, q \in \mathbf{Z}, \omega_1 \in \mathbf{R}_*^+\}, \\ z_0 \in \omega_1 + y\pi i, \quad 0 \le y \le 1. \end{cases}$$

Furthermore if X is the minimal immersion of C onto $M \subset \mathbf{R}^3$, X must satisfy:

- *X sends the vertical lines $\{\mathrm{Re}\,(z) = c, c \neq k\omega_1, k \in \mathbf{Z}\}$ to horizontal Jordan curves in $\mathbf{R}^3$.*
- *X sends the vertical lines $\{\mathrm{Re}\,(z) = k\omega_1, k \in \mathbf{Z}\}$ to horizontal straight lines in $\mathbf{R}^3$ and particularly X sends $\{\mathrm{Re}\,(z) = 0\}$ to the x_1-axis D_1 and $\{\mathrm{Re}\,(z) = \omega_1\}$ to D_2.*

At last we have:

$$\forall z \in C, \quad X(z + 2\pi i) = X(z), \tag{1}$$

$$X(2\omega_1 - \bar{z}) = S_{D_2}[X(z)], \tag{2}$$

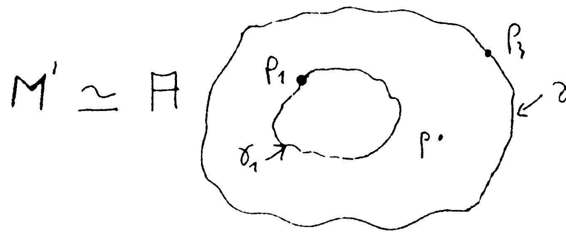
$$X(z + 2\omega_1) = S_{2\theta}[X(z)], \tag{3}$$

where S_{D_2} is the reflection about D_2 and $S_{2\theta} = S_{D_1} \circ S_{D_2}$.

Proof of Lemma 1. Let M_1 be the piece of M bounded by $D_1 \cup D_2$ and let M_2 be the reflection of M_1 about D_2, i.e. $M_2 = S_{D2}(M_1)$. Let D_3 be $S_{D2}(D_1)$, so that M_2 is bounded by D_2 and D_3.

By construction $M' = M_1 \cup M_2$ is homeomorphic to a one-punctured planar annulus A bounded by two Jordan curves γ_1, γ_3 each one with a point p_i, $i = 1, 3$ removed.

Let Z be the minimal immersion of A onto M', let A have the conformal structure induced by Z. As each end of M is conformally a one-punctured disc we deduce that A is conformally a one-punctured annulus bounded by two Jordan curves γ_1, γ_3 each one with a point p_i, $i = 1, 3$ removed.



Assume that $Z(\gamma_i) = D_i$, $i = 1, 3$. Z sends a neighbourhood of p to the end passing through the line D_2. Assume that D_2 belongs to the plane $P_2 = \{x_3 = c, c > 0\}$. Then D_3 belongs to the horizontal plane $P_3 = \{x_3 = 2c\}$ (recall that $P_1 = \{x_3 = 0\}$).

Let Z_3 be the third coordinate function of Z, Z_3 is a harmonic function. By construction of M, the line D_2 is the only part of M in the plane P_2, that is:

$$\mathrm{M} \cap \mathrm{P}_2 = \mathrm{D}_2,$$

as $Z_3(\gamma_{1}) = 0$ and $Z_3(\gamma_3) = 2c$, we deduced that $\{Z_3^{-1}(c) \cup p\}$ is an embedded closed curve in the interior of $\bar{A}$, and by the maximum principle applied to the function Z_3, this curve must be connected, so that $\{Z_3^{-1}(c) \cup p\}$ is a Jordan curve γ_2 in the interior of A.

Now let α be a real number with $\alpha \neq c$ and $0 \leq \alpha \leq 2c$, again by the maximum principle we get that $Z_3^{-1}(\alpha)$ is a Jordan curve γ of $\bar{A}$. Hence we have a foliation $(\gamma_i)_{1 \leq i \leq 3}$ of $\bar{A}$ with Jordan curves γ_i, so that Z_3 is constant over each curve γ_i, i.e.:

$$\forall i \in [1, 3], \quad Z_3(\gamma_i) = c_i \qquad \text{with: } 0 \leq c_i \leq 2c \quad \text{and} \quad c_1 = 0;\ c_2 = c;\ c_3 = 2c.$$

Let Z_3^* be a harmonic conjugate of Z_3 over $\bar{A}$, we have locally:

$$Z_3^*(z) = \int_\alpha *dZ_3,$$

where α is a path between a base point z_0 and z, and $*dZ_3$ is defined by:

$$*dZ_3 = \frac{\partial Z_3}{\partial x}\,dy - \frac{\partial Z_3}{\partial y}\,dx, \qquad z = x + iy.$$

It can happen that Z_3^* is not globally defined on $\bar{A}$, i.e. if γ is any Jordan curve in $\bar{A}$ generating $\Pi_1(\bar{A})$, let a be the real number defined by:

$$a = \int_\gamma *dZ_3.$$

Then Z_3^* is well defined on $\bar{A}$ if and only if $a = 0$. In case $a \neq 0$ let us consider the function:

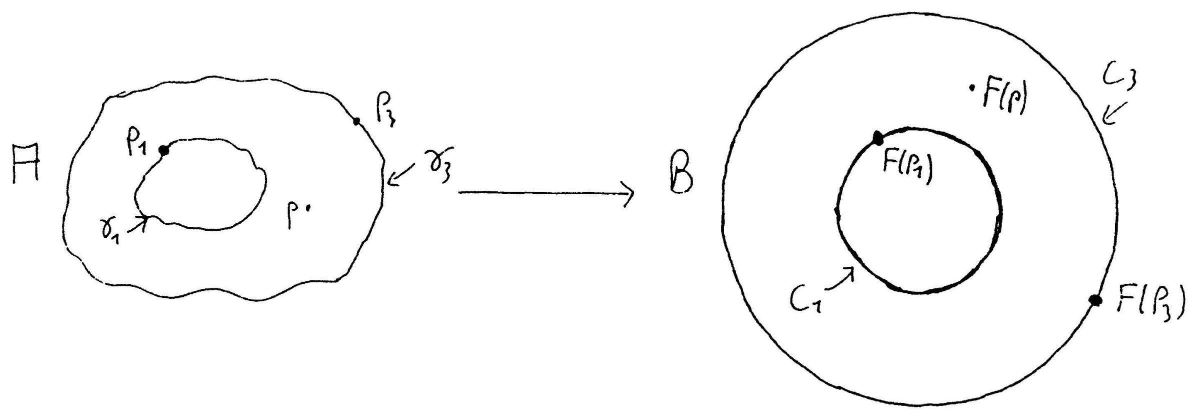
$$F(z) = \exp\left[\frac{2\pi}{a}(Z_3(z) + iZ_3^*(z))\right].$$

In case $a = 0$ we put:

$$F(z) = \exp\,[Z_3(z) + iZ_3^*(z)],$$

in any case F is a well defined map of $\bar{A}$ into C, F maps the Jordan curves $\{\gamma_i\}_{1 \le i \le 3}$ to concentric circles of C: $\{|u| = \exp\,[(2\Pi/a)c_i]$ or $\exp\,[c_i]$ depending on the expression of $F\}$. F maps $\bar{A}$ on an annulus B of C bounded by two concentric circles C_1, C_3 with

$$F(\gamma_i) = C_i, \qquad 1 \le i \le 3.$$



It is easy to see that F is a n-covering map of $\bar{A}$ onto B in $\mathbf{C}$, so after composing F with the nth root map, we can assume that F is an embedding of $\bar{A}$ onto B.

Let us call Y the composed map $Z \circ F^{-1}$,

$$B \xrightarrow{Y = Z \circ F^{-1}} \mathbf{R}^3$$

we clearly have $Y(C_i) = D_i$, $i = 1, 2, 3$ so by the reflection principle if I_C denotes the reflection in $\mathbf{C}$ about a circle C we have:

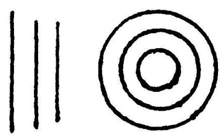
$$C_3 = I_{C_2}(C_1) \quad (\text{because } Y(C_3) = D_3 = S_{D_2}(D_1))$$

and then $F(P_3) = I_{C_2}(F(P_1))$.

Now by reflection principle we extend Y to a minimal immersion of $\mathbf{C} - E$ onto M in $\mathbf{R}^3$, where E is the infinite discrete set of $\mathbf{C}$ obtained by taking $F(P_1)$, $F(P)$ and all the successive images of those points by the reflections about the circles $C_1, C_2, I_{C_1}(C_2), I_{C_2}(C_1)\ldots$

Then the exponential map $z \to u = \exp(z)$ of $\mathbf{C}$ sends the vertical lines $\{\mathrm{Re}(z) = a\}$ of the z-complex plane on the concentric circles $\{|u| = e^a\}$ of the u-complex plane.

$$(\mathbf{C}, z) \xrightarrow{e^z} (\mathbf{C}, u)$$



Up to a homothety of the u-plane we can assume that C_1 is the circle of radius one. So that if we call X the composed map: $X = Y \circ \exp$, X sends the imaginary axis $\{\mathrm{Re}(z) = 0\}$ to D_1 in $\mathbf{R}^3$. Furthermore up to a rotation in the u-plane, we can assume that $F(p_1) = 1$, so that $\exp(0) = F(p_1) = 1$. Let denote z_0 the inverse image, by the exponential map, of $F(p)$ with $0 \leq \mathrm{Im}\, z_0 \leq 2\pi$, and ω_1 the real part of z_0.

$$\omega_1 = Re\,(z_0).$$

Then X sends the line $\{\mathrm{Re}(z) = \omega_1\}$ to the horizontal line D_2 in $\mathbf{R}^3$.

It is clear that the inverse image of E by the exponential map is just $\Gamma \cup (z_0 + \Gamma)$, with

$$\Gamma = \{2\omega_1 p + 2\pi i q, p, q \in \mathbf{Z}\}.$$

Furthermore by construction of Y and by the geometric properties of M in $\mathbf{R}^3$ we must have:

$$\forall z \in C = \mathbf{C} - (\Gamma \cup (z_0 + \Gamma)), \quad X(z + 2\pi i) = X(z), \tag{1}$$

$$X(2\omega_1 - \bar{z}) = S_{D_2}[X(z)], \tag{2}$$

$$X(4\omega_1 - \bar{z}) = S_{D_3}[X(z)]. \tag{2'}$$

(1) comes from the fact that $X = Y \circ \exp$, and (2) (resp. (2)′) comes from the fact that M is invariant by the reflection about D_2 (resp. D_3).

So that combining (2) and (2)′ we have:

$$\forall z \in C, \quad X(z + 2\omega_1) = S_{2\theta}[X(z)]. \tag{3}$$

Finally considering the map $z \to -z$ of the z-plane keeping fixed the point 0, and because of (1) we can assume that:

$$0 \leq \operatorname{Re}(z_0) \quad \text{and} \quad \operatorname{Im}(z_0) = y\pi, \qquad 0 \leq y \leq 1.$$

Remarks

(1) From Lemma 1 we see that X induces a minimal embedding of $C/_{2\pi i \mathbf{Z}}$ onto M in $\mathbf{R}^3$ and a minimal embedding of $T^2 - \{0, z_0\} = C/_{\Gamma \cup (z_0 + \Gamma)}$ onto $M_{/S_{2\theta}}$ in $\mathbf{R}^3_{/S_{2\theta}}$.

(2) Until now we don't know if ω_1 is bigger or not than π, so from now we assume π bigger than ω_1, the rest of the proof will not be affected.

Let $\omega_3 = i\pi$, so that the lattice Γ is:

$$\Gamma = \{2\omega_1 p + 2\pi i q, p, q \in \mathbf{Z}\}.$$

LEMMA 2. *With the hypothesis of Theorem* 1 *we have* $\theta = 0$ *and the Weierstrass representation of the minimal immersion* X *of* C *in* $\mathbf{R}^3$ *in Lemma* 1 *is*:

$$\forall z \in C, \quad g(z) = \lambda[P(z) - P(\omega_2)],$$

$$\eta = \frac{dz}{g(z)} = \frac{dz}{\lambda[P(z) - P(\omega_2)]},$$

where P *is the* P*-function of Weierstrass associated to the torus* $T^2 = \mathbf{C}/\Gamma$, $\omega_2 = \omega_1 + \omega_3$ *and* $\lambda = i\sqrt{-2/P''(\omega_2)} \in i\mathbf{R}$.

Proof of Lemma 2. We are going to use some basic facts about the σ and ζ functions associated with the P-function of Weierstrass (we recall these facts in the Appendix).

Let us call (Φ_1, Φ_2, Φ_3) the coordinate forms of X, i.e.

$$X(z) = (X_1(z), X_2(z), X_3(z)) = \operatorname{Re} \int_{z_0}^{z} (\Phi_1, \Phi_2, \Phi_3)$$

with z_0 a base point in C and:

$$\Phi_1 = \frac{1}{2}\eta(1-g^2); \qquad \Phi_2 = \frac{i}{2}\eta(1+g^2); \qquad \Phi_3 = \eta g.$$

As the third coordinate function X_3 is constant on vertical lines $\{\mathrm{Re}(z) = c\}$ we can assume $\Phi_3 = 1$ and then $\eta = dz/g(z)$.

As $M_{/S_{2\theta}}$ has two flat horizontal ends parameterized by 0 and z_0 (with $\mathrm{Re}(z_0) = \omega_1$, see Lemma 1), we can assume that

$$g(0) = \infty \quad \text{and} \quad g(z_0) = 0,$$

also g is a 2 to 1 map near each end.

Furthermore on a fundamental domain of T^2 in $\mathbf{C}$ there is no other point where g is vertical, if not, up to a rotation in $\mathbf{R}^3$, another such point would be an n-pole of g, $n \geq 1$, but this point is a regular point of M, so it must be a $2n$-zero of η and then it is an n-pole of $\Phi_3 = g\eta$, but $\Phi_3 = 1$ as we saw before.

As g is the Gauss map of M, up to the stereographic projection of S^2 to $\mathbf{C} \cup \{\infty\}$, by the geometric properties of M in $\mathbf{R}^3$, g must satisfy:

$$\forall z \in \mathbf{C} \quad \begin{cases} g(z+2\omega_3) = g(z), & (1)\\ g(2\omega_1 - \bar{z}) = -e^{2i\theta}\overline{g(z)}, & (2)\\ g(z+2\omega_1) = e^{2i\theta}g(z). & (3) \end{cases}$$

Those properties (1), (2), (3) of g come from respectively the properties (1), (2), (3) of X established in Lemma 1.

From (3) we deduce that the map $(g'/g)(z)$ is well defined on the torus $T^2 = \mathbf{C}/_\Gamma$. Furthermore $(g'/g)(z)$ is an elliptic function on T^2 and has a single pole at a point z in T^2 if and only if z is a pole or a zero of g, and $(g'/g)(z)$ has no other pole on T^2. From what we saw before we see that $(g'/g)(z)$ has two single poles on T^2, so it is an elliptic function of degree two on T^2 with a single pole at 0 and z_0. The most general form of such a function is:

$$\frac{g'}{g}(z) = \frac{a_1 P(z) + a_0 + bP'(z)}{P(z) - P(z_0)}, \qquad a_1, a_0, b \in \mathbf{C}$$

with $a_1 P(z_0) + a_0 = bP'(z_0)$ because $-z_0$ is not a pole of $(g'/g)(z)$.

From (2) we get that

$$\forall z \in \mathbf{C}, \quad 2\omega_1 - \bar{z} = z \Rightarrow \frac{g'}{g}(z) \in i\mathbf{R}.$$

So $(g'/g)(z)$ must have purely imaginary values on the line $\{\mathrm{Re}(z) = \omega_1\}$ in $\mathbf{C}$. But as $\mathrm{Re}(z_0) = \omega_1$, $P(z_0) \in \mathbf{R}$ and then $P(z) - P(z_0)$ has real values on the line $\{\mathrm{Re}(z) = \omega_1\}$ also $P'(z)$ has purely imaginary values on this line, so we have:

$$\forall y \in \mathbf{R}, \quad a_1 P(\omega_1 + y\omega_3) + a_0 + bP'(\omega_1 + y\omega_3) \in i\mathbf{R}.$$

For $y = 0$ and $y = 1$ we deduce (as $P'(\omega_1) = 0 = P'(\omega_1 + \omega_3)$)

$$a_1 P(\omega_1) + a_0 \in i\mathbf{R},$$

$$a_1 P(\omega_2) + a_0 \in i\mathbf{R},$$

as $P(\omega_1)$ and $P(\omega_2)$ are two distinct real numbers, we get that a_1 and a_0 are purely imaginary numbers, $a_1, a_0 \in i\mathbf{R}$, so that:

$$\forall y \in \mathbf{R}, \quad bP'(\omega_1 + y\omega_3) \in i\mathbf{R}$$

and then b is a real number:

$$a_1, a_0 \in i\mathbf{R}, \quad b \in \mathbf{R}.$$

Now let us look for g, we have:

$$\forall z \in \mathbf{C}, \quad \frac{g'}{g}(z) = a_1 + \frac{a_1 P(z_0) + a_0}{P(z) - P(z_0)} + b\frac{P'(z)}{P(z) - P(z_0)}.$$

We want to show that $z_0 = \omega_2$. Let us suppose now that $z_0 \neq \omega_i$, $i = 1, 2$, that is:

$$z_0 = \omega_1 + y\omega_3, \qquad 0 < y < 1.$$

As $P'(z_0) \neq 0$ we have

$$\frac{1}{P(z) - P(z_0)} = \frac{1}{P'(z_0)}[\zeta(z - z_0) - \zeta(z + z_0) + 2\zeta(z_0)]$$

as $a_1 P(z_0) + a_0 = bP'(z_0)$ we get:

$$\frac{g'}{g}(z) = a_1 + 2\zeta(z_0) + b\left[\zeta(z - z_0) - \zeta(z + z_0) + \frac{P'(z)}{P(z) - P(z_0)}\right]$$

and so

$$\forall z \in C, \quad g(z) = \lambda\, e^{[a_1 + 2b\zeta(z_0)]z} \cdot \left[\frac{\sigma(z - z_0)}{\sigma(z + z_0)}(P(z) - P(z_0))\right]^b$$

with $\lambda \in \mathbf{C}$, $a_1 \in i\mathbf{R}$, $b \in \mathbf{R}$.

But g must have a double pole at 0 and a double zero at z_0, now we remark that the function in brackets $(\sigma(z - z_0)/\sigma(z + z_0))(P(z) - P(z_0))$, has the same zeros and poles, so we must have $b = 1$, so:

$$\forall z \in \mathbf{C}, \quad g(z) = \lambda\, e^{Az}\frac{\sigma(z - z_0)}{\sigma(z + z_0)}(P(z) - P(z_0))$$

where, $\lambda \in \mathbf{C}$, $A = a_1 + 2\zeta(z_0)$.

From the property (#) of the σ-function, see the Appendix, we have:

$$\forall z \in \mathbf{C}, \quad g(z + 2\omega_i) = e^{2\omega_i A - 4\eta_i z_0} g(z), \qquad i = 1, 2, 3,$$

from conditions (1) and (3) on g we get:

$$\begin{cases} 2\omega_1 A - 4\eta_1 z_0 = 2i\theta + 2p\pi i, & p \in \mathbf{Z}, \\ 2\omega_3 A - 4\eta_3 z_0 = 2q\pi i, & q \in \mathbf{Z} \end{cases}$$

and then

$$z_0 = z_0(\theta) = -\frac{2\theta}{2\pi}\omega_3 - p\omega_3 + q\omega_1.$$

By considering the real part and the imaginary part of z_0, as we know that $\operatorname{Re}(z_0) = \omega_1$, and $0 < \operatorname{Im}(z_0) < -i\omega_3$ we have $q = 1$ and $p = -1$, so:

$$z_0(\theta) = \omega_1 + \left(1 - \frac{\theta}{\pi}\omega_3\right), \qquad \theta \in [0, \pi]$$

and

$$2\omega_1 A = 4\eta_1 z_0(\theta) + 2i\theta - 2\pi i. \tag{$*$}$$

On the other hand as the map X is well defined on C we must have:

$$\forall \gamma \in \Pi_1(C), \quad \operatorname{Re} \int_\gamma (\Phi_1, \Phi_2, \Phi_3) = 0.$$

In particular if γ is a small circle around 0, we must have:

$$\operatorname{Re} \int_\gamma (\Phi_1, \Phi_2) = 0,$$

that is

$$\int_\gamma \eta = \overline{\int_\gamma \eta g^2}$$

and as $\eta = dz/g(z)$ then:

$$-2\pi \overline{\operatorname{Re} s(g, 0)} = 2\pi \operatorname{Re} s\left(\frac{1}{g}, 0\right) = 0 \quad \left(\text{because 0 is a zero of } \frac{1}{g}\right)$$

and so we must have

$$\operatorname{Re} s(g, 0) = 0.$$

A computation shows that:

$$\operatorname{Re} s(g, 0) = -\lambda(A - 2\zeta(z_0)).$$

So we get that $A = 2\zeta(z_0)$. In view of $(*)$ we have

$$2\omega_1 \zeta(z_0(\theta)) = 2\eta_1 z_0(\theta) + i\theta - i\pi. \tag{T}$$

Let $h(\theta)$ be the function on $[0, \pi]$ defined by:

$$\forall \theta \in [0, \pi], \quad h(\theta) = \frac{1}{i}[2\omega_1 \zeta(z_0(\theta)) - 2\eta_1 z_0(\theta) - i\theta + i\pi].$$

As T^2 is a rectangular torus we have:

$$\forall y \in \mathbf{R}, \quad \mathrm{Re}\,(\zeta(\omega_1 + y\omega_3)) = \eta_1 = \zeta(\omega_1),$$

and then:

$$\forall \theta \in [0, \pi], \quad \mathrm{Re}\,[\zeta(z_0(\theta))] = \eta_1;$$

we deduce that $h(\theta)$ is a real-valued function.

Now let us look for solutions θ of (T), i.e. zeros of the h-function, we have:

$$\begin{aligned} h(0) &= \frac{1}{i}[2\omega_1\zeta(\omega_2) - 2\eta_1\omega_2 + \pi i] \\ &= \frac{1}{i}[2\omega_1\eta_2 - 2\eta_1\omega_2 + \pi i] \\ &= 0 \quad \text{by Legendre relation, (see the Appendix)} \end{aligned}$$

and

$$h(\pi) = \frac{1}{i}[2\omega_1\zeta(\omega_1) - 2\eta_1\omega_1] = \frac{1}{i}[2\omega_1\eta_1 - 2\eta_1\omega_1] = 0.$$

So 0 and π are solutions of (T). Let us show there is no other solution in $[0, \pi]$. Suppose there is another solution θ in $]0, \pi[$, then by Role's theorem the function $h'(\theta)$ would have at least two distinct zeros in $]0, \pi[$, but we have

$$h'(\theta) = \frac{1}{i}\left[\frac{2\omega_1\omega_3}{\pi} P(z_0(\theta)) + 2\frac{\eta_1\omega_3}{\pi}\right].$$

$z_0(\theta)$ is a strictly monotone function of θ with values in $L = \{\omega_1 + y\omega_3, 0 \le y \le 1\}$. Furthermore P is a strictly monotone and real-valued function on L. We deduce that $h'(\theta)$ is a strictly monotone function on $[0, \pi]$ and then $h'(\theta)$ cannot have two zeros on $]0, \pi[$.

So the only solutions of (T) are $\theta = 0$ and $\theta = \pi$, but we then get $z_0 = \omega_2$ or $z_0 = \omega_1$. Let us show that $z_0 = \omega_1$ does not work. For that let us assume that $z_0 = \omega_1$. Then $(g'/g)(z)$ must have a single pole at 0 and ω_1, so:

$$\forall z \in \mathbf{C}, \quad \frac{g'}{g}(z) = \frac{a_1 P(z) + a_0 + bP'(z)}{P(z) - P(\omega_1)}, \qquad a_1, a_0, b \in \mathbf{C},$$

with: $a_1 P(\omega_1) + a_0 = 0$, we deduce that:

$$\frac{g'}{g}(z) = a_1 + b\frac{P'(z)}{P(z) - P(\omega_1)}.$$

As before we must have $a_1 \in i\mathbf{R}$ and $b \in \mathbf{R}$, furthermore:

$$\forall z \in \mathbf{C}, \quad g(z) = \lambda\, e^{a_1 z}[P(z) - P(\omega_1)]^b.$$

As g must have a double pole at 0 and a double zero at ω_1 we deduce $b = 1$, and

$$\forall z \in \mathbf{C}, \quad g(z) = \lambda\, e^{a_1 z}[P(z) - P(\omega_1)], \qquad \lambda \in \mathbf{C}, \quad a_1 \in i\mathbf{R}$$

but as we must have $g(z + 2\omega_3) = g(z)$, we deduce that $a_1 \cdot 2\omega_3 = 2\pi i q$, $q \in \mathbf{Z}$ and so $a_1 = 0$ because $a_1 \in i\mathbf{R}$, at last we have:

$$\begin{cases} g(z) = \lambda(P(z) - P(\omega_1)), & \lambda \in \mathbf{C}, \\ \eta = \dfrac{dz}{g(z)}. \end{cases}$$

As we have (property (1) of Lemma 1 with $\omega_3 = i\pi$)

$$\forall z \in C, \quad X(z + 2\omega_3) = X(z)$$

we deduce that $\operatorname{Re}\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} (\Phi_1, \Phi_2) = 0$, and then

$$\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} \eta = \overline{\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} \eta g^2}.$$

So

$$\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} \frac{dz}{P(z) - P(\omega_1)} = \lambda\bar{\lambda}\, \overline{\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} (P(z) - P(\omega_1))\, dz} \tag{4}$$

as we have:

$$\frac{1}{P(z) - P(\omega_1)} = \frac{2}{P''(\omega_1)}(P(z - \omega_1) - P(\omega_1)), \quad \text{and} \quad \zeta' = -P$$

we conclude:

$$\Leftrightarrow -\frac{4}{P''(\omega_1)}(\eta_3+\omega_3 P(\omega_1)) = -2\lambda\bar{\lambda}\overline{(\eta_3+\omega_3 P(\omega_1))} \tag{4}$$

and then: (as $\eta_3, \omega_3 \in i\mathbf{R}$ and $P(\omega_1) \in \mathbf{R}$)

$$\lambda\bar{\lambda} = -\frac{2}{P''(\omega_1)} \tag{5}$$

but as the P-function satisfies:

$$P'^2(z) = 4(P(z)-e_1)(P(z)-e_2)(P(z)-e_3), \qquad e_i = P(\omega_i), \quad i = 1, 2, 3$$

with e_1, e_2, e_3 real numbers, $e_1+e_2+e_3=0$, $e_3<e_2<e_1$, we deduce:

$$\begin{aligned} P''(\omega_1) &= 2(3e_1^2+e_1e_2+e_1e_3+e_2e_3) \\ &= 4\left(e_1+\frac{e_3}{2}\right)(e_1-e_3) = 2(e_1-e_2)(e_1-e_3) \end{aligned}$$

and then $P''(\omega_1)$ is a positive number (as $e_1>e_2$ and $e_1>e_3$) so (5) cannot be satisfied, and $z_0 \neq \omega_1$.

The only case remaining is $z_0 = \omega_2 = \omega_1 + \omega_3$, as before we have:

$$\begin{cases} g(z) = \lambda(P(z)-P(\omega_2)), \\ \eta = \dfrac{dz}{g(z)} \end{cases}$$

and the condition (5) becomes

$$\lambda\bar{\lambda} = \frac{-2}{P''(\omega_2)}$$

but now we have

$$\begin{aligned} P''(\omega_2) &= 2(3e_2^2+e_1e_2+e_1e_3+e_2e_3) \\ &= 2(e_1+2e_3)(2e_1+e_3) = 2(e_3-e_2)(e_1-e_2) \end{aligned}$$

and $P''(\omega_2)$ is a negative number because $e_1>e_2$ and $e_3<e_2$.

At last as we assume that X sends the line $\{\operatorname{Re}(z) = 0\}$ to the x_1-axis of $\mathbf{R}^3$, g must have purely imaginary values on the line $\{\operatorname{Re}(z) = 0\}$, so λ must be in $i\mathbf{R}$ and we conclude:

$$\lambda = \pm i\sqrt{\frac{-2}{P''(\omega_2)}},$$

and without loss of generality we can assume

$$\lambda = i\sqrt{\frac{-2}{P''(\omega_2)}} \in i\mathbf{R}.$$

This concludes the proof of Lemma 2.

Remark. A computation shows that the minimal surface given by the data (g, η) in Lemma 2 is invariant by the u-translation where u is given by:

$$\begin{aligned} u &= \operatorname{Re}\int_{\omega_3/2}^{\omega_3/2+2\omega_1} (\Phi_1, \Phi_2, \Phi_3) \\ &= \left(0, \sqrt{\frac{-2}{P''(\omega_2)}}(2\eta_1 + e_2), 2\omega_1\right). \end{aligned}$$

LEMMA 3. *The data (g, η) given in Lemma 2 is the Weierstrass representation of Riemann's example.*

Proof. Riemann [9] constructed for every rectangular torus an embedded minimal annulus bounded by two parallel horizontal straight lines such that every intersection with any horizontal plane is a circle. Inversely we just saw in Lemmas 1 and 2 that if such a surface exists then its Weierstrass representation is necessarily the one given in Lemma 2; this concludes the proof of lemma 3 and the proof of Theorem 1.

Remarks

(1) In [3], Hoffman, Karcher and Rosenberg gave an explicit proof that each horizontal intersection of the surface defined by the data (g, η) of Lemma 2 is a circle.

(2) Darboux [1] gave explicitly the equations of Riemann's examples in terms of a parameter k, $0 \leq k \leq 1$ associated to a rectangular torus T^2 (that is for every rectangular torus we can associate a real number k, $0 \leq k \leq 1$ and conversely for

each such number there is a corresponding rectangular torus) and the Jacobi function associated to T^2. For more details about these concepts see Gerretsen and Sansone [2] p. 286.

(3) In [11] Appendix C we gave the following Weierstrass representation for Riemann's examples using the equations given by Darboux, see Remark 2 above). Following the notations of [11], we set $2K' = \omega_1$, $2iK = \omega_3$, $2K' + 2iK = \omega_2$ and we use the variable z instead of $(z - K')$ in [11]. Let Γ denote the lattice generated by $(2\omega_1, 2\omega_3) = (4K', 4iK)$ and let P stand for the Weierstrass P-function on the torus $\mathbf{C}/\Gamma$. Set

$$\begin{cases} g_1(z) = \dfrac{kk'}{4} \dfrac{1}{P(z) - P(\omega_2)}, \\ \eta_1 = \dfrac{2}{k}(P(z) - P(\omega_2))\, dz = \dfrac{k'}{2} \dfrac{dz}{g(z)}. \end{cases}$$

We want to show that the data (g_1, η_1) defines the same surface as the data given in Lemma 2, up to a rigid motion of $\mathbf{R}^3$. This will give a new proof of Lemma 3.

To see this recall that (see again [11] p. 60):

$$\begin{cases} k^2 + k'^2 = 1; \quad h^2 = \dfrac{e_2 - e_3}{e_1 - e_3}; \quad k, k' \in [0, 1], \\ e_1 - e_3 = \dfrac{1}{4} \quad \left(\text{since } K' = h\omega_1 \text{ with } h = \dfrac{1}{2}\right). \end{cases} \tag{$*$}$$

We have (since $1/(P(z) - P(\omega_2)) = (2/P''(\omega_2))(P(z - \omega_2) - P(\omega_2))$:

$$g_1(z) = \frac{kk'}{2P''(\omega_2)}(P(z - \omega_2) - P(\omega_2)),$$

but again we can make a change of variable and put z instead of $(z - \omega_2)$, so:

$$g_1(z) = \frac{kk'}{2P''(\omega_2)}(P(z) - P(\omega_2)).$$

Using $(*)$ and since $P''(\omega_2) = 2(e_3 - e_2)(e_1 - e_2)$ a simple computation shows that:

$$\frac{kk'}{2P''(\omega_2)} = -\sqrt{\frac{-2}{P''(\omega_2)}}.$$

So we have:

$$\begin{cases} g_1(z) = -\sqrt{\dfrac{-2}{P''(\omega_2)}}(P(z) - P(\omega_2)), \\ \eta_1 = \dfrac{k'}{2}\dfrac{dz}{g_1(z)}. \end{cases}$$

Finally the $3\pi/2$ rotation about the x_3-axis in $\mathbf{R}^3$ applied to the surface defined by (g_1, η_1) gives the following data for the rotated surface:

$$\begin{cases} g_2(z) = i\sqrt{\dfrac{-2}{P''(\omega_2)}}(P(z) - P(\omega_2)), \\ \eta_2 = \dfrac{k'}{2}\dfrac{dz}{g_2(z)}, \end{cases}$$

that is:

$$\begin{cases} g_2(z) = g(z), \\ \eta_2 = \dfrac{k'}{2}\eta, \end{cases}$$

where (g, η) is the data given in Lemma 2. Then (g_2, η_2) defines the same surface as (g, η) up to the $k'/2$ homothety in $\mathbf{R}^3$, and so the date (g, η) given in Lemma 2 defines Riemann's examples.

(4) In fact in Lemmas 1 and 2 we just need $M/_{S_{20}}$ to be an immersed two-punctured torus in $\mathbf{R}^3/_{S_{20}}$ with finite total curvature and two embedded planar ends. That is we can remove the hypothesis "M is properly embedded" by "$M/_{S_{20}}$ has finite total curvature and embedded planar ends".

§3. Proof of Theorem 2

To show that the data (g_k, η_k) defines an immersed mineral surface in $\mathbf{R}^3/_z$ we just need to verify the period conditions, that is if γ and μ are the paths on T^2 defined by:

$$\gamma(t) = \frac{\omega_1}{2} + 2t\omega_3, \qquad t \in [0, 1],$$

$$\mu(t) = 2t\omega_1 + \frac{\omega_3}{2}, \qquad t \in [0, 1],$$

it is enough to show that:

$$\mathrm{Re}\int_\gamma (\Phi_1, \Phi_2, \Phi_3) = (0, 0, 0), \tag{1}$$

where:

$$\Phi_1 = \frac{\eta_k}{2}(1 - g_k^2), \qquad \Phi_2 = i\frac{\eta_k}{2}(1 + g_k^2), \qquad \Phi_3 = \eta_k g_k.$$

This claim holds since the forms $\Phi_i, i = 1, 2, 3$ have no residue (because the P-function of Weierstrass is an even function). Let us remark that if the conditions (1) are satisfied then the forms (Φ_1, Φ_2, Φ_3) must have periods on the path μ, that is:

$$\mathrm{Re}\int_\mu (\Phi_1, \Phi_2, \Phi_3) \neq (0, 0, 0),$$

otherwise the data (g_k, η_k) defines an immersed minimal surface with finite total curvature and two parallel flat ends in $\mathbf{R}^3$, but the "Half space theorem" of Hoffman and Meeks [4] shows this situation is impossible.

Let us assume for a while that (1) is satisfied. Let I be the map on T^2 defined by:

$$\forall z \in T^2, \quad I(z) = -\bar{z}.$$

Let us remark that, as $P''(\omega_2)$ is a negative real number (see the proof of Lemma 2 in §2), $\lambda_k \in i\mathbf{R}$. We deduce that:

$$\begin{aligned}\forall z \in T^2, \quad g_k(I(z)) &= \lambda_k[P(-\bar{z}) - P(\omega_2)]^{2k+1} \\ &= \overline{-g_k(z)}.\end{aligned}$$

Then:

$$I_*(\eta_k) = -\frac{d\bar{z}}{g_k(I(z))} = \frac{d\bar{z}}{\overline{g_k(z)}} = \bar{\eta}_k,$$

so that:

$$I_*(\Phi_1) = \bar{\Phi}_1; \qquad I_*(\Phi_2) = -\bar{\Phi}_2; \qquad I_*(\Phi_3) = -\bar{\Phi}_3.$$

Calling X_k the minimal immersion defined, up to a translation in $\mathbf{R}^3$, by (g_k, η_k), we deduce that:

$$\forall z \in T^2 - \{0, \omega_2\}, \quad X_k(I(z)) = S_1[X_k(z)],$$

where S_1 is the reflection about the x_1-axis. This shows that X_k sends the line C_1 defined by:

$$C_1(t) = 2t\omega_3, \qquad t \in]0, 1[$$

to D_1 which is the x_1-axis in the horizontal plane $P_1 = \{x_3 = 0\}$.

In the same way if J is the map on T^2 defined by:

$$\forall z \in T^2, \quad J(z) = 2\omega_1 - \bar{z},$$

we can show that:

$$\forall z \in T^2 - \{0, \omega_2\}, \quad X_k(J(z)) = S_2[X_k(z)],$$

where S_2 is the reflection about D_2, a straight line parallel to D_1 in a horizontal plane P_2, P_2 distinct from P_1. Then X_k sends the line C_2 of T^2 on D_2 in $\mathbf{R}^3$ where

$$C_2(t) = \omega_1 + 2t\omega_3, \qquad t \in [0, 1], \quad t \neq \frac{1}{2}.$$

Let

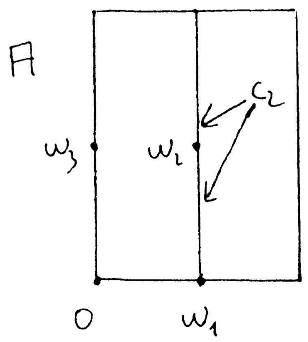
$$A = \{z \in T^2 \,/\, 0 \leq \operatorname{Re}(z) \leq \omega_1, \ z \neq 0, \ z \neq \omega_2\}.$$

As $\Phi_3 = 1$ and hence:

$$(X_k)_3(z) = \operatorname{Re}(z), \qquad \forall z \in T^2 - \{0, \omega_2\},$$

we see that X_k sends $A \cap T^2$ to the slab of $\mathbf{R}^3$ bounded by P_1 and P_2.

Of course $X_k(A)$ is a minimal immersed annulus in $\mathbf{R}^3$ between P_1 and P_2 and bounded by $D_1 \cup D_2$.



Finally denoting $(X_k)_i$ by X_i, $1 = 1, 2, 3$ for simplicity, we have:

$$\forall z \in T^2 - \{0, \omega_2\}, \quad (X_1 - iX_2)(z) = \int_{z_0}^{z} \eta_k - \overline{\int_{z_0}^{z} \eta_k g_k^2},$$

therefore, for z near 0 we have

$$\begin{aligned}(X_1 - iX_2)(z) &\simeq \overline{\int_{z_0}^{z} \lambda_k \cdot \frac{dz}{z^{2(2k+1)}}} \\ &\simeq -\frac{\overline{\lambda_k}}{4k+1}\left(\overline{\frac{1}{z^{4k+1}}}\right)\end{aligned}$$

and so the projection of the end near 0 on a horizontal plane is a $(4k+1)$ to 1 map.

Near the other end ω_2 we also have:

$$\begin{aligned}(X_1 - iX_2)(z) &\simeq \int_{z_0}^{z} \left(\frac{2}{p''(\omega_2)}\right)^{2k+1} \frac{1}{\lambda_k} \cdot \frac{dz}{(z-\omega_2)^{2(2k+1)}} \\ &\simeq \frac{\lambda k}{3k+1} \cdot \frac{1}{(z-\omega_2)^{4k+1}},\end{aligned}$$

and again the projection of the end ω_2 on a horizontal plane is a $(4k+1)$ to 1 map.

So it remains to show that conditions (1) hold. As we have $\Phi_3 = 1$ we deduce:

$$\begin{aligned}&\Leftrightarrow \operatorname{Re}\int_\gamma \Phi_1 = 0 \quad \text{and} \quad \operatorname{Re}\int_\gamma \Phi_2 = 0 \\ &\Leftrightarrow \int_\gamma \eta_k = \overline{\int_\gamma \eta_k g_k^2}.\end{aligned} \tag{1}$$

As $\eta_k = dz/g_k(z)$ we have:

$$\Leftrightarrow \int_\gamma \frac{dz}{g_k(z)} = \overline{\int_\gamma g_k(z)\,dz}$$
$$\Leftrightarrow \int_\gamma \frac{dz}{(P(z) - P(\omega_2))^{2k+1}} = \lambda_k \bar{\lambda}_k \overline{\int_\gamma (P(z) - P(\omega_2))^{2k+1}\,dz}, \tag{1}$$

but:

$$\frac{1}{P(z) - P(\omega_2)} = \frac{2}{P''(\omega_2)}(P(z - \omega_2) - P(\omega_2)).$$

So:

$$\Leftrightarrow \lambda_k \bar{\lambda}_k \overline{\int_\gamma (P(z) - P(\omega_2))^{2k+1}\,dz} = \left(\frac{2}{P''(\omega_2)}\right)^{2k+1} \int_\gamma (P(z - \omega_2) - P(\omega_2))^{2k+1}\,dz$$

$$\Leftrightarrow \lambda_k \bar{\lambda}_k \sum_{q=0}^{2k+1} C_{2k+1}^q (-P(\omega_2))^{2k+1-q} \overline{\int_\gamma P(z)^q\,dz} \tag{1}$$
$$= \left(\frac{2}{P''(\omega_2)}\right)^{2k+1} \sum_{q=0}^{2k+1} C_{2k+1}^q (-P(\omega_2))^{2k+1-q} \cdot \int_\gamma P(z - \omega_2)^q\,dz.$$

The following lemma shows the last equality is true.

LEMMA 1. *For every positive integer q we have*:

$$\int_\gamma P(z)^q\,dz = -\overline{\int_\gamma P^q(z - \omega_2)\,dz}.$$

Assuming Lemma 1 we have:

$$\Leftrightarrow \lambda_k \bar{\lambda}_k = -\left(\frac{2}{P''(\omega_2)}\right)^{2k+1}$$

which is true as $\lambda_k = i\sqrt{(-2/P''(\omega_2))^{2k+1}}$. So it just remains to prove Lemma 1.

Proof of Lemma 1. As T^2 is a rectangular torus we have the following Laurent series for the Weierstrass P-function, see [2], [8] or the Appendix.

$$P(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} a_n z^n$$

with a_n real numbers.

We deduce that for any positive integer q the Laurent series of $P^q(z)$ has the following type:

$$P^q(z) = \sum_{n=1}^{q} b_{-n} z^{-2n} + \sum_{n=0}^{+\infty} b_n z^n,$$

with b_j real numbers, $j = 1, \dots, n$.

Also if $P^{(k)}(z)$ is the kth-derivative of P we have:

$$P^{(k)}(z) = (-1)^k \frac{(k+1)!}{z^{k+2}} + F_k(z),$$

where F_k is a holomorphic map near 0. We deduce that:

$$P^q(z) - \sum_{n=1}^{q} \frac{b_{-n}}{(2n-1)!} P^{(2n-2)}(z)$$

is an elliptic function on T^2 without poles, so this function is constant and taking $z = \omega_1$, we see that this constant c must be real, so:

$$P^q(z) = \sum_{n=1}^{q} \frac{b_{-n}}{(2n-1)!} P^{(2n-2)}(z) + c$$

with b_{-n} and c real numbers. Then:

$$\int_{\gamma} P^q(z)\,dz = \sum_{n=1}^{q} \frac{b_{-n}}{(2n-1)!} \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} P^{(2n-2)}(z)\,dz + c \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} dz$$

$$= b_{-1} \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} P(z)\,dz + 2\omega_3 c$$

since for $k \geq 2$:

$$\int_{\omega_1/2}^{\omega_1/2+2\omega_3} P^{(k)}(z)\,dz = P^{(k-1)}\left(\frac{\omega_1}{2}+2\omega_3\right) - P^{(k-1)}\left(\frac{\omega_1}{2}\right) = 0.$$

Then, as $\zeta'(z) = -P(z)$ we have:

$$\begin{aligned}\int_\gamma P^q(z)\,dz &= b_{-1}\left[-\zeta\left(\frac{\omega_1}{2}+2\omega_3\right)+\zeta\left(\frac{\omega_1}{2}\right)\right]+2\omega_3 c\\ &= 2\eta_3 b_{-1} + 2\omega_3 c.\end{aligned}$$

In the same way, as:

$$P^q(z-\omega_2) = \sum_{n=1}^{q} \frac{b_{-n}}{(2n-1)!} P^{(2n-2)}(z-\omega_2) + c,$$

we also have:

$$\int_\gamma P^q(z-\omega_2)\,dz = -2\eta_3 b_{-1} + 2\omega_3 c.$$

This concludes the proof because c and b_{-1} are real numbers and ω_3 and η_3 are purely imaginary numbers.

Remarks

(1) Following the arguments of Lemmas 1 and 2 in §2, it is easy to show that the surfaces S_k are the only minimal immersed surfaces between P_1 and P_2 bounded by $D_1 \cup D_2$ with finite total curvature.

(2) We do not know if there exists surfaces like S_k which are bounded by two horizontal lines D_1, D_2 and make a non zero angle θ. Of course for $k = 0$, Theorem 1 shows that such a surface does not exist.

§4. Appendix

Let T^2 be a torus $\mathbf{C}/_\Gamma$ where Γ is the lattice of $\mathbf{C}$ given by:

$$\Gamma = \{p \cdot 2\omega_1 + q \cdot 2\omega_3, p, q \in \mathbf{Z}, \omega_1 \in \mathbf{R}_*^+, \operatorname{Im}(\omega_3) \in \mathbf{R}_*^+\}.$$

The Weirstrass P-function is a special function defined on T^2 which is a meromorphic function on $\mathbf{C}$ such that:

P has a pole of order two at each point of Γ and:

$$\forall z \in \mathbf{C} - \Gamma, \quad P(z + 2\omega_i) = P(z), \qquad i = 1, 3.$$

If $\omega_2 = \omega_1 + \omega_3$, $P(\omega_1) + P(\omega_2) + P(\omega_3) = 0$.

P has the following Laurent series:

$$P(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma - \{0\}} \left\{\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right\}.$$

P is an even function: $\forall z \in \mathbf{C} - \Gamma$, $P(-z) = P(z)$.

There are two other functions related to P, namely the ζ and σ-functions: The ζ-function satisfies:

$$\forall z \in \mathbf{C} - \Gamma, \quad \zeta'(z) = -P(z).$$

If $\eta_i = \zeta(\omega_i)$, $i = 1, 2, 3$, then $\eta_2 = \eta_1 + \eta_3$, so ζ is uniquely defined on $\mathbf{C} - \Gamma$. We have:

$$\forall z \in \mathbf{C} - \Gamma, \quad \zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i, \qquad i = 1, 2, 3,$$

and then ζ is not defined on T^2 because z and $(z + 2\omega_i)$ represent the same point on T^2.

ζ is an odd function: $\forall z \in \mathbf{C} - \Gamma, \zeta(-z) = -\zeta(z)$.

The following Legendre relation holds.

$$\omega_2\eta_1 - \omega_1\eta_2 = i\frac{\pi}{2}.$$

The σ-function satisfies:

$$\forall z \in \mathbf{C} - \Gamma, \quad \frac{\sigma'}{\sigma}(z) = \zeta(z),$$

and then σ is an holomorphic function on $\mathbf{C}$, furthermore:

$$\lim_{z\to 0}\frac{\sigma(z)}{z}=1,$$

so that σ is uniquely determined on $\mathbf{C}$.

We also have:

$$\forall z\in\mathbf{C},\quad \sigma(z+2\omega_i)=-e^{2\eta_i(z+\omega_i)}\sigma(z). \qquad (\#)$$

σ is an odd function: $\forall z\in\mathbf{C}, \sigma(-z)=-\sigma(z)$.

Furthermore if T^2 is a rectangular torus, that is if $\omega_3\in i\mathbf{R}_*^+$, we have:

$$\forall z\in\mathbf{C}-\Gamma,\quad P(\bar z)=\overline{P(z)};\qquad \zeta(\bar z)=\overline{\zeta(z)}.$$

$$\forall z\in\mathbf{C},\quad \sigma(\bar z)=\overline{\sigma(z)}.$$

For more details about those functions see Gerretsen–Sansone [2] or Molk–Tannery [8].

REFERENCES

[1] G. Darboux. *Théorie des surfaces.* Vol I, 2nd édition, Paris, 1914, pp. 461–644, §252.
[2] J. Gerretsen, G. Sansone. *Lectures on the theory of functions of a complex variable.* P. Noordhoff–Groningen, 1960.
[3] D. Hoffman, H. Karcher, H. Rosenberg. *A characterization of Riemann examples.* Preprint.
[4] D. Hoffman, W. H. Meeks. *The global theory of properly embedded minimal surfaces, Theorem* 10.1. Preprint, University of Massachusetts at Amherst.
[5] H. B. Lawson. *Lectures on minimal submanifolds.* Vol I. Publish or Perish.
[6] W. H. Meeks, H. Rosenberg. *The geometry of periodic minimal surfaces.* Preprint.
[7] W. H. Meeks, B. White. *Minimal surfaces bounded by convex curves in parallel planes.* Preprint.
[8] J. Molk, J. Tannery. *Eléments de la théorie des fonctions elliptiques.* Bronx, N.Y., Chelsea, 1972.
[9] B. Riemann. *Oeuvres complètes.* Tome XIII des mémoires de la société royale de Goettingen (1867), p. 305.
[10] M. Schiffman. *On surfaces of stationary area bounded by two circles, or convex curves, in parallel planes.* Annals of Math *63* (1956), 77–90.
[11] E. Toubiana. *Géométrie des surfaces minimales de* $\mathbf{R}^3$. Thèse de doctorat, Université Paris VII, 1986.

203 rue du Faubourg
Saint-Antoine
75011 Paris

Received April 15, 1991