

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 67 (1992)

Artikel: Invariants on three-manifolds with spin structure.
Autor: Blanchet, Christian
DOI: <https://doi.org/10.5169/seals-51104>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 25.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Invariants on three-manifolds with spin structure

CHRISTIAN BLANCHET

Introduction

New invariants on three-manifolds associated to certain roots of unity were recently obtained by Witten and constructed by Reshetkin and Turaev ([W], [RT]). These invariants are combinations of generalised Jones polynomials, calculated on a framed link in S^3 , representing a surgery presentation of the manifold.

Following Lickorish, one can construct the invariants corresponding to $SU(2)$ using the one variable Kauffman bracket. See [L1–L2–L3] for the roots $A = e^{i\pi/2r}$ and [BHMV] for a generalisation.

Kirby and Melvin have shown that the invariant corresponding to the root $A = e^{i\pi/8k}$ decomposes as a sum of spin invariants, over all spin structures on a given manifold ([KM2]; see also [T2]). For $A = e^{i\pi/(8k+4)}$, they give an analogous decomposition as a sum over all the modulo 2 cohomology classes on the manifold.

In this article, we use the elementary methods of Lickorish's, as refined in [BHMV], and a spin version of the Kirby calculus, to construct an invariant on three-manifold with spin structure for each root of unity A of order ρ , $\rho \not\equiv 8$ modulo 16, in a convenient ring. Up to normalisation this invariant appears as a generalisation of the spin invariants of [KM2]–[T2]. We show that for a given ρ , the invariant is essentially unique.

In a similar way, one can construct an invariant for three-manifolds M^3 equipped with a cohomology class $c \in H^1(M^3, \mathbb{Z}/2)$, for each root of unity A of order ρ , $\rho \not\equiv 0$ modulo 16.

Different relations between these invariants are given: If $\rho \equiv 0$ modulo 16, we prove that the sum of the spin invariants, for all spin structures on a given manifold, is equal to the invariant of the manifold without structure, which will be called the ‘unspun’ invariant. (This word was suggested by the referee). If $\rho \equiv 8$ modulo 16, the unspun invariant decomposes as a sum over all the cohomology classes in an analogous way. In all other cases, the invariants defined for a given A can be expressed using only the invariant associated to the zero cohomology class, which we will call the even invariant.

We show that, for A of order 16, the spin invariant is equivalent to the Rochlin invariant (compare with [KM2], theorem 7.1). It enables one to distinguish the two spin structures on the lens space $L(n, 1)$ n even and $n \not\equiv 0$ modulo 16. The two spin structures on $L(16k, 1)$, $k > 0$, are distinguished by the invariant corresponding to a root of unity A of order $32k$.

§I. Kirby calculus and spin structure

It is well known that every oriented three-manifold M^3 can be obtained by surgery on a framed link $L = (L_i, p_i)_{1 \leq i \leq n}$ in S^3 . M^3 is diffeomorphic to $M_L = \partial W_L$, where W_L is the manifold which is obtained by glueing a handle $A_i = D^2 \times D^2$ along a tubular neighbourhood V_i of each component L_i . $(S^1 \times D^2, S^1 \times 1)$ is identified with (V_i, l_i) ; here l_i is the preferred longitude: $lk(L_i, l_i) = p_i$. To each spin structure s on M_L , is associated an obstruction:

$$w_2(s) \in H^2(W_L, M_L, \mathbb{Z}/2).$$

(See [M] or [Ki2] for generalities about spin structures.) Using Poincaré duality, the class corresponding to $w_2(s)$ can be written in the basis of $H_2(W_L, \mathbb{Z}/2)$ corresponding to the handles:

$$\sum c_i [L_i] = [K(s)] \quad (c_i \in \mathbb{Z}/2).$$

This gives the characteristic sublink $K(s)$, which is known to satisfy:

$$\forall x \in H_2(W_L, \mathbb{Z}/2) \quad x \cdot x = [K(s)] \cdot x \quad (\text{in } \mathbb{Z}/2).$$

This condition is equivalent to the system:

$$B(c_i) = (p_i) \quad (\text{modulo 2}).$$

Here B is the linking matrix:

$$B = (b_{ij}) \quad \text{with: } b_{ii} = p_i \quad \text{and} \quad b_{ij} = lk(L_i, L_j) \quad \text{for } i \neq j.$$

This gives a one to one correspondence between the spin structures and the solutions of the condition above. We denote by $M_{(L, K)}$ the manifold M_L equipped with the spin structure corresponding to the characteristic sublink K : every closed three-manifold with spin structure admits such a characteristic surgery presentation.

The components of a framed link $L = (L_i, p_i)$, with characteristic sublink K are denoted by the triple (L_i, p_i, c_i) . Here $c_i \in \mathbb{Z}/2$ is one whenever L_i belongs to K .

The following theorem is a refinement of the well known Kirby calculus.

THEOREM (I.1). *The two spin manifolds $M_{(L, K)}$ and $M_{(L', K')}$ are spin diffeomorphic if and only if (L, K) and (L', K') are equivalent under the relation generated by the following moves:*

KS1: *Add to (L, K) an unknotted component $(L_i, \varepsilon, 1)$, isolated from the others by an embedded two-sphere, with $\varepsilon = \pm 1$.*

KS2: *Modify a component (L_i, p_i, c_i) adding (using a band) a push-off of another one (L_j, p_j, c_j) . The new component indexed by i , is: (L'_i, p'_i, c'_i) , with $p'_i = p_i + p_j + 2lk(L_i, L_j)$. The linking number is calculated with orientations of L_i and L_j , coherent with an orientation of L'_i ; $c'_i = c_i$, and $c'_j = c_j + c_i \pmod{2}$.*

The only changes needed to be made to the proof given in [Kil] concerns the characteristic sublink:

The **KS2** move (handle slide) does not modify W_L ; the basis of $H_2(W_L, \mathbb{Z}/2)$ is changed, and it is easy to calculate the new characteristic coefficients.

The **KS1** move (stabilization) adds $\pm \mathbb{C}P^2$ to W_L . The new coefficient is determined by the characteristic condition; the c_i coefficients, which determine the obstruction on the other handles, are not modified.

Remarks (I.2)

1. Following [FR], one can replace the two moves by:

KS: Add an unknotted component (L_i, ε, c_i) with

$$\varepsilon = \pm 1 \quad \text{and} \quad c_i = 1 + \sum_{j \neq i} c_j lk(L_i, L_j).$$

One must then add to each p_j the number $\varepsilon(lk(L_i, L_j))^2$.

2. The framed link can be represented by a diagram in which the coefficient p_i is given by the longitude which is parallel to the component L_i in the plane. The positive **KS** move (**KS+**) is described by the Figure 1. An integer n near a curve means the presence of n parallels in the plane.

3. An analogous statement holds for three-manifolds M^3 equipped with a cohomology class $c \in H^1(M^3, \mathbb{Z}/2)$. If L is a surgery presentation of M^3 , then c corresponds to a sublink C such that the modulo 2 homology class $[C]$ is the kernel

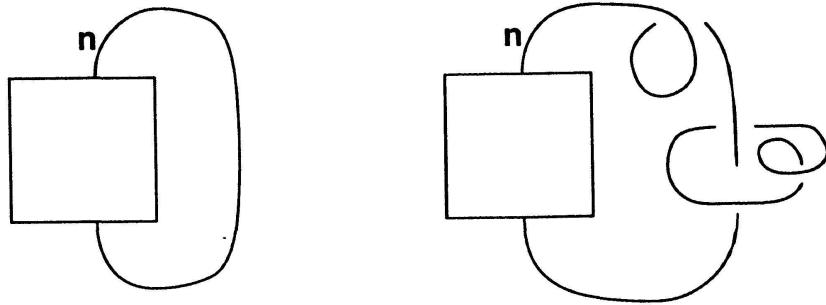


Figure 1

of B (interpreted modulo 2). The move of Remark I.2.1 above becomes:

KC: Add an unknotted component (L_i, ε, c_i) with

$$\varepsilon = \pm 1 \quad \text{and} \quad c_i = \sum_{j \neq i} c_j \text{lk}(L_i, L_j).$$

Add to each p_j the number $\varepsilon(\text{lk}(L_i, L_j))^2$.

§II. The spin invariants

(1) The bracket and the metabracket

We first recall a few results about the Jones–Kauffman module, of the solid torus, which is a key ingredient in the construction. Links in the solid torus $S^1 \times D^2 = S^1 \times I \times I$ can be represented by diagrams in the annulus $S^1 \times I$. $K(S^1 \times D^2)$ is the $\mathbb{Z}[A, A^{-1}]$ -module freely generated by these link diagrams, quotiented by the relations:

- (a) $(D \cup o) = (-A^2 - A^{-2})(D) = \delta(D)$,
- (b) $(\backslash) = A(\asymp) + A^{-1}(\backslash)(\backslash)$.

A product is defined in $K(S^1 \times D^2)$ by the union of two annuli along a component of their boundary. $K(S^1 \times D^2)$ is the polynomial algebra $\mathbb{Z}[A, A^{-1}][z]$, z representing a simple curve, essential in the annulus. The degree gives a $\mathbb{Z}/2$ -graduation. $K(S^1 \times D^2) = K^0(S^1 \times D^2) \oplus K^1(S^1 \times D^2)$.

The Kauffman bracket is defined on link diagrams in the plane by:

- (i) $\langle \emptyset \rangle = 1$,
- (ii) $\langle D \cup o \rangle = (-A^2 - A^{-2})\langle D \rangle = \delta\langle D \rangle$,
- (iii) $\langle \backslash \rangle = A\langle \asymp \rangle + A^{-1}\langle \backslash \rangle(\backslash)$.

This polynomial is an invariant of unoriented framed links in S^3 . We recall that on a link diagram the framing of a component is given by the parallel in the plane.

Given a link diagram, one can replace each of the n components L_i by k_i parallels in the plane, and then evaluate the Kauffman bracket of this cabled link, which only depends on the framed link L , represented by the given diagram. This yields a multilinear form:

$$\langle , \dots, \rangle_L : (K(S^1 \times D^2))^{\otimes n} \rightarrow \mathbb{Z}[A, A^{-1}],$$

which we call the meta-bracket (cf: [L3], [BHMV]).

We shall simply use $\langle \rangle$ for the linear form associated with the unknot with zero framing, and \langle , \rangle_k for the bilinear form associated with the Hopf link, with framing coefficient k on each component.

As in [L3] and [BHMV], we will use the Chebyshev polynomials: $(e_n)_{n \in \mathbb{Z}}$ is the family of polynomials in $K(S^1 \times D^2)$ defined by

$$e_{-1} = 0, e_0 = 1 \text{ and for every } n: ze_n = e_{n+1} + e_{n-1}.$$

e_n is odd or even depending on the parity of n , and: $e_{-n} = -e_{-2+n}$. It is easily shown that for every integer n :

$$\langle e_n \rangle = \frac{(-1)^n (A^{2n+2} - A^{-2n-2})}{A^2 - A^{-2}}.$$

The operators c , t and τ are defined on $K(S^1 \times D^2)$ by Figure 2.

Using relations (a) and (b) one can check the following lemma:

LEMMA (II.1). *For every x :*

- (a) $tzx = -A^3 \tau tx$,
- (b) $\tau zx = A^2 z \tau x + (1 - A^{-4}) cx$,
- (c) $c zx = A^{-2} z cx + (1 - A^4) \tau x$.

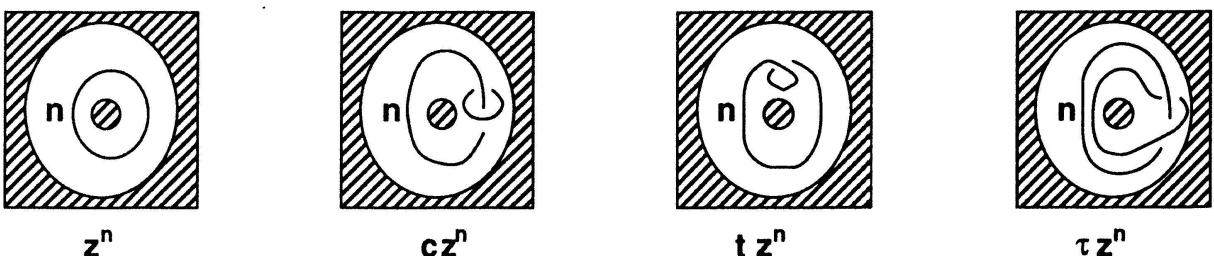


Figure 2

The proposition below follows by induction (see also [L3] and [BHMV]).

PROPOSITION (II.2). *The basis $(e_n)_{n \geq 0}$ diagonalizes the operators c and t :*

$$ce_n = \lambda_n e_n, \quad \text{with } \lambda_n = -A^{2n+2} - A^{-2n-2},$$

and

$$te_n = \mu_n e_n, \quad \text{with } \mu_n = (-1)^n A^{n^2+2n}.$$

(2) How to construct invariants?

The following proposition gives a way to construct invariants on three-manifolds with spin structure. The indeterminate A is evaluated in a ring Λ , and we set:

$$K_A(S^1 \times D^2) = K(S^1 \times D^2) \otimes \Lambda.$$

(Λ must be invertible and will be a root of unity.)

PROPOSITION (II.3). *If $\omega_0 \in K_A^0(S^1 \times D^2)$ and $\omega_1 \in K_A^1(S^1 \times D^2)$ satisfy the following conditions:*

- (a) $\forall x_0 \in K_A^0(S^1 \times D^2) \langle x_0, \omega_1 \rangle_\varepsilon = \langle x_0 \rangle \langle t^\varepsilon \omega_1 \rangle$, for $\varepsilon = \pm 1$;
- (b) $\forall x_1 \in K_A^1(S^1 \times D^2) \langle x_1, \omega_0 \rangle_\varepsilon = \langle x_1 \rangle \langle t^\varepsilon \omega_1 \rangle$, for $\varepsilon = \pm 1$;
- (c) $\langle t^\varepsilon \omega_1 \rangle$ is invertible in Λ , for $\varepsilon = \pm 1$;

then there exists an invariant θ_A of three-manifolds with spin structure such that for any characteristic surgery presentation $(L, K) = (L_i, p_i, c_i)_{1 \leq i \leq n}$, one has the following expression:

$$\theta_A(M_{(L, K)}) = \frac{\langle \omega_{c_1}, \dots, \omega_{c_n} \rangle_L}{\langle t\omega_1 \rangle^{b_+} \langle t^{-1}\omega_1 \rangle^{b_-}}.$$

Here (b_+, b_-) is the signature of the quadratic form associated with the linking matrix B , which is the intersection form on $H_2(W_L, \mathbb{Z})$.

Proof. The only thing to show is that the expression given above is not changed by a KS -move. If we add a new unknotted component, either $(L_{n+1}, \varepsilon, 1)$ or $(L_{n+1}, \varepsilon, 0)$, we make use of (a) or (b) respectively. In each case we get a multiplicative factor $\langle t^\varepsilon \omega_1 \rangle$ for the meta-bracket, which is canceled because either b_+ or b_- is modified.

Remarks (II.4)

1. Multiplication by z is a symmetric operator for $\langle , \rangle_\varepsilon$, $\varepsilon = \pm 1$. It follows that if one requires that $z\omega_0 = \delta\omega_1$, then conditions (a) and (b) are equivalent.

2. Suppose Λ is equipped with a conjugation which send A on A^{-1} , one has on $K_A(S^1 \times D^2)$ a compatible involution defined by taking the mirror image of a diagram. If ω_0 and ω_1 are fixed by this involution, then the conditions corresponding to $\varepsilon = 1$ and $\varepsilon = -1$ are equivalent.

3. In a similar way, one can define an invariant on three-manifolds with cohomology class by the formula:

$$\theta_A(M_{(L,C)}) = \frac{\langle \omega_{c_1}, \dots, \omega_{c_n} \rangle_L}{\langle t\omega_0 \rangle^{b_+} \langle t^{-1}\omega_0 \rangle^{b_-}}$$

if:

- (a) $\forall x_0 \in K_A^0(S^1 \times D^2)$ $\langle x_0, \omega_0 \rangle_\varepsilon = \langle x_0 \rangle \langle t^\varepsilon \omega_0 \rangle$, for $\varepsilon = \pm 1$;
- (b) $\forall x_1 \in K_A^1(S^1 \times D^2)$ $\langle x_1, \omega_1 \rangle_\varepsilon = \langle x_1 \rangle \langle t^\varepsilon \omega_0 \rangle$, for $\varepsilon = \pm 1$;
- (c) $\langle t^\varepsilon \omega_0 \rangle$ is invertible in Λ , for $\varepsilon = \pm 1$.

4. An ‘unspun’ invariant is defined ([BHMV]) by the formula:

$$\theta_A(M_L) = \frac{\langle \omega, \dots, \omega \rangle_L}{\langle t\omega \rangle^{b_+} \langle t^{-1}\omega \rangle^{b_-}}$$

if $\forall x \langle x, \omega \rangle_\varepsilon = \langle x \rangle \langle t^\varepsilon \omega \rangle$ and $\langle t^\varepsilon \omega \rangle$ is invertible, $\varepsilon = \pm 1$.

In §IV we will systematically study the conditions (a), (b) and (c) of the proposition above, and the use of roots of unity will be justified there. However, in order to construct invariants we only need to exhibit solutions to (a), (b) and (c). We will now do so.

(3) Existence of the invariants

Let A be a root of unity in an integral domain Λ , for which the order r of $q = A^4$ is not congruent to 2, modulo 4.

We will assume that

- if $r = 1$, then 2 is invertible, and
- if $r > 1$, then r is invertible.

Let $\omega = \omega_0 + \omega_1$ be the decomposition corresponding to the $\mathbb{Z}/2$ -graduation of the element ω defined as follows:

- (a) If $r = 1$, then $\omega = 1 + \delta^{-1}z$.
- (b) If $r > 1$ is odd, then $\omega = \sum_{i=0}^{r-2} \langle e_i \rangle e_i$.
- (c) If $r = 4k$, then $\omega = \frac{1}{2} \sum_{i=0}^{r-2} \langle e_i \rangle e_i$.

We shall see in §IV that in the case $r = 4k$, the same invariant is obtained, with ω_0 and ω_1 replaced by the reduced elements $\tilde{\omega}_0$ and $\tilde{\omega}_1$, defined by: $\tilde{\omega}_0 = \sum_{i=0}^{k-1} \langle e_{2i} \rangle e_{2i}$, $z\tilde{\omega}_0 = \delta\tilde{\omega}_1$.

Remark. When the order of A is even, the element ω satisfy the condition of Remark II.4.4, and is (up to a coefficient 2 in cases (a) and (b)) the one which is used in [BHMV].

THEOREM (II.5). *There exists an invariant of three-manifolds with spin structure such that for any characteristic surgery presentation $(L, K) = (L_i, p_i, c_i)_{1 \leq i \leq n}$, one has the following expression:*

$$\theta_A(M_{(L, K)}) = \frac{\langle \omega_{c_1}, \dots, \omega_{c_n} \rangle_L}{\langle t\omega_1 \rangle^{b_+} \langle t^{-1}\omega_1 \rangle^{b_-}}.$$

This invariant is 1 for S^3 . It is multiplicative for connected sum. And if A has a conjugation sending A on A^{-1} , changing orientation corresponds to this conjugation.

Remark (II.6). Let:

$$\begin{aligned} A_\rho &= (\mathbb{Z}[A, A^{-1}]/\varphi_\rho) \left[\frac{1}{r} \right], & \text{if } r > 1 \text{ is odd and } \rho \in \{r, 2r, 4r\}, \\ A_\rho &= (\mathbb{Z}[A, A^{-1}]/\varphi_\rho) \left[\frac{1}{r} \right], & \text{if } r = 4k = 16\rho, \\ A_1 &= A_2 = \mathbb{Z} \left[\frac{1}{2} \right] \quad \text{and} \quad A_4 = (\mathbb{Z}[A, A^{-1}]/\varphi_4) \left[\frac{1}{2} \right]. \end{aligned}$$

Here φ_ρ is the ρ -th cyclotomic polynomial. The invariant above is defined in A_ρ , it is denoted by θ_ρ and we shall make it clear in §IV in which sense θ_ρ is universal.

The case $r = 1$ will be studied at the end of §II. To prove existence of this invariant for $r > 1$, we are going to show that ω_0 and ω_1 satisfy the conditions of

Proposition II.1. In order to do that we need to define a quotient of the algebra $K_A(S^1 \times D^2)$: $V_A = K_A(S^1 \times D^2)/(e_{r-1})$. Here (e_{r-1}) denote the ideal generated by e_{r-1} . We observe that:

$$(e_{r-1}) = \text{Span} \{e_{r-1-k} + e_{r-1+k}, k \in \mathbb{N}\}.$$

LEMMA (II.7). (a) *In the quotient algebra V_A :*

$$e_{n+2r} = e_n, \quad \text{for every } n.$$

(b) *The operators c and t , the linear form $\langle \cdot \rangle$, and the scalar products $\langle \cdot, \cdot \rangle_i$, are well defined on the quotient V_A .*

The proof is straightforward. For (b), observe that:

$$\langle z^k, z^{k'} \rangle_0 = \langle c^k z^{k'} \rangle, \quad \text{and} \quad \langle x, y \rangle_i = \langle t^i x, t^i y \rangle_0.$$

Let G be defined on V_A by:

$$G(x) = \sum_{j=0}^{r-1} e_{2j+1} t(xe_{2j+1}).$$

PROPOSITION (II.8). *For every x_0 in V_A^0 : $G(x_0) = G(1)x_0$.*

Proof. It follows from: $e_2 e_n = e_{n-2} + e_n + e_{n+2}$, that for every x :

$$G(xe_2) = \sum_{j=0}^{r-1} e_{2j+1} t(xe_{2j-1}) + \sum_{j=0}^{r-1} e_{2j+1} t(xe_{2j+1}) + \sum_{j=0}^{r-1} e_{2j+1} t(xe_{2j+3}).$$

Using Lemma II.7.(a):

$$\begin{aligned} G(xe_2) &= \sum_{j=0}^{r-1} e_{2j+3} t(xe_{2j+1}) + \sum_{j=0}^{r-1} e_{2j+1} t(xe_{2j+1}) + \sum_{j=0}^{r-1} e_{2j-1} t(xe_{2j+1}), \\ G(xe_2) &= \sum_{j=0}^{r-1} e_2 e_{2j+1} t(xe_{2j+1}) = G(x)e_2. \end{aligned}$$

One can deduce that

- (1) $G(e_2) = G(1)e_2$.
- (2) for every i , $G(e_2 e_{2i}) = e_2 G(e_{2i})$.

Using this, the proposition can be shown, by induction, for all e_{2i} , which generate V_A^0 .

Proof of Theorem II.5

Let:

$$\Omega_1 = \sum_{j=0}^{r-1} \langle e_{2j+1} \rangle e_{2j+1}.$$

In V_A : $\Omega_1 = 2\omega_1$, if r is odd, and $\Omega_1 = 4\omega_1$ if $r = 4k$.

For x_0 in V_A^0 :

$$\langle x_0, \Omega_1 \rangle_1 = \langle t(x_0 \Omega_1) \rangle = \langle G(x_0) \rangle = \langle x_0 \rangle \langle t\Omega_1 \rangle.$$

We can deduce that ω_1 satisfies the condition (a) of II.3.

We can show that in V_A : $z\omega_0 = \delta\omega_1$, so ω_0 satisfies condition II.3.(b). We will now prove that $\langle t\omega_1 \rangle$ is invertible.

$$\begin{aligned} \langle t\Omega_1 \rangle &= \sum_{j=0}^{r-1} \mu_{2j+1} \langle e_{2j+1} \rangle^2, \\ (A^2 - A^{-2})^2 \langle t\Omega_1 \rangle &= - \sum_{j=0}^{r-1} A^{4j^2 + 4j + 1 + 4j + 2} (A^{4j+4} - A^{-4j-4})^2 \\ &= -A^3 \sum_{j=0}^{r-1} q^{j^2 + 2j} (q^{2j+2} - 2 + q^{-2j-2}) \\ &= -A^3 \sum_{j=0}^{r-1} q^{(j+2)^2 - 2} - 2q^{(j+1)^2 - 1} + q^{j^2 - 2} \\ &= (q^{-2} - q^{-1}) 2A^3 g, \\ \langle t\Omega_1 \rangle (A^2 - A^{-2}) &= 2A^{-3} g. \end{aligned}$$

Here g is the Gauss sum:

$$g = \sum_{j=0}^{r-1} q^{j^2}.$$

g is well known for $q = e^{2i\pi/r} = \zeta_r$:

$$g = \frac{1}{2} \sqrt{r} (1+i)(1+e^{-i\pi r/2}) \quad (\text{cf [BE]}).$$

Now we consider the morphism: $\varphi: \mathbb{Z}[\zeta] \rightarrow \Lambda$, which sends ζ_r to q . We claim:

$$-\langle t\Omega_1 \rangle \langle t^{-1}\Omega_1 \rangle (A^2 - A^{-2})^2 = 4\varphi(g\bar{g}) = \begin{cases} 4r & \text{if } r \text{ is odd} \\ 8r & \text{if } r = 4k \end{cases}.$$

For r odd, $\langle t\omega_1 \rangle \langle t^{-1}\omega_1 \rangle (A^2 - A^{-2})^2 = -r$ is invertible.

For $r = 4k$, $\langle t\omega_1 \rangle \langle t^{-1}\omega_1 \rangle (A^2 - A^{-2})^2 = -2k$ is also invertible.

Thus $\langle t\omega_1 \rangle$ is invertible in every case.

(4) Some special cases

If $\Lambda = \mathbb{C}$, we can write: $\langle t\omega_1 \rangle = \beta\gamma$, with: $\beta > 0$ and $|\gamma| = 1$. The invariant can then be expressed by:

$$\theta_A(M_{(L,K)}) = \beta^{-n+v}\gamma^{-\sigma}\langle \omega_{c_1}, \dots, \omega_{c_n} \rangle_L.$$

Here $\sigma = b_+ - b_-$ is the index of W_L , and v is the nullity of the linking matrix (the first Betti number of M_L).

We are going to study the cases: $A = \zeta_{12}$ and $A = \zeta_{16}$ ($\zeta_n = e^{2i\pi/n}$). We denote by $\mu \in \mathbb{Z}/16$ the Rochlin invariant of (M^3, s) .

PROPOSITION (II.9). *For $A = \zeta_{12}$, $\theta_A(M^3, s) = i^\mu$, and for $A = \zeta_{16}$, $\theta_A(M^3, s) = (-\zeta_{16})^{-3\mu}$.*

Proof. For $A = \zeta_{12}$, $r = 3$, $\omega_0 = 1$, $\omega_1 = -z$, $\langle t\omega_1 \rangle = -i$. For $A = \zeta_{16}$, $r = 4$, $\tilde{\omega}_0 = 1$, $\tilde{\omega}_1 = -(1/\sqrt{2})z$, $\langle t\tilde{\omega}_1 \rangle = -(\zeta_{16})^3$. One can evaluate the invariant using a link with empty characteristic sublink; in this case, W_L is spin and its index gives μ .

Remark. For $A = -\zeta_{16}$, $\theta_A(M^3, s) = (\zeta_{16})^{-3\mu}$ is exactly the invariant $\tau_4(M^3, s)$ of Kirby–Melvin (see [KM2], theorem 7.1).

Case $r = 1$. For all j , $\lambda_{2j} = \delta$, $\mu_{2j} = 1$ and $\mu_{2j+1} = -A^3$; one can easily deduce that (ω_0, ω_1) satisfies conditions II.3.

Using a presentation (L, \emptyset) as above, we show that the invariant is: $(-A)^\mu$.

§III. Some symmetry formulas and their consequences

(1) *The odd case*

A good question is: What is the influence of the spin structure s when one calculates the invariant $\theta_A(M^3, s)$ defined in II.5? Here is the answer for odd r :

THEOREM (III.1). *For odd r , the following reduction holds for the spin invariant defined in II.5:*

$$\theta_A(M^3, s) = \alpha^\mu \tilde{\theta}_A(M^3),$$

with $\alpha = -A^{r^2}$ and

$$\tilde{\theta}_A(M^3) = \frac{\langle \omega_0, \dots, \omega_0 \rangle_L}{\langle t\omega_0 \rangle^{b_+} \langle t^{-1}\omega_0 \rangle^{b_-}}.$$

We recall that μ is the Rochlin invariant. We observe that $\alpha^4 = 1$; more precisely, the order of α is 2, 1 or 4 according to the order of A , which may be r , $2r$ or $4r$.

If one constructs invariants for manifolds M^3 equipped with a cohomology class in $H^1(M^3, \mathbb{Z}/2)$ as indicated in the Remark 3 of §II.4.(3), the invariant $\tilde{\theta}_A(M^3)$ is the one obtained for the zero cohomology class. The formula above can be written:

$$\theta_A(M^3, s) = \theta_{A^{r^2}}(M^3, s) \tilde{\theta}_A(M^3).$$

We can show that the same formula holds for the ‘unspun’ invariant constructed with ω when the order of A is $4r$, r odd:

$$\theta_A(M^3) = \theta_{A^{r^2}}(M^3) \tilde{\theta}_A(M^3).$$

When the order of A is $2r$, r odd, the ‘unspun’ invariant $\theta_A(M^3)$ is exactly $\tilde{\theta}_A(M^3)$.

The following lemma gives some needed symmetry formulas, and is easily established.

LEMMA (III.2). *For every integer i :*

- (a) $\langle e_{r-2-i} \rangle = A^{2r} \langle e_i \rangle$,
- (b) $\lambda_{r-2-i} = A^{2r} \lambda_i$,
- (c) $\mu_{r-2-i} = -A^{r^2 + 2r(1+i)} \mu_i$.

Proof of theorem III.1

Using Lemma III.2, we can show that: $\langle t\omega_1 \rangle = -A^{-r^2} \langle t\omega_0 \rangle$. We claim that: $\forall x_0 \in K_A^0(S^1 \times D^2)$ $\langle tx_0, t\omega_0 \rangle = \langle x_0 \rangle \langle t\omega_0 \rangle$, so that $\tilde{\theta}_A$ is well defined.

Now we can evaluate the two invariants using a presentation (L, \emptyset) ...

(2) *The case $r = 4k$*

Let us begin by giving an example which shows that no reduction analogous to III.1 holds in this case.

The lens space $L(n, 1)$ is obtained with an unknotted circle weighted by n . For even n , this space has two spin structures, and the respective Rochlin invariants are: $\text{sgn}(n)$ and $\text{sgn}(n) - n$ (here $\text{sgn}(n)$ is the sign of n : 0 or ± 1). This can be seen using $\theta_{\zeta_{16}}$.

For $n = 0$, the two spin structures are equivalent (the space is $S^1 \times S^2$). The Rochlin invariant distinguishes the two spin structures if n is not in $16\mathbb{Z}$.

For $n = 16k$, $k \neq 0$, the invariants for $A = \zeta_{32k}$ are opposite and not zero, and so the two spin structures are distinguished.

Proof. These two invariants are:

$$I_0 = \langle t\omega_1 \rangle^{-\text{sgn}(k)} \langle t^{16k} \omega_0 \rangle = \langle t\omega_1 \rangle^{-\text{sgn}(k)} \langle \omega_0 \rangle,$$

and

$$I_1 = \langle t\omega_1 \rangle^{-\text{sgn}(k)} \langle t^{16k} \omega_1 \rangle = \langle t\omega_1 \rangle^{-\text{sgn}(k)} \langle -\omega_1 \rangle.$$

We have: $\langle \omega_0 \rangle = \langle \omega_1 \rangle = (-r/2) \langle A^2 - A^{-2} \rangle^{-2}$, so $I_0 = -I_1 \neq 0$.

We now give the decomposition theorem. A similar proposition was announced by Turaev in [T2] and appears in [KM2]. The hypotheses are those of II.3.

THEOREM (III.3). *For $r = 4k$, the global invariant (see II.4.3) decomposes as following:*

$$\theta_A(M_L) = \sum_{s \in \text{Spin}(M_L)} \theta_A(M_L, s).$$

We need some symmetry formulas again.

LEMMA (III.4). *If $r = 4k$, for all integers i, j :*

- (a) $\langle e_{r-2-i} \rangle = \langle e_i \rangle$.
- (b) $\lambda_{r-2-i} = -\lambda_i$.
- (c) $\mu_{r-2-i} = (-1)^{i+1} \mu_i$.
- (d) $\langle z^j, e_{r-2-i} \rangle = (-1)^j \langle z^j, e_i \rangle$.

In V_A we set

$$N_+ = \text{Span} \{e_{2k-1-i} + e_{2k-1+i}, 0 \leq i \leq 2k-1\} = N_+^0 \oplus N_+^1,$$

$$N_- = \text{Span} \{e_{2k-1-i} + e_{2k-1+i}, 1 \leq i \leq 2k-1\} = N_-^0 \oplus N_-^1.$$

COROLLARY (III.5). (a) *For the scalar product $\langle \cdot, \cdot \rangle_0$:*

$$V_A^0 = (N_-)^{\perp} \quad \text{and} \quad V_A^1 = (N_+)^{\perp}.$$

$$(b) \quad t(N_+^1) = N_+^1, \quad t(N_-^1) = N_-^1, \quad t(N_+^0) = N_-^0, \quad t(N_-^0) = N_+^0.$$

Proof of Theorem III.3

We can deduce from III.5 that for every k :

$$t^k \omega_1 \text{ is in } N_+^1, \quad t^{2k} \omega_0 \text{ is in } N_+^0 \text{ and } t^{2k+1} \omega_0 \text{ is in } N_-^0.$$

Now we claim that we can use a link L , with only unknotted components: any surgery presentation can be reduced to such a link, using K -moves.

Let us write:

$$\langle \omega, \dots, \omega \rangle_L = \sum_{K \subset L} \langle \omega_{a_1}, \dots, \omega_{a_n} \rangle_L.$$

Here a_i is one if and only if the component L_i is in the sublink K . We are going to show that, in this sum, the contribution of a term corresponding to a sublink which is not characteristic is zero. Let K be a sublink of L , which contains a component L_i , such that:

$$[L_i] \cdot [K] \neq [L_i] \cdot [L_i] \pmod{2}.$$

(a) If L_i is not in K , then: $lk(L_i, K) = p_i + 1 \pmod{2}$, p_i being the coefficient of the component L_i . If p_i is even, the term corresponding to K can be written:

$$\langle t^{2k}\omega_0, x_1 \rangle_0, \quad \text{with } x_1 \text{ odd, and is zero, according to III.5.}$$

If p_i is odd the term is written:

$$\langle t^{2k+1}\omega_0, x_0 \rangle_0, \quad \text{with } x_0 \text{ even...}$$

(b) If L_i is in K , then $lk(L_i, K - L_i) = 1 \pmod{2}$.

The corresponding term is:

$$\langle t^k\omega_1, x_1 \rangle_0, \quad \text{with } x_1 \text{ odd...}$$

A particular case is that, for $\varepsilon = \pm 1$, $\langle t^\varepsilon\omega \rangle = \langle t^\varepsilon\omega_1 \rangle$.

The theorem is proven.

§IV. Uniqueness

We are going to study the conditions of Proposition II.3. This will justify the choice of roots of unity, and will be used to prove the following uniqueness theorem.

THEOREM (IV.1). *Suppose Λ is an integral domain containing a root of unity A , of order ρ , and θ_A is the invariant associated to a solution (w_0, w_1) of conditions II.3.*

There exists a unit λ in Λ , a homomorphism $f: \Lambda_\rho \rightarrow \Lambda$ induced by $\mathbb{Z}[A, A^{-1}] \rightarrow \Lambda$ such that:

$$\theta_A(M, s) = \lambda^v f(\theta_\rho(M, s)).$$

Here v is the first Betti number of M .

See II.6 for the definition of Λ_ρ and θ_ρ .

(1) *Diagonalisation of the bilinear form*

$$\langle \cdot, \cdot \rangle_0 : K^0(S^1 \times D^2) \otimes K^1(S^1 \times D^2) \rightarrow \mathbb{Z}[A, A^{-1}].$$

Recall that the eigenvalues of the operator c are

$$\lambda_n = -A^{2n+2} - A^{-2n-2} \quad (n \geq 0).$$

We define the elements Q_{2k} and Q_{2k+1} ($k \geq 0$) respectively in $K^0(S^1 \times D^2)$ and $K^1(S^1 \times D^2)$ by:

$$Q_0 = 1, \quad Q_1 = z$$

and for $k \geq 1$:

$$Q_{2k} = \prod_{i=0}^{k-1} (z^2 - \lambda_{2i+1}^2),$$

$$Q_{2k+1} = z \prod_{i=0}^{k-1} (z^2 - \lambda_{2i}^2).$$

PROPOSITION (IV.2)

(a) If $l \neq k$, $\langle Q_{2k}, Q_{2l+1} \rangle_0 = 0$.

$$(b) d_k = \langle Q_{2k}, Q_{2k+1} \rangle_0 = \frac{-1}{A^2 - A^{-2}} \prod_{i=1}^{2k+1} (A^{4i} - A^{-4i}).$$

Proof. (a) For $l < k$:

$$\begin{aligned} \langle Q_{2k}, e_{2l+1} \rangle_0 &= \langle Q_{2k}(c) \cdot e_{2l+1} \rangle \\ &= Q_{2k}(\lambda_{2l+1}) \langle e_{2l+1} \rangle = 0. \end{aligned}$$

Thus Q_{2k} is orthogonal to $\text{Span}\{z^{2l+1}, l < k\}$. In the same way, Q_{2k+1} is orthogonal to $\text{Span}\{z^{2l}, l < k-1\}$.

$$(b) d_k = \langle Q_{2k}, e_{2k+1} \rangle_0,$$

$$d_k = \langle Q_{2k}(c) \cdot e_{2k+1} \rangle = Q_{2k}(\lambda_{2k+1}) \langle e_{2k+1} \rangle,$$

$$d_k = (-1)^{2k+1} \frac{A^{4k+4} - A^{-4k-4}}{A^2 - A^{-2}} \prod_{l < k} (\lambda_{2k+1}^2 - \lambda_{2l+1}^2).$$

One can deduce (b) from the identity:

$$\lambda_{2k+1}^2 - \lambda_{2l+1}^2 = (A^{4(k+l)+8} - A^{-4(k+l)-8})(A^{4(k-l)} - A^{-4(k-l)}).$$

In order to discuss the conditions of II.3, we need the following lemma:

LEMMA (IV.3)

$$\langle t^{-1}Q_{2k} \rangle = q^{-k^2 - 2k} \prod_{l=1}^k (1 + q^{2l})(1 - q^{2l+1}).$$

We first prove the following technical lemma:

LEMMA (IV.4)

$$(a) \tau Q_{2k} = A^{4k} Q_{2k+1}.$$

$$(b) \tau^2 Q_{2k} = A^{8k+2} Q_{2k+2} + A^{-6}(2 + A^{16k+16} + A^{16k+8})Q_{2k} \\ + A^{8k-2}(A^{8k+4} - A^{-8k-4})(A^{8k} - A^{-8k})Q_{2k-2}.$$

Proof of Lemma IV.4

$$(a) \tau Q_{2k} = A^{4k} Q_{2k+1} + \sum_{l=0}^{k-1} \zeta_l Q_{2l+1} \quad (\text{see Lemma II.1}).$$

For $l < k$:

$$\langle Q_{2l}, \tau Q_{2k} \rangle_0 = \zeta_l d_1 = \langle \tau Q_{2l}, Q_{2k} \rangle_0 = 0.$$

(b) In a similar way:

$$\tau Q_{2k+1} = A^{4k+2} Q_{2k+2} + \xi_k Q_{2k} + v_{k-1} Q_{2k-2} \quad (v_{-1} = 0).$$

ξ_k and v_{k-1} satisfy $\xi_k d_k = \langle \tau Q_{2k+1}, Q_{2k+1} \rangle$, and

$$v_{k-1} d_{k-1} = \langle \tau Q_{2k+1}, Q_{2k-1} \rangle = A^{4k-2} d_k.$$

One easily obtains:

$$v_{k-1} = A^{4k-2}(A^{8k+4} - A^{-8k-4})(A^{8k} - A^{-8k}).$$

To calculate ξ_k , we shall use e_{2k} and e_{2k-1} :

$$e_{2k} = Q_{2k} + \eta_{2k-2} Q_{2k-2} + \dots,$$

$$e_{2k+1} = Q_{2k+1} + \eta_{2k-1} Q_{2k-1} + \dots.$$

The dots indicate a polynomial which has degree smaller than the preceding term.

$$\begin{aligned}\langle Q_{2k-1}, e_{2k} \rangle &= \eta_{2k-2} d_{k-1} \\ &= Q_{2k-1}(\lambda_{2k}) \langle e_{2k} \rangle \\ &= (\lambda_{2k}^2 - \lambda_{2k-2}^2)^{-1} d_k.\end{aligned}$$

One deduces:

$$\eta_{2k-2} = \frac{A^{8k+4} - A^{-8k-4}}{A^4 - A^{-4}}.$$

In a similar way:

$$\eta_{2k-1} = \frac{A^{8k} - A^{-8k}}{A^4 - A^{-4}}.$$

Now, we can calculate ξ_k :

$$\begin{aligned}\tau Q_{2k+1} &= \tau e_{2k+1} - \eta_{2k-1} \tau Q_{2k-1} \\ &= A^{4k+2} e_{2k+2} + A^{-4k-6} e_{2k} - \eta_{2k-1} A^{4k-2} Q_{2k} + \dots \\ &= A^{4k+2} Q_{2k+2} + A^{4k+2} \eta_{2k} Q_{2k} + A^{-4k-6} Q_{2k} - \eta_{2k-1} A^{4k-2} Q_{2k} + \dots.\end{aligned}$$

One deduces:

$$\xi_k = 2A^{-4k-6} + A^{12k+10} + A^{12k+2}$$

and

$$\begin{aligned}\tau^2 Q_{2k} &= A^{4k} \tau Q_{2k+1} \\ &= A^{4k} (A^{4k+2} Q_{2k+2} + \xi_k Q_{2k} + \eta_{k-1} Q_{2k-2}), \\ \tau^2 Q_{2k} &= A^{8k+2} Q_{2k+2} + A^{-6} (2 + A^{16k+16} + A^{16k+8}) Q_{2k} \\ &\quad + A^{8k-2} (A^{8k+4} - A^{-8k-4}) (A^{8k} - A^{-8k}) Q_{2k-2}.\end{aligned}$$

Proof of Lemma IV.3

Let: $P_{2k} = t^{-1} Q_{2k}$. Using Lemma II.1, IV.4(b) can be written:

$$\begin{aligned}A^{-6} z^2 P_{2k+2} &= A^{8k+2} P_{2k+2} + A^{-6} (2 + A^{16k+16} + A^{16k+8}) P_{2k} \\ &\quad + A^{8k-2} (A^{8k+4} - A^{-8k-4}) (A^{8k} - A^{-8k}) P_{2k-2}.\end{aligned}$$

Let $u_k = P_{2k}(\delta)$.

$$\begin{aligned} u_{k+1} &= A^{-8k-8}(\delta^2 - 2 - A^{16k+16} - A^{16k+8})u_k \\ &\quad - A^{-4}(A^{8k+4} - A^{-8k-4})(A^{8k} - A^{-8k})u_{k-1}. \end{aligned}$$

Lemma IV.3 can now be proven by induction.

(2) Condition II.3 studied

Suppose A is in a field Λ (for example, the quotient field of an integral domain). We want to study the following equation:

$$\forall x_0 \in K_A^0(S^1 \times D^2) \quad \langle x_0, v_1 \rangle_0 = \langle t^{-1}x_0 \rangle \langle v_1 \rangle. \quad (E_1)$$

We are looking for a solution $v_1 \in K_A^1(S^1 \times D^2)$ such that: $\langle v_1 \rangle \neq 0$. Let: $w = \langle v_1 \rangle^{-1}v_1$. (E_1) becomes:

$$\forall x_0 \in K_A^0(S^1 \times D^2) \quad \langle x_0, w \rangle_0 = \langle t^{-1}x_0 \rangle. \quad (E_2)$$

It is clear that w determines v_1 , up to a multiplicative coefficient.

Write: $w = \sum w_k Q_{2k+1}$. (E_2) becomes:

$$\forall k \quad w_k d_k = u_k.$$

A polynomial solution can exist only if u_k is zero for k big. Thus q must be a root of unity, the order of which is again denoted by r . The smallest integer k for which d_k is zero is the smallest for which:

$$q^{4k} = 1 \quad \text{or} \quad q^{4k+2} = 1.$$

The order of q^2 is $2k$ or $2k+1$.

Case 1: r is odd, so $r = 2k+1$.

(E_2) has a unique solution v of minimal degree: $\deg(v) = 2k-1$. The other solutions are: $v + x$ with $x \in N_k = \text{Span} \{Q_{2l+1}, l \geq k\}$.

Case 2: r is congruent to 2 modulo 4.

The order of q^2 is: $r/2 = 2k+1$, but u_k is not zero: $q^{2k+1} = -1$. (E_2) has no solution.

Case 3: r is congruent to 0 modulo 4.

The order of q^2 is: $r/2 = 2k$, $q^{2k} = -1$, and for $l \geq k$, $u_l = 0$. The conclusion is the same as in case 1.

Conclusion

The discussion above shows that an invariant can be defined, using Proposition II.3, only if A is a root of unity whose order is not congruent to 8, modulo 16. Furthermore, the solution ω_1 , given in theorem II.5, has minimal degree for odd r ; for $r = 4k$, a solution of minimal degree is $\tilde{\omega}_1$.

(3) *Proof of Theorem IV.1*

Recall the hypothesis: A is a root of unity in an integral domain \mathcal{A} , and (w_0, w_1) is a solution of condition II.3. The discussion above shows that the order ρ of A is not congruent to eight modulo 16, or equivalently, the order r of $q = A^4$ is not congruent to 2 modulo 4. In the following we suppose $r > 1$; the case $r = 1$ is left to the reader.

Let us work with the quotient field $Q(\mathcal{A})$. First of all, we reduce the problem, using the quotient space

$$V_A = K(S^1 \times D^2) \otimes Q(\mathcal{A}) / (e_{r-1}).$$

LEMMA (IV.5). *The meta-bracket is well defined on V_A .*

Proof. Suppose x is in the ideal (e_{r-1}) , we want to show that:

$$\langle \dots, x, \dots \rangle_L = 0.$$

Let L_i be the component where x is satellized. Note that changing any crossing on the diagram, adding around it an unknotted circle, weighted by ± 1 , and satellized with ω_1 , does not change the nullity of the meta-bracket. Hence we can suppose that L_i is unknotted.

Then we can write: $\langle \dots, x, \dots \rangle_L = \langle t^k(x), y \rangle_0 = 0$.

(a) *Case $r = 2k + 1$.* The diagonalization of §IV.1 shows that the bilinear form:

$$\langle \cdot, \cdot \rangle_0 : V_A^0 \otimes V_A^1 \rightarrow Q(\mathcal{A}) \text{ is non-singular.}$$

Observe that $(Q_{2j})_{0 \leq j \leq k-1}$ and $(Q_{2j+1})_{0 \leq j \leq k-1}$ are bases of V_A^0 and V_A^1 , respectively, and that $(e_{r-1}) = (e_{2k}) = \text{Span} \{Q_l, l \geq 2k\} = (Q_{2k})$. Using the unique solution of (E_2) , one can see that in V_A^1

$$\langle tw_1 \rangle \omega_1 = \langle t\omega_1 \rangle w_1.$$

We can write $w_1 = \lambda \omega_1$, and also $w_0 = \lambda \omega_0$. λ is the constant of w_0 (reduced in V_A^0), so is in Λ .

$$\langle tw_1 \rangle = \lambda \langle t\omega_1 \rangle \text{ is invertible, so } \lambda \text{ is invertible.}$$

(b) *Case $r = 4k$.* Now the bilinear form: $\langle \cdot, \cdot \rangle_0: V_A^0 \otimes V_A^1 \rightarrow Q(\Lambda)$, is singular.

$$\text{Let: } W_A^1 = V_A^1 / N_-^1, \text{ and } W_A^0 = V_A^0 / N_+^0.$$

We can deduce from III.5 that the bilinear form $\langle \cdot, \cdot \rangle_0$ is defined on $W_A^0 \otimes W_A^1$; the diagonalisation of IV.1) shows that it is non singular. In W_A^1 and V_A^0 / N_-^0 , we have respectively: $w_1 = \lambda \omega_1 = \lambda \tilde{\omega}_1$, and $w_0 = \lambda \omega_0 = \lambda \tilde{\omega}_0$. λ is the constant of w_0 , so is in Λ .

$\langle tw_1 \rangle = \lambda \langle t\tilde{\omega}_1 \rangle$ is invertible, so λ is invertible. One can check that $\langle t\tilde{\omega}_1 \rangle$ lives in Λ , more precisely in the homomorphic image of $\mathbb{Z}[A, A^{-1}]$.

The proof is achieved as in the odd case using the following lemma. This lemma justifies the use of the reduced elements $\tilde{\omega}_0$ and $\tilde{\omega}_1$.

LEMMA (IV.6). *Suppose $(L, K) = (L_i, p_i, c_i)_{1 \leq i \leq n}$ is a link with characteristic sublink. Every expression $\langle x_1, \dots, x_n \rangle_L$, in which each x_i has the parity of c_i , is zero if for some j , x_j is in N_-^0 , or in N_-^1 .*

Proof. As already seen, we can suppose that L_j is unknotted.

(a) Suppose L_j is in K : $lk(L_j, K - L_j) \equiv 0 \pmod{2}$. The expression can be written: $\langle y, t^\alpha x_j \rangle_0$ with y even. $t^\alpha x_j$ is in N_-^1 , so the scalar product is zero.

(b) Suppose L_j is not in K : $lk(L_j, K) \equiv p_j \pmod{2}$. The expression can be written $\langle t^{p_j} x_j, y \rangle_0$. The parity of y is given by p_j ; the scalar product is zero again.

Acknowledgements

The author thanks his advisor Pr P. Vogel, and also Pr N. Habegger and G. Masbaum for many discussions.

REFERENCES

- [BE] C. BERNDT and J. EVANS, 'The determination of Gauss sums', Bull. of the AMS, Vol. 5, Sept 1981, 107–129.
- [B] C. BLANCHET, 'Une famille d'invariants sur les variétés spin de dimension trois', thèse, Université de NANTES, fév. 1991.
- [BHMV] C. BLANCHET, N. HABEGGER, G. MASBAUM and P. VOGEL, 'Invariants on three-manifolds derived from the Kauffman bracket', preprint.
- [FR] R. A. FENN and C. P. ROURKE, 'On Kirby's calculus of links', Topology 18 (1979), 1–15.
- [K] L. H. KAUFFMAN, 'New invariants in the theory of knots', Amer. Math. Monthly, Vol. 19 (1988), 195–242.
- [Ka] S. J. KAPLAN, 'Constructing framed 4-manifolds with given almost framed boundaries', Transactions of the AMS 254 (1979), 237–263.
- [Kil] R. KIRBY, 'A calculus for framed links', Invent. Math. 45 (1978), 35–55.
- [Ki2] R. KIRBY, 'The topology of 4-manifolds', Lect. Notes in Math. Springer Verlag.
- [KM1] R. KIRBY and P. MELVIN, 'Evaluation of the 3-manifolds invariants of Witten and Reshetkin–Turaev for $sl(2, \mathbb{C})$ ', in: Geometry of Low-dim. Manifolds. Durham 1989. (LMS Lect. Note Ser., Vol. 151), Cambridge Univ. Press 1990.
- [KM2] R. KIRBY and P. MELVIN, 'The 3-manifolds invariants of Witten and Reshetkin–Turaev for $sl(2, \mathbb{C})$ ', Inv. Math. 105 (1991), 473–545.
- [L1] W. B. R. LICKORISH, 'Three-manifolds invariants from the combinatorics of the Jones polynomial', Pac. J. Math. 149, No. 2 (1991), 337–347.
- [L2] W. B. R. LICKORISH, 'Three-manifolds and the Temperley–Lieb algebra', Math. Ann. 290 (1991), 657–670.
- [L3] W. B. R. LICKORISH, 'Calculations with the Temperley–Lieb algebra', preprint.
- [M] J. MILNOR, 'Spin structures on manifolds', l'Ens. Math. 9 (1963), 198–203.
- [RT] N. Y. RESHETIKIN and V. G. TURAEV, 'Invariants of three-manifolds via links polynomials and quantum groups', preprint.
- [T1] V. G. TURAEV, 'The Conway and Kauffman modules of the solid torus', LOMI (1989).
- [T2] V. G. TURAEV, 'State sum models in low-dimensional topology', Proc. ICM Kyoto 1990, Vol. I, 689–698.
- [W] E. WITTEN, 'Quantum field theory and Jones polynomial', Comm. Math. Phys. 121 (1989), 351–399.

Département de Mathématiques
 2 rue de la Houssinière
 44072 NANTES Cedex 03
 FRANCE

Received July 10, 1991