

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 67 (1992)

Artikel: Covering homotopy 3-spheres.
Autor: Piergallini, R.
DOI: <https://doi.org/10.5169/seals-51095>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 25.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Covering homotopy 3-spheres

R. PIERGALLINI

Abstract. We prove the following theorem: for any closed orientable 3-manifold M and any homotopy 3-sphere Σ , there exists a simple 3-fold branched covering $p : M \rightarrow \Sigma$.

We also propose the conjecture that, for any primitive branched covering $p : M \rightarrow N$ between orientable 3-manifolds, $g(M) \geq g(N)$, where g denotes the Heegaard genus. By the above mentioned result, the genus 0 case of such conjecture is equivalent to the Poincaré conjecture.

Introduction

An n -fold branched covering $p : M \rightarrow N$ between connected pl 3-manifolds, is a non-degenerate pl map for which exist two links $S(p) \subset M$ and $B(p) \subset N$ (the *branch link*), such that the restriction $p|_{M - S(p)} : M - S(p) \rightarrow N - B(p)$ is an ordinary n -fold covering.

The branched covering p can be described in terms of the monodromy ω of $p|_{M - S(p)}$ (with respect to any base-point), which represents $\pi_1(N - B(p))$ into Σ_n , the symmetric group of degree n .

We say that p is *simple* if $\omega(\mu)$ is a transposition for each meridian μ around $B(p)$. Moreover, p is *primitive* if it cannot be factored as a branched covering followed by an ordinary covering, that is if it induces a surjection between fundamental groups.

A well known theorem proved by Hilden [2], Hirsch [3] and Montesinos [5], says that every closed orientable 3-manifold can be presented as a 3-fold simple branched cover of S^3 .

In this paper we adapt the proof of that theorem given in [6], in order to prove that in fact S^3 can be replaced with any homotopy 3-sphere Σ .

In particular, we have that any homotopy 3-sphere is covered by S^3 . This fact allows us to see the Poincaré conjecture as a special case of the much more general conjecture that primitive branched coverings cannot increase the Heegaard genus.

1. Branched coverings of homotopy 3-spheres

The following theorem is the version for homotopy 3-spheres, of the Hilden–Hirsch–Montesinos theorem on branched coverings of S^3 mentioned in the introduction. The proof of the theorem follows essentially the same line of [6] (cf. also [7]).

THEOREM. *For any closed orientable 3-manifold M and any homotopy 3-sphere Σ , there exists a simple 3-fold branched covering $p : M \rightarrow \Sigma$.*

Proof. Let Σ be a homotopy 3-sphere, and $p_\Sigma : \Sigma \# \Sigma \# \Sigma \rightarrow \Sigma$ a simple 3-fold covering branched over $B(p_\Sigma) =$ two unlinked trivial knots, with monodromy respectively (12) and (23).

If $C \subset \Sigma$ is a 3-cell and $\Gamma = \Sigma - C$, then $p_\Sigma = p_C \cup_{\text{Bd}} p_\Gamma$, where $p_C : C \rightarrow C$ is the simple 3-fold branched covering shown in Figure 1, and $p_\Gamma : \Gamma_1 \#_{\text{Bd}} \Gamma_2 \#_{\text{Bd}} \Gamma_3 \rightarrow \Gamma$ is obtained by substituting a 3-cell in the interior of C with a copy of Γ , as shown in Figure 2.

Now, let M be a closed orientable 3-manifold and $L_1, L_2, L_3 \subset \Gamma$ be disjoint links, such that M can be obtained from Σ by surgery on L_2 , and there are surgeries on L_1 and L_3 giving S^3 .

Since Γ is simply connected, L_i bounds a singular union of disks F_i that can be assumed, by using a well-known piping technique, to have only clasp singularities (cf. Figure 3), for $i = 1, 2, 3$. Moreover, since such F_i 's separately collapse into graphs, we can also easily assume that they are disjoint from each other.

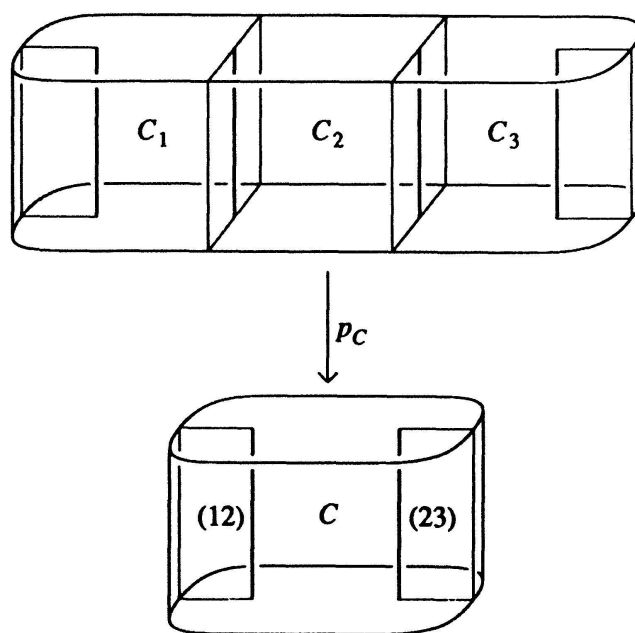


Figure 1

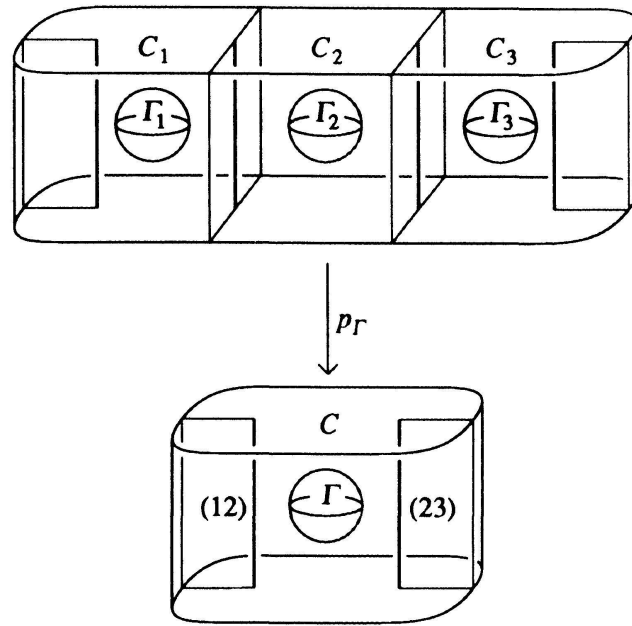


Figure 2

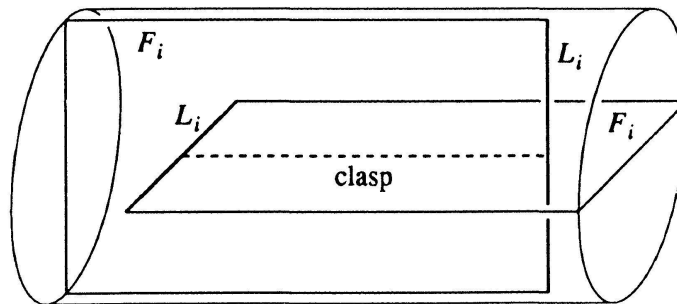


Figure 3

In Figure 3 is represented (up to homeomorphism) a neighborhood of a clasp of F_i . Now, we can change the surgery link L_i without changing the resulting manifold, by using the operations introduced by Kirby in [4]. In particular we perform the operation described in Prop. 1A of [4], inside the neighborhood of Figure 3, in order to eliminate the clasp, as shown in Figure 4.

Let L'_i be the modified link L_i after all the clasps of F_i have been removed as said above, and L''_i be the union of all the trivial loops λ added performing such modifications (cf. Figure 4), for $i = 1, 2, 3$. These new links have the following properties:

- (1) M can be obtained from Σ by surgery on $L'_2 \cup L''_2$;
- (2) S^3 can be obtained from Σ by surgery on $L'_1 \cup L''_1$ and $L'_3 \cup L''_3$;

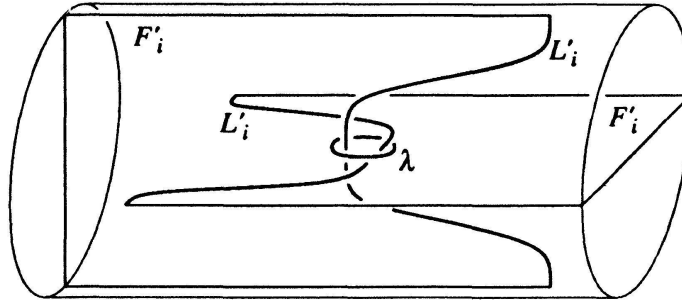


Figure 4

- (3) L'_i and L''_i are trivial links in $\Gamma - (L'_j \cup L''_j \cup L'_k \cup L''_k)$, that is they respectively bound two non-singular union of disks F'_i and F''_i in $\Gamma - (L'_j \cup L''_j \cup L'_k \cup L''_k)$, for $\{i, j, k\} = \{1, 2, 3\}$.

Now, let $\alpha'_i, \alpha''_i \subset C$ be the arcs between $\text{Bd } \Gamma$ and the branch link $B(p_\Sigma)$ of the covering p_Σ , shown in Figure 5.

We can assume, up to an ambient isotopy of Σ , that each component G of F'_i (resp. F''_i) meets $B(p_\Sigma)$ exactly in a small arc of its boundary, in such a way that $G - \Gamma$ consists of a band parallel to the arc α'_i (resp. α''_i), for $i = 1, 2, 3$.

Then $A = \text{Cl}(\bigcup_{i=1,2,3} (L'_i \cup L''_i) - B(p_\Sigma))$ consists of finitely many arcs in Σ , whose end-points are in $B(p_\Sigma)$. Moreover, by property 3, $p_\Sigma^{-1}(A)$ is isotopically equivalent to $\tilde{L}_1 \cup \tilde{L}_2 \cup \tilde{L}_3 \cup \tilde{A} \subset \Sigma \# \Sigma \# \Sigma$, where: \tilde{L}_i is a copy of $L'_i \cup L''_i$ inside Γ_i for $i = 1, 2, 3$, and \tilde{A} consists of finitely many arcs (cf. Figure 5).

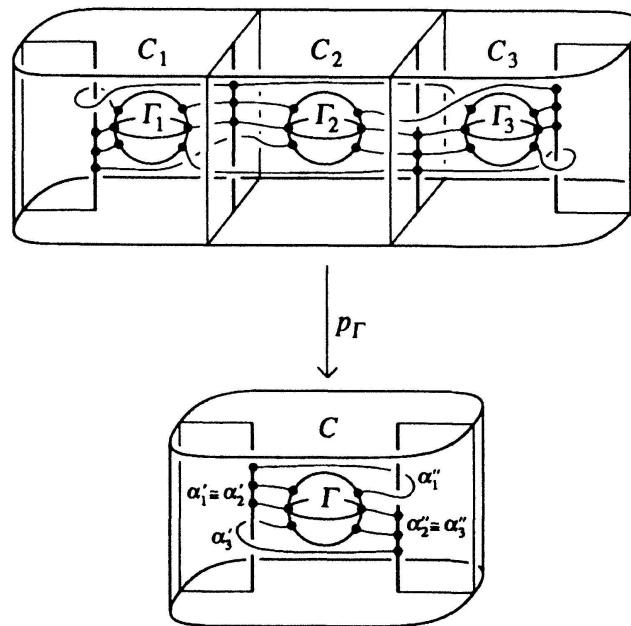


Figure 5

Now, following [6], it is easy to prove, that the surgeries of properties 1 and 2, performed on the links \tilde{L}_i , can be realized lifting (by means of p_Σ) trivial surgeries (= removing 3-cells from Σ and sewing them back differently). In such a way we get $M \cong S^3 \# M \# S^3$ as a branched covering of Σ .

We conclude this section with a corollary, which perhaps might be useful in order to prove the Poincaré conjecture. In the next section we will discuss a conjecture in this direction.

COROLLARY. *For any homotopy 3-sphere Σ there exists a simple 3-fold branched covering $p : S^3 \rightarrow \Sigma$.*

2. Branched coverings and Heegaard genus

In this section we propose and briefly discuss a conjecture about Heegaard genus of branched coverings, which is related to various other conjectures about 3-manifolds.

First of all, we observe that, in the light of the corollary above, the Poincaré conjecture is equivalent to the following one: if $p : S^3 \rightarrow M$ is a primitive branched covering, then $M \cong S^3$. By the positive solution of the Smith conjecture ([8]), this is known to be true when p is cyclic.

Now, as a generalization of such a formulation of the Poincaré conjecture, we conjecture that primitive branched coverings cannot increase the Heegaard genus. More precisely, we propose the following:

Conjecture. If $p : M \rightarrow N$ is a primitive branched covering between closed orientable 3-manifolds, then $g(M) \geq g(N)$, where g denotes the Heegaard genus.

Of course, this conjecture becomes trivially true if the Heegaard genus is replaced by the rank of the fundamental group of the manifolds. So it can be thought as a weakening of the Waldhausen conjecture about the coincidence of the rank with the Heegaard genus.

In [1], Boileau and Zieschang proved that the Waldhausen conjecture is true for most of the closed orientable Seifert manifolds. Then, the same holds for our conjecture.

In the same paper, Boileau and Zieschang also give a counterexample to the Waldhausen conjecture, proving that, for the Seifert manifolds $N = S(0; e_0; 1/2, 1/2, 1/2, \beta/2\lambda + 1)$, with $\lambda > 0$, $\gcd(\beta, 2\lambda + 1) = 1$ and $e_0 \neq \pm 1/2(2\lambda + 1)$, one has $\text{rk}(N) = 2$ and $g(N) = 3$. Then a possible counterexample to our conjecture, could be given by a genus 2 primitive branched covering of such an N .

REFERENCES

- [1] BOILEAU, C. M. and ZIESCHANG, H., *Heegaard genus of closed orientable Seifert 3-manifolds*, Invent. Math. 76 (1984), 455–468.
- [2] HILDEN, H. M., *Every closed orientable 3-manifold is a 3-fold branched covering space of S^3* , Bull. Amer. Math. Soc. 80 (1974), 1243–1244.
- [3] HIRSCH, U., *Über offene Abbildungen auf die 3-Sphäre*, Math. Zeitschrift 140 (1974), 203–230.
- [4] KIRBY, R., *A calculus for framed links in S^3* , Invent. Math. 45 (1978), 35–56.
- [5] MONTESINOS, J. M., *A representation of closed, orientable 3-manifolds as 3-folds branched coverings of S^3* , Bull. Amer. Math. Soc. 80 (1974), 845–846.
- [6] MONTESINOS, J. M., *Three-manifolds as 3-fold branched covers of S^3* , Quart. J. Math. Oxford 27 (1976), 85–94.
- [7] MONTESINOS, J. M., *Lecture on 3-fold simple coverings and 3-manifolds*, Contemporary Mathematics 44 (1985), 157–177.
- [8] MORGAN, J. W. and BASS, H., *The Smith Conjecture*, Academic Press 1984.

*Dipartimento di Matematica
Università di Perugia
I-06100 Perugia
Italy*

Received February 25, 1991; March 26, 1992