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Autor: Dorfmeister, J. / Guan, Zhuang-Dan
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Classification of compact homogeneous pseudo-Kähler manifolds

JOSFF DORFMEISTER* and ZHUANG-DAN GUAN

Introduction

Compact homogeneous Kähler manifolds have been classified by Borel [1] and Matsushima [11] (see also Borel–Remmert [2]). Together with the flat homogeneous Kähler manifolds and the bounded homogeneous domains they form the building blocks of an arbitrary homogeneous Kähler manifold [4]. Since the proof of the Fundamental Conjecture for homogeneous Kähler manifolds [4] the structure of these manifolds is known. We are interested in considering more general classes of homogeneous complex manifolds.

One of the most natural generalizations of Kähler manifolds are pseudo-Kähler manifolds (see 1.1 for a definition).

In [5] and [6] we have classified all homogeneous pseudo-Kähler manifolds admitting a reductive transitive group of automorphisms.

In this note we classify all compact homogeneous pseudo-Kähler manifolds. Note that by an automorphism of a pseudo-Kähler manifold we always mean a biholomorphic map which leaves the pseudo-metric invariant. We prove

THEOREM A. *Let M be a compact homogeneous pseudo-Kähler manifold and G an effective transitive group of automorphisms of M . Then G is reductive, and its semisimple part is compact.*

This and results from [5] and [6] then yield the main result of this paper.

THEOREM B. *Let M be a compact homogeneous pseudo-Kähler manifold and G an effective and transitive group of automorphisms of M . Then*

(a) *$G = C \times S$ where C is a complex torus and S is a compact semisimple Lie group with trivial center. In particular, G is compact.*

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(b) *The isotropy subgroup H of a base point in M is contained in S and we have*

$$M = G/H = C \times S/H$$

as a product of pseudo-Kähler manifolds where S/H is a rational homogeneous space.

(c) *The pseudo-Kähler structures on C and S/H are a difference of Kähler structures.*

To prove that transitive groups of automorphisms of a compact pseudo-Kähler manifold are reductive we consider two natural fibrations of M , the Huckleberry–Oeljeklaus–Tits fibration and the Hano–Kobayashi fibration (see 1.2 and 1.4 for definitions). We show

THEOREM C. *Let M be a compact homogeneous manifold admitting an invariant volume form. Then the Huckleberry–Oeljeklaus–Tits fibration and the Hano–Kobayashi fibration of M are the same.*

This last theorem is the main result of §1. In §2 we prove part of the main result of this paper (Theorem B) under the assumption that M is homogeneous under a reductive group of holomorphic transformations. In the last section (§3) we show that transitive groups of automorphisms of a pseudo-Kähler manifold are reductive and prove the main result quoted above.

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§1. Two fibrations

1.1. Let M be a complex manifold and j its complex structure tensor. Let φ be a (real) closed non-degenerate two-form on M , i.e. (M, φ) is a *symplectic manifold*. If φ is j -invariant, the (M, φ) is called a *pseudo-Kähler manifold*. In this case

$$(X, Y) = \varphi(jX, Y) + i\varphi(X, Y)$$

is a *non-degenerate sesqui-linear form* on M , \mathbb{C} -linear in the first argument and \mathbb{C} -antilinear in the second argument.

A pseudo-Kähler manifold (M, φ) is called *homogeneous* if there exists a Lie group $G \subset \text{Aut}(M, \varphi)$ that acts transitively on M . Here by $\text{Aut}(M, \varphi)$ we denote the group of biholomorphic maps of M leaving φ invariant. As usual, if (M, φ) is homogeneous we identify $M = G/H$ and we say that G acts *effectively* if H does not contain any normal subgroup of G . We say G acts *almost effectively* if $\{g \in G; g \cdot p = p \text{ for all } p \in M\}$ is discrete in G .

1.2. In this section we recall some basic results on a generalization of the Tits fibration, introduced by A. Huckleberry and E. Oeljeklaus [9]. It coincides with a fibration considered by Hano [7] in case the isotropy group is connected. Using the initials of the authors involved in the development of this fibration we will talk about the *HOT-fibration* (instead of the *g-anticanonical fibration* [9]).

Denoting by H_0 the connected component of the identity in H and by $\text{Norm}_G(H_0)$ the *normalizer* of H_0 in G we have

THEOREM ([9]). *Let G be a connected real Lie group acting almost effectively and transitively as a group of holomorphic transformations on the complex manifold $M = G/H$ and let $G/H \rightarrow G/J$ be the HOT-fibration.*

Then

(a) $J = \{k \in \text{Norm}_G(H_0); R(k) : G/H_0 \rightarrow G/H_0, gH_0 \rightarrow gkH_0, \text{ is holomorphic}\}$
where G/H_0 carries the complex structure induced by $G/H_0 \rightarrow G/H$.

In particular we have $J \subset \text{Norm}_G(H_0)$.

(b) J/H_0 is a complex Lie group and $G/H_0 \rightarrow G/J$ is a holomorphic J/H_0 -principal fiber bundle.

In particular, the fibering $G/H \rightarrow G/J$ is locally holomorphically trivial.

(c) If G is a connected complex Lie group and H a closed complex subgroup, then $J = \text{Norm}_G(H_0)$.

Thus for a complex Lie group G the HOT-fibration coincides with the Tits fibration.

1.3. For later use we will recall Tits's result on the fibration of compact homogeneous spaces

THEOREM ([13]). *Let G be a connected complex Lie group and H a closed complex subgroup such that G/H is compact.*

Then $G/\text{Norm}_G(H_0)$ is a rational homogeneous space and $\text{Norm}_G(H_0)/H$ is connected and parallelizable. Moreover, if $G/H \rightarrow G/R$ is a holomorphic fibration with parallelizable fiber R/H , then $R \subset \text{Norm}_G(H_0)$; if in addition the base G/R is rational homogeneous, then $R = \text{Norm}_G(H_0)$.

For definitions and results on rational homogeneous spaces we refer to the literature cited in [13]. We would like to point out however, that rational homogeneous spaces are simply connected. Moreover, if G is a real Lie group such that G/H is a compact complex manifold with G acting holomorphically, then there exists a connected complex Lie group $G^{\mathbb{C}}$ such that $G \subset G^{\mathbb{C}}$ and $G/H = G^{\mathbb{C}}/H^{\mathbb{C}}$.

We would like to point out that in general $G^{\mathbb{C}}$ is *not* a complexification of G . But we can – and will – assume from now on that $\text{Lie } G^{\mathbb{C}} = \text{Lie } G + i \text{ Lie } G$ holds.

From the definition of the HOT-fibration [9; §1.7] it is easy to see that G/H and $G^{\mathbb{C}}/H^{\mathbb{C}}$ have the same HOT-fibration. We rephrase this more precisely in

PROPOSITION. *Let G be a connected real Lie group acting almost effectively and transitively as a group of holomorphic transformations on the compact, complex manifold $G/H \cong G^{\mathbb{C}}/H^{\mathbb{C}}$.*

Let $G/H \rightarrow G/J$ denote the HOT-fibration of G/H . Then the action of $G^{\mathbb{C}}$ on G/H preserves this fibration. Moreover, let $G^{\mathbb{C}}/H^{\mathbb{C}} \rightarrow G^{\mathbb{C}}/J^{\mathbb{C}}$ denote the Tits fibration. Then $J = J^{\mathbb{C}} \cap G$, i.e., $G/J \cong G^{\mathbb{C}}/J^{\mathbb{C}}$. Thus for compact G/H the HOT-fibration and the Tits-fibration are the same. In particular, J is connected and G/J is rational homogeneous.

1.4. Next we want to discuss the Hano–Kobayashi fibration. We will call this the *HK-fibration*. Let M be a complex manifold and ω a volume form on M . Then locally we have $\omega = K(z, \bar{z}) dz^1 \wedge \cdots \wedge dz^n \wedge d\bar{z}^1 \wedge \cdots \wedge d\bar{z}^n$. We also set

$$R_{i\bar{j}} = \frac{\partial^2 \log K}{\partial z^i \partial \bar{z}^j}$$

and

$$\chi = i \sum R_{i\bar{j}} dz^i \wedge d\bar{z}^j.$$

Then χ is called the *Ricci form* of M . For later use we recall the main result on the HK-fibration for homogeneous complex manifolds.

THEOREM ([8]). *Let M be a connected complex manifold and G a connected real Lie group acting holomorphically on M . Assume moreover that $M = G/H$ admits a G -invariant volume element ω and denote by χ the associated Ricci form of M .*

Then there exists a unique closed subgroup I of G containing H and a non-degenerate closed two-form $\hat{\chi}$ on G/I such that

- (a) *G/I is a homogeneous symplectic manifold with respect to $\hat{\chi}$ and the projection $G/H \rightarrow G/I$ is G -invariant.*

- (b) *The fiber I/H of this fibration is a complex connected submanifold of G/H and $\chi|_{I/H} = 0$.*
- (c) *The pull-back of $\hat{\chi}$ to M is equal to χ .*
- (d) *If I/H is compact, then it is (complex) parallelizable.*

The fibration described in this Theorem will be called the HK-fibration.

1.5. In the rest of this paper we will use frequently arguments on the Lie algebra level.

First we recall the following result due to Koszul ([10]).

PROPOSITION. *Let G be a real Lie group and H a closed subgroup. Then G/H admits a G -invariant complex structure if and only if there exists an endomorphism j of $\mathfrak{g} = \text{Lie } G$ such that for all $x, y \in \mathfrak{g}, r \in H$ we have ($\mathfrak{h} = \text{Lie } H$)*

$$j\mathfrak{h} \subset \mathfrak{h}, \tag{1.5.1}$$

$$j^2x = -x(\text{mod } \mathfrak{h}), \tag{1.5.2}$$

$$\text{Ad } r \cdot (jx) = j \text{Ad } r \cdot x(\text{mod } \mathfrak{h}), \tag{1.5.3}$$

$$[jx, jy] = j[jx, y] + j[x, jy] + [x, y](\text{mod } \mathfrak{h}). \tag{1.5.4}$$

Note that j is only determined modulo \mathfrak{h} . In what follows we will always assume $j\mathfrak{h} = 0$.

1.6. We retain the notation and the assumptions of Proposition 1.5. In addition we assume that $M = G/H$ has a G -invariant volume form ω . We set

$$\psi(x) = \text{trace}_{\mathfrak{g}/\mathfrak{h}}(\text{ad } jx - j \text{ad } x), \quad x \in \mathfrak{g}. \tag{1.6.1}$$

Then

THEOREM ([10]). *The Ricci form associated with ω is given by the formula*

$$\chi(x, y) = \psi([x, y]), \quad x, y \in \mathfrak{g}. \tag{1.6.2}$$

Moreover, the Ricci form satisfies for $x, y, z \in \mathfrak{g}$

$$\chi(jx, jy) = \chi(x, y), \tag{1.6.3}$$

$$\chi([x, y], z) + \chi([y, z], x) + \chi([z, x], y) = 0, \tag{1.6.4}$$

$$\chi(\mathfrak{g}, \mathfrak{h}) = 0. \tag{1.6.5}$$

Remark. If $M = G/H$ is a homogeneous pseudo-Kähler manifold, then M has a G -invariant volume element and the results above apply to the associated Ricci form.

1.7. In the rest of this chapter we will compare the subgroups I and J associated with the HK-fibration (see 1.4) and the HOT-fibration (see 1.2) respectively. To be able to do this we consider a connected complex homogeneous manifold $M = G/H$, where G is a real Lie group acting holomorphically on M . We also assume that M admits a G -invariant volume form ω . We set $\underline{g} = \text{Lie } G$ and $\underline{h} = \text{Lie } H$. From Theorem 1.2 and Theorem 1.4 it is easy to derive

$$\underline{j} = \text{Lie } J = \{x \in \underline{g}; [x, jy] = j[x, y](\text{mod } \underline{h}) \text{ for all } y \in \underline{g}\}, \quad (1.7.1)$$

$$\underline{i} = \text{Lie } I = \{x \in \underline{g}; \chi(x, \underline{g}) = 0\}. \quad (1.7.2)$$

From [7] we know that \underline{j} can also be described as follows: Let $\underline{g}^{\mathbb{C}}$ denote the complexification of \underline{g} and set $\underline{g}_- = \{x + ijx; x \in \underline{g}\}$. Then $\underline{h} = \underline{g} \cap \underline{g}_-$ and

$$\underline{j} = \underline{g} \cap \text{norm}_{\underline{g}^{\mathbb{C}}}(\underline{g}_-). \quad (1.7.3)$$

Moreover, since we assume $j\underline{h} = 0$, (1.7.1) implies

$$\underline{j} \subset \text{norm}_{\underline{g}}(\underline{h}).$$

In particular, \underline{h} is an ideal of \underline{j} .

1.8. We retain the notation and assumptions of the last section.

LEMMA. *Under the above assumptions we have $\underline{j} \subset \underline{i}$.*

Proof. Let $x \in \underline{j}$ and $y \in \underline{g}$. Then (1.7.1) implies $j[x, y] = [x, jy] + h$. Therefore

$$\begin{aligned} \text{ad}(j[x, y]) - j \text{ad}[x, y] &= \text{ad}[x, jy] + \text{ad } h - j[\text{ad } x, \text{ad } y] \\ &= [\text{ad } x, \text{ad } jy] + \text{ad } h - [\text{ad } x, j \text{ad } y] + [\text{ad } x, j] \text{ad } y \\ &= [\text{ad } x, \text{ad } jy - j \text{ad } y] + \text{ad } h + [\text{ad } x, j] \text{ad } y. \end{aligned}$$

We note that $\text{ad}(jy) - j \text{ad } y$ and $\text{ad } x$ leave \underline{h} invariant. Therefore the trace of the first summand vanishes on $\underline{g}/\underline{h}$. Since M admits an invariant volume form, we know $\text{trace}_{\underline{g}/\underline{h}} \text{ad } h = 0$ for all $h \in \underline{h}$. Finally, (1.7.1) implies $[\text{ad } x, j]\underline{g} \subset \underline{h}$, whence the last term vanishes on $\underline{g}/\underline{h}$. Altogether this shows $\chi(\underline{j}, \underline{g}) = 0$, proving the assertion.

1.9. In this section we prove the first main result of this paper (Theorem C of the introduction).

THEOREM. *Let M be a connected complex compact manifold and let G be a connected real Lie group acting transitively and holomorphically on M . Assume that $M = G/H$ admits a G -invariant volume element.*

Then the Lie groups I and J defining the HK-fibration and the HOT-fibration are connected and equal.

In particular, the fiber of this fibration is complex parallelizable.

Proof. From Proposition 1.3 we know that J is connected. Hence Lemma 1.8 implies $H \subset J \subset I_0 \subset I$, where I_0 is the identity component of I . From 1.7 we know that \underline{h} is an ideal of \underline{j} and [14; Theorem 1] implies that \underline{h} is an ideal of \underline{i} . Hence J/H_0 is a Lie subgroup of the Lie group I_0/H_0 , where H_0 denotes the identity component of H . Moreover, from Theorem 1.4 and [9; §1.7, Corollary 5] it follows that J/H_0 and I_0/H_0 are actually complex Lie groups. Hence $I_0/J \subset G/J$ is a closed complex submanifold and therefore a projective manifold. Since G/J is projective algebraic it embeds equivariantly into \mathbb{P}_N [9; Chapter I, Theorem 6]. This implies that the maximal solvable subgroups of $I_0^\mathbb{C}$ have a fixed point in I_0/J by Borel's Fixed Point Theorem [9; Chapter I]. Therefore the stabilizer of $I_0^\mathbb{C}$ at e/J is parabolic and [9; Chapter I, Theorem 6] implies that I_0/J is a rational homogeneous space. Finally, we consider the two complex fibrations $I_0/H_0 \rightarrow I_0/J$ and $I_0/H_0 \rightarrow I_0/I_0$. Both fibrations have rational homogeneous spaces as bases and parallelizable homogeneous spaces as fibers. Therefore, by the uniqueness of the Tits-fibration (1.2) we get $J = I_0$. From Part (b) of Theorem 1.4 we know that I/H is connected. Since $H \subset I_0$, this implies $I = I_0 = J$.

COROLLARY. $\underline{i} = \text{Lie } I = \underline{j} = \text{Lie } J$.

§2. The case of a reductive group action

The main goal of this section is to prove

THEOREM. *Let (M, φ) be a connected compact symplectic manifold and let G be a connected reductive Lie group acting transitively and effectively on M . Assume moreover that G leaves φ invariant.*

Then $M = G/H$ and H is connected and compact. Moreover, $\text{Lie } G' = [\text{Lie } G, \text{Lie } G]$ is a semisimple compact subalgebra of $\underline{\mathfrak{g}}$, $\text{Lie } H \subset \text{Lie } G'$ and there exists some $w \in \text{Lie } G'$ such that $\text{Lie } H = \{x \in \text{Lie } G'; [x, w] = 0\}$.

Proof. Let \tilde{G} be the universal covering group of G and $\pi : \tilde{G} \rightarrow G$ the covering homomorphism. Set $\tilde{H} = \pi^{-1}(H)$. Since $\tilde{G}/\tilde{H} = G/H = M$ is compact and symplectic, we know that M admits a finite invariant measure. Hence, by a result of Selberg (see e.g. [12; Lemma 5.4]), \tilde{H} has “property (S) in \tilde{G} ”, i.e. for any neighborhood \tilde{M} of the identity of \tilde{G} and for any element $g \in \tilde{G}$, there exists an integer $n > 0$ such that $g^n \in \tilde{M}\tilde{H}\tilde{M}$.

Next, since \tilde{G} is simply connected and reductive, we obtain $\tilde{G} \cong \tilde{G}_n \times \tilde{C} \times \tilde{G}_c$, where \tilde{G}_n corresponds to the sum of the non-compact factors in $\text{Lie } G$, \tilde{G}_c to the sum of the compact factors and \tilde{C} to the center in $\text{Lie } G$. Let $\pi_n : \tilde{G} \rightarrow \tilde{G}_n$ be the canonical projection. Then $\pi_n(\tilde{H})$ is a subgroup of \tilde{G}_n having property (S) in \tilde{G}_n . Since \tilde{G}_n has no compact factors we can apply Borel’s Density Theorem (see e.g. [12; Corollary 5.16]) and obtain that the Lie algebra $\underline{h}_n = d\pi_n(\text{Lie } \tilde{H})$ is an ideal of $\underline{g}_n = \text{Lie } \tilde{G}_n$. On the other hand we know $\text{Lie } G = \underline{g} = \underline{g}_n + \underline{c} + \underline{g}_c = \text{Lie } G_n + \text{Lie } C + \text{Lie } G_c$. Moreover, from a result of Matsushima [11; Theorem 1] we know that the identity component H_0 of H is contained in the maximal semisimple subgroup S of G and that there exists an element $w \in \underline{s} = \text{Lie } S = \underline{g}_n + \underline{g}_c$ such that $\underline{h} = \text{Lie } H = \{x \in \underline{s}; [x, w] = 0\}$. Therefore, splitting $w = w_n + w_c$, $w_n \in \underline{g}_n$, $w_c \in \underline{g}_c$, we obtain that $\underline{h}_n = d\pi_n(\text{Lie } \tilde{H})$ is the centralizer of w_n in \underline{g}_n . From this it is easy to derive, since \underline{g} is reductive, that $\underline{h}_n \subset \underline{h}$ is an ideal of \underline{g} . Since G acts effectively, $\underline{h}_n = 0$. This implies $\underline{g}_n = 0$. Therefore G itself has no non-compact factor. Matsushima’s result thus implies that H_0 is contained in the (maximal) compact factor of G . In particular, H_0 is compact. Hence, again using [11; Theorem 1] we see that H is connected, whence also compact. This finishes the proof of the Theorem.

§3. Reductivity of G

3.1. In this section we consider a compact pseudo-Kähler manifold (M, φ) . We assume that there exists a connected real Lie group G acting holomorphically, effectively and transitively on M .

The goal of this chapter is to prove that G is reductive.

To fix some notation we note that we have $M = G/H$, where H is some closed subgroup of G .

We set $\underline{g} = \text{Lie } G$ and $\underline{h} = \text{Lie } H$. In what follows we will use intensively the Lie algebras \underline{i} and \underline{j} as described in section 1.7.

We also set $\underline{r} = \text{rad } (\underline{g})$ and denote by \underline{s} a maximal semisimple subalgebra of \underline{g} . Moreover, by \underline{s}_n and \underline{s}_c we denote the sum of all noncompact and all compact summands of \underline{s} respectively.

3.2. In this section we prove

LEMMA. *With the notation and under the assumptions of 3.1 we have*

- (a) $\underline{i} = \underline{r} + \underline{s}_0 + \underline{i}_c$, where $\underline{s} = \underline{s}_0 + \underline{s}''$, $\underline{s}_0 = \underline{s}_n + \underline{s}'_c$ and \underline{s}'_c and \underline{s}''_c are ideals of \underline{s}_c .
- (b) \underline{i}_c is the centralizer of some $w_c \in \underline{i}_c$ in \underline{s}_c .

Proof. From Theorem 1.9 we know that the HOT-fibration and the HK-fibration are the same. Therefore G/I is a rational homogeneous, compact, pseudo-Kählerian manifold relative to $\hat{\chi}$, the two-form on G/I induced from the Ricci form χ on $M = G/H$. Moreover, from [13; Theorem 4.1] we know $\text{rad}(\text{Lie } G^{\mathbb{C}}) \subset \text{Lie } J^{\mathbb{C}}$, whence $\underline{r} \subset \underline{i} = \underline{j}$ holds. Let \underline{q} denote the maximal ideal of \underline{g} contained in \underline{i} and Q the maximal (normal) subgroup of G satisfying $\text{Lie } Q = \underline{q}$. Then G/Q acts transitively and effectively on G/I . Since $\underline{r} \subset \underline{q}$, we know that $\underline{g}/\underline{q}$ is semisimple. Thus the Theorem in §2 implies that $\underline{g}/\underline{q}$ is a semisimple and compact Lie algebra. Moreover, $\underline{h}/\underline{q}$ is the centralizer of some element $[w] \in \underline{g}/\underline{q}$. From this the Lemma follows.

COROLLARY. *With the notation and under the assumption of 3.1 the algebra \underline{i}_c is reductive, i.e. $\underline{i}_c = \underline{c}_c + \underline{c}_s$, where \underline{c}_s is semisimple and \underline{c}_c is abelian.*

3.3. Our assumption always was that G be a real Lie group. In case G is actually a complex Lie group, we have

LEMMA. *We retain the notation and the assumptions of 3.1. Moreover we assume that G is a complex Lie group. Then G/H is a complex abelian Lie group.*

Proof. Let φ denote the pullback of the given pseudo-Kähler form on G/H . This can be written $\varphi = \sum_{i=1}^n c_i \omega_i \wedge \bar{\omega}_i$ where $\omega_1, \dots, \omega_n$ is a basis for the Maurer–Cartan forms of \underline{g} . Let us assume that $\omega_1, \dots, \omega_k$ are a basis for the Maurer–Cartan forms of \underline{h} . Since φ is pseudo-Kählerian, we know $c_i = 0$ for $i \leq k$, and $c_i \neq 0$ for $i > k$. The closedness condition of φ implies $0 = d\varphi = \sum c_i (\omega_i \wedge d\bar{\omega}_i + d\omega_i \wedge \bar{\omega}_i)$. Note that here the first term is of type (1, 2) and the second is of type (2, 1). Therefore $0 = \sum c_i \omega_i \wedge d\bar{\omega}_i$ and $0 = \sum c_i d\omega_i \wedge \bar{\omega}_i$. But $d\omega_i = \frac{1}{2} \sum_{r,s} c^i_{rs} \omega_r \wedge \omega_s$, where c^i_{rs} denotes the structure constants of \underline{g} (see [3; §IV]). Therefore, $c^i_{rs} = 0$ for all $i > k$ and all r, s . This implies $[\underline{g}, \underline{g}] \subset \underline{h}$, and the assertion follows.

3.4. Next we want to restrict our attention to the subalgebra \underline{i} of \underline{g} . We set

$$\underline{h}' = \{x \in \underline{i}; \varphi(x, \underline{i}) = 0\}. \tag{3.4.1}$$

It is easy to see that \underline{h}' is j -invariant. From 1.7 it follows that $\underline{h}'/\underline{h}$ is a complex subalgebra of the complex Lie algebra $\underline{i}/\underline{h}$. Moreover, the two form $\hat{\varphi}$ induced from

φ on $\underline{i}/\underline{h}$ is non-degenerate and j -invariant modulo $\underline{h}'/\underline{h}$. Therefore, from Lemma 3.3, we obtain

$$\hat{\nu} = (\underline{i}/\underline{h})/(\underline{h}'/\underline{h}) \text{ is abelian.} \quad (3.4.2)$$

This implies in particular

$$\underline{h}' \text{ is an ideal of } \underline{i}. \quad (3.4.3)$$

We set $\underline{r}' = \text{rad}(\underline{i})$. Then

$$\underline{r}' = \underline{r} + \underline{c}_c. \quad (3.4.4)$$

Moreover, since \underline{h}' is an ideal of \underline{i} , we have

$$\underline{h}' = \underline{r}' \cap \underline{h}' + (\underline{s}_0 + \underline{c}_s) \cap \underline{h}'. \quad (3.4.5)$$

We also know that \underline{h} is an ideal of \underline{i} , consequently

$$\underline{h} = \underline{r}' \cap \underline{h} + (\underline{s}_0 + \underline{c}_s) \cap \underline{h}. \quad (3.4.6)$$

More precisely, $(\underline{s}_0 + \underline{c}_s) \cap \underline{h} = \underline{s}'_0 + \underline{c}'_s$, where \underline{s}'_0 and \underline{c}'_s is a direct summand of \underline{s}_0 and \underline{c}_s respectively. Therefore, $\underline{i}/\underline{h} \cong \underline{r}'/\underline{r}' \cap \underline{h} + \underline{s}_0/\underline{s}'_0 + \underline{c}_s/\underline{c}'_s$. But since $\underline{i}/\underline{h}$ is a complex Lie algebra and $\underline{c}_s/\underline{c}'_s$ is a semisimple compact Lie algebra (or $=0$), we obtain $\underline{c}_s = \underline{c}'_s \subset \underline{h}$. Thus

$$\underline{h} = \underline{r}' \cap \underline{h} + \underline{s}'_0 + \underline{c}_s. \quad (3.4.7)$$

By the same argument we see $\underline{s}'_c = \underline{s}_0 \cap \underline{s}_c \subset \underline{s}'_0$. Next we look at \underline{h}' . We know $(\underline{s}_0 + \underline{c}) \cap \underline{h}' = \underline{s}''_0 + \underline{c}_s$, where \underline{s}''_0 is an ideal of \underline{s}_0 containing \underline{s}'_0 . Then $\underline{h}'/\underline{h} \cong \underline{r}' \cap \underline{h}'/\underline{r}' \cap \underline{h} + \underline{s}''_0/\underline{s}'_0$ and $\underline{i}/\underline{h} \cong \underline{r}'/\underline{r}' \cap \underline{h}' + \underline{s}_0/\underline{s}''_0$. Therefore $\hat{\nu} = (\underline{i}/\underline{h})/(\underline{h}'/\underline{h}) \cong \underline{r}'/\underline{r}' \cap \underline{h}' + \underline{s}_0/\underline{s}''_0$. But $\hat{\nu}$ is abelian by (3.4.2), whence $\underline{s}_0 = \underline{s}''_0 \subset \underline{h}'$. We thus have shown

$$\underline{h}' = \underline{r}' \cap \underline{h}' + \underline{s}_0 + \underline{c}_s. \quad (3.4.8)$$

3.5. In the following sections we will use the decompositions derived above to clarify the structures of \underline{i} . As usual, by $\text{nil}(\underline{i})$ we denote the nilradical of \underline{i} . We retain the notation and the assumptions used above.

LEMMA. $\text{nil}(\underline{i}) \subset \underline{r}' \cap \underline{h}'$.

Proof. Consider the action of the semisimple Lie algebra $\underline{s}_0 + \underline{c}_s$ on \underline{i} . Then $\underline{i} = \underline{r}' \cap \underline{h}' + \underline{a} + \underline{s}_0 + \underline{c}_s$, where \underline{a} is invariant under $\underline{s}_0 + \underline{c}_s$. But since \underline{h}' is an ideal of \underline{i} and $\underline{s}_0 + \underline{c}_s \subset \underline{h}'$, this implies $[\underline{s}_0 + \underline{c}_s, \underline{a}] = 0$. Also, since $\hat{v} \cong \underline{i}/\underline{h}'$ is abelian, $[\underline{a}, \underline{a}] \subset \underline{h}'$. From this it follows $[\underline{i}, \underline{r}'] \subset \underline{h}'$, thus the claim.

COROLLARY 1. $[\underline{r}, [\underline{r}, \underline{r}]] = 0$.

Proof. As usual, by \underline{s} we denote a maximal semisimple subalgebra of \underline{g} . Then $\varphi([\underline{r}, [\underline{r}, \underline{r}]], \underline{s}) \subset \varphi(\underline{r}, [\underline{r}, \underline{r}]) = 0$, since $\underline{r} \subset \underline{i}$ and $[\underline{r}, \underline{r}] \subset \text{nil}(\underline{i}) \subset \underline{h}'$. Since $[\underline{r}, [\underline{r}, \underline{r}]] \subset \text{nil}(\underline{i}) \subset \underline{h}'$ and $\underline{r} \subset \underline{i}$ we also have $\varphi([\underline{r}, [\underline{r}, \underline{r}]], \underline{r}) = 0$, therefore $[\underline{r}, [\underline{r}, \underline{r}]] \subset \underline{h}$. But $[\underline{r}, [\underline{r}, \underline{r}]]$ is an ideal of \underline{g} , whence the claim.

COROLLARY 2. $\text{ad } \underline{r}$ consists of nilpotent endomorphisms of \underline{g} .

3.6. The goal of this section is to show (still under the usual assumptions of this chapter)

LEMMA. $\underline{s}_0 = 0$.

Proof. Since $\underline{s}_0 \subset \underline{h}'$ and $\underline{r} \subset \underline{i}$, we have $\varphi(\underline{s}_0, \underline{r}) = 0$. Moreover, using the notation of 3.1 we have $\varphi(\underline{s}_0, \underline{s}_c) = \varphi(\underline{s}_0, [\underline{s}_c, \underline{s}_c]) = 0$. This shows that φ is nondegenerate on $\underline{s}_0/\underline{s}'_0$. From the closedness condition of φ we obtain $\varphi(x, y) = \beta(b, [x, y])$ for all $x, y \in \underline{s}_0$, where β denotes the Killing form of \underline{s}_0 . From this we derive $\underline{s}'_0 = \{x \in \underline{s}_0; [x, b] = 0\}$. But \underline{s}'_0 is an ideal of \underline{s}_0 , hence $\underline{s}_0 = \underline{s}'_0$. Since we know now $\underline{s}_0 \subset \underline{h}$ and $\underline{r} \subset \underline{i}$, clearly $[\underline{s}_0, \underline{r}] \subset \underline{h} \cap \underline{r}$. It is easy to see that $[\underline{s}_0, \underline{r}]$ is invariant under $\underline{s} = \underline{s}_0 + \underline{i}_c$. Therefore, the ideal of \underline{g} generated by $[\underline{s}_0, \underline{r}]$ is contained in \underline{h} , whence $[\underline{s}_0, \underline{r}] = 0$. Thus \underline{s}_0 is an ideal of \underline{g} , but $\underline{s}_0 \subset \underline{h}$ and $\underline{s}_0 = 0$ follows.

3.7. In this section we prove a result that will be used frequently in the rest of this chapter. We retain the notation and the assumptions of this chapter.

LEMMA. Let $x_0 \in \underline{g}$ and assume $[x_0, \underline{r}] \subset \underline{h}$. Moreover assume that $\text{ad } x_0$ is semisimple on $\underline{g}/\underline{r}$. Then $S_{\underline{r}} = 0$, where S denotes the semisimple part of $\text{ad } x_0$.

Proof. Let $\text{ad } x_0 = S + N$ the decomposition of $\text{ad } x_0$ into its semisimple part S and its nilpotent part N . We can assume that S leaves \underline{s} invariant [4; Appendix]. Moreover, since S and N are polynomials in $\text{ad } x_0$ without constant term, $S\underline{r} \subset \underline{h} \cap \text{nil}(\underline{g})$ and $N\underline{r} \subset \underline{h} \cap \text{nil}(\underline{g})$. Let $\underline{r}^c = \bigoplus \underline{r}_\alpha^c$ be the decomposition of \underline{r}^c , the complexification of \underline{r} , into eigenspaces relative to S . Then

$$\underline{r}_\alpha^c \subset (\underline{h} \cap \text{nil}(\underline{g}))^c \quad \text{for all } \alpha \neq 0. \tag{3.7.1}$$

Suppose there exists some $\alpha \neq 0$. In what follows we fix such an α . Let $\underline{s}_\beta^{\mathbb{C}}$ be any eigenspace of S in $\underline{s}^{\mathbb{C}}$. Then

$$[\underline{s}_\beta^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}] \subset \underline{r}_{\alpha+\beta}^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}} \quad \text{if } \alpha + \beta \neq 0. \quad (3.7.2)$$

If $\beta = -\alpha$, then

$$\varphi(\underline{s}_\gamma^{\mathbb{C}}, [\underline{s}_{-\alpha}^{\mathbb{C}}, \underline{r}_{-\alpha}^{\mathbb{C}}]) = 0 \quad \text{if } \gamma + \alpha \neq 0. \quad (3.7.3)$$

Indeed, $\varphi(x_\gamma, [y_{-\alpha}, z_\alpha]) = -\varphi([x_\gamma, z_\alpha], y_{-\alpha}) = 0$ if $x_\gamma \in \underline{s}_\gamma^{\mathbb{C}}$, $y_{-\alpha} \in \underline{s}_{-\alpha}^{\mathbb{C}}$, $z_\alpha \in \underline{r}_\alpha^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}}$, and $\gamma + \alpha \neq 0$ since in this case $[\underline{s}_\gamma^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}] \subset \underline{r}_{\alpha+\gamma}^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}}$ by (3.7.1).

Consider now the case $\gamma = -\alpha$. From our assumption we obtain $\underline{s}_{-\alpha}^{\mathbb{C}} = S\underline{s}_{-\alpha}^{\mathbb{C}} \subset [x_0, \underline{s}_{-\alpha}^{\mathbb{C}}] + \text{nil}(\underline{g})^{\mathbb{C}}$. Hence, $\varphi(\underline{s}_{-\alpha}^{\mathbb{C}}, [\underline{s}_{-\alpha}^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}]) \subset \varphi([x_0, \underline{s}_{-\alpha}^{\mathbb{C}}] + \text{nil}(\underline{g})^{\mathbb{C}}, [\underline{s}_{-\alpha}^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}]) \subset \varphi(\underline{s}_{-\alpha}^{\mathbb{C}}, [x_0, \text{nil}(\underline{g})^{\mathbb{C}}]) + \varphi([\text{nil}(\underline{g})^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}], \underline{s}_{-\alpha}^{\mathbb{C}}) = 0$, since $[x_0, \text{nil}(\underline{g})^{\mathbb{C}}] \subset \underline{h}^{\mathbb{C}}$ and $[\text{nil}(\underline{g})^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}] \subset \underline{h}^{\mathbb{C}}$. Therefore we have

$$\varphi(\underline{s}_{-\alpha}^{\mathbb{C}}, [\underline{s}_{-\alpha}^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}]) = 0. \quad (3.7.4)$$

As a consequence of the above results we obtain

$$\varphi(\underline{s}^{\mathbb{C}}, [\underline{s}^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}]) = 0 \quad \text{if } \alpha \neq 0. \quad (3.7.5)$$

Since $\underline{r}_\gamma^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}}$ for $\gamma \neq 0$, we clearly have $\varphi(\underline{r}_\gamma^{\mathbb{C}}, \underline{g}) = 0$ in this case. If $\gamma = 0$, then $\varphi(\underline{r}_0^{\mathbb{C}}, [\underline{s}^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}]) = \varphi([\underline{r}_0^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}], \underline{s}^{\mathbb{C}}) = 0$, since $[\underline{r}_0^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}] \subset \underline{r}_\alpha^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}}$. Thus, altogether we have shown

$$\varphi(\underline{r}^{\mathbb{C}}, [\underline{s}^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}]) = 0. \quad (3.7.6)$$

Equations (3.7.5) and (3.7.6) together imply

$$[\underline{s}^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}] \subset \underline{h}^{\mathbb{C}}. \quad (3.7.7)$$

Next we consider the vector space $\underline{q}^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}} \cap \underline{r}^{\mathbb{C}}$ spanned by the subspaces $\underline{r}_\alpha^{\mathbb{C}}$ and $[\underline{s}^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}]$, $\alpha \neq 0$. It is easy to see that $\underline{q}^{\mathbb{C}}$ is invariant under complex conjugation relative to \underline{q} .

$$\underline{q}^{\mathbb{C}} \text{ is an } \underline{s}\text{-module.} \quad (3.7.8)$$

Indeed, consider $A = [\underline{s}_\gamma^{\mathbb{C}}, [\underline{s}_\beta^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}]]$. If $\beta + \alpha \neq 0$, then the inner commutator is contained in $\underline{r}_{\alpha+\beta}^{\mathbb{C}}$, whence $A \subset \underline{q}^{\mathbb{C}}$. If $\beta + \alpha = 0$, then we use $A = [[\underline{s}_{-\gamma}^{\mathbb{C}}, \underline{s}_{-\alpha}^{\mathbb{C}}], \underline{r}_\alpha^{\mathbb{C}}] + [\underline{s}_{-\alpha}^{\mathbb{C}}, [\underline{s}_{-\gamma}^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}]]$. Clearly, the first summand is in $\underline{q}^{\mathbb{C}}$. In the second summand we

have $[\underline{s}_\gamma^{\mathbb{C}}, \underline{r}_\alpha^{\mathbb{C}}] \subset \underline{r}_{\alpha+\beta}^{\mathbb{C}} \subset \underline{h}^{\mathbb{C}}$ if $\alpha + \gamma \neq 0$. If $\alpha + \gamma = 0$, then the whole second summand is contained in $\underline{r}_{-\alpha}^{\mathbb{C}}$, finishing the proof of (3.7.8). Now it is straight forward to verify that the ideal of \underline{g} generated by $\underline{q}^{\mathbb{C}} \cap \underline{g}$ is actually contained in \underline{h} . But since the transitive group G in question acts effectively, this ideal is trivial. In particular we have $\underline{r}_\alpha^{\mathbb{C}} = 0$ for all $\alpha \neq 0$. Therefore $S\underline{r} = 0$, proving the assertion.

3.8. In this section we continue our investigation of \underline{s} . Since we know from 3.6 that $\underline{s}_0 = 0$ holds, \underline{s} is compact. We split $\underline{s} = \underline{s}_a + \underline{s}_b$, where

$$\underline{s}_a = \{x \in \underline{s}; [x, r] = 0\} \tag{3.8.1}$$

and \underline{s}_b is a complementary ideal of \underline{s}_a in \underline{s} . Since \underline{i}_c is the centralizer of some element in \underline{s} ,

$$\underline{i}_c = \underline{i}_a + \underline{i}_b, \quad \text{where } \underline{i}_* = \underline{i}_c \cap \underline{s}_*, \quad * = a, b. \tag{3.8.2}$$

Since \underline{i}_a and \underline{i}_b are reductive, with obvious notation we have

$$\underline{i}_a = \underline{c}_a + \underline{c}_{sa} \quad \text{and} \quad \underline{i}_b = \underline{c}_b + \underline{c}_{sb}. \tag{3.8.3}$$

LEMMA. (a) \underline{s}_a and $\underline{s}_b + \underline{r}$ are ideals of \underline{g} .

(b) $\underline{h} = \underline{h} \cap \underline{r} + \underline{c}_b + \underline{c}_a + \underline{c}_{sa}$.

Proof. Clearly, \underline{s}_a and $\underline{s}_b + \underline{r}$ are ideals of \underline{g} . Moreover, we have $\varphi(\underline{s}_a, \underline{r}) = \varphi([\underline{s}_a, \underline{s}_a], \underline{r}) = 0$ and similarly $\varphi(\underline{s}_a, \underline{s}_b) = 0$. Therefore, \underline{s}_a and $\underline{s}_b + \underline{r}$ are perpendicular. This implies $\underline{h} = \underline{h} \cap \underline{s}_a + \underline{h} \cap (\underline{s}_b + \underline{r})$. From Lemma 3.1 it follows that $\underline{h} \cap \underline{s}_a$ is the centralizer of some $w_a \in \underline{s}_a$.

Now let $x_0 \in \underline{h} \cap (\underline{r} + \underline{c}_a + \underline{c}_b)$. Clearly, $[x_0, r] \subset \underline{h}$, since \underline{h} is an ideal of \underline{i} and $\underline{r} \subset \underline{i}$. Moreover, $\text{ad } x_0$ is semisimple on $\underline{g}/\underline{r}$. Therefore, by the last lemma $S\underline{r} = 0$, where S denotes the semisimple part of $\text{ad } x_0$. In view of Corollary 3.7.1 we can write $\underline{r} = \underline{a} + [\underline{a}, \underline{a}]$ where $[\underline{s}, \underline{a}] \subset \underline{a}$. Hence $x_0 = c + a + n$ with $c \in \underline{c}_a + \underline{c}_b$, $a \in \underline{a}$, and $n \in [\underline{a}, \underline{a}]$. Note $[n, r] = 0$ by Corollary 3.5.1. Therefore $\text{ad } x_0 | \underline{r} = \text{ad } (c + a) | \underline{r}$. Since we know that the semisimple part of $\text{ad } x_0$ vanishes on \underline{r} , the endomorphism $A = \text{ad } (c + a) | \underline{r}$ is nilpotent. But $\text{ad } c | \underline{r}$ is semisimple and leaves \underline{a} and $[\underline{a}, \underline{a}]$ invariant, while $\text{ad } \underline{a}$ maps \underline{a} into $[\underline{a}, \underline{a}]$ and annihilates $[\underline{a}, \underline{a}]$. This shows $\text{ad } c | \underline{a} = 0$ and $\text{ad } c | [\underline{a}, \underline{a}] = 0$, whence $[c, r] = 0$. Therefore, $c \in \underline{c}_a$ and the assertion follows.

3.9. Clearly, to show that \underline{g} is reductive, we have to prove $\underline{s}_b = 0$. This is the goal of this section.

LEMMA. $\underline{s}_b = 0$.

Proof. From Corollary 3.7 we know that $\text{ad } \underline{r}$ consists of nilpotent endomorphisms of \underline{g} . Moreover, $\text{ad } \underline{c}$, $\underline{c} \in \underline{c}_b$, is semisimple on \underline{g} and has only purely imaginary eigenvalues. Restricting $\text{ad } (\underline{r} + \underline{c}_b)$ to the complex Lie algebra $\underline{i}/\underline{h}$ we obtain the radical of $\underline{i}/\underline{h}$. But this is a complex solvable Lie algebra, whence $\text{ad } \underline{c}_b \mid \underline{i}/\underline{h} = 0$. In particular we get $[\underline{c}_b, \underline{r}] \subset \underline{h}$. From Lemma 3.7 we thus obtain $[\underline{c}_b, \underline{r}] = 0$, i.e. $\underline{c}_b = 0$. From Lemma 3.7 it follows easily that $\underline{c}_{bs} = 0$ holds. Thus $\underline{s}_b = 0$.

3.10. With the results of the previous sections it will be easy now to prove (Theorem A of the introduction).

THEOREM. *Let (M, φ) be a compact connected pseudo-Kähler manifold and G an effective transitive group of automorphisms of M . Then G is reductive and its semisimple part is compact.*

Proof. From Lemma 3.9 it follows that $\underline{g} = \underline{r} + \underline{s}_a$, where $[\underline{r}, \underline{s}_a] = 0$ and \underline{s}_a is semisimple. Moreover, $\underline{h} = \underline{h} \cap \underline{r} + \underline{h} \cap \underline{s}_a$. Therefore, the radical of $\underline{i}/\underline{h}$ is $\underline{r}/\underline{h} \cap \underline{r}$. Since this is j -invariant we can assume $j\underline{r} \subset \underline{r}$. Also, $\underline{h} \cap \underline{r}$ is an ideal of \underline{g} contained in \underline{h} , hence $\underline{h} \cap \underline{r} = 0$. This implies $\underline{h} = \underline{c}_a + \underline{c}_{as}$, by Lemma 3.8, and $\underline{i}/\underline{h} \cong \underline{r}$. In particular, \underline{r} is a complex Lie algebra and $\varphi(\underline{r}, \underline{s}_a) = 0$ shows that $(\underline{r}, 0, j, \varphi)$ is a pseudo-Kähler algebra. Thus Lemma 3.3 shows that \underline{r} is abelian. Therefore \underline{g} is reductive, proving the assertion.

3.11. In this section we will give the *proof of Theorem B* of the introduction.

First we note that Theorem A (see 3.10) shows that G is reductive and its semisimple part S is compact. From the Theorem in §2 we thus obtain that the isotropy subgroup H of G is connected, compact and contained in the maximal semisimple Lie subgroup S of G . From [11; Theorem 1] it thus follows that S has trivial center and that $G = C \times S$ holds. Clearly, $G/H = C \times S/H$. Since G/H is compact, we see that C is a complex torus. In particular, G is compact. It is easy to see that $\text{Lie } C$ and $\text{Lie } S$ are perpendicular relative to the given pseudo-Kählerian structure. Thus $G/H = C \times S/H$ is the product of pseudo-Kähler manifolds.

Therefore it only remains to prove that S/H is a rational homogeneous manifold and that the given pseudo-Kähler structures on C and S/H are a difference of Kähler structures. The first statement follows from 3.9, since $\underline{i} = \underline{r} + \underline{c}_{as}$, where $\underline{h} = \underline{c}_a = \underline{c}_{as}$ and $\underline{r} = \text{Lie } C$. The second statement follows from [6].

Added in proof. Recently we received the preprint: A. T. Huckleberry, Homogeneous pseudo-Kählerian manifolds: A hamiltonian viewpoint. In this paper Theorem B is proven by a different method.

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Department of Mathematics
University of Kansas
Lawrence, KS 66045
USA

Department of Mathematics
University of California
Berkeley, CA 94720
USA

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