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On the topological equivalence between Anosov flows on three-manifolds

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1. Introduction

In this paper we shall be concerned with transitive Anosov flows on a closed, three dimensional, manifold M. If $\phi_t : M \to M$ is such a flow then, by a theorem of Fried ([Fri1]), it admits a surface of section: there exist a compact surface with boundary Σ and embedding $j : \Sigma \to M$ such that $j(\text{int } \Sigma)$ is transverse to ϕ_t, $j(\partial\Sigma)$ is a union of closed orbits $\gamma_1, \ldots, \gamma_N$ of ϕ_t, and every flowline intersects $j(\Sigma)$ in a uniformly bounded time. If $\tilde{M}$ denotes the blowing-up of M along $\gamma_1, \ldots, \gamma_N$ and $\tilde{\phi}_t$ is the lifted flow, then there exists a fibration $\Sigma \hookrightarrow \tilde{M} \to S^1$ whose fibres are transverse to $\tilde{\phi}_t$. Then $\tilde{\phi}_t$ induces a first return map $f : \Sigma \to \Sigma$ which is topologically conjugate to a pseudo-Anosov diffeomorphism, with semi-saddle singularities on the boundary $\partial\Sigma$ ([FLP], [Thu]).

A transitive Anosov flow admits many surfaces of section. The genus and the number of holes of a surface of section are not uniquely defined.

When M is a bundle over S^1 with fibre T^2 Plante proved ([Pla1], see also [Arm]), that any Anosov flow is topologically equivalent to the suspension of a hyperbolic automorphism A of T^2 (A represents the monodromy of the bundle). Recalling that the complement of the boundary of a surface of section is a bundle over S^1, our result may be seen as an extension of Plante's result:

THEOREM. *Let $\phi_t : M \to M$ be a transitive Anosov flow on a closed 3-manifold M and let $\Sigma \hookrightarrow M$ be a surface of section, with $\partial\Sigma = \{\gamma_1, \ldots, \gamma_N\}$. Let $\psi_t : M \to M$ be another transitive Anosov flow. Suppose that there exists a homeomorphism $h : M \to M$ and closed orbits $\bar{\gamma}_1, \ldots, \bar{\gamma}_N$ of ψ_t such that for all $j = 1, \ldots, N$ one has $h(\gamma_j) = \bar{\gamma}_j$ and preserving the orientations given by the flows. Assume further that h maps the germs along γ_j of the stable manifolds $W^s_{\phi_t}(\gamma_j)$ for ϕ_t to the germs along $\bar{\gamma}_j$ of the stable manifolds $W^s_{\psi_t}(\bar{\gamma}_j)$ for ψ_t. Then ϕ_t and ψ_t are topologically equivalent.*

The hypotheses of the theorem imply easily that h may be chosen so that it realizes a topological equivalence between the restriction of ϕ_t to a neighbourhood of $\bigcup_{j=1}^{N} \gamma_j$ and the restriction of ψ_t to a neighbourhood of $\bigcup_{j=1}^{N} \bar{\gamma}_j$, and hence that ψ_t is topologically equivalent to an Anosov flow whose oriented orbits on a neighbourhood of $\bigcup_{j=1}^{N} \gamma_j$ are equal to the oriented orbits of ϕ_t. Hence we will assume, without loss of generality, that ψ_t and ϕ_t have the same orbit-structure in such a neighbourhood.

The proof of this theorem is in the spirit of the proof of the main result of [Ghy-Ser] (see also [Pla2]). Firstly, using the technique of Roussarie ([Rou1], [Rou2]) we show that a fibre of the fibration $\Sigma \hookrightarrow \tilde{M} \to S^1$ transverse to $\tilde{\phi}_t$ is isotopic rel $(\partial \tilde{M})$ to a surface transverse to the stable foliation of $\tilde{\psi}_t$, where $\tilde{\psi}_t$ is the lifting of ψ_t on $\tilde{M}$. Then we cut $\tilde{M}$ along this surface, obtaining a manifold diffeomorphic to $\Sigma \times [0, 1]$ with a foliation transverse to $\Sigma \times \{0, 1\} \subset \partial(\Sigma \times [0, 1])$. The analysis of this foliation will show that every closed orbit of $\tilde{\psi}_t$ represents a non-trivial element of $\pi_1(S^1) \subset \pi_1(\tilde{M})$, and a theorem of [Ver], together with standard facts in three-dimensional topology, will imply that the initial fibration $\Sigma \hookrightarrow \tilde{M} \to S^1$ is isotopic to a fibration transverse to $\tilde{\psi}_t$. Finally, the rigidity of pseudo-Anosov maps ([FLP]) will complete the proof.

We remark that we require the transitivity of ψ_t (i.e. the density of the leaves of the stable foliation); this hypothesis is avoided in Plante's theorem, thanks to the solvability of $\pi_1(M)$. However, we don't know counterexamples to a possible extension of our theorem to the non-transitive case.

I am grateful to A. Verjovsky for introducing me to this subject. I thank also the referee for useful remarks and for pointing to me an imprecision in the original proof of lemma 5.

2. Preliminaries

As remarked before, we may suppose without loss of generality that the oriented orbits of ϕ_t and ψ_t on a neighbourhood of $\bigcup_{j=1}^{N} \gamma_j$ are equal. The lifted flows $\tilde{\phi}_t, \tilde{\psi}_t : \tilde{M} \to \tilde{M}$, where $\tilde{M}$ is the blow-up of M along $\gamma_1, \ldots, \gamma_N$, have the same orbits in a neighbourhood of the boundary $\partial \tilde{M}$, which consists of N copies of T^2 (an orbit in the boundary of a surface of section has always an orientable neighborhood, diffeomorphic to $D^2 \times S^1$). On any component of $\partial \tilde{M}$ these flows are Morse-Smale flows, with 4 or 2 closed orbits, depending on the orientability or non-orientability of the stable and unstable manifolds of the closed orbit of ϕ_t or ψ_t corresponding to the component.

We denote by $\tilde{\Sigma} \subset \tilde{M}$ the lifting of $\Sigma \subset M$. The surface $\tilde{\Sigma}$ is a cross section for $\tilde{\phi}_t$, and there is a fibration $\Sigma \hookrightarrow \tilde{M} \xrightarrow{p} S^1$ transverse to $\tilde{\phi}_t$ and with $\tilde{\Sigma} = p^{-1}(0)$. Let $f : \tilde{\Sigma} \to \tilde{\Sigma}$ be the first return map of $\tilde{\phi}_t$. Let us remark that f preserves the components of the boundary $\partial \tilde{\Sigma}$.

The lifting $\tilde{\mathscr{F}}^s$ of the stable foliation $\mathscr{F}^s$ of ϕ_t is transverse to $\tilde{\Sigma}$ and defines on $\tilde{\Sigma}$ a foliation (by lines) with semi-saddle singularities on $\partial\tilde{\Sigma}$. The lifting $\tilde{\mathscr{G}}^s$ of the stable foliation $\mathscr{G}^s$ of ψ_t defines on $\tilde{\Sigma}$ a foliation with singularities $\mathscr{H}$.

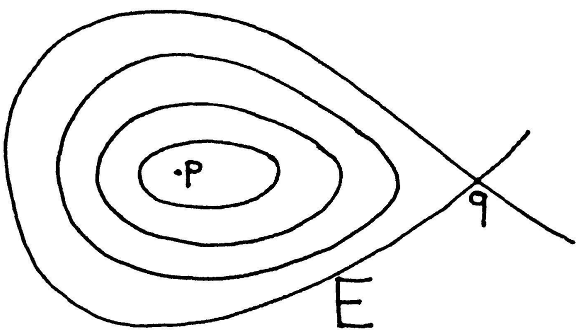
A small deformation of Σ (fixing $\partial\Sigma$) will ensure that the only singularities of $\mathscr{H}$ are saddles, centers (both of Morse type) and semi-saddles on $\partial\tilde{\Sigma}$, and that there are no connections between two saddles or one saddle and a semi-saddle ([Fra], p. 82–84).

LEMMA 1. *The foliation with singularities $\mathscr{H}$ has no connections between two semi-saddle singularities.*

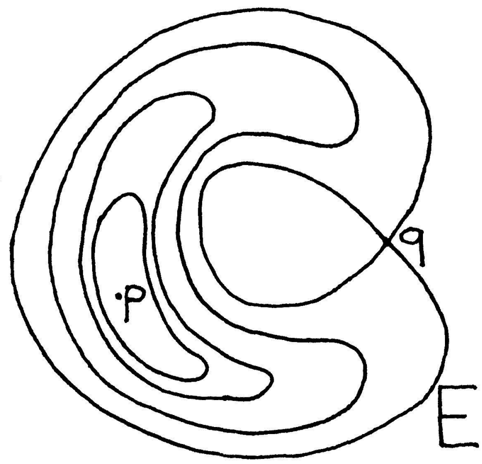
Proof. Assume that there is a connection $l \in \mathscr{H}$ between two semi-saddles $r, q \in \partial\tilde{\Sigma}$. Then $\tilde{\psi}_t(l)$ is a segment which joins $\tilde{\psi}_t(r)$ and $\tilde{\psi}_t(q)$. For $t \to +\infty$ the length of $\tilde{\psi}_t(l)$ goes to zero but the distance between $\tilde{\psi}_t(r)$ and $\tilde{\psi}_t(q)$ cannot go to zero, hence we arrive to a contradiction. □

3. Deformation of $\tilde{\Sigma}$ to a surface transverse to $\tilde{\mathscr{G}}^s$

Let $\tilde{\Sigma}$ be as above and let $p \in \tilde{\Sigma}$ be a center for $\mathscr{H}$. Let $E \subset \tilde{\Sigma}$ be the closure of the union of leaves of $\mathscr{H}$ which are circles bounding on $\tilde{\Sigma}$ a (unique) disk containing p as unique singularity. As in [Rou1], ∂E is formed by singularities of $\mathscr{H}$ of saddle type or of semi-saddle type and by leaves of $\mathscr{H}$ joining these singularities. The absence of connections between two semi-saddles, or a semi-saddle and a saddle means that E is bounded only by one or two homoclinic trajectories at a single saddle point q:



type (a)



type (b)

In particular, E is contained in int $\tilde{\Sigma}$.

The foliation $\mathscr{H}$ has the same number of semi-saddles on $\partial\tilde{\Sigma}$ as the foliation induced by $\tilde{\mathscr{F}}^s$, which is without singularities in int $\tilde{\Sigma}$. A Poincaré-Hopf argument shows that if we isotope $\tilde{\Sigma}$ (fixing $\partial\tilde{\Sigma}$) in such a way that we eliminate the centers of $\mathscr{H}$, then automatically we eliminate the saddles.

LEMMA 2. *$\tilde{\Sigma}$ is isotopic* rel $\partial\tilde{\Sigma}$ *to a surface $\tilde{\Sigma}'$ whose interior is transverse to $\tilde{\mathscr{G}}^s$.*

Proof. The embedding $\tilde{\Sigma} \hookrightarrow \tilde{M}$ induces an injective map $\pi_1(\tilde{\Sigma}) \to \pi_1(\tilde{M})$, the foliation $\tilde{\mathscr{G}}^s$ has no limit cycles ([Nov]), and $\tilde{M}$ is irreducible: every sphere embedded in $\tilde{M}$ bounds a ball. This is sufficient in order to apply the method of Roussarie ([Rou1], p. 49–52) for the elimination of the centers of type (a). So we suppose now that all the centers of $\mathscr{H}$ are of type (b).

Let $p \in \tilde{\Sigma}$ be one of these centers and let E and q be as above. If η_1, η_2 are the homoclinic orbits at the saddle point q, then η_1 and η_2 are not contractible in the leaf L of $\tilde{\mathscr{G}}^s$ containing them, because $\mathscr{H}$ has no centers of type (a) ([Rou2], p. 109). In particular, L is a leaf diffeomorphic to a cylinder or to a Moebius strip. We may deform a little $\tilde{\Sigma}$ in such a way that all the saddles of $\mathscr{H}$ belong to leaves of $\tilde{\mathscr{G}}^s$ diffeomorphic to planes. Then there are no more centers of type (b), but only centers of type (a) which can be eliminated as before. $\square$

Remark. Here and in the following we make an extensive use of the observation by Plante ([Pla2]) that many standard facts in foliation theory (in particular, the works of Roussarie on isotopies of surfaces in foliated 3-manifolds) are true also for C^1 foliations, thanks to the general position argument of [Fra], p. 82–84.

4. Half Reeb components and half limit cycles

Let $(N, \mathscr{F})$ be a foliated 3-manifold, with $\partial N \neq \varnothing$ and $\mathscr{F}$ transverse to ∂N; if A is a connected component of ∂N, we shall denote by $(N \cup_A N, \mathscr{F} \cup_A \mathscr{F})$ the foliated manifold obtained by gluing two copies of $(N, \mathscr{F})$ along A with the identity diffeomorphism (everything can be done in such a way that $\mathscr{F} \cup_A \mathscr{F}$ is as smooth as $\mathscr{F}$).

An *orientable* (*non-orientable*) *half Reeb component* ([Ghy–Ser], [Mou–Rou]) of $\mathscr{F}$ is a saturated set $\Omega \subset N$ bounded by a leaf $L \simeq S^1 \times [0, 1]$ ($L \simeq$ Moebius strip) and an annulus (a Moebius strip) $K \subset A \subset \partial N$ with $\partial K = \partial L$, such that the double $\Omega \cup_K \Omega \subset N \cup_A N$ is an orientable (non orientable) Reeb component of $\mathscr{F} \cup_A \mathscr{F}$.

Remark that if Ω is a half Reeb component of $\mathscr{F}$, then K is a planar Reeb component for the foliation induced by $\mathscr{F}$ on ∂N.

A *half limit cycle* of $\mathscr{F}$ is a differentiable map $\Gamma : [0, 1) \times [0, 1] \to N$ such that:

(1) $\forall t \in [0, 1), s \mapsto \Gamma(t, s)$ is a curve on a leaf L_t with ends on a component $A \subset \partial N$.
(2) $s \mapsto \Gamma(0, s)$ defines a non trivial element of $\pi_1(L_0, \partial L_0)$, but for $t \in (0, 1)$ $s \mapsto \Gamma(t, s)$ defines a trivial element of $\pi_1(L_t, \partial L_t)$.

When we pass to the double $N \cup_A N$ we see that a half limit cycle Γ determines a limit cycle $\tilde{\Gamma} : [0, 1) \times S^1 \to N \cup_A N$. If N is compact, then a half limit cycle is associated to a half Reeb component Ω, with $\partial\Omega = L_0 \cup K, K \subset A$. The following lemma is a relative version of Novikov's theorem ([Nov], see also [Rou1]):

LEMMA 3. *Let $\gamma : [0, 1] \to N$ be a curve with image in a leaf $L \in \mathscr{F}$ such that $\gamma(0), \gamma(1) \in A$, where A is a connected component of ∂N; suppose that γ represents a non trivial element of $\pi_1(L, \partial L)$ and a trivial element of $\pi_1(N, A)$; suppose also that $\mathscr{F}$ has no limit cycles. Then $\mathscr{F}$ has a half limit cycle.*

Proof. Passing to $(N \cup_A N, \mathscr{F} \cup_A \mathscr{F})$ we have a cycle $\tilde{\gamma} = \gamma \cup_A \gamma$ on a leaf which is contractible in $N \cup_A N$ but non contractible in the leaf. Take a disk in $N \cup_A N$ with boundary $\tilde{\gamma}$ and symmetric with respect to A; this disk contains a limit cycle by Novikov theorem and the absence of limit cycles of $\mathscr{F}$ implies that this limit cycle is the double of a half limit cycle of $\mathscr{F}$. □

Now we return to our foliation $\tilde{\mathscr{G}}^s$ on $\tilde{M}$. Let $\tilde{\Sigma}'$ be the surface given by lemma 2; if we cut $\tilde{M}$ along $\tilde{\Sigma}'$ we obtain a manifold $\tilde{\Sigma}' \times [0, 1]$ with a foliation $\tilde{\mathscr{G}}^{s\prime}$ on $(\operatorname{int} \tilde{\Sigma}') \times [0, 1]$ which is transverse to $(\operatorname{int} \tilde{\Sigma}') \times \{0, 1\}$. The foliation $\tilde{\mathscr{G}}^{s\prime}$ has no limit cycles, but it may have half limit cycles. It will be useful to avoid that situation, by an appropriate choice of the surface along which we cut $\tilde{M}$.

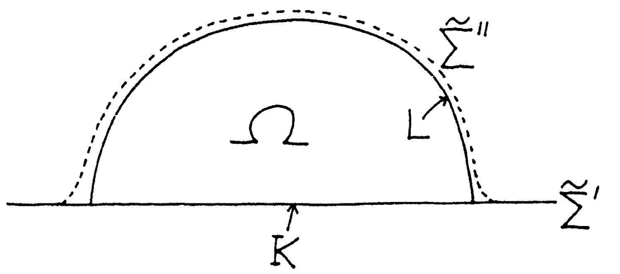
LEMMA 4. *$\tilde{\Sigma}'$ is isotopic* rel $\partial\tilde{\Sigma}'$ *to a surface $\tilde{\Sigma}''$ whose interior is again transverse to $\tilde{\mathscr{G}}^s$ and such that the foliation $\tilde{\mathscr{G}}^{s\prime\prime}$ induced on* $(\operatorname{int} \tilde{\Sigma}'') \times [0, 1]$ *has no half limit cycles.*

Proof. Let N be the 3-manifold obtained from $\tilde{\Sigma}' \times [0, 1]$ by collapsing to points the circles $c \times \{t\}$, where c is a connected component of $\partial\tilde{\Sigma}'$ and $t \in [0, 1]$; N is a compact manifold with boundary equipped with a foliation with "line prong" singularities ([Ina–Mat]) $\mathscr{G}_N$, induced by $\tilde{\mathscr{G}}^{s\prime}$. If we pass to the double $N \cup_{\partial N} N$ we obtain a closed 3-manifold ($\simeq$surface $\times S^1$) with a foliation $2\mathscr{G}_N$ with circle prong singularities, such that every circle injects its fundamental group in

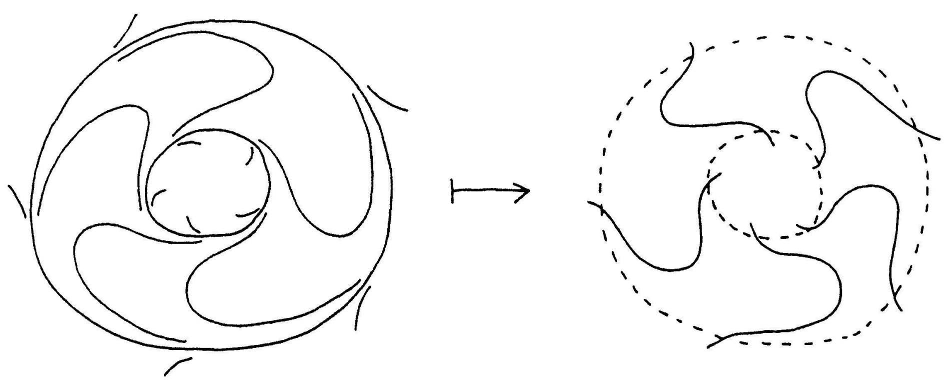
$\pi_1(N \cup_{\partial N} N)$, every separatrix has only one singular end and injects its fundamental group in that of the corresponding extended leaf. The separatrices of $2\mathscr{G}_N$ are non compact and they may be different from cylinders, furthermore there may be singular circles of $2\mathscr{G}_N$ with only one prong.

Suppose that $\bar{\mathscr{G}}^{s'}$ has a half limit cycle, then $2\mathscr{G}_N$ has a limit cycle. The same arguments of [Ina–Mat] show that this limit cycle is associated to a Reeb component of $2\mathscr{G}_N$ and hence to a half Reeb component of $\mathscr{G}_N$, i.e. of $\bar{\mathscr{G}}^{s'}$. Hence, it is sufficient to prove that $\tilde{\Sigma}'$ is isotopic to another surface $\tilde{\Sigma}''$, such that the foliation $\bar{\mathscr{G}}^{s''}$ has no half Reeb components.

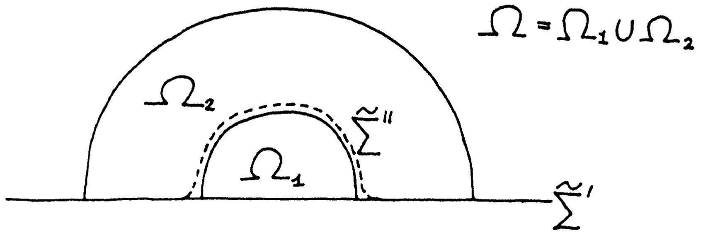
Let Ω be a half Reeb component of $\bar{\mathscr{G}}^{s'}$, with $\partial\Omega = L \cup K$, where L is a leaf of $\bar{\mathscr{G}}^{s'}$, $K \subset \operatorname{int} \tilde{\Sigma}' \times \{0\}$ (or $\operatorname{int} \tilde{\Sigma}' \times \{1\}$); L has hyperbolic holonomy, and in particular has non trivial holonomy on both sides. This is enough to guarantee the existence of an isotopy of $\tilde{\Sigma}'$, fixing the complement of a neighbourhood of K and moving $\tilde{\Sigma}'$ to a surface $\tilde{\Sigma}''$ which does not intersect Ω and is again transverse to $\bar{\mathscr{G}}^{s}$:



Such an isotopy cancels a half Reeb component. From the point of view of the foliation on $\tilde{\Sigma}'$ this corresponds to the cancellation of a planar Reeb component:



After the cancellation of a half Reeb component it may happen that a new half Reeb component appears. This is the case if we start from a *turbulized half Reeb component*, i.e. a saturated set Ω which is obtained from a half Reeb component Ω_0 by a half turbulization ([Ghy–Ser]) along a closed curve contained in the boundary:



The hyperbolicity (hence, the finiteness) of the limit cycles on $\tilde{\Sigma}'$ that bound planar Reeb components or turbulized planar Reeb components implies that this phenomenon can occur only a finite number of times and so after a finite number of isotopies of the previous type we obtain a surface satisfying the conclusion of the lemma. To this regard, observe also that the elimination of a planar Reeb component in the foliation on $\tilde{\Sigma}'$ may produce new (perhaps hyberbolic) limit cycles, but these cycles cannot be in the boundary of other new Reeb components, because they admit closed transversals. □

5. Structure of the foliation $\tilde{\mathscr{G}}^s$

Let $\tilde{\Sigma} \subset \tilde{M}$ be the surface obtained in the previous section: int $\tilde{\Sigma}$ is transverse to $\tilde{\mathscr{G}}^s$ and the foliation $\tilde{\mathscr{G}}^s$ induced on (int $\tilde{\Sigma}$) $\times [0, 1]$ has no half limit cycles (and, of course, no limit cycles).

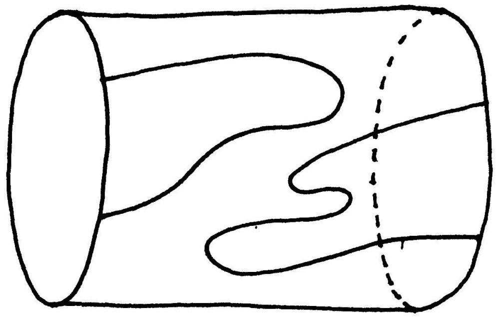
LEMMA 5. *Let L be a leaf of $\tilde{\mathscr{G}}^s$ containing a closed orbit of $\tilde{\psi}_t$. Then*: (i) *$L \cap \tilde{\Sigma}$ is formed by a finite set of lines, and each line does not separate L*; (ii) *the leaves of $\tilde{\mathscr{G}}^s$ originating from L are all diffeomorphic to* $R \times [0, 1]$, *with* $R \times \{0\} \subset \text{int}\, \tilde{\Sigma} \times \{0\}$ *and* $R \times \{1\} \subset \text{int}\, \tilde{\Sigma} \times \{1\}$.

Proof. Consider firstly a leaf $L \in \tilde{\mathscr{G}}^s$ diffeomorphic to a (open) cylinder; $L \cap \tilde{\Sigma}$ is a closed, non-empty, 1-dimensional submanifold of L. It is composed of:

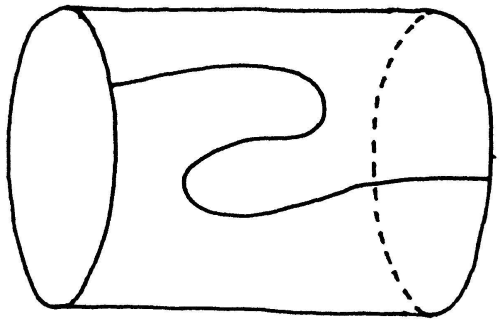
(a) circles non homotopic to zero (because they are non homotopic to zero in $\tilde{\Sigma}$ and $\pi_1(\tilde{\Sigma}) \hookrightarrow \pi_1(\tilde{M})$ is injective);

(b) lines from $+\infty$ to $+\infty$ or from $-\infty$ to $-\infty$ (we denote by $+\infty$ and $-\infty$ the two ends of L);

(c) lines from $-\infty$ to $+\infty$.



(b)

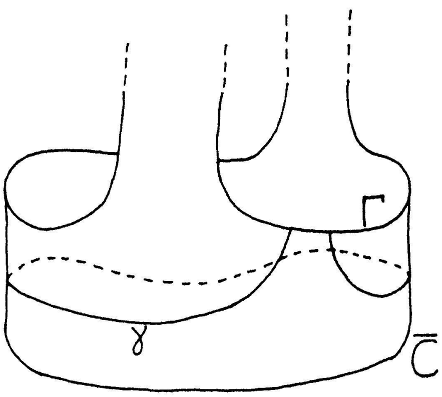


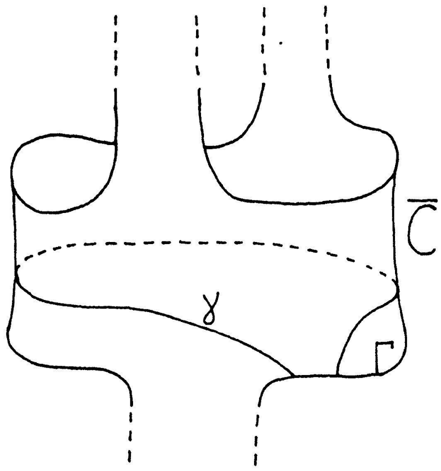
(c)

We have to show that (a) and (b) cannot occur.

The density of L in $\tilde{M}$ (due to the transitivity of ψ_t) implies that $\tilde{\Sigma} \cap L$ contains at least one line, because $\tilde{\Sigma} \cap L$ is dense in $\tilde{\Sigma}$ and every circle correspond to a hyperbolic limit cycle for the foliation induced on $\tilde{\Sigma}$.

Suppose that there are no lines of type (c), then there is a connected component C of $L \backslash (L \cap \tilde{\Sigma})$ which is non simply connected and bounded by a finite number (≥ 1) or a countable set of lines of type (b) and at most one circle:





This component gives origin to a leaf $\bar{C}$ of $\bar{\mathscr{G}}^s$, with $\partial \bar{C} \subset (\operatorname{int} \tilde{\Sigma}) \times \{0, 1\}$. Take a path $\gamma : [0, 1] \to \bar{C}$ such that:

(1) $\gamma(0), \gamma(1)$ are on the same line $\Gamma \subset \partial \bar{C}$;

(2) γ defines a non-trivial element of $\pi_1(\bar{C}, \partial \bar{C})$.

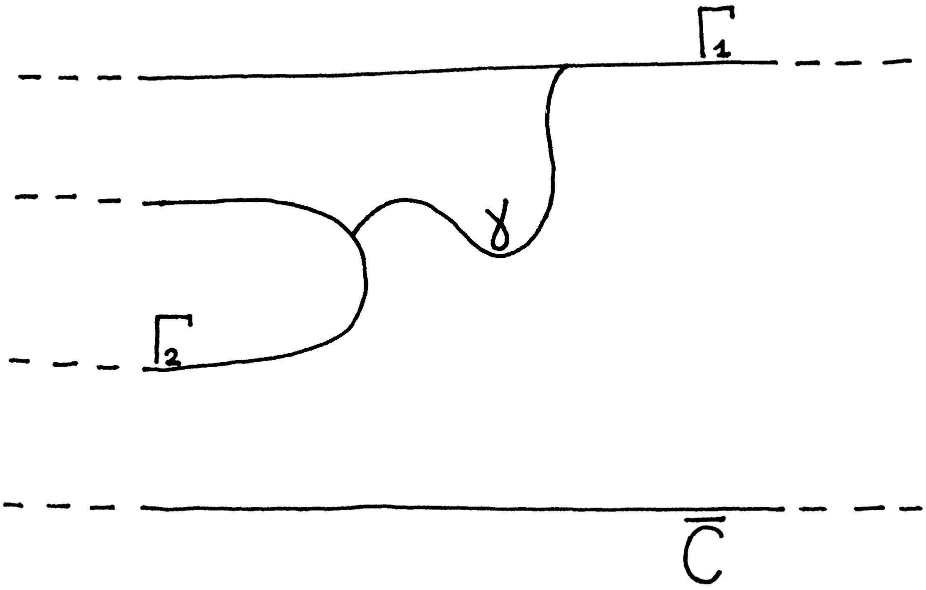
Such a γ surely exists and represents a trivial element of $\pi_1((\operatorname{int} \tilde{\Sigma}) \times [0, 1], A)$ (which is the trivial group) where $A = \operatorname{int} \tilde{\Sigma} \times \{0\}$ or $\operatorname{int} \tilde{\Sigma} \times \{1\}$ is the component containing Γ. By lemma 3, $\bar{\mathscr{G}}^s$ would have a half limit cycle, but this is precluded by the hypotheses.

Hence there is in $\tilde{\Sigma} \cap L$ at least one line from $-\infty$ to $+\infty$. This implies that there are no circles.

Assume now that $\tilde{\Sigma} \cap L$ contains a line of type (b), then $L \backslash (L \cap \tilde{\Sigma})$ has a connected component C such that the corresponding leaf $\bar{C} \in \tilde{\mathscr{G}}^s$ has at least three components in the boundary. Then we may find a path $\gamma : [0, 1] \to \bar{C}$ such that:

(1) $\gamma(0) \in \Gamma_1, \gamma(1) \in \Gamma_2$, where Γ_1, Γ_2 are two different lines of $\partial\bar{C}$;

(2) Γ_1, Γ_2 are on the same component of $(\operatorname{int} \tilde{\Sigma}) \times \{0, 1\}$.



γ represents a non-trivial element of $\pi_1(\bar{C}, \partial\bar{C})$ (by (1)) and a trivial element of $\pi_1((\operatorname{int} \tilde{\Sigma}) \times [0, 1], A)$, where $A = \operatorname{int} \tilde{\Sigma} \times \{0\}$ or $\operatorname{int} \tilde{\Sigma} \times \{1\}$ is the component containing Γ. We arrive again to a contradiction, by lemma 3, and this means that $\tilde{\Sigma} \cap L$ does not contain lines of type (b), and contains only lines from $-\infty$ to $+\infty$.

The same argument shows that if C is a connected component of $L \backslash (L \cap \tilde{\Sigma})$ and $\bar{C} \simeq R \times [0, 1] \in \tilde{\mathscr{G}}^s$ is the associated leaf, then the two components of $\partial\bar{C}$ are on different components of $\partial((\operatorname{int} \tilde{\Sigma}) \times [0, 1])$, since otherwise we could find a path $\gamma : [0, 1] \to \bar{C}$ with properties (1) and (2).

This completes the proof in the case of a cylindrical leaf; in the case of a leaf diffeomorphic to a Moebius strip the proof is completely similar. □

We could prove that *any* leaf of $\tilde{\mathscr{G}}^s$ is of the type $R \times [0, 1]$, with $R \times \{0\} \subset \operatorname{int} \tilde{\Sigma} \times \{0\}$, $R \times \{1\} \subset \operatorname{int} \tilde{\Sigma} \times \{1\}$, and then that $\tilde{\mathscr{G}}^s$ is isotopic rel $(\operatorname{int} \tilde{\Sigma} \times \{0\})$ to the product foliation $\mathscr{H} \times [0, 1]$, where $\mathscr{H}$ is the foliation (by lines) on $\operatorname{int} \tilde{\Sigma} \times \{0\}$ induced by $\tilde{\mathscr{G}}^s$ (cfr. e.g. [Rou–Mou]). However, we shall use the lemma only for the following

COROLLARY. *Under the projection $\pi_1(\tilde{M}) \to \pi_1(S^1)$ induced by the fibration $\tilde{M} \to S^1$ every closed orbit of $\tilde{\psi}_t$ defines a non-trivial element of $\pi_1(S^1)$.* □

6. End of the proof

The cooriented surface $\tilde{\Sigma} \subset \tilde{M}$ defines a cohomology class $\omega \in H^1(\tilde{M}, Z)$, which is also represented by the homotopy class of the fibration $p : \tilde{M} \to S^1$ (see §2). This class, by the above corollary, is different from 0 when evaluated on the homology class represented by a closed orbit of $\tilde{\psi}_t$.

LEMMA 6. *The fibration $p : \tilde{M} \to S^1$ is isotopic to a fibration $q : \tilde{M} \to S^1$ with fibres transverse to $\tilde{\psi}_t$.*

Proof. The above remark on the class ω represented by p implies that if $\tilde{\Sigma} \times R$ is the cyclic covering of $\tilde{M}$ defined by p and if $\hat{\psi}_t : \tilde{\Sigma} \times R \to \tilde{\Sigma} \times R$ is the lifting of the flow $\tilde{\psi}_t$, then every orbit of $\hat{\psi}_t$ which projects to a closed orbit of $\tilde{\psi}_t$ goes from $-\infty$ to $+\infty$ or from $+\infty$ to $-\infty$. However, as observed by Verjovsky ([Ver], p. 74; the result given there holds also in the case of manifolds with boundary), the transitivity of $\tilde{\psi}_t$ implies that all the orbits of $\hat{\psi}_t$ go from $-\infty$ to $+\infty$ (we have chosen the covering $\tilde{\Sigma} \times R \to \tilde{M}$ in such a way that the lifting $\hat{\phi}_t$ of $\tilde{\phi}_t$ has orbits from $-\infty$ to $+\infty$).

Then ([Ful], [Ver]) there exists a fibration $q : \tilde{M} \to S^1$ homotopic to p and with fibres transverse to $\tilde{\psi}_t$. A theorem due essentially to Waldhausen ([Wal], cfr. also [Lau], [Fri2]) implies that q is in fact isotopic to p. Alternatively, p and q define two cohomologous closed non-singular 1-forms $[p^*(d\theta)]$, $[q^*(d\theta)]$, which are isotopic by [Kup–Quê]. The presence of the boundary does not give any trouble, because the fibrations induced by p and q on $\partial\tilde{M}$ are clearly isotopic. Hence, we may firstly isotope $p^*(d\theta)$ to a form β equal to $q^*(d\theta)$ in a neighborhood of $\partial\tilde{M}$. Then we collapse $\partial\tilde{M}$ along the fibres of $\beta|_{\partial\tilde{M}} = q^*(d\theta)|_{\partial\tilde{M}}$ and we obtain a closed 3-manifold M_0 with two closed non-singular cohomologous 1-forms β_0 and $q^*(d\theta)_0$, equal in a neighborhood of the set of circles $\Gamma \subset M_0$ arising from $\partial\tilde{M}$. Finally, the proof of [Kup–Quê] gives an isotopy from β_0 to $q^*(d\theta)_0$ which can be assumed to preserve Γ, and hence which lifts to an isotopy from β to $q^*(d\theta)$. □

The isotopy of Lemma 6 may be chosen so that it is equal to the identity on a neighborhood of $\partial\tilde{M}$, hence there is an isotopy of M, equal to the identity on a neighborhood of $\partial\Sigma$, which transforms ψ_t in a flow (again denoted by ψ_t) such that the induced flow on $\tilde{M}$ (again denoted by $\tilde{\psi}_t$) is transverse to the fibration $p : \tilde{M} \to S^1$.

LEMMA 7. *$\tilde{\phi}_t, \tilde{\psi}_t : \tilde{M} \to \tilde{M}$ are topologically equivalent.*

Proof. Fix a fibre $\tilde{\Sigma} = p^{-1}(0) \subset \tilde{M}$ and let $f, g : \tilde{\Sigma} \to \tilde{\Sigma}$ be the first return maps of $\tilde{\phi}_t, \tilde{\psi}_t$; f and g are topologically conjugate to pseudo-Anosov diffeomorphisms

([Fri1]). They are also isotopic, because $\tilde{\phi}_t$ and $\tilde{\psi}_t$ are transverse to the fibration and hence the corresponding vector fields are homotopic within vector fields transverse to the fibration. So by [FLP] there is $h : \tilde{\Sigma} \to \tilde{\Sigma}$, homeomorphism isotopic to the identity, such that $h \circ f = g \circ h$. This h generates a homeomorphism $\tilde{h} : \tilde{M} \to \tilde{M}$, mapping oriented orbits of $\tilde{\phi}_t$ to oriented orbits of $\tilde{\psi}_t$. $\square$

Recall that the blow-up projection $\tilde{M} \to \tilde{M}$ gives a fibration $\mathscr{L}_0$ by circles of $\partial \tilde{M}$, which is a union of N tori. A fibre is projected to a point of one of the closed orbits $\gamma_1, \ldots, \gamma_N$. On $\partial \tilde{M}$ there is another fibration by circles $\mathscr{L}_1$, induced by $\tilde{\Sigma} \hookrightarrow \tilde{M} \to S^1$. The homeomorphism $\tilde{h}$ constructed in Lemma 7 preserves $\mathscr{L}_1$ (when restricted to $\partial \tilde{M}$) but not necessarily $\mathscr{L}_0$.

On $\partial \tilde{M}$ there is also a foliation given by the orbits of $\tilde{\phi}_t$ (equal to $\tilde{\psi}_t$); this foliation is transverse to $\mathscr{L}_0$ and to $\mathscr{L}_1$, and it is preserved by $\tilde{h}|_{\partial \tilde{M}}$. It is then possible to compose $\tilde{h}$ with a homeomorphism k of the form $k(x) = \tilde{\psi}_{t(x)}(x)$, $x \in \tilde{M}$, $t : \tilde{M} \to R$ a suitable function, in such a way that $k \circ \tilde{h}|_{\partial \tilde{M}}$ preserves $\mathscr{L}_0$ instead of $\mathscr{L}_1$.

Then $\tilde{h}' = k \circ \tilde{h}$ is again a topological equivalence between $\tilde{\phi}_t$ and $\tilde{\psi}_t$, and the preservation of $\mathscr{L}_0$ allows to "blow-down" $\tilde{h}'$ to a homeomorphism $\hat{h} : M \to M$ realizing a topological equivalence between ϕ_t and ψ_t.

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