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On the minimal surfaces of Riemann

ERIC TOUBIANA

§1. Introduction

Let γ_i , $i = 1, 2$ be plane Jordan curves in horizontal planes P_i , $i = 1, 2$, $P_1 \neq P_2$, we know that under conditions on γ_i , for example γ_1 not too far from γ_2 , γ_1 and γ_2 bound a least area minimal annulus M between P_1 and P_2 . Meeks and White [7] were able to prove that when the γ_i are convex there are at most two minimal annuli bounded by $\gamma_1 \cup \gamma_2$. Assuming that the γ_i are convex, Shiffman [10] proved that if M is a minimal annulus bounded by $\gamma_1 \cup \gamma_2$ then for each horizontal plane P between P_1 and P_2 , the intersection $P \cap M$ is again a convex Jordan curve, furthermore if γ_1 and γ_2 are circles, then $P \cap M$ is also a circle.

In view of this last result it is natural to ask what happens when two straight lines replace the Jordan curves. Namely, let D_i , $i = 1, 2$, be straight lines in horizontal planes P_i , $i = 1, 2$, $P_1 \neq P_2$. Let us assume that D_1 makes an angle θ with D_2 , $\theta \in [0, \pi]$. Now let M be a minimal annulus between P_1 and P_2 bounded by $D_1 \cup D_2$. If P is a horizontal plane between P_1 and P_2 , what can we say about the intersection $P \cap M$?

For example, if $\theta = 0$ i.e. D_1 and D_2 are parallel, Riemann [9] has constructed a minimal embedded annulus S between P_1 and P_2 bounded by $D_1 \cup D_2$. Furthermore the intersection with any horizontal plane is a circle, see [3] for a detailed description of Riemann's examples.

In the same paper [3], Hoffman, Karcher and Rosenberg proved that if D_1 and D_2 are parallel, i.e. $\theta = 0$, then the only minimal properly embedded annulus between P_1 and P_2 , bounded by $D_1 \cup D_2$ is precisely Riemann's example.

Here we shall prove that the case $\theta \neq 0$ does not occur, namely we show the following.

THEOREM 1. *Let D_i , $i = 1, 2$ be straight lines in horizontal planes P_i , $i = 1, 2$, $P_1 \neq P_2$, let $\theta \in [0, \pi]$ be the angle that D_1 makes with D_2 . Let us assume there exists a minimal properly embedded annulus M between P_1 and P_2 bounded by $D_1 \cup D_2$.*

Then necessarily $\theta = 0$ and M is Riemann's example S .

Let us assume now $\theta = 0$, i.e. D_1 and D_2 parallel. We can generalize Riemann's examples to yield a family of minimal surfaces S_k with the following properties:

- (1) $S_0 = S$ where S is Riemann's example.
- (2) For every integer $k, k \geq 0$, S_k is a minimal immersed annulus between P_1 and P_2 , bounded by $D_1 \cup D_2$ such that after reflection about the lines D_1, D_2, \dots we get a complete minimal surface in \mathbf{R}^3 , which we call again S_k . Furthermore S_k is invariant under the translation $X \rightarrow X + 2u$ where u is the vector orthogonal to D_1 translating D_1 to D_2 . Also each end of S_k in \mathbf{R}^3 is a flat horizontal end, i.e. an end asymptotic to a horizontal plane, and the projection of every end over any horizontal plane is a $(4k + 1)$ to 1 map.

The following gives the Weierstrass representation of this family.

THEOREM 2. Let T be a rectangular torus, i.e. $T^2 = C/\Gamma$ where $\Gamma = \{2\omega_1 p + 2\omega_3 q, p, q \in \mathbf{Z}, \omega_1 \in \mathbf{R}_*^+, \omega_3 \in i\mathbf{R}_*^+\}$.

Then for every $k, k \in \mathbf{N}$ the following data (g_k, η_k) is the Weierstrass representation of the surface S_k described above.

$$g_k = \lambda_k [P(z) - P(\omega_2)]^{2k+1},$$

$$\eta_k = \frac{dz}{g_k(z)} = \frac{dz}{\lambda_k [P(z) - P(\omega_2)]^{2k+1}},$$

where $\omega_2 = \omega_1 + \omega_3$, $z \in T^2 - \{0, \omega_2\}$, $\lambda_k = i\sqrt{(-2/P''(\omega_2))^{2k+1}}$ and P is the P -function of Weierstrass.

We prove Theorems 1 and 2 in §2 and §3 respectively.

§2. Proof of Theorem 1

We are going to use the Weierstrass representation for a minimal surface, see Lawson [5] p. 113, and the reflection principle, namely if a minimal surface in \mathbf{R}^3 has a piece γ of a straight line in its boundary then we can extend minimally this surface along γ by the reflection about the line defined by γ , see [5] p. 82.

Let us take the notation of Theorem 1. By the reflection principle we can extend M to a complete properly embedded minimal surface in \mathbf{R}^3 , let us call M again this surface. As the composition of the two reflections about D_1 and D_2 is a screw-motion $S_{2\theta}$, i.e. the 2θ -rotation with respect to the x_3 -axis composed with a vertical

translation, we get that M is globally invariant by the screw-motion $S_{2\theta}$:

$$S_{2\theta}(M) = M.$$

Then we get a quotient surface $M/S_{2\theta}$ in $\mathbf{R}^3/S_{2\theta}$. By construction this surface is topologically a two-punctured torus properly and minimally embedded in $\mathbf{R}^3/S_{2\theta}$. By a theorem of Meeks and Rosenberg [6] we deduce that $M/S_{2\theta}$ has finite total curvature. From this we get that each end of M is conformally a punctured disc, see [5] p. 130. At last we deduce that the ends of M are parallel, flat, embedded and also the Gauss map g of M has order two at each end (this last claim comes from the fact that if E_1 is the end of M passing through the straight line D_1 then, by construction, $E_1 - P_1$ has two connected components). Subsequently we assume that D_1 is the x_1 -axis in the plane $P_1 = \{x_3 = 0\}$.

LEMMA 1. *The extended complete minimal surface M in \mathbf{R}^3 is parameterized by $C = C - (\Gamma \cup (z_0 + \Gamma))$ where:*

$$\begin{cases} \Gamma = \{2\omega_1 \cdot p + 2\pi i \cdot q, p, q \in \mathbf{Z}, \omega_1 \in \mathbf{R}_*^+\}, \\ z_0 \in \omega_1 + y\pi i, \quad 0 \leq y \leq 1. \end{cases}$$

Furthermore if X is the minimal immersion of C onto $M \subset \mathbf{R}^3$, X must satisfy:

- X sends the vertical lines $\{\operatorname{Re}(z) = c, c \neq k\omega_1, k \in \mathbf{Z}\}$ to horizontal Jordan curves in \mathbf{R}^3 .
- X sends the vertical lines $\{\operatorname{Re}(z) = k\omega_1, k \in \mathbf{Z}\}$ to horizontal straight lines in \mathbf{R}^3 and particularly X sends $\{\operatorname{Re}(z) = 0\}$ to the x_1 -axis D_1 and $\{\operatorname{Re}(z) = \omega_1\}$ to D_2 .

At last we have:

$$\forall z \in C, \quad X(z + 2\pi i) = X(z), \quad (1)$$

$$X(2\omega_1 - \bar{z}) = S_{D_2}[X(z)], \quad (2)$$

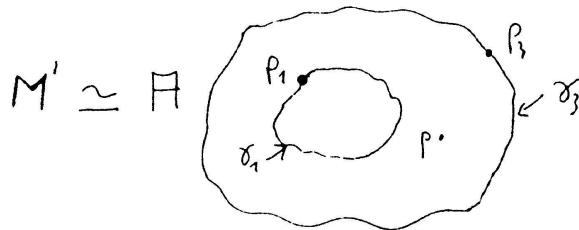
$$X(z + 2\omega_1) = S_{2\theta}[X(z)], \quad (3)$$

where S_{D_2} is the reflection about D_2 and $S_{2\theta} = S_{D_1} \circ S_{D_2}$.

Proof of Lemma 1. Let M_1 be the piece of M bounded by $D_1 \cup D_2$ and let M_2 be the reflection of M_1 about D_2 , i.e. $M_2 = S_{D_2}(M_1)$. Let D_3 be $S_{D_2}(D_1)$, so that M_2 is bounded by D_2 and D_3 .

By construction $M' = M_1 \cup M_2$ is homeomorphic to a one-punctured planar annulus A bounded by two Jordan curves γ_1, γ_3 each one with a point $p_i, i = 1, 3$ removed.

Let Z be the minimal immersion of A onto M' , let A have the conformal structure induced by Z . As each end of M is conformally a one-punctured disc we deduce that A is conformally a one-punctured annulus bounded by two Jordan curves γ_1, γ_3 each one with a point $p_i, i = 1, 3$ removed.



Assume that $Z(\gamma_i) = D_i, i = 1, 3$. Z sends a neighbourhood of p to the end passing through the line D_2 . Assume that D_2 belongs to the plane $P_2 = \{x_3 = c, c > 0\}$. Then D_3 belongs to the horizontal plane $P_3 = \{x_3 = 2c\}$ (recall that $P_1 = \{x_3 = 0\}$).

Let Z_3 be the third coordinate function of Z , Z_3 is a harmonic function. By construction of M , the line D_2 is the only part of M in the plane P_2 , that is:

$$M \cap P_2 = D_2,$$

as $Z_3(\gamma_1) = 0$ and $Z_3(\gamma_3) = 2c$, we deduced that $\{Z_3^{-1}(c) \cup p\}$ is an embedded closed curve in the interior of \bar{A} , and by the maximum principle applied to the function Z_3 , this curve must be connected, so that $\{Z_3^{-1}(c) \cup p\}$ is a Jordan curve γ_2 in the interior of A .

Now let α be a real number with $\alpha \neq c$ and $0 \leq \alpha \leq 2c$, again by the maximum principle we get that $Z_3^{-1}(\alpha)$ is a Jordan curve γ of \bar{A} . Hence we have a foliation $(\gamma_i)_{1 \leq i \leq 3}$ of \bar{A} with Jordan curves γ_i , so that Z_3 is constant over each curve γ_i , i.e.:

$$\forall i \in [1, 3], \quad Z_3(\gamma_i) = c_i \quad \text{with: } 0 \leq c_i \leq 2c \quad \text{and} \quad c_1 = 0; c_2 = c; c_3 = 2c.$$

Let Z_3^* be a harmonic conjugate of Z_3 over \bar{A} , we have locally:

$$Z_3^*(z) = \int_{\alpha} *dZ_3,$$

where α is a path between a base point z_0 and z , and $*dZ_3$ is defined by:

$$*dZ_3 = \frac{\partial Z_3}{\partial x} dy - \frac{\partial Z_3}{\partial y} dx, \quad z = x + iy.$$

It can happen that Z_3^* is not globally defined on \bar{A} , i.e. if γ is any Jordan curve in \bar{A} generating $\Pi_1(\bar{A})$, let a be the real number defined by:

$$a = \int_{\gamma} *dZ_3.$$

Then Z_3^* is well defined on \bar{A} if and only if $a = 0$. In case $a \neq 0$ let us consider the function:

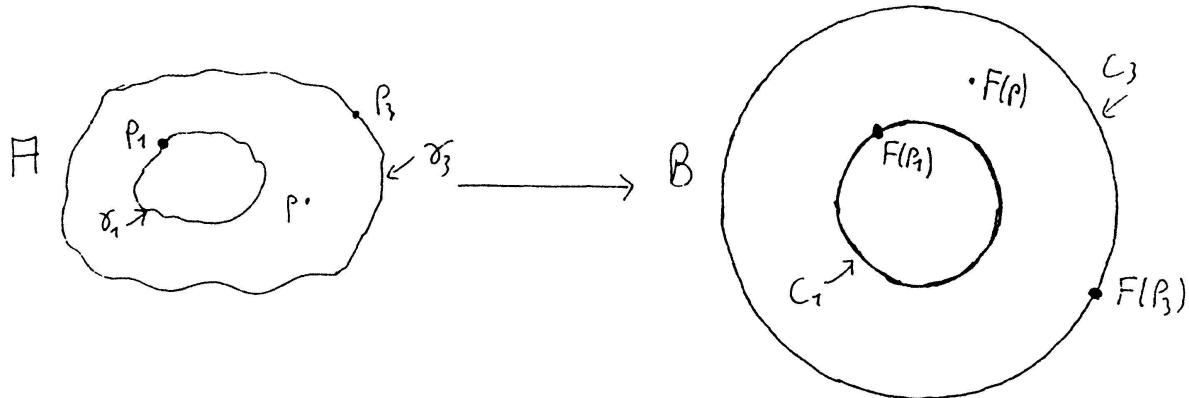
$$F(z) = \exp \left[\frac{2\pi}{a} (Z_3(z) + iZ_3^*(z)) \right].$$

In case $a = 0$ we put:

$$F(z) = \exp [Z_3(z) + iZ_3^*(z)],$$

in any case F is a well defined map of \bar{A} into C , F maps the Jordan curves $\{\gamma_i\}_{1 \leq i \leq 3}$ to concentric circles of C : $\{|u| = \exp [(2\pi/a)c_i]\}$ or $\exp [c_i]$ depending on the expression of F . F maps \bar{A} on an annulus B of C bounded by two concentric circles C_1, C_3 with

$$F(\gamma_i) = C_i, \quad 1 \leq i \leq 3.$$



It is easy to see that F is a n -covering map of \bar{A} onto B in C , so after composing F with the n th root map, we can assume that F is an embedding of \bar{A} onto B .

Let us call Y the composed map $Z \circ F^{-1}$,

$$B \xrightarrow{Y = Z \circ F^{-1}} \mathbf{R}^3$$

we clearly have $Y(C_i) = D_i$, $i = 1, 2, 3$ so by the reflection principle if I_C denotes the reflection in \mathbf{C} about a circle C we have:

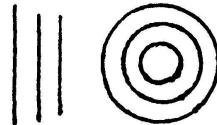
$$C_3 = I_{C_2}(C_1) \quad (\text{because } Y(C_3) = D_3 = S_{D_2}(D_1))$$

and then $F(P_3) = I_{C_2}(F(P_1))$.

Now by reflection principle we extend Y to a minimal immersion of $\mathbf{C} - E$ onto M in \mathbf{R}^3 , where E is the infinite discrete set of \mathbf{C} obtained by taking $F(P_1), F(P)$ and all the successive images of those points by the reflections about the circles $C_1, C_2, I_{C_1}(C_2), I_{C_2}(C_1) \dots$

Then the exponential map $z \rightarrow u = \exp(z)$ of \mathbf{C} sends the vertical lines $\{\operatorname{Re}(z) = a\}$ of the z -complex plane on the concentric circles $\{|u| = e^a\}$ of the u -complex plane.

$$(\mathbf{C}, z) \xrightarrow{e^z} (\mathbf{C}, u)$$



Up to a homothety of the u -plane we can assume that C_1 is the circle of radius one. So that if we call X the composed map: $X = Y \circ \exp$, X sends the imaginary axis $\{\operatorname{Re}(z) = 0\}$ to D_1 in \mathbf{R}^3 . Furthermore up to a rotation in the u -plane, we can assume that $F(p_1) = 1$, so that $\exp(0) = F(p_1) = 1$. Let denote z_0 the inverse image, by the exponential map, of $F(p)$ with $0 \leq \operatorname{Im} z_0 \leq 2\pi$, and ω_1 the real part of z_0 .

$$\omega_1 = \operatorname{Re}(z_0).$$

Then X sends the line $\{\operatorname{Re}(z) = \omega_1\}$ to the horizontal line D_2 in \mathbf{R}^3 .

It is clear that the inverse image of E by the exponential map is just $\Gamma \cup (z_0 + \Gamma)$, with

$$\Gamma = \{2\omega_1 p + 2\pi i q, p, q \in \mathbf{Z}\}.$$

Furthermore by construction of Y and by the geometric properties of M in \mathbf{R}^3 we must have:

$$\forall z \in C = \mathbf{C} - (\Gamma \cup (z_0 + \Gamma)), \quad X(z + 2\pi i) = X(z), \quad (1)$$

$$X(2\omega_1 - \bar{z}) = S_{D_2}[X(z)], \quad (2)$$

$$X(4\omega_1 - \bar{z}) = S_{D_3}[X(z)]. \quad (2)'$$

(1) comes from the fact that $X = Y \circ \exp$, and (2) (resp. (2)') comes from the fact that M is invariant by the reflection about D_2 (resp. D_3).

So that combining (2) and (2)' we have:

$$\forall z \in C, \quad X(z + 2\omega_1) = S_{2\theta}[X(z)]. \quad (3)$$

Finally considering the map $z \rightarrow -z$ of the z -plane keeping fixed the point 0, and because of (1) we can assume that:

$$0 \leq \operatorname{Re}(z_0) \quad \text{and} \quad \operatorname{Im}(z_0) = y\pi, \quad 0 \leq y \leq 1.$$

Remarks

(1) From Lemma 1 we see that X induces a minimal embedding of $C/_{2\pi i\mathbb{Z}}$ onto M in \mathbf{R}^3 and a minimal embedding of $T^2 - \{0, z_0\} = C/_{\Gamma \cup (z_0 + \Gamma)}$ onto $M_{/S_{2\theta}}$ in $\mathbf{R}^3_{/S_{2\theta}}$.

(2) Until now we don't know if ω_1 is bigger or not than π , so from now we assume π bigger than ω_1 , the rest of the proof will not be affected.

Let $\omega_3 = i\pi$, so that the lattice Γ is:

$$\Gamma = \{2\omega_1 p + 2\pi i q, p, q \in \mathbb{Z}\}.$$

LEMMA 2. *With the hypothesis of Theorem 1 we have $\theta = 0$ and the Weierstrass representation of the minimal immersion X of C in \mathbf{R}^3 in Lemma 1 is:*

$$\forall z \in C, \quad g(z) = \lambda[P(z) - P(\omega_2)],$$

$$\eta = \frac{dz}{g(z)} = \frac{dz}{\lambda[P(z) - P(\omega_2)]},$$

where P is the P -function of Weierstrass associated to the torus $T^2 = \mathbf{C}/\Gamma$, $\omega_2 = \omega_1 + \omega_3$ and $\lambda = i\sqrt{-2/P''(\omega_2)} \in i\mathbf{R}$.

Proof of Lemma 2. We are going to use some basic facts about the σ and ζ functions associated with the P -function of Weierstrass (we recall these facts in the Appendix).

Let us call (Φ_1, Φ_2, Φ_3) the coordinate forms of X , i.e.

$$X(z) = (X_1(z), X_2(z), X_3(z)) = \operatorname{Re} \int_{z_0}^z (\Phi_1, \Phi_2, \Phi_3)$$

with z_0 a base point in C and:

$$\Phi_1 = \frac{1}{2} \eta(1 - g^2); \quad \Phi_2 = \frac{i}{2} \eta(1 + g^2); \quad \Phi_3 = \eta g.$$

As the third coordinate function X_3 is constant on vertical lines $\{\operatorname{Re}(z) = c\}$ we can assume $\Phi_3 = 1$ and then $\eta = dz/g(z)$.

As $M_{/S_{2\theta}}$ has two flat horizontal ends parameterized by 0 and z_0 (with $\operatorname{Re}(z_0) = \omega_1$, see Lemma 1), we can assume that

$$g(0) = \infty \quad \text{and} \quad g(z_0) = 0,$$

also g is a 2 to 1 map near each end.

Furthermore on a fundamental domain of T^2 in \mathbf{C} there is no other point where g is vertical, if not, up to a rotation in \mathbf{R}^3 , another such point would be an n -pole of g , $n \geq 1$, but this point is a regular point of M , so it must be a $2n$ -zero of η and then it is an n -pole of $\Phi_3 = g\eta$, but $\Phi_3 = 1$ as we saw before.

As g is the Gauss map of M , up to the stereographic projection of S^2 to $\mathbf{C} \cup \{\infty\}$, by the geometric properties of M in \mathbf{R}^3 , g must satisfy:

$$\forall z \in \mathbf{C} \quad \begin{cases} g(z + 2\omega_3) = g(z), \\ g(2\omega_1 - \bar{z}) = -e^{2i\theta} \overline{g(z)}, \\ g(z + 2\omega_1) = e^{2i\theta} g(z). \end{cases} \quad \begin{array}{l} (1) \\ (2) \\ (3) \end{array}$$

Those properties (1), (2), (3) of g come from respectively the properties (1), (2), (3) of X established in Lemma 1.

From (3) we deduce that the map $(g'/g)(z)$ is well defined on the torus $T^2 = \mathbf{C}/\Gamma$. Furthermore $(g'/g)(z)$ is an elliptic function on T^2 and has a single pole at a point z in T^2 if and only if z is a pole or a zero of g , and $(g'/g)(z)$ has no other pole on T^2 . From what we saw before we see that $(g'/g)(z)$ has two single poles on T^2 , so it is an elliptic function of degree two on T^2 with a single pole at 0 and z_0 . The most general form of such a function is:

$$\frac{g'}{g}(z) = \frac{a_1 P(z) + a_0 + b P'(z)}{P(z) - P(z_0)}, \quad a_1, a_0, b \in \mathbf{C}$$

with $a_1 P(z_0) + a_0 = b P'(z_0)$ because $-z_0$ is not a pole of $(g'/g)(z)$.

From (2) we get that

$$\forall z \in \mathbf{C}, \quad 2\omega_1 - \bar{z} = z \Rightarrow \frac{g'}{g}(z) \in i\mathbf{R}.$$

So $(g'/g)(z)$ must have purely imaginary values on the line $\{\operatorname{Re}(z) = \omega_1\}$ in \mathbf{C} . But as $\operatorname{Re}(z_0) = \omega_1$, $P(z_0) \in \mathbf{R}$ and then $P(z) - P(z_0)$ has real values on the line $\{\operatorname{Re}(z) = \omega_1\}$ also $P'(z)$ has purely imaginary values on this line, so we have:

$$\forall y \in \mathbf{R}, \quad a_1 P(\omega_1 + y\omega_3) + a_0 + bP'(\omega_1 + y\omega_3) \in i\mathbf{R}.$$

For $y = 0$ and $y = 1$ we deduce (as $P'(\omega_1) = 0 = P'(\omega_1 + \omega_3)$)

$$a_1 P(\omega_1) + a_0 \in i\mathbf{R},$$

$$a_1 P(\omega_2) + a_0 \in i\mathbf{R},$$

as $P(\omega_1)$ and $P(\omega_2)$ are two distinct real numbers, we get that a_1 and a_0 are purely imaginary numbers, $a_1, a_0 \in i\mathbf{R}$, so that:

$$\forall y \in \mathbf{R}, \quad bP'(\omega_1 + y\omega_3) \in i\mathbf{R}$$

and then b is a real number:

$$a_1, a_0 \in i\mathbf{R}, \quad b \in \mathbf{R}.$$

Now let us look for g , we have:

$$\forall z \in \mathbf{C}, \quad \frac{g'}{g}(z) = a_1 + \frac{a_1 P(z_0) + a_0}{P(z) - P(z_0)} + b \frac{P'(z)}{P(z) - P(z_0)}.$$

We want to show that $z_0 = \omega_2$. Let us suppose now that $z_0 \neq \omega_i$, $i = 1, 2$, that is:

$$z_0 = \omega_1 + y\omega_3, \quad 0 < y < 1.$$

As $P'(z_0) \neq 0$ we have

$$\frac{1}{P(z) - P(z_0)} = \frac{1}{P'(z_0)} [\zeta(z - z_0) - \zeta(z + z_0) + 2\zeta(z_0)]$$

as $a_1 P(z_0) + a_0 = bP'(z_0)$ we get:

$$\frac{g'}{g}(z) = a_1 + 2\zeta(z_0) + b \left[\zeta(z - z_0) - \zeta(z + z_0) + \frac{P'(z)}{P(z) - P(z_0)} \right]$$

and so

$$\forall z \in C, \quad g(z) = \lambda e^{[a_1 + 2b\zeta(z_0)]z} \cdot \left[\frac{\sigma(z - z_0)}{\sigma(z + z_0)} (P(z) - P(z_0)) \right]^b$$

with $\lambda \in \mathbf{C}$, $a_1 \in i\mathbf{R}$, $b \in \mathbf{R}$.

But g must have a double pole at 0 and a double zero at z_0 , now we remark that the function in brackets $(\sigma(z - z_0)/\sigma(z + z_0))(P(z) - P(z_0))$, has the same zeros and poles, so we must have $b = 1$, so:

$$\forall z \in \mathbf{C}, \quad g(z) = \lambda e^{Az} \frac{\sigma(z - z_0)}{\sigma(z + z_0)} (P(z) - P(z_0))$$

where, $\lambda \in \mathbf{C}$, $A = a_1 + 2\zeta(z_0)$.

From the property (#) of the σ -function, see the Appendix, we have:

$$\forall z \in \mathbf{C}, \quad g(z + 2\omega_i) = e^{2\omega_i A - 4\eta_i z_0} g(z), \quad i = 1, 2, 3,$$

from conditions (1) and (3) on g we get:

$$\begin{cases} 2\omega_1 A - 4\eta_1 z_0 = 2i\theta + 2p\pi i, & p \in \mathbf{Z}, \\ 2\omega_3 A - 4\eta_3 z_0 = 2q\pi i, & q \in \mathbf{Z} \end{cases}$$

and then

$$z_0 = z_0(\theta) = -\frac{2\theta}{2\pi} \omega_3 - p\omega_3 + q\omega_1.$$

By considering the real part and the imaginary part of z_0 , as we know that $\operatorname{Re}(z_0) = \omega_1$, and $0 < \operatorname{Im}(z_0) < -i\omega_3$ we have $q = 1$ and $p = -1$, so:

$$z_0(\theta) = \omega_1 + \left(1 - \frac{\theta}{\pi} \omega_3 \right), \quad \theta \in [0, \pi]$$

and

$$2\omega_1 A = 4\eta_1 z_0(\theta) + 2i\theta - 2\pi i. \quad (*)$$

On the other hand as the map X is well defined on C we must have:

$$\forall \gamma \in \Pi_1(C), \quad \operatorname{Re} \int_{\gamma} (\Phi_1, \Phi_2, \Phi_3) = 0.$$

In particular if γ is a small circle around 0, we must have:

$$\operatorname{Re} \int_{\gamma} (\Phi_1, \Phi_2) = 0,$$

that is

$$\int_{\gamma} \eta = \overline{\int_{\gamma} \eta g^2}$$

and as $\eta = dz/g(z)$ then:

$$-2\pi \overline{\operatorname{Re} s(g, 0)} = 2\pi \operatorname{Re} s\left(\frac{1}{g}, 0\right) = 0 \quad \left(\text{because } 0 \text{ is a zero of } \frac{1}{g}\right)$$

and so we must have

$$\operatorname{Re} s(g, 0) = 0.$$

A computation shows that:

$$\operatorname{Re} s(g, 0) = -\lambda(A - 2\zeta(z_0)).$$

So we get that $A = 2\zeta(z_0)$. In view of $(*)$ we have

$$2\omega_1 \zeta(z_0(\theta)) = 2\eta_1 z_0(\theta) + i\theta - i\pi. \quad (T)$$

Let $h(\theta)$ be the function on $[0, \pi]$ defined by:

$$\forall \theta \in [0, \pi], \quad h(\theta) = \frac{1}{i} [2\omega_1 \zeta(z_0(\theta)) - 2\eta_1 z_0(\theta) - i\theta + i\pi].$$

As T^2 is a rectangular torus we have:

$$\forall y \in \mathbf{R}, \quad \operatorname{Re}(\zeta(\omega_1 + y\omega_3)) = \eta_1 = \zeta(\omega_1),$$

and then:

$$\forall \theta \in [0, \pi], \quad \operatorname{Re}[\zeta(z_0(\theta))] = \eta_1;$$

we deduce that $h(\theta)$ is a real-valued function.

Now let us look for solutions θ of (T) , i.e. zeros of the h -function, we have:

$$\begin{aligned} h(0) &= \frac{1}{i} [2\omega_1 \zeta(\omega_2) - 2\eta_1 \omega_2 + \pi i] \\ &= \frac{1}{i} [2\omega_1 \eta_2 - 2\eta_1 \omega_2 + \pi i] \\ &= 0 \quad \text{by Legendre relation, (see the Appendix)} \end{aligned}$$

and

$$h(\pi) = \frac{1}{i} [2\omega_1 \zeta(\omega_1) - 2\eta_1 \omega_1] = \frac{1}{i} [2\omega_1 \eta_1 - 2\eta_1 \omega_1] = 0.$$

So 0 and π are solutions of (T) . Let us show there is no other solution in $[0, \pi]$. Suppose there is another solution θ in $]0, \pi[$, then by Rolle's theorem the function $h'(\theta)$ would have at least two distinct zeros in $]0, \pi[$, but we have

$$h'(\theta) = \frac{1}{i} \left[\frac{2\omega_1 \omega_3}{\pi} P(z_0(\theta)) + 2 \frac{\eta_1 \omega_3}{\pi} \right].$$

$z_0(\theta)$ is a strictly monotone function of θ with values in $L = \{\omega_1 + y\omega_3, 0 \leq y \leq 1\}$. Furthermore P is a strictly monotone and real-valued function on L . We deduce that $h'(\theta)$ is a strictly monotone function on $[0, \pi]$ and then $h'(\theta)$ cannot have two zeros on $]0, \pi[$.

So the only solutions of (T) are $\theta = 0$ and $\theta = \pi$, but we then get $z_0 = \omega_2$ or $z_0 = \omega_1$. Let us show that $z_0 = \omega_1$ does not work. For that let us assume that $z_0 = \omega_1$. Then $(g'/g)(z)$ must have a single pole at 0 and ω_1 , so:

$$\forall z \in \mathbf{C}, \quad \frac{g'}{g}(z) = \frac{a_1 P(z) + a_0 + bP'(z)}{P(z) - P(\omega_1)}, \quad a_1, a_0, b \in \mathbf{C},$$

with: $a_1 P(\omega_1) + a_0 = 0$, we deduce that:

$$\frac{g'}{g}(z) = a_1 + b \frac{P'(z)}{P(z) - P(\omega_1)}.$$

As before we must have $a_1 \in i\mathbf{R}$ and $b \in \mathbf{R}$, furthermore:

$$\forall z \in \mathbf{C}, \quad g(z) = \lambda e^{a_1 z} [P(z) - P(\omega_1)]^b.$$

As g must have a double pole at 0 and a double zero at ω_1 we deduce $b = 1$, and

$$\forall z \in \mathbf{C}, \quad g(z) = \lambda e^{a_1 z} [P(z) - P(\omega_1)], \quad \lambda \in \mathbf{C}, \quad a_1 \in i\mathbf{R}$$

but as we must have $g(z + 2\omega_3) = g(z)$, we deduce that $a_1 \cdot 2\omega_3 = 2\pi i q$, $q \in \mathbf{Z}$ and so $a_1 = 0$ because $a_1 \in i\mathbf{R}$, at last we have:

$$\begin{cases} g(z) = \lambda(P(z) - P(\omega_1)), \quad \lambda \in \mathbf{C}, \\ \eta = \frac{dz}{g(z)}. \end{cases}$$

As we have (property (1) of Lemma 1 with $\omega_3 = i\pi$)

$$\forall z \in \mathbf{C}, \quad X(z + 2\omega_3) = X(z)$$

we deduce that $\operatorname{Re} \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} (\Phi_1, \Phi_2) = 0$, and then

$$\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} \eta = \overline{\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} \eta g^2}.$$

So

$$\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} \frac{dz}{P(z) - P(\omega_1)} = \lambda \overline{\lambda} \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} (P(z) - P(\omega_1)) dz \quad (4)$$

as we have:

$$\frac{1}{P(z) - P(\omega_1)} = \frac{2}{P''(\omega_1)} (P(z - \omega_1) - P(\omega_1)), \quad \text{and} \quad \zeta' = -P$$

we conclude:

$$\Leftrightarrow -\frac{4}{P''(\omega_1)}(\eta_3 + \omega_3 P(\omega_1)) = -2\lambda\bar{\lambda}(\overline{\eta_3 + \omega_3 P(\omega_1)}) \quad (4)$$

and then: (as $\eta_3, \omega_3 \in i\mathbf{R}$ and $P(\omega_1) \in \mathbf{R}$)

$$\lambda\bar{\lambda} = -\frac{2}{P''(\omega_1)} \quad (5)$$

but as the P -function satisfies:

$$P'^2(z) = 4(P(z) - e_1)(P(z) - e_2)(P(z) - e_3), \quad e_i = P(\omega_i), \quad i = 1, 2, 3$$

with e_1, e_2, e_3 real numbers, $e_1 + e_2 + e_3 = 0$, $e_3 < e_2 < e_1$, we deduce:

$$\begin{aligned} P''(\omega_1) &= 2(3e_1^2 + e_1e_2 + e_1e_3 + e_2e_3) \\ &= 4\left(e_1 + \frac{e_3}{2}\right)(e_1 - e_3) = 2(e_1 - e_2)(e_1 - e_3) \end{aligned}$$

and then $P''(\omega_1)$ is a positive number (as $e_1 > e_2$ and $e_1 > e_3$) so (5) cannot be satisfied, and $z_0 \neq \omega_1$.

The only case remaining is $z_0 = \omega_2 = \omega_1 + \omega_3$, as before we have:

$$\begin{cases} g(z) = \lambda(P(z) - P(\omega_2)), \\ \eta = \frac{dz}{g(z)} \end{cases}$$

and the condition (5) becomes

$$\lambda\bar{\lambda} = \frac{-2}{P''(\omega_2)}$$

but now we have

$$\begin{aligned} P''(\omega_2) &= 2(3e_2^2 + e_1e_2 + e_1e_3 + e_2e_3) \\ &= 2(e_1 + 2e_3)(2e_1 + e_3) = 2(e_3 - e_2)(e_1 - e_2) \end{aligned}$$

and $P''(\omega_2)$ is a negative number because $e_1 > e_2$ and $e_3 < e_2$.

At last as we assume that X sends the line $\{\operatorname{Re}(z) = 0\}$ to the x_1 -axis of \mathbf{R}^3 , g must have purely imaginary values on the line $\{\operatorname{Re}(z) = 0\}$, so λ must be in $i\mathbf{R}$ and we conclude:

$$\lambda = \pm i \sqrt{\frac{-2}{P''(\omega_2)}},$$

and without loss of generality we can assume

$$\lambda = i \sqrt{\frac{-2}{P''(\omega_2)}} \in i\mathbf{R}.$$

This concludes the proof of Lemma 2.

Remark. A computation shows that the minimal surface given by the data (g, η) in Lemma 2 is invariant by the u -translation where u is given by:

$$\begin{aligned} u &= \operatorname{Re} \int_{\omega_3/2}^{\omega_3/2 + 2\omega_1} (\Phi_1, \Phi_2, \Phi_3) \\ &= \left(0, \sqrt{\frac{-2}{P''(\omega_2)}} (2\eta_1 + e_2), 2\omega_1 \right). \end{aligned}$$

LEMMA 3. *The data (g, η) given in Lemma 2 is the Weierstrass representation of Riemann's example.*

Proof. Riemann [9] constructed for every rectangular torus an embedded minimal annulus bounded by two parallel horizontal straight lines such that every intersection with any horizontal plane is a circle. Inversely we just saw in Lemmas 1 and 2 that if such a surface exists then its Weierstrass representation is necessarily the one given in Lemma 2; this concludes the proof of lemma 3 and the proof of Theorem 1.

Remarks

(1) In [3], Hoffman, Karcher and Rosenberg gave an explicit proof that each horizontal intersection of the surface defined by the data (g, η) of Lemma 2 is a circle.

(2) Darboux [1] gave explicitly the equations of Riemann's examples in terms of a parameter k , $0 \leq k \leq 1$ associated to a rectangular torus T^2 (that is for every rectangular torus we can associate a real number k , $0 \leq k \leq 1$ and conversely for

each such number there is a corresponding rectangular torus) and the Jacobi function associated to T^2 . For more details about these concepts see Gerretsen and Sansone [2] p. 286.

(3) In [11] Appendix C we gave the following Weierstrass representation for Riemann's examples using the equations given by Darboux, see Remark 2 above). Following the notations of [11], we set $2K' = \omega_1$, $2iK = \omega_3$, $2K' + 2iK = \omega_2$ and we use the variable z instead of $(z - K')$ in [11]. Let Γ denote the lattice generated by $(2\omega_1, 2\omega_3) = (4K', 4iK)$ and let P stand for the Weierstrass P -function on the torus \mathbf{C}/Γ . Set

$$\begin{cases} g_1(z) = \frac{kk'}{4} \frac{1}{P(z) - P(\omega_2)}, \\ \eta_1 = \frac{2}{k} (P(z) - P(\omega_2)) dz = \frac{k'}{2} \frac{dz}{g(z)}. \end{cases}$$

We want to show that the data (g_1, η_1) defines the same surface as the data given in Lemma 2, up to a rigid motion of \mathbf{R}^3 . This will give a new proof of Lemma 3.

To see this recall that (see again [11] p. 60):

$$\begin{cases} k^2 + k'^2 = 1; \quad h^2 = \frac{e_2 - e_3}{e_1 - e_3}; \quad k, k' \in [0, 1], \\ e_1 - e_3 = \frac{1}{4} \quad \left(\text{since } K' = h\omega_1 \text{ with } h = \frac{1}{2} \right). \end{cases} \quad (*)$$

We have (since $1/(P(z) - P(\omega_2)) = (2/P''(\omega_2))(P(z - \omega_2) - P(\omega_2))$):

$$g_1(z) = \frac{kk'}{2P''(\omega_2)} (P(z - \omega_2) - P(\omega_2)),$$

but again we can make a change of variable and put z instead of $(z - \omega_2)$, so:

$$g_1(z) = \frac{kk'}{2P''(\omega_2)} (P(z) - P(\omega_2)).$$

Using $(*)$ and since $P''(\omega_2) = 2(e_3 - e_2)(e_1 - e_2)$ a simple computation shows that:

$$\frac{kk'}{2P''(\omega_2)} = -\sqrt{\frac{-2}{P''(\omega_2)}}.$$

So we have:

$$\begin{cases} g_1(z) = -\sqrt{\frac{-2}{P''(\omega_2)}} (P(z) - P(\omega_2)), \\ \eta_1 = \frac{k'}{2} \frac{dz}{g_1(z)}. \end{cases}$$

Finally the $3\pi/2$ rotation about the x_3 -axis in \mathbf{R}^3 applied to the surface defined by (g_1, η_1) gives the following data for the rotated surface:

$$\begin{cases} g_2(z) = i \sqrt{\frac{-2}{P''(\omega_2)}} (P(z) - P(\omega_2)), \\ \eta_2 = \frac{k'}{2} \frac{dz}{g_2(z)}, \end{cases}$$

that is:

$$\begin{cases} g_2(z) = g(z), \\ \eta_2 = \frac{k'}{2} \eta, \end{cases}$$

where (g, η) is the data given in Lemma 2. Then (g_2, η_2) defines the same surface as (g, η) up to the $k'/2$ homothety in \mathbf{R}^3 , and so the date (g, η) given in Lemma 2 defines Riemann's examples.

(4) In fact in Lemmas 1 and 2 we just need $M_{/S_{2\theta}}$ to be an immersed two-punctured torus in $\mathbf{R}^3_{/S_{2\theta}}$ with finite total curvature and two embedded planar ends. That is we can remove the hypothesis " M is properly embedded" by " $M_{/S_{2\theta}}$ has finite total curvature and embedded planar ends".

§3. Proof of Theorem 2

To show that the data (g_k, η_k) defines an immersed mineral surface in $\mathbf{R}^3_{/z}$ we just need to verify the period conditions, that is if γ and μ are the paths on T^2 defined by:

$$\gamma(t) = \frac{\omega_1}{2} + 2t\omega_3, \quad t \in [0, 1],$$

$$\mu(t) = 2t\omega_1 + \frac{\omega_3}{2}, \quad t \in [0, 1],$$

it is enough to show that:

$$\operatorname{Re} \int_{\gamma} (\Phi_1, \Phi_2, \Phi_3) = (0, 0, 0), \quad (1)$$

where:

$$\Phi_1 = \frac{\eta_k}{2} (1 - g_k^2), \quad \Phi_2 = i \frac{\eta_k}{2} (1 + g_k^2), \quad \Phi_3 = \eta_k g_k.$$

This claim holds since the forms $\Phi_i, i = 1, 2, 3$ have no residue (because the P -function of Weierstrass is an even function). Let us remark that if the conditions (1) are satisfied then the forms (Φ_1, Φ_2, Φ_3) must have periods on the path μ , that is:

$$\operatorname{Re} \int_{\mu} (\Phi_1, \Phi_2, \Phi_3) \neq (0, 0, 0),$$

otherwise the data (g_k, η_k) defines an immersed minimal surface with finite total curvature and two parallel flat ends in \mathbf{R}^3 , but the “Half space theorem” of Hoffman and Meeks [4] shows this situation is impossible.

Let us assume for a while that (1) is satisfied. Let I be the map on T^2 defined by:

$$\forall z \in T^2, \quad I(z) = -\bar{z}.$$

Let us remark that, as $P''(\omega_2)$ is a negative real number (see the proof of Lemma 2 in §2), $\lambda_k \in i\mathbf{R}$. We deduce that:

$$\begin{aligned} \forall z \in T^2, \quad g_k(I(z)) &= \lambda_k [P(-\bar{z}) - P(\omega_2)]^{2k+1} \\ &= \overline{-g_k(z)}. \end{aligned}$$

Then:

$$I_*(\eta_k) = -\frac{d\bar{z}}{g_k(I(z))} = \frac{d\bar{z}}{\overline{g_k(z)}} = \bar{\eta}_k,$$

so that:

$$I_*(\Phi_1) = \bar{\Phi}_1; \quad I_*(\Phi_2) = -\bar{\Phi}_2; \quad I_*(\Phi_3) = -\bar{\Phi}_3.$$

Calling X_k the minimal immersion defined, up to a translation in \mathbf{R}^3 , by (g_k, η_k) , we deduce that:

$$\forall z \in T^2 - \{0, \omega_2\}, \quad X_k(I(z)) = S_1[X_k(z)],$$

where S_1 is the reflection about the x_1 -axis. This shows that X_k sends the line C_1 defined by:

$$C_1(t) = 2t\omega_3, \quad t \in]0, 1[$$

to D_1 which is the x_1 -axis in the horizontal plane $P_1 = \{x_3 = 0\}$.

In the same way if J is the map on T^2 defined by:

$$\forall z \in T^2, \quad J(z) = 2\omega_1 - \bar{z},$$

we can show that:

$$\forall z \in T^2 - \{0, \omega_2\}, \quad X_k(J(z)) = S_2[X_k(z)],$$

where S_2 is the reflection about D_2 , a straight line parallel to D_1 in a horizontal plane P_2 , P_2 distinct from P_1 . Then X_k sends the line C_2 of T^2 on D_2 in \mathbf{R}^3 where

$$C_2(t) = \omega_1 + 2t\omega_3, \quad t \in [0, 1], \quad t \neq \frac{1}{2}.$$

Let

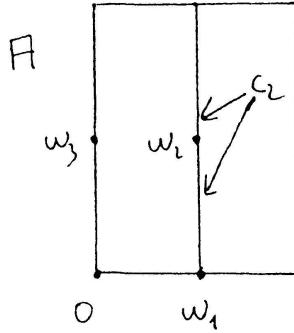
$$A = \{z \in T^2 / 0 \leq \operatorname{Re}(z) \leq \omega_1, z \neq 0, z \neq \omega_2\}.$$

As $\Phi_3 = 1$ and hence:

$$(X_k)_3(z) = \operatorname{Re}(z), \quad \forall z \in T^2 - \{0, \omega_2\},$$

we see that X_k sends $A \cap T^2$ to the slab of \mathbf{R}^3 bounded by P_1 and P_2 .

Of course $X_k(A)$ is a minimal immersed annulus in \mathbf{R}^3 between P_1 and P_2 and bounded by $D_1 \cup D_2$.



Finally denoting $(X_k)_i$ by X_i , $1 = 1, 2, 3$ for simplicity, we have:

$$\forall z \in T^2 - \{0, \omega_2\}, \quad (X_1 - iX_2)(z) = \int_{z_0}^z \eta_k - \overline{\int_{z_0}^z \eta_k g_k^2},$$

therefore, for z near 0 we have

$$\begin{aligned} (X_1 - iX_2)(z) &\simeq \int_{z_0}^z \lambda_k \cdot \frac{dz}{z^{2(2k+1)}} \\ &\simeq -\frac{\overline{\lambda_k}}{4k+1} \left(\frac{1}{z^{4k+1}} \right) \end{aligned}$$

and so the projection of the end near 0 on a horizontal plane is a $(4k+1)$ to 1 map.

Near the other end ω_2 we also have:

$$\begin{aligned} (X_1 - iX_2)(z) &\simeq \int_{z_0}^z \left(\frac{2}{p''(\omega_2)} \right)^{2k+1} \frac{1}{\lambda_k} \cdot \frac{dz}{(z - \omega_2)^{2(2k+1)}} \\ &\simeq \frac{\lambda k}{3k+1} \cdot \frac{1}{(z - \omega_2)^{4k+1}}, \end{aligned}$$

and again the projection of the end ω_2 on a horizontal plane is a $(4k+1)$ to 1 map.

So it remains to show that conditions (1) hold. As we have $\Phi_3 = 1$ we deduce:

$$\begin{aligned} &\Leftrightarrow \operatorname{Re} \int_{\gamma} \Phi_1 = 0 \quad \text{and} \quad \operatorname{Re} \int_{\gamma} \Phi_2 = 0 \\ &\Leftrightarrow \int_{\gamma} \eta_k = \overline{\int_{\gamma} \eta_k g_k^2}. \end{aligned} \tag{1}$$

As $\eta_k = dz/g_k(z)$ we have:

$$\begin{aligned} &\Leftrightarrow \int_{\gamma} \frac{dz}{g_k(z)} = \overline{\int_{\gamma} g_k(z) dz} \\ &\Leftrightarrow \int_{\gamma} \frac{dz}{(P(z) - P(\omega_2))^{2k+1}} = \lambda_k \bar{\lambda}_k \overline{\int_{\gamma} (P(z) - P(\omega_2))^{2k+1} dz}, \end{aligned} \quad (1)$$

but:

$$\frac{1}{P(z) - P(\omega_2)} = \frac{2}{P''(\omega_2)} (P(z - \omega_2) - P(\omega_2)).$$

So:

$$\begin{aligned} &\Leftrightarrow \lambda_k \bar{\lambda}_k \overline{\int_{\gamma} (P(z) - P(\omega_2))^{2k+1} dz} = \left(\frac{2}{P''(\omega_2)} \right)^{2k+1} \int_{\gamma} (P(z - \omega_2) - P(\omega_2))^{2k+1} dz \\ &\Leftrightarrow \lambda_k \bar{\lambda}_k \sum_{q=0}^{2k+1} C_{2k+1}^q (-P(\omega_2))^{2k+1-q} \overline{\int_{\gamma} P(z)^q dz} \\ &= \left(\frac{2}{P''(\omega_2)} \right)^{2k+1} \sum_{q=0}^{2k+1} C_{2k+1}^q (-P(\omega_2))^{2k+1-q} \cdot \int_{\gamma} P(z - \omega_2)^q dz. \end{aligned} \quad (1)$$

The following lemma shows the last equality is true.

LEMMA 1. *For every positive integer q we have:*

$$\int_{\gamma} P(z)^q dz = - \overline{\int_{\gamma} P^q(z - \omega_2) dz}.$$

Assuming Lemma 1 we have:

$$\Leftrightarrow \lambda_k \bar{\lambda}_k = - \left(\frac{2}{P''(\omega_2)} \right)^{2k+1}$$

which is true as $\lambda_k = i\sqrt{(-2/P''(\omega_2))^{2k+1}}$. So it just remains to prove Lemma 1.

Proof of Lemma 1. As T^2 is a rectangular torus we have the following Laurent series for the Weierstrass P -function, see [2], [8] or the Appendix.

$$P(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} a_n z^n$$

with a_n real numbers.

We deduce that for any positive integer q the Laurent series of $P^q(z)$ has the following type:

$$P^q(z) = \sum_{n=1}^q b_{-n} z^{-2n} + \sum_{n=0}^{+\infty} b_n z^n,$$

with b_j real numbers, $j = 1, \dots, n$.

Also if $P^{(k)}(z)$ is the k th-derivative of P we have:

$$P^{(k)}(z) = (-1)^k \frac{(k+1)!}{z^{k+2}} + F_k(z),$$

where F_k is a holomorphic map near 0. We deduce that:

$$P^q(z) - \sum_{n=1}^q \frac{b_{-n}}{(2n-1)!} P^{(2n-2)}(z)$$

is an elliptic function on T^2 without poles, so this function is constant and taking $z = \omega_1$, we see that this constant c must be real, so:

$$P^q(z) = \sum_{n=1}^q \frac{b_{-n}}{(2n-1)!} P^{(2n-2)}(z) + c$$

with b_{-n} and c real numbers. Then:

$$\begin{aligned} \int_{\gamma} P^q(z) dz &= \sum_{n=1}^q \frac{b_{-n}}{(2n-1)!} \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} P^{(2n-2)}(z) dz + c \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} dz \\ &= b_{-1} \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} P(z) dz + 2\omega_3 c \end{aligned}$$

since for $k \geq 2$:

$$\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} P^{(k)}(z) dz = P^{(k-1)}\left(\frac{\omega_1}{2} + 2\omega_3\right) - P^{(k-1)}\left(\frac{\omega_1}{2}\right) = 0.$$

Then, as $\zeta'(z) = -P(z)$ we have:

$$\begin{aligned} \int_{\gamma} P^q(z) dz &= b_{-1} \left[-\zeta\left(\frac{\omega_1}{2} + 2\omega_3\right) + \zeta\left(\frac{\omega_1}{2}\right) \right] + 2\omega_3 c \\ &= 2\eta_3 b_{-1} + 2\omega_3 c. \end{aligned}$$

In the same way, as:

$$P^q(z - \omega_2) = \sum_{n=1}^q \frac{b_{-n}}{(2n-1)!} P^{(2n-2)}(z - \omega_2) + c,$$

we also have:

$$\int_{\gamma} P^q(z - \omega_2) dz = -2\eta_3 b_{-1} + 2\omega_3 c.$$

This concludes the proof because c and b_{-1} are real numbers and ω_3 and η_3 are purely imaginary numbers.

Remarks

(1) Following the arguments of Lemmas 1 and 2 in §2, it is easy to show that the surfaces S_k are the only minimal immersed surfaces between P_1 and P_2 bounded by $D_1 \cup D_2$ with finite total curvature.

(2) We do not know if there exists surfaces like S_k which are bounded by two horizontal lines D_1, D_2 and make a non zero angle θ . Of course for $k = 0$, Theorem 1 shows that such a surface does not exist.

§4. Appendix

Let T^2 be a torus \mathbf{C}/Γ where Γ is the lattice of \mathbf{C} given by:

$$\Gamma = \{p \cdot 2\omega_1 + q \cdot 2\omega_3, p, q \in \mathbf{Z}, \omega_1 \in \mathbf{R}_*^+, \operatorname{Im}(\omega_3) \in \mathbf{R}_*^+\}.$$

The Weierstrass P -function is a special function defined on T^2 which is a meromorphic function on \mathbf{C} such that:

P has a pole of order two at each point of Γ and:

$$\forall z \in \mathbf{C} - \Gamma, \quad P(z + 2\omega_i) = P(z), \quad i = 1, 3.$$

$$\text{If } \omega_2 = \omega_1 + \omega_3, \quad P(\omega_1) + P(\omega_2) + P(\omega_3) = 0.$$

P has the following Laurent series:

$$P(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma - \{0\}} \left\{ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right\}.$$

$$P \text{ is an even function: } \forall z \in \mathbf{C} - \Gamma, \quad P(-z) = P(z).$$

There are two other functions related to P , namely the ζ and σ -functions: The ζ -function satisfies:

$$\forall z \in \mathbf{C} - \Gamma, \quad \zeta'(z) = -P(z).$$

If $\eta_i = \zeta(\omega_i)$, $i = 1, 2, 3$, then $\eta_2 = \eta_1 + \eta_3$, so ζ is uniquely defined on $\mathbf{C} - \Gamma$. We have:

$$\forall z \in \mathbf{C} - \Gamma, \quad \zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i, \quad i = 1, 2, 3,$$

and then ζ is not defined on T^2 because z and $(z + 2\omega_i)$ represent the same point on T^2 .

$$\zeta \text{ is an odd function: } \forall z \in \mathbf{C} - \Gamma, \quad \zeta(-z) = -\zeta(z).$$

The following Legendre relation holds.

$$\omega_2\eta_1 - \omega_1\eta_2 = i\frac{\pi}{2}.$$

The σ -function satisfies:

$$\forall z \in \mathbf{C} - \Gamma, \quad \frac{\sigma'}{\sigma}(z) = \zeta(z),$$

and then σ is an holomorphic function on \mathbf{C} , furthermore:

$$\lim_{z \rightarrow 0} \frac{\sigma(z)}{z} = 1,$$

so that σ is uniquely determined on \mathbf{C} .

We also have:

$$\forall z \in \mathbf{C}, \quad \sigma(z + 2\omega_i) = -e^{2\eta_i(z + \omega_i)} \sigma(z). \quad (\#)$$

σ is an odd function: $\forall z \in \mathbf{C}, \sigma(-z) = -\sigma(z)$.

Furthermore if T^2 is a rectangular torus, that is if $\omega_3 \in i\mathbf{R}_*^+$, we have:

$$\forall z \in \mathbf{C} - \Gamma, \quad P(\bar{z}) = \overline{P(z)}; \quad \zeta(\bar{z}) = \overline{\zeta(z)}.$$

$$\forall z \in \mathbf{C}, \quad \sigma(\bar{z}) = \overline{\sigma(z)}.$$

For more details about those functions see Gerretsen–Sansone [2] or Molk–Tannery [8].

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