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# On the minimal surfaces of Riemann

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## §1. Introduction

Let  $\gamma_i, i = 1, 2$  be plane Jordan curves in horizontal planes  $P_i, i = 1, 2, P_1 \neq P_2$ , we know that under conditions on  $\gamma_i$ , for example  $\gamma_1$  not too far from  $\gamma_2$ ,  $\gamma_1$  and  $\gamma_2$  bound a least area minimal annulus  $M$  between  $P_1$  and  $P_2$ . Meeks and White [7] were able to prove that when the  $\gamma_i$  are convex there are at most two minimal annuli bounded by  $\gamma_1 \cup \gamma_2$ . Assuming that the  $\gamma_i$  are convex, Shiffman [10] proved that if  $M$  is a minimal annulus bounded by  $\gamma_1 \cup \gamma_2$  then for each horizontal plane  $P$  between  $P_1$  and  $P_2$ , the intersection  $P \cap M$  is again a convex Jordan curve, furthermore if  $\gamma_1$  and  $\gamma_2$  are circles, then  $P \cap M$  is also a circle.

In view of this last result it is natural to ask what happens when two straight lines replace the Jordan curves. Namely, let  $D_i, i = 1, 2$ , be straight lines in horizontal planes  $P_i, i = 1, 2, P_1 \neq P_2$ . Let us assume that  $D_1$  makes an angle  $\theta$  with  $D_2, \theta \in [0, \pi]$ . Now let  $M$  be a minimal annulus between  $P_1$  and  $P_2$  bounded by  $D_1 \cup D_2$ . If  $P$  is a horizontal plane between  $P_1$  and  $P_2$ , what can we say about the intersection  $P \cap M$ ?

For example, if  $\theta = 0$  i.e.  $D_1$  and  $D_2$  are parallel, Riemann [9] has constructed a minimal embedded annulus  $S$  between  $P_1$  and  $P_2$  bounded by  $D_1 \cup D_2$ . Furthermore the intersection with any horizontal plane is a circle, see [3] for a detailed description of Riemann's examples.

In the same paper [3], Hoffman, Karcher and Rosenberg proved that if  $D_1$  and  $D_2$  are parallel, i.e.  $\theta = 0$ , then the only minimal properly embedded annulus between  $P_1$  and  $P_2$ , bounded by  $D_1 \cup D_2$  is precisely Riemann's example.

Here we shall prove that the case  $\theta \neq 0$  does not occur, namely we show the following.

**THEOREM 1.** *Let  $D_i, i = 1, 2$  be straight lines in horizontal planes  $P_i, i = 1, 2, P_1 \neq P_2$ , let  $\theta \in [0, \pi]$  be the angle that  $D_1$  makes with  $D_2$ . Let us assume there exists a minimal properly embedded annulus  $M$  between  $P_1$  and  $P_2$  bounded by  $D_1 \cup D_2$ .*

*Then necessarily  $\theta = 0$  and  $M$  is Riemann's example  $S$ .*

Let us assume now  $\theta = 0$ , i.e.  $D_1$  and  $D_2$  parallel. We can generalize Riemann's examples to yield a family of minimal surfaces  $S_k$  with the following properties:

- (1)  $S_0 = S$  where  $S$  is Riemann's example.
- (2) For every integer  $k, k \geq 0$ ,  $S_k$  is a minimal immersed annulus between  $P_1$  and  $P_2$ , bounded by  $D_1 \cup D_2$  such that after reflection about the lines  $D_1, D_2, \dots$  we get a complete minimal surface in  $\mathbb{R}^3$ , which we call again  $S_k$ . Furthermore  $S_k$  is invariant under the translation  $X \rightarrow X + 2u$  where  $u$  is the vector orthogonal to  $D_1$  translating  $D_1$  to  $D_2$ . Also each end of  $S_k$  in  $\mathbb{R}^3$  is a flat horizontal end, i.e. an end asymptotic to a horizontal plane, and the projection of every end over any horizontal plane is a  $(4k + 1)$  to 1 map.

The following gives the Weierstrass representation of this family.

**THEOREM 2.** Let  $T$  be a rectangular torus, i.e.  $T^2 = \mathbb{C}/\Gamma$  where  $\Gamma = \{2\omega_1 p + 2\omega_3 q, p, q \in \mathbb{Z}, \omega_1 \in \mathbb{R}_*^+, \omega_3 \in i\mathbb{R}_*^+\}$ .

Then for every  $k, k \in \mathbb{N}$  the following data  $(g_k, \eta_k)$  is the Weierstrass representation of the surface  $S_k$  described above.

$$g_k = \lambda_k [P(z) - P(\omega_2)]^{2k+1},$$

$$\eta_k = \frac{dz}{g_k(z)} = \frac{dz}{\lambda_k [P(z) - P(\omega_2)]^{2k+1}},$$

where  $\omega_2 = \omega_1 + \omega_3$ ,  $z \in T^2 - \{0, \omega_2\}$ ,  $\lambda_k = i\sqrt{(-2/P''(\omega_2))^{2k+1}}$  and  $P$  is the  $P$ -function of Weierstrass.

We prove Theorems 1 and 2 in §2 and §3 respectively.

## §2. Proof of Theorem 1

We are going to use the Weierstrass representation for a minimal surface, see Lawson [5] p. 113, and the reflection principle, namely if a minimal surface in  $\mathbb{R}^3$  has a piece  $\gamma$  of a straight line in its boundary then we can extend minimally this surface along  $\gamma$  by the reflection about the line defined by  $\gamma$ , see [5] p. 82.

Let us take the notation of Theorem 1. By the reflection principle we can extend  $M$  to a complete properly embedded minimal surface in  $\mathbb{R}^3$ , let us call  $M$  again this surface. As the composition of the two reflections about  $D_1$  and  $D_2$  is a screw-motion  $S_{2\theta}$ , i.e. the  $2\theta$ -rotation with respect to the  $x_3$ -axis composed with a vertical

translation, we get that  $M$  is globally invariant by the screw-motion  $S_{2\theta}$ :

$$S_{2\theta}(M) = M.$$

Then we get a quotient surface  $M/S_{2\theta}$  in  $\mathbf{R}^3/S_{2\theta}$ . By construction this surface is topologically a two-punctured torus properly and minimally embedded in  $\mathbf{R}^3/S_{2\theta}$ . By a theorem of Meeks and Rosenberg [6] we deduce that  $M/S_{2\theta}$  has finite total curvature. From this we get that each end of  $M$  is conformally a punctured disc, see [5] p. 130. At last we deduce that the ends of  $M$  are parallel, flat, embedded and also the Gauss map  $g$  of  $M$  has order two at each end (this last claim comes from the fact that if  $E_1$  is the end of  $M$  passing through the straight line  $D_1$  then, by construction,  $E_1 - P_1$  has two connected components). Subsequently we assume that  $D_1$  is the  $x_1$ -axis in the plane  $P_1 = \{x_3 = 0\}$ .

**LEMMA 1.** *The extended complete minimal surface  $M$  in  $\mathbf{R}^3$  is parameterized by  $C = C - (\Gamma \cup (z_0 + \Gamma))$  where:*

$$\begin{cases} \Gamma = \{2\omega_1 \cdot p + 2\pi i \cdot q, p, q \in \mathbf{Z}, \omega_1 \in \mathbf{R}_*^+\}, \\ z_0 \in \omega_1 + y\pi i, \quad 0 \leq y \leq 1. \end{cases}$$

Furthermore if  $X$  is the minimal immersion of  $C$  onto  $M \subset \mathbf{R}^3$ ,  $X$  must satisfy:

- $X$  sends the vertical lines  $\{\operatorname{Re}(z) = c, c \neq k\omega_1, k \in \mathbf{Z}\}$  to horizontal Jordan curves in  $\mathbf{R}^3$ .
- $X$  sends the vertical lines  $\{\operatorname{Re}(z) = k\omega_1, k \in \mathbf{Z}\}$  to horizontal straight lines in  $\mathbf{R}^3$  and particularly  $X$  sends  $\{\operatorname{Re}(z) = 0\}$  to the  $x_1$ -axis  $D_1$  and  $\{\operatorname{Re}(z) = \omega_1\}$  to  $D_2$ .

At last we have:

$$\forall z \in C, \quad X(z + 2\pi i) = X(z), \tag{1}$$

$$X(2\omega_1 - \bar{z}) = S_{D_2}[X(z)], \tag{2}$$

$$X(z + 2\omega_1) = S_{2\theta}[X(z)], \tag{3}$$

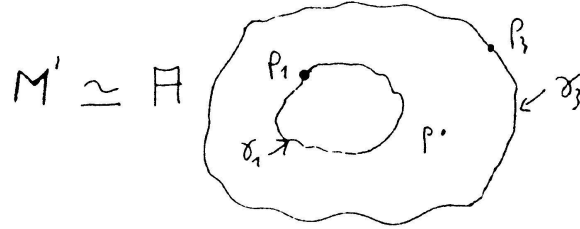
where  $S_{D_2}$  is the reflection about  $D_2$  and  $S_{2\theta} = S_{D_1} \circ S_{D_2}$ .

*Proof of Lemma 1.* Let  $M_1$  be the piece of  $M$  bounded by  $D_1 \cup D_2$  and let  $M_2$  be the reflection of  $M_1$  about  $D_2$ , i.e.  $M_2 = S_{D_2}(M_1)$ . Let  $D_3$  be  $S_{D_2}(D_1)$ , so that  $M_2$  is bounded by  $D_2$  and  $D_3$ .

By construction  $M' = M_1 \cup M_2$  is homeomorphic to a one-punctured planar annulus  $A$  bounded by two Jordan curves  $\gamma_1, \gamma_3$  each one with a point  $p_i, i = 1, 3$  removed.



Let  $Z$  be the minimal immersion of  $A$  onto  $M'$ , let  $A$  have the conformal structure induced by  $Z$ . As each end of  $M$  is conformally a one-punctured disc we deduce that  $A$  is conformally a one-punctured annulus bounded by two Jordan curves  $\gamma_1, \gamma_3$  each one with a point  $p_i, i = 1, 3$  removed.



Assume that  $Z(\gamma_i) = D_i, i = 1, 3$ .  $Z$  sends a neighbourhood of  $p$  to the end passing through the line  $D_2$ . Assume that  $D_2$  belongs to the plane  $P_2 = \{x_3 = c, c > 0\}$ . Then  $D_3$  belongs to the horizontal plane  $P_3 = \{x_3 = 2c\}$  (recall that  $P_1 = \{x_3 = 0\}$ ).

Let  $Z_3$  be the third coordinate function of  $Z$ ,  $Z_3$  is a harmonic function. By construction of  $M$ , the line  $D_2$  is the only part of  $M$  in the plane  $P_2$ , that is:

$$M \cap P_2 = D_2,$$

as  $Z_3(\gamma_1) = 0$  and  $Z_3(\gamma_3) = 2c$ , we deduced that  $\{Z_3^{-1}(c) \cup p\}$  is an embedded closed curve in the interior of  $\bar{A}$ , and by the maximum principle applied to the function  $Z_3$ , this curve must be connected, so that  $\{Z_3^{-1}(c) \cup p\}$  is a Jordan curve  $\gamma_2$  in the interior of  $A$ .

Now let  $\alpha$  be a real number with  $\alpha \neq c$  and  $0 \leq \alpha \leq 2c$ , again by the maximum principle we get that  $Z_3^{-1}(\alpha)$  is a Jordan curve  $\gamma$  of  $\bar{A}$ . Hence we have a foliation  $(\gamma_i)_{1 \leq i \leq 3}$  of  $\bar{A}$  with Jordan curves  $\gamma_i$ , so that  $Z_3$  is constant over each curve  $\gamma_i$ , i.e.:

$$\forall i \in [1, 3], \quad Z_3(\gamma_i) = c_i \quad \text{with: } 0 \leq c_i \leq 2c \quad \text{and} \quad c_1 = 0; c_2 = c; c_3 = 2c.$$

Let  $Z_3^*$  be a harmonic conjugate of  $Z_3$  over  $\bar{A}$ , we have locally:

$$Z_3^*(z) = \int_{\alpha} *dZ_3,$$

where  $\alpha$  is a path between a base point  $z_0$  and  $z$ , and  $*dZ_3$  is defined by:

$$*dZ_3 = \frac{\partial Z_3}{\partial x} dy - \frac{\partial Z_3}{\partial y} dx, \quad z = x + iy.$$

It can happen that  $Z_3^*$  is not globally defined on  $\bar{A}$ , i.e. if  $\gamma$  is any Jordan curve in  $\bar{A}$  generating  $\Pi_1(\bar{A})$ , let  $a$  be the real number defined by:

$$a = \int_{\gamma} *dZ_3.$$

Then  $Z_3^*$  is well defined on  $\bar{A}$  if and only if  $a = 0$ . In case  $a \neq 0$  let us consider the function:

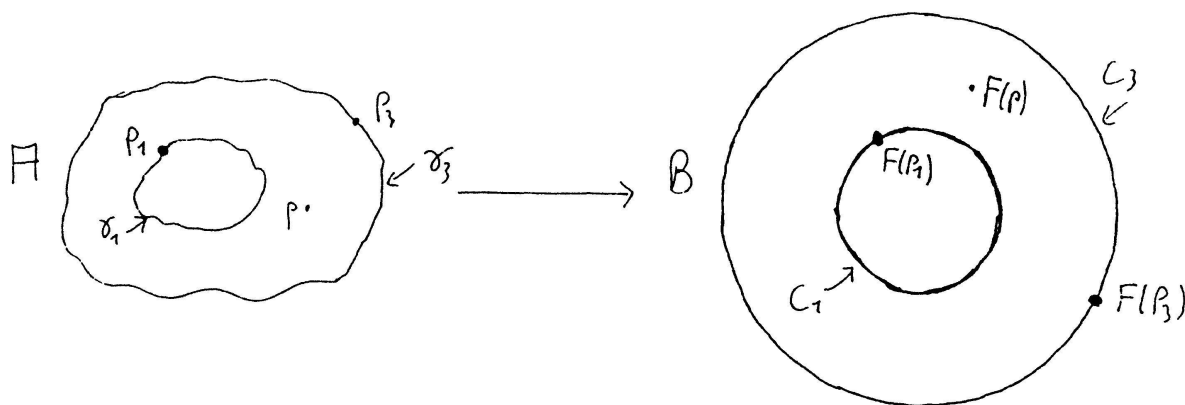
$$F(z) = \exp \left[ \frac{2\pi}{a} (Z_3(z) + iZ_3^*(z)) \right].$$

In case  $a = 0$  we put:

$$F(z) = \exp [Z_3(z) + iZ_3^*(z)],$$

in any case  $F$  is a well defined map of  $\bar{A}$  into  $\mathbb{C}$ ,  $F$  maps the Jordan curves  $\{\gamma_i\}_{1 \leq i \leq 3}$  to concentric circles of  $\mathbb{C}$ :  $\{|u| = \exp [(2\pi/a)c_i]$  or  $\exp [c_i]$  depending on the expression of  $F\}$ .  $F$  maps  $\bar{A}$  on an annulus  $B$  of  $\mathbb{C}$  bounded by two concentric circles  $C_1, C_3$  with

$$F(\gamma_i) = C_i, \quad 1 \leq i \leq 3.$$



It is easy to see that  $F$  is a  $n$ -covering map of  $\bar{A}$  onto  $B$  in  $\mathbb{C}$ , so after composing  $F$  with the  $n$ th root map, we can assume that  $F$  is an embedding of  $\bar{A}$  onto  $B$ .

Let us call  $Y$  the composed map  $Z \circ F^{-1}$ ,

$$B \xrightarrow{Y = Z \circ F^{-1}} \mathbb{R}^3$$

we clearly have  $Y(C_i) = D_i$ ,  $i = 1, 2, 3$  so by the reflection principle if  $I_C$  denotes the reflection in  $\mathbf{C}$  about a circle  $C$  we have:

$$C_3 = I_{C_2}(C_1) \quad (\text{because } Y(C_3) = D_3 = S_{D_2}(D_1))$$

and then  $F(P_3) = I_{C_2}(F(P_1))$ .

Now by reflection principle we extend  $Y$  to a minimal immersion of  $\mathbf{C} - E$  onto  $M$  in  $\mathbf{R}^3$ , where  $E$  is the infinite discrete set of  $\mathbf{C}$  obtained by taking  $F(P_1)$ ,  $F(P)$  and all the successive images of those points by the reflections about the circles  $C_1, C_2, I_{C_1}(C_2), I_{C_2}(C_1) \dots$

Then the exponential map  $z \rightarrow u = \exp(z)$  of  $\mathbf{C}$  sends the vertical lines  $\{\operatorname{Re}(z) = a\}$  of the  $z$ -complex plane on the concentric circles  $\{|u| = e^a\}$  of the  $u$ -complex plane.

$$(\mathbf{C}, z) \xrightarrow{e^z} (\mathbf{C}, u)$$



Up to a homothety of the  $u$ -plane we can assume that  $C_1$  is the circle of radius one. So that if we call  $X$  the composed map:  $X = Y \circ \exp$ ,  $X$  sends the imaginary axis  $\{\operatorname{Re}(z) = 0\}$  to  $D_1$  in  $\mathbf{R}^3$ . Furthermore up to a rotation in the  $u$ -plane, we can assume that  $F(p_1) = 1$ , so that  $\exp(0) = F(p_1) = 1$ . Let denote  $z_0$  the inverse image, by the exponential map, of  $F(p)$  with  $0 \leq \operatorname{Im} z_0 \leq 2\pi$ , and  $\omega_1$  the real part of  $z_0$ .

$$\omega_1 = \operatorname{Re}(z_0).$$

Then  $X$  sends the line  $\{\operatorname{Re}(z) = \omega_1\}$  to the horizontal line  $D_2$  in  $\mathbf{R}^3$ .

It is clear that the inverse image of  $E$  by the exponential map is just  $\Gamma \cup (z_0 + \Gamma)$ , with

$$\Gamma = \{2\omega_1 p + 2\pi i q, p, q \in \mathbf{Z}\}.$$

Furthermore by construction of  $Y$  and by the geometric properties of  $M$  in  $\mathbf{R}^3$  we must have:

$$\forall z \in C = \mathbf{C} - (\Gamma \cup (z_0 + \Gamma)), \quad X(z + 2\pi i) = X(z), \quad (1)$$

$$X(2\omega_1 - \bar{z}) = S_{D_2}[X(z)], \quad (2)$$

$$X(4\omega_1 - \bar{z}) = S_{D_3}[X(z)]. \quad (2)'$$

(1) comes from the fact that  $X = Y \circ \exp$ , and (2) (resp. (2)') comes from the fact that  $M$  is invariant by the reflection about  $D_2$  (resp.  $D_3$ ).

So that combining (2) and (2)' we have:

$$\forall z \in C, \quad X(z + 2\omega_1) = S_{2\theta}[X(z)]. \quad (3)$$

Finally considering the map  $z \rightarrow -z$  of the  $z$ -plane keeping fixed the point 0, and because of (1) we can assume that:

$$0 \leq \operatorname{Re}(z_0) \quad \text{and} \quad \operatorname{Im}(z_0) = y\pi, \quad 0 \leq y \leq 1.$$

### Remarks

(1) From Lemma 1 we see that  $X$  induces a minimal embedding of  $C/_{2\pi i\mathbb{Z}}$  onto  $M$  in  $\mathbb{R}^3$  and a minimal embedding of  $T^2 - \{0, z_0\} = C/_{\Gamma \cup (z_0 + \Gamma)}$  onto  $M/_{S_{2\theta}}$  in  $\mathbb{R}^3/_{S_{2\theta}}$ .

(2) Until now we don't know if  $\omega_1$  is bigger or not than  $\pi$ , so from now we assume  $\pi$  bigger than  $\omega_1$ , the rest of the proof will not be affected.

Let  $\omega_3 = i\pi$ , so that the lattice  $\Gamma$  is:

$$\Gamma = \{2\omega_1 p + 2\pi i q, p, q \in \mathbb{Z}\}.$$

**LEMMA 2.** *With the hypothesis of Theorem 1 we have  $\theta = 0$  and the Weierstrass representation of the minimal immersion  $X$  of  $C$  in  $\mathbb{R}^3$  in Lemma 1 is:*

$$\forall z \in C, \quad g(z) = \lambda[P(z) - P(\omega_2)],$$

$$\eta = \frac{dz}{g(z)} = \frac{dz}{\lambda[P(z) - P(\omega_2)]},$$

where  $P$  is the  $P$ -function of Weierstrass associated to the torus  $T^2 = \mathbb{C}/\Gamma$ ,  $\omega_2 = \omega_1 + \omega_3$  and  $\lambda = i\sqrt{-2/P''(\omega_2)} \in i\mathbb{R}$ .

*Proof of Lemma 2.* We are going to use some basic facts about the  $\sigma$  and  $\zeta$  functions associated with the  $P$ -function of Weierstrass (we recall these facts in the Appendix).

Let us call  $(\Phi_1, \Phi_2, \Phi_3)$  the coordinate forms of  $X$ , i.e.

$$X(z) = (X_1(z), X_2(z), X_3(z)) = \operatorname{Re} \int_{z_0}^z (\Phi_1, \Phi_2, \Phi_3)$$

with  $z_0$  a base point in  $C$  and:

$$\Phi_1 = \frac{1}{2} \eta(1 - g^2); \quad \Phi_2 = \frac{i}{2} \eta(1 + g^2); \quad \Phi_3 = \eta g.$$

As the third coordinate function  $X_3$  is constant on vertical lines  $\{\operatorname{Re}(z) = c\}$  we can assume  $\Phi_3 = 1$  and then  $\eta = dz/g(z)$ .

As  $M_{/S_{20}}$  has two flat horizontal ends parameterized by 0 and  $z_0$  (with  $\operatorname{Re}(z_0) = \omega_1$ , see Lemma 1), we can assume that

$$g(0) = \infty \quad \text{and} \quad g(z_0) = 0,$$

also  $g$  is a 2 to 1 map near each end.

Furthermore on a fundamental domain of  $T^2$  in  $\mathbb{C}$  there is no other point where  $g$  is vertical, if not, up to a rotation in  $\mathbb{R}^3$ , another such point would be an  $n$ -pole of  $g$ ,  $n \geq 1$ , but this point is a regular point of  $M$ , so it must be a  $2n$ -zero of  $\eta$  and then it is an  $n$ -pole of  $\Phi_3 = g\eta$ , but  $\Phi_3 = 1$  as we saw before.

As  $g$  is the Gauss map of  $M$ , up to the stereographic projection of  $S^2$  to  $\mathbb{C} \cup \{\infty\}$ , by the geometric properties of  $M$  in  $\mathbb{R}^3$ ,  $g$  must satisfy:

$$\forall z \in \mathbb{C} \quad \begin{cases} g(z + 2\omega_3) = g(z), & (1) \\ g(2\omega_1 - \bar{z}) = -e^{2i\theta} \overline{g(z)}, & (2) \\ g(z + 2\omega_1) = e^{2i\theta} g(z). & (3) \end{cases}$$

Those properties (1), (2), (3) of  $g$  come from respectively the properties (1), (2), (3) of  $X$  established in Lemma 1.

From (3) we deduce that the map  $(g'/g)(z)$  is well defined on the torus  $T^2 = \mathbb{C}/\Gamma$ . Furthermore  $(g'/g)(z)$  is an elliptic function on  $T^2$  and has a single pole at a point  $z$  in  $T^2$  if and only if  $z$  is a pole or a zero of  $g$ , and  $(g'/g)(z)$  has no other pole on  $T^2$ . From what we saw before we see that  $(g'/g)(z)$  has two single poles on  $T^2$ , so it is an elliptic function of degree two on  $T^2$  with a single pole at 0 and  $z_0$ . The most general form of such a function is:

$$\frac{g'}{g}(z) = \frac{a_1 P(z) + a_0 + bP'(z)}{P(z) - P(z_0)}, \quad a_1, a_0, b \in \mathbb{C}$$

with  $a_1 P(z_0) + a_0 = bP'(z_0)$  because  $-z_0$  is not a pole of  $(g'/g)(z)$ .

From (2) we get that

$$\forall z \in \mathbf{C}, \quad 2\omega_1 - \bar{z} = z \Rightarrow \frac{g'}{g}(z) \in i\mathbf{R}.$$

So  $(g'/g)(z)$  must have purely imaginary values on the line  $\{\operatorname{Re}(z) = \omega_1\}$  in  $\mathbf{C}$ . But as  $\operatorname{Re}(z_0) = \omega_1$ ,  $P(z_0) \in \mathbf{R}$  and then  $P(z) - P(z_0)$  has real values on the line  $\{\operatorname{Re}(z) = \omega_1\}$  also  $P'(z)$  has purely imaginary values on this line, so we have:

$$\forall y \in \mathbf{R}, \quad a_1 P(\omega_1 + y\omega_3) + a_0 + bP'(\omega_1 + y\omega_3) \in i\mathbf{R}.$$

For  $y = 0$  and  $y = 1$  we deduce (as  $P'(\omega_1) = 0 = P'(\omega_1 + \omega_3)$ )

$$a_1 P(\omega_1) + a_0 \in i\mathbf{R},$$

$$a_1 P(\omega_2) + a_0 \in i\mathbf{R},$$

as  $P(\omega_1)$  and  $P(\omega_2)$  are two distinct real numbers, we get that  $a_1$  and  $a_0$  are purely imaginary numbers,  $a_1, a_0 \in i\mathbf{R}$ , so that:

$$\forall y \in \mathbf{R}, \quad bP'(\omega_1 + y\omega_3) \in i\mathbf{R}$$

and then  $b$  is a real number:

$$a_1, a_0 \in i\mathbf{R}, \quad b \in \mathbf{R}.$$

Now let us look for  $g$ , we have:

$$\forall z \in \mathbf{C}, \quad \frac{g'}{g}(z) = a_1 + \frac{a_1 P(z_0) + a_0}{P(z) - P(z_0)} + b \frac{P'(z)}{P(z) - P(z_0)}.$$

We want to show that  $z_0 = \omega_2$ . Let us suppose now that  $z_0 \neq \omega_i$ ,  $i = 1, 2$ , that is:

$$z_0 = \omega_1 + y\omega_3, \quad 0 < y < 1.$$

As  $P'(z_0) \neq 0$  we have

$$\frac{1}{P(z) - P(z_0)} = \frac{1}{P'(z_0)} [\zeta(z - z_0) - \zeta(z + z_0) + 2\zeta(z_0)]$$

as  $a_1 P(z_0) + a_0 = bP'(z_0)$  we get:

$$\frac{g'}{g}(z) = a_1 + 2\zeta(z_0) + b \left[ \zeta(z - z_0) - \zeta(z + z_0) + \frac{P'(z)}{P(z) - P(z_0)} \right]$$

and so

$$\forall z \in \mathbb{C}, \quad g(z) = \lambda e^{[a_1 + 2b\zeta(z_0)]z} \cdot \left[ \frac{\sigma(z - z_0)}{\sigma(z + z_0)} (P(z) - P(z_0)) \right]^b$$

with  $\lambda \in \mathbb{C}$ ,  $a_1 \in i\mathbb{R}$ ,  $b \in \mathbb{R}$ .

But  $g$  must have a double pole at 0 and a double zero at  $z_0$ , now we remark that the function in brackets  $(\sigma(z - z_0)/\sigma(z + z_0))(P(z) - P(z_0))$ , has the same zeros and poles, so we must have  $b = 1$ , so:

$$\forall z \in \mathbb{C}, \quad g(z) = \lambda e^{Az} \frac{\sigma(z - z_0)}{\sigma(z + z_0)} (P(z) - P(z_0))$$

where,  $\lambda \in \mathbb{C}$ ,  $A = a_1 + 2\zeta(z_0)$ .

From the property (#) of the  $\sigma$ -function, see the Appendix, we have:

$$\forall z \in \mathbb{C}, \quad g(z + 2\omega_i) = e^{2\omega_i A - 4\eta_i z_0} g(z), \quad i = 1, 2, 3,$$

from conditions (1) and (3) on  $g$  we get:

$$\begin{cases} 2\omega_1 A - 4\eta_1 z_0 = 2i\theta + 2p\pi i, & p \in \mathbb{Z}, \\ 2\omega_3 A - 4\eta_3 z_0 = 2q\pi i, & q \in \mathbb{Z} \end{cases}$$

and then

$$z_0 = z_0(\theta) = -\frac{2\theta}{2\pi} \omega_3 - p\omega_3 + q\omega_1.$$

By considering the real part and the imaginary part of  $z_0$ , as we know that  $\operatorname{Re}(z_0) = \omega_1$ , and  $0 < \operatorname{Im}(z_0) < -i\omega_3$  we have  $q = 1$  and  $p = -1$ , so:

$$z_0(\theta) = \omega_1 + \left( 1 - \frac{\theta}{\pi} \right) \omega_3, \quad \theta \in [0, \pi]$$

and

$$2\omega_1 A = 4\eta_1 z_0(\theta) + 2i\theta - 2\pi i. \quad (*)$$

On the other hand as the map  $X$  is well defined on  $C$  we must have:

$$\forall \gamma \in \Pi_1(C), \quad \operatorname{Re} \int_{\gamma} (\Phi_1, \Phi_2, \Phi_3) = 0.$$

In particular if  $\gamma$  is a small circle around 0, we must have:

$$\operatorname{Re} \int_{\gamma} (\Phi_1, \Phi_2) = 0,$$

that is

$$\int_{\gamma} \eta = \overline{\int_{\gamma} \eta g^2}$$

and as  $\eta = dz/g(z)$  then:

$$-2\pi \overline{\operatorname{Re} s(g, 0)} = 2\pi \operatorname{Re} s\left(\frac{1}{g}, 0\right) = 0 \quad \left(\text{because } 0 \text{ is a zero of } \frac{1}{g}\right)$$

and so we must have

$$\operatorname{Re} s(g, 0) = 0.$$

A computation shows that:

$$\operatorname{Re} s(g, 0) = -\lambda(A - 2\zeta(z_0)).$$

So we get that  $A = 2\zeta(z_0)$ . In view of  $(*)$  we have

$$2\omega_1 \zeta(z_0(\theta)) = 2\eta_1 z_0(\theta) + i\theta - i\pi. \quad (T)$$

Let  $h(\theta)$  be the function on  $[0, \pi]$  defined by:

$$\forall \theta \in [0, \pi], \quad h(\theta) = \frac{1}{i} [2\omega_1 \zeta(z_0(\theta)) - 2\eta_1 z_0(\theta) - i\theta + i\pi].$$



As  $T^2$  is a rectangular torus we have:

$$\forall y \in \mathbf{R}, \quad \operatorname{Re} (\zeta(\omega_1 + y\omega_3)) = \eta_1 = \zeta(\omega_1),$$

and then:

$$\forall \theta \in [0, \pi], \quad \operatorname{Re} [\zeta(z_0(\theta))] = \eta_1;$$

we deduce that  $h(\theta)$  is a real-valued function.

Now let us look for solutions  $\theta$  of  $(T)$ , i.e. zeros of the  $h$ -function, we have:

$$\begin{aligned} h(0) &= \frac{1}{i} [2\omega_1 \zeta(\omega_2) - 2\eta_1 \omega_2 + \pi i] \\ &= \frac{1}{i} [2\omega_1 \eta_2 - 2\eta_1 \omega_2 + \pi i] \\ &= 0 \quad \text{by Legendre relation, (see the Appendix)} \end{aligned}$$

and

$$h(\pi) = \frac{1}{i} [2\omega_1 \zeta(\omega_1) - 2\eta_1 \omega_1] = \frac{1}{i} [2\omega_1 \eta_1 - 2\eta_1 \omega_1] = 0.$$

So 0 and  $\pi$  are solutions of  $(T)$ . Let us show there is no other solution in  $[0, \pi]$ . Suppose there is another solution  $\theta$  in  $]0, \pi[$ , then by Rolle's theorem the function  $h'(\theta)$  would have at least two distinct zeros in  $]0, \pi[$ , but we have

$$h'(\theta) = \frac{1}{i} \left[ \frac{2\omega_1 \omega_3}{\pi} P(z_0(\theta)) + 2 \frac{\eta_1 \omega_3}{\pi} \right].$$

$z_0(\theta)$  is a strictly monotone function of  $\theta$  with values in  $L = \{\omega_1 + y\omega_3, 0 \leq y \leq 1\}$ . Furthermore  $P$  is a strictly monotone and real-valued function on  $L$ . We deduce that  $h'(\theta)$  is a strictly monotone function on  $[0, \pi]$  and then  $h'(\theta)$  cannot have two zeros on  $]0, \pi[$ .

So the only solutions of  $(T)$  are  $\theta = 0$  and  $\theta = \pi$ , but we then get  $z_0 = \omega_2$  or  $z_0 = \omega_1$ . Let us show that  $z_0 = \omega_1$  does not work. For that let us assume that  $z_0 = \omega_1$ . Then  $(g'/g)(z)$  must have a single pole at 0 and  $\omega_1$ , so:

$$\forall z \in \mathbf{C}, \quad \frac{g'}{g}(z) = \frac{a_1 P(z) + a_0 + b P'(z)}{P(z) - P(\omega_1)}, \quad a_1, a_0, b \in \mathbf{C},$$

with:  $a_1 P(\omega_1) + a_0 = 0$ , we deduce that:

$$\frac{g'}{g}(z) = a_1 + b \frac{P'(z)}{P(z) - P(\omega_1)}.$$

As before we must have  $a_1 \in i\mathbf{R}$  and  $b \in \mathbf{R}$ , furthermore:

$$\forall z \in \mathbf{C}, \quad g(z) = \lambda e^{a_1 z} [P(z) - P(\omega_1)]^b.$$

As  $g$  must have a double pole at 0 and a double zero at  $\omega_1$  we deduce  $b = 1$ , and

$$\forall z \in \mathbf{C}, \quad g(z) = \lambda e^{a_1 z} [P(z) - P(\omega_1)], \quad \lambda \in \mathbf{C}, \quad a_1 \in i\mathbf{R}$$

but as we must have  $g(z + 2\omega_3) = g(z)$ , we deduce that  $a_1 \cdot 2\omega_3 = 2\pi i q$ ,  $q \in \mathbf{Z}$  and so  $a_1 = 0$  because  $a_1 \in i\mathbf{R}$ , at last we have:

$$\begin{cases} g(z) = \lambda(P(z) - P(\omega_1)), & \lambda \in \mathbf{C}, \\ \eta = \frac{dz}{g(z)}. \end{cases}$$

As we have (property (1) of Lemma 1 with  $\omega_3 = i\pi$ )

$$\forall z \in \mathbf{C}, \quad X(z + 2\omega_3) = X(z)$$

we deduce that  $\operatorname{Re} \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} (\Phi_1, \Phi_2) = 0$ , and then

$$\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} \eta = \overline{\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} \eta g^2}.$$

So

$$\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} \frac{dz}{P(z) - P(\omega_1)} = \lambda \overline{\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} (P(z) - P(\omega_1)) dz} \quad (4)$$

as we have:

$$\frac{1}{P(z) - P(\omega_1)} = \frac{2}{P''(\omega_1)} (P(z - \omega_1) - P(\omega_1)), \quad \text{and} \quad \zeta' = -P$$

we conclude:

$$\Leftrightarrow -\frac{4}{P''(\omega_1)}(\eta_3 + \omega_3 P(\omega_1)) = -2\lambda\bar{\lambda}(\eta_3 + \omega_3 P(\omega_1)) \quad (4)$$

and then: (as  $\eta_3, \omega_3 \in i\mathbf{R}$  and  $P(\omega_1) \in \mathbf{R}$ )

$$\lambda\bar{\lambda} = -\frac{2}{P''(\omega_1)} \quad (5)$$

but as the  $P$ -function satisfies:

$$P'^2(z) = 4(P(z) - e_1)(P(z) - e_2)(P(z) - e_3), \quad e_i = P(\omega_i), \quad i = 1, 2, 3$$

with  $e_1, e_2, e_3$  real numbers,  $e_1 + e_2 + e_3 = 0$ ,  $e_3 < e_2 < e_1$ , we deduce:

$$\begin{aligned} P''(\omega_1) &= 2(3e_1^2 + e_1e_2 + e_1e_3 + e_2e_3) \\ &= 4\left(e_1 + \frac{e_3}{2}\right)(e_1 - e_3) = 2(e_1 - e_2)(e_1 - e_3) \end{aligned}$$

and then  $P''(\omega_1)$  is a positive number (as  $e_1 > e_2$  and  $e_1 > e_3$ ) so (5) cannot be satisfied, and  $z_0 \neq \omega_1$ .

The only case remaining is  $z_0 = \omega_2 = \omega_1 + \omega_3$ , as before we have:

$$\begin{cases} g(z) = \lambda(P(z) - P(\omega_2)), \\ \eta = \frac{dz}{g(z)} \end{cases}$$

and the condition (5) becomes

$$\lambda\bar{\lambda} = \frac{-2}{P''(\omega_2)}$$

but now we have

$$\begin{aligned} P''(\omega_2) &= 2(3e_2^2 + e_1e_2 + e_1e_3 + e_2e_3) \\ &= 2(e_1 + 2e_3)(2e_1 + e_3) = 2(e_3 - e_2)(e_1 - e_2) \end{aligned}$$

and  $P''(\omega_2)$  is a negative number because  $e_1 > e_2$  and  $e_3 < e_2$ .

At last as we assume that  $X$  sends the line  $\{\operatorname{Re}(z) = 0\}$  to the  $x_1$ -axis of  $\mathbf{R}^3$ ,  $g$  must have purely imaginary values on the line  $\{\operatorname{Re}(z) = 0\}$ , so  $\lambda$  must be in  $i\mathbf{R}$  and we conclude:

$$\lambda = \pm i \sqrt{\frac{-2}{P''(\omega_2)}},$$

and without loss of generality we can assume

$$\lambda = i \sqrt{\frac{-2}{P''(\omega_2)}} \in i\mathbf{R}.$$

This concludes the proof of Lemma 2.

*Remark.* A computation shows that the minimal surface given by the data  $(g, \eta)$  in Lemma 2 is invariant by the  $u$ -translation where  $u$  is given by:

$$\begin{aligned} u &= \operatorname{Re} \int_{\omega_3/2}^{\omega_3/2 + 2\omega_1} (\Phi_1, \Phi_2, \Phi_3) \\ &= \left( 0, \sqrt{\frac{-2}{P''(\omega_2)}} (2\eta_1 + e_2), 2\omega_1 \right). \end{aligned}$$

**LEMMA 3.** *The data  $(g, \eta)$  given in Lemma 2 is the Weierstrass representation of Riemann's example.*

*Proof.* Riemann [9] constructed for every rectangular torus an embedded minimal annulus bounded by two parallel horizontal straight lines such that every intersection with any horizontal plane is a circle. Inversely we just saw in Lemmas 1 and 2 that if such a surface exists then its Weierstrass representation is necessarily the one given in Lemma 2; this concludes the proof of lemma 3 and the proof of Theorem 1.

### Remarks

(1) In [3], Hoffman, Karcher and Rosenberg gave an explicit proof that each horizontal intersection of the surface defined by the data  $(g, \eta)$  of Lemma 2 is a circle.

(2) Darboux [1] gave explicitly the equations of Riemann's examples in terms of a parameter  $k$ ,  $0 \leq k \leq 1$  associated to a rectangular torus  $T^2$  (that is for every rectangular torus we can associate a real number  $k$ ,  $0 \leq k \leq 1$  and conversely for

each such number there is a corresponding rectangular torus) and the Jacobi function associated to  $T^2$ . For more details about these concepts see Gerretsen and Sansone [2] p. 286.

(3) In [11] Appendix C we gave the following Weierstrass representation for Riemann's examples using the equations given by Darboux, see Remark 2 above). Following the notations of [11], we set  $2K' = \omega_1$ ,  $2iK = \omega_3$ ,  $2K' + 2iK = \omega_2$  and we use the variable  $z$  instead of  $(z - K')$  in [11]. Let  $\Gamma$  denote the lattice generated by  $(2\omega_1, 2\omega_3) = (4K', 4iK)$  and let  $P$  stand for the Weierstrass  $P$ -function on the torus  $\mathbb{C}/\Gamma$ . Set

$$\begin{cases} g_1(z) = \frac{kk'}{4} \frac{1}{P(z) - P(\omega_2)}, \\ \eta_1 = \frac{2}{k} (P(z) - P(\omega_2)) dz = \frac{k'}{2} \frac{dz}{g(z)}. \end{cases}$$

We want to show that the data  $(g_1, \eta_1)$  defines the same surface as the data given in Lemma 2, up to a rigid motion of  $\mathbf{R}^3$ . This will give a new proof of Lemma 3.

To see this recall that (see again [11] p. 60):

$$\begin{cases} k^2 + k'^2 = 1; & h^2 = \frac{e_2 - e_3}{e_1 - e_3}; & k, k' \in [0, 1], \\ e_1 - e_3 = \frac{1}{4} & \left( \text{since } K' = h\omega_1 \text{ with } h = \frac{1}{2} \right). \end{cases} \quad (*)$$

We have (since  $1/(P(z) - P(\omega_2)) = (2/P''(\omega_2))(P(z - \omega_2) - P(\omega_2))$ ):

$$g_1(z) = \frac{kk'}{2P''(\omega_2)} (P(z - \omega_2) - P(\omega_2)),$$

but again we can make a change of variable and put  $z$  instead of  $(z - \omega_2)$ , so:

$$g_1(z) = \frac{kk'}{2P''(\omega_2)} (P(z) - P(\omega_2)).$$

Using (\*) and since  $P''(\omega_2) = 2(e_3 - e_2)(e_1 - e_2)$  a simple computation shows that:

$$\frac{kk'}{2P''(\omega_2)} = -\sqrt{\frac{-2}{P''(\omega_2)}}.$$

So we have:

$$\begin{cases} g_1(z) = -\sqrt{\frac{-2}{P''(\omega_2)}} (P(z) - P(\omega_2)), \\ \eta_1 = \frac{k'}{2} \frac{dz}{g_1(z)}. \end{cases}$$

Finally the  $3\pi/2$  rotation about the  $x_3$ -axis in  $\mathbf{R}^3$  applied to the surface defined by  $(g_1, \eta_1)$  gives the following data for the rotated surface:

$$\begin{cases} g_2(z) = i\sqrt{\frac{-2}{P''(\omega_2)}} (P(z) - P(\omega_2)), \\ \eta_2 = \frac{k'}{2} \frac{dz}{g_2(z)}, \end{cases}$$

that is:

$$\begin{cases} g_2(z) = g(z), \\ \eta_2 = \frac{k'}{2} \eta, \end{cases}$$

where  $(g, \eta)$  is the data given in Lemma 2. Then  $(g_2, \eta_2)$  defines the same surface as  $(g, \eta)$  up to the  $k'/2$  homothety in  $\mathbf{R}^3$ , and so the data  $(g, \eta)$  given in Lemma 2 defines Riemann's examples.

(4) In fact in Lemmas 1 and 2 we just need  $M/S_{20}$  to be an immersed two-punctured torus in  $\mathbf{R}^3/S_{20}$  with finite total curvature and two embedded planar ends. That is we can remove the hypothesis " $M$  is properly embedded" by " $M/S_{20}$  has finite total curvature and embedded planar ends".

### §3. Proof of Theorem 2

To show that the data  $(g_k, \eta_k)$  defines an immersed minimal surface in  $\mathbf{R}^3/z$  we just need to verify the period conditions, that is if  $\gamma$  and  $\mu$  are the paths on  $T^2$  defined by:

$$\gamma(t) = \frac{\omega_1}{2} + 2t\omega_3, \quad t \in [0, 1],$$

$$\mu(t) = 2t\omega_1 + \frac{\omega_3}{2}, \quad t \in [0, 1],$$

it is enough to show that:

$$\operatorname{Re} \int_{\gamma} (\Phi_1, \Phi_2, \Phi_3) = (0, 0, 0), \quad (1)$$

where:

$$\Phi_1 = \frac{\eta_k}{2} (1 - g_k^2), \quad \Phi_2 = i \frac{\eta_k}{2} (1 + g_k^2), \quad \Phi_3 = \eta_k g_k.$$

This claim holds since the forms  $\Phi_i, i = 1, 2, 3$  have no residue (because the  $P$ -function of Weierstrass is an even function). Let us remark that if the conditions (1) are satisfied then the forms  $(\Phi_1, \Phi_2, \Phi_3)$  must have periods on the path  $\mu$ , that is:

$$\operatorname{Re} \int_{\mu} (\Phi_1, \Phi_2, \Phi_3) \neq (0, 0, 0),$$

otherwise the data  $(g_k, \eta_k)$  defines an immersed minimal surface with finite total curvature and two parallel flat ends in  $\mathbf{R}^3$ , but the “Half space theorem” of Hoffman and Meeks [4] shows this situation is impossible.

Let us assume for a while that (1) is satisfied. Let  $I$  be the map on  $T^2$  defined by:

$$\forall z \in T^2, \quad I(z) = -\bar{z}.$$

Let us remark that, as  $P''(\omega_2)$  is a negative real number (see the proof of Lemma 2 in §2),  $\lambda_k \in i\mathbf{R}$ . We deduce that:

$$\begin{aligned} \forall z \in T^2, \quad g_k(I(z)) &= \lambda_k [P(-\bar{z}) - P(\omega_2)]^{2k+1} \\ &= \overline{-g_k(z)}. \end{aligned}$$

Then:

$$I_*(\eta_k) = -\frac{d\bar{z}}{g_k(I(z))} = \frac{d\bar{z}}{g_k(z)} = \bar{\eta}_k,$$

so that:

$$I_*(\Phi_1) = \bar{\Phi}_1; \quad I_*(\Phi_2) = -\bar{\Phi}_2; \quad I_*(\Phi_3) = -\bar{\Phi}_3.$$

Calling  $X_k$  the minimal immersion defined, up to a translation in  $\mathbf{R}^3$ , by  $(g_k, \eta_k)$ , we deduce that:

$$\forall z \in T^2 - \{0, \omega_2\}, \quad X_k(I(z)) = S_1[X_k(z)],$$

where  $S_1$  is the reflection about the  $x_1$ -axis. This shows that  $X_k$  sends the line  $C_1$  defined by:

$$C_1(t) = 2t\omega_3, \quad t \in ]0, 1[$$

to  $D_1$  which is the  $x_1$ -axis in the horizontal plane  $P_1 = \{x_3 = 0\}$ .

In the same way if  $J$  is the map on  $T^2$  defined by:

$$\forall z \in T^2, \quad J(z) = 2\omega_1 - \bar{z},$$

we can show that:

$$\forall z \in T^2 - \{0, \omega_2\}, \quad X_k(J(z)) = S_2[X_k(z)],$$

where  $S_2$  is the reflection about  $D_2$ , a straight line parallel to  $D_1$  in a horizontal plane  $P_2$ ,  $P_2$  distinct from  $P_1$ . Then  $X_k$  sends the line  $C_2$  of  $T^2$  on  $D_2$  in  $\mathbf{R}^3$  where

$$C_2(t) = \omega_1 + 2t\omega_3, \quad t \in [0, 1], \quad t \neq \frac{1}{2}.$$

Let

$$A = \{z \in T^2 / 0 \leq \operatorname{Re}(z) \leq \omega_1, z \neq 0, z \neq \omega_2\}.$$

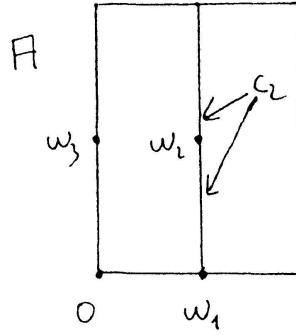
As  $\Phi_3 = 1$  and hence:

$$(X_k)_3(z) = \operatorname{Re}(z), \quad \forall z \in T^2 - \{0, \omega_2\},$$

we see that  $X_k$  sends  $A \cap T^2$  to the slab of  $\mathbf{R}^3$  bounded by  $P_1$  and  $P_2$ .

Of course  $X_k(A)$  is a minimal immersed annulus in  $\mathbf{R}^3$  between  $P_1$  and  $P_2$  and bounded by  $D_1 \cup D_2$ .





Finally denoting  $(X_k)_i$  by  $X_i$ ,  $i = 1, 2, 3$  for simplicity, we have:

$$\forall z \in T^2 - \{0, \omega_2\}, \quad (X_1 - iX_2)(z) = \int_{z_0}^z \eta_k - \overline{\int_{z_0}^z \eta_k g_k^2},$$

therefore, for  $z$  near 0 we have

$$\begin{aligned} (X_1 - iX_2)(z) &\simeq \int_{z_0}^z \lambda_k \cdot \frac{dz}{z^{2(2k+1)}} \\ &\simeq -\frac{\overline{\lambda_k}}{4k+1} \left( \frac{1}{z^{4k+1}} \right) \end{aligned}$$

and so the projection of the end near 0 on a horizontal plane is a  $(4k+1)$  to 1 map.

Near the other end  $\omega_2$  we also have:

$$\begin{aligned} (X_1 - iX_2)(z) &\simeq \int_{z_0}^z \left( \frac{2}{p''(\omega_2)} \right)^{2k+1} \frac{1}{\lambda_k} \cdot \frac{dz}{(z - \omega_2)^{2(2k+1)}} \\ &\simeq \frac{\lambda_k}{3k+1} \cdot \frac{1}{(z - \omega_2)^{4k+1}}, \end{aligned}$$

and again the projection of the end  $\omega_2$  on a horizontal plane is a  $(4k+1)$  to 1 map.

So it remains to show that conditions (1) hold. As we have  $\Phi_3 = 1$  we deduce:

$$\begin{aligned} &\Leftrightarrow \operatorname{Re} \int_{\gamma} \Phi_1 = 0 \quad \text{and} \quad \operatorname{Re} \int_{\gamma} \Phi_2 = 0 \\ &\Leftrightarrow \int_{\gamma} \eta_k = \overline{\int_{\gamma} \eta_k g_k^2}. \end{aligned} \tag{1}$$

As  $\eta_k = dz/g_k(z)$  we have:

$$\begin{aligned} \Leftrightarrow \int_{\gamma} \frac{dz}{g_k(z)} &= \overline{\int_{\gamma} g_k(z) dz} \\ \Leftrightarrow \int_{\gamma} \frac{dz}{(P(z) - P(\omega_2))^{2k+1}} &= \lambda_k \bar{\lambda}_k \overline{\int_{\gamma} (P(z) - P(\omega_2))^{2k+1} dz}, \end{aligned} \quad (1)$$

but:

$$\frac{1}{P(z) - P(\omega_2)} = \frac{2}{P''(\omega_2)} (P(z - \omega_2) - P(\omega_2)).$$

So:

$$\begin{aligned} \Leftrightarrow \lambda_k \bar{\lambda}_k \overline{\int_{\gamma} (P(z) - P(\omega_2))^{2k+1} dz} &= \left( \frac{2}{P''(\omega_2)} \right)^{2k+1} \overline{\int_{\gamma} (P(z - \omega_2) - P(\omega_2))^{2k+1} dz} \\ \Leftrightarrow \lambda_k \bar{\lambda}_k \sum_{q=0}^{2k+1} C_{2k+1}^q (-P(\omega_2))^{2k+1-q} \overline{\int_{\gamma} P(z)^q dz} &= \left( \frac{2}{P''(\omega_2)} \right)^{2k+1} \sum_{q=0}^{2k+1} C_{2k+1}^q (-P(\omega_2))^{2k+1-q} \cdot \int_{\gamma} P(z - \omega_2)^q dz. \end{aligned} \quad (1)$$

The following lemma shows the last equality is true.

**LEMMA 1.** *For every positive integer  $q$  we have:*

$$\int_{\gamma} P(z)^q dz = - \overline{\int_{\gamma} P^q(z - \omega_2) dz}.$$

Assuming Lemma 1 we have:

$$\Leftrightarrow \lambda_k \bar{\lambda}_k = - \left( \frac{2}{P''(\omega_2)} \right)^{2k+1}$$

which is true as  $\lambda_k = i\sqrt{(-2/P''(\omega_2))^{2k+1}}$ . So it just remains to prove Lemma 1.

*Proof of Lemma 1.* As  $T^2$  is a rectangular torus we have the following Laurent series for the Weierstrass  $P$ -function, see [2], [8] or the Appendix.

$$P(z) = \frac{1}{z^2} + \sum_{n=1}^{\infty} a_n z^n$$

with  $a_n$  real numbers.

We deduce that for any positive integer  $q$  the Laurent series of  $P^q(z)$  has the following type:

$$P^q(z) = \sum_{n=1}^q b_{-n} z^{-2n} + \sum_{n=0}^{+\infty} b_n z^n,$$

with  $b_j$  real numbers,  $j = 1, \dots, n$ .

Also if  $P^{(k)}(z)$  is the  $k$ th-derivative of  $P$  we have:

$$P^{(k)}(z) = (-1)^k \frac{(k+1)!}{z^{k+2}} + F_k(z),$$

where  $F_k$  is a holomorphic map near 0. We deduce that:

$$P^q(z) - \sum_{n=1}^q \frac{b_{-n}}{(2n-1)!} P^{(2n-2)}(z)$$

is an elliptic function on  $T^2$  without poles, so this function is constant and taking  $z = \omega_1$ , we see that this constant  $c$  must be real, so:

$$P^q(z) = \sum_{n=1}^q \frac{b_{-n}}{(2n-1)!} P^{(2n-2)}(z) + c$$

with  $b_{-n}$  and  $c$  real numbers. Then:

$$\begin{aligned} \int_{\gamma} P^q(z) dz &= \sum_{n=1}^q \frac{b_{-n}}{(2n-1)!} \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} P^{(2n-2)}(z) dz + c \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} dz \\ &= b_{-1} \int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} P(z) dz + 2\omega_3 c \end{aligned}$$

since for  $k \geq 2$ :

$$\int_{\omega_1/2}^{\omega_1/2 + 2\omega_3} P^{(k)}(z) dz = P^{(k-1)}\left(\frac{\omega_1}{2} + 2\omega_3\right) - P^{(k-1)}\left(\frac{\omega_1}{2}\right) = 0.$$

Then, as  $\zeta'(z) = -P(z)$  we have:

$$\begin{aligned} \int_{\gamma} P^q(z) dz &= b_{-1} \left[ -\zeta\left(\frac{\omega_1}{2} + 2\omega_3\right) + \zeta\left(\frac{\omega_1}{2}\right) \right] + 2\omega_3 c \\ &= 2\eta_3 b_{-1} + 2\omega_3 c. \end{aligned}$$

In the same way, as:

$$P^q(z - \omega_2) = \sum_{n=1}^q \frac{b_{-n}}{(2n-1)!} P^{(2n-2)}(z - \omega_2) + c,$$

we also have:

$$\int_{\gamma} P^q(z - \omega_2) dz = -2\eta_3 b_{-1} + 2\omega_3 c.$$

This concludes the proof because  $c$  and  $b_{-1}$  are real numbers and  $\omega_3$  and  $\eta_3$  are purely imaginary numbers.

### Remarks

(1) Following the arguments of Lemmas 1 and 2 in §2, it is easy to show that the surfaces  $S_k$  are the only minimal immersed surfaces between  $P_1$  and  $P_2$  bounded by  $D_1 \cup D_2$  with finite total curvature.

(2) We do not know if there exists surfaces like  $S_k$  which are bounded by two horizontal lines  $D_1, D_2$  and make a non zero angle  $\theta$ . Of course for  $k = 0$ , Theorem 1 shows that such a surface does not exist.

## §4. Appendix

Let  $T^2$  be a torus  $\mathbf{C}/\Gamma$  where  $\Gamma$  is the lattice of  $\mathbf{C}$  given by:

$$\Gamma = \{p \cdot 2\omega_1 + q \cdot 2\omega_3, p, q \in \mathbf{Z}, \omega_1 \in \mathbf{R}_*^+, \operatorname{Im}(\omega_3) \in \mathbf{R}_*^+\}.$$

The Weierstrass  $P$ -function is a special function defined on  $T^2$  which is a meromorphic function on  $\mathbf{C}$  such that:

$P$  has a pole of order two at each point of  $\Gamma$  and:

$$\forall z \in \mathbf{C} - \Gamma, \quad P(z + 2\omega_i) = P(z), \quad i = 1, 3.$$

$$\text{If } \omega_2 = \omega_1 + \omega_3, \quad P(\omega_1) + P(\omega_2) + P(\omega_3) = 0.$$

$P$  has the following Laurent series:

$$P(z) = \frac{1}{z^2} + \sum_{\omega \in \Gamma - \{0\}} \left\{ \frac{1}{(z - \omega)^2} - \frac{1}{\omega^2} \right\}.$$

$$P \text{ is an even function: } \forall z \in \mathbf{C} - \Gamma, \quad P(-z) = P(z).$$

There are two other functions related to  $P$ , namely the  $\zeta$  and  $\sigma$ -functions: The  $\zeta$ -function satisfies:

$$\forall z \in \mathbf{C} - \Gamma, \quad \zeta'(z) = -P(z).$$

If  $\eta_i = \zeta(\omega_i)$ ,  $i = 1, 2, 3$ , then  $\eta_2 = \eta_1 + \eta_3$ , so  $\zeta$  is uniquely defined on  $\mathbf{C} - \Gamma$ . We have:

$$\forall z \in \mathbf{C} - \Gamma, \quad \zeta(z + 2\omega_i) = \zeta(z) + 2\eta_i, \quad i = 1, 2, 3,$$

and then  $\zeta$  is not defined on  $T^2$  because  $z$  and  $(z + 2\omega_i)$  represent the same point on  $T^2$ .

$$\zeta \text{ is an odd function: } \forall z \in \mathbf{C} - \Gamma, \quad \zeta(-z) = -\zeta(z).$$

The following Legendre relation holds.

$$\omega_2 \eta_1 - \omega_1 \eta_2 = i \frac{\pi}{2}.$$

The  $\sigma$ -function satisfies:

$$\forall z \in \mathbf{C} - \Gamma, \quad \frac{\sigma'}{\sigma}(z) = \zeta(z),$$

and then  $\sigma$  is an holomorphic function on  $\mathbb{C}$ , furthermore:

$$\lim_{z \rightarrow 0} \frac{\sigma(z)}{z} = 1,$$

so that  $\sigma$  is uniquely determined on  $\mathbb{C}$ .

We also have:

$$\forall z \in \mathbb{C}, \quad \sigma(z + 2\omega_i) = -e^{2\eta_i(z + \omega_i)} \sigma(z). \quad (\#)$$

$\sigma$  is an odd function:  $\forall z \in \mathbb{C}, \sigma(-z) = -\sigma(z)$ .

Furthermore if  $T^2$  is a rectangular torus, that is if  $\omega_3 \in i\mathbb{R}_*^+$ , we have:

$$\forall z \in \mathbb{C} - \Gamma, \quad P(\bar{z}) = \overline{P(z)}; \quad \zeta(\bar{z}) = \overline{\zeta(z)}.$$

$$\forall z \in \mathbb{C}, \quad \sigma(\bar{z}) = \overline{\sigma(z)}.$$

For more details about those functions see Gerretsen–Sansone [2] or Molk–Tannery [8].

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