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# On the trigonometry of symmetric spaces

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## 1. Introduction

Symmetric spaces were introduced, classified and intensively studied by Élie Cartan more than sixty years ago (cf. e.g. [Ca1], [Ca2], [Ca3]). However, the understanding of their geometry is still in an infancy state, with the striking exception of the spaces of constant curvature. In the study of Euclidean, spherical and hyperbolic geometry triangles play a crucial rôle. Actually, these three geometries are characterized by the same congruence theorem, "side-angle-side", for triangles, which is equivalent to the fact that the underlying space has constant sectional curvature. The theorem states that the congruence class of a geodesic triangle $\mathscr{A}\mathscr{B}\mathscr{C}$ is completely determined by two of its sides, e.g. $a = \mathscr{B}\mathscr{C}$, $b = \mathscr{A}\mathscr{C}$, and the enclosed angle $\gamma$ at $\mathscr{C}$. The third side $c$ and the angles $\alpha$ and $\beta$ can be computed from $a$, $b$ and $\gamma$. This is the essence of the "laws of trigonometry", the law of cosines and the law of sines.

Various applications of spherical and hyperbolic geometry rest on the effectiveness of these formulae. To understand and to generalize them is one of the aims of the present work. We shall develop a "trigonometry" for arbitrary (irreducible) Riemannian symmetric spaces of *non-compact type*. This involves a detailed study of (generic) triangles together with their congruence classes in such spaces.

The Riemannian symmetric spaces of rank 1 are precisely the non-Euclidean two-point homogeneous spaces. Their trigonometry has been studied by B. A. Rozenfeld and more recently by W.-Y. Hsiang, cf. [Roz] and [Hsi]. A different approach using models in projective spaces has been given by U. Brehm, cf. [Bre].

A Riemannian symmetric space $S$ is centrally symmetric with respect to any of its points. This implies that the group of isometries $I(S)$ operates transitively. The geometry of $S$ is therefore intimately connected with the geometry of its group of isometries, which in turn is a semisimple Lie group. Thus to study symmetric spaces we have at our disposal on the one hand Riemannian geometry and on the other hand the rich structure theory of semisimple Lie groups.

Let $S$ be a Riemannian symmetric space of *non-compact type*. A *flat* in $S$ is a maximal, totally geodesic submanifold with sectional curvature zero. It is also the orbit of a group conjugate to a certain abelian subgroup $A$ of $G = I_0(S)$, the connected component of the identity in $I(S)$. A *Weyl chamber* in $S$ is an (open) cone in a flat of $S$. We shall define the set $CS = G/M$ of all Weyl chambers of $S$ and will show that it is a trivial homogeneous bundle over $S : K/M \to G/M \to S = G/K$. Here $M$ is the subgroup of the isotropy group $K$ consisting of those elements which fix a chosen basic Weyl chamber $c_0$.

The approach taken in our work is to translate geometric configurations given by points of $S$ into group theoretic relations. A key idea is to relate the geometry of $S$ to that of the Weyl chamber bundle $CS$.

We call a geodesic triangle $\mathscr{A}\mathscr{B}\mathscr{C}$ *regular* if the geodesic segments $\mathscr{A}\mathscr{B}$, $\mathscr{B}\mathscr{C}$, $\mathscr{C}\mathscr{A}$ lie on regular geodesics, i.e. geodesics which are contained in precisely one flat. A regular geodesic triangle $\mathscr{A}\mathscr{B}\mathscr{C}$ in $S$ with a distinguished segment $\mathscr{A}\mathscr{B}$ is called *marked.* In order to obtain *geometric quantities*, i.e. quantities independent of a particular representative of a congruence class, we determine the space of invariants for the relative position of two points and also that of two Weyl chambers with a common apex. These invariants (which actually are points in certain orbit spaces) will allow us to define appropriate notions of *side* and *angle* for a congruence class of marked, regular geodesic triangles.

To every congruence class of marked triangles we shall thus associate six geometric quantities, namely three sides $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and three angles $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$. However, these quantities cannot be arbitrary. In order to derive from a triangle their representatives must satisfy a characteristic relation, namely $a_1 k_1 a_2 k_2 a_3 k_3 M = M$, which is a kind of "closing condition" in the Weyl chamber bundle. It is called *fundamental relation* and will be one of our basic tools.

It turns out that two marked, regular geodesic triangles are congruent essentially if two sides and the enclosed angle of one triangle coincide with the corresponding elements of the other. This is a generalization of the above mentioned classical congruence theorem *SAS* in hyperbolic geometry. We thus obtain a set of *defining quantities*. By *trigonometry* of $S$ is meant a (minimal) set of functional relations which allows us to deduce from the three defining quantities of a triangle the three remaining ones. In classical geometry these are the law of cosines and the law of sines.

We obtain the *generalized laws of cosines* from the fundamental relation by using Invariant Theory for symmetric spaces and the Cartan decomposition of $G$.

In hyperbolic geometry the laws of sines can be interpreted as an integral of the geodesic flow (the "angle of parallelism"). The corresponding generalizations are integrals of the Weyl chamber flow which is a certain action of the abelian subgroup $A$ of $G$ (determined by the choice of $c_0$) that generalizes the geodesic flow. We construct such integrals (the "subtended angle of a flat") and deduce the *generalized laws of sines* from them.

Finally, we compute in detail the trigonometry of the space of Euclidean structures on $\mathbf{R}^n$, i.e. of $SL(n, \mathbf{R})/SO(n)$, to illustrate the general concepts.

We emphasize that we are dealing *only* with symmetric spaces of *non-compact type*. Since there is no Iwasawa decomposition for compact groups, our global method to derive the laws of sines does *not* work for the dual spaces of *compact type*.

There is only one symmetric space of compact type and higher rank whose trigonometry has been investigated, the Lie group $SU(3)$, cf. [Asl]. It would be interesting to have a geometric analogue of the subtended angle for symmetric spaces of compact type. Such an object should lead to a global formulation of the laws of sines also for these spaces.

The present paper is part of the author's dissertation [Leu]. I wish to express my gratitude to Hans-Christoph Im Hof from the University of Basel for his friendly support and criticism.

## 2. The Weyl chamber bundle of a symmetric space of non-compact type

In this section we describe some (essentially) known aspects of the geometry of symmetric spaces (of non-compact type) and of the structure of semisimple Lie groups that will be used later. Further we introduce the Weyl chamber bundle and relate its geometry to various Lie group decompositions.

By $S$ we denote an irreducible Riemannian symmetric space of non-compact type with base point $x_0$.

### 2.1. *Flats and Weyl chambers*

DEFINITIONS. A *flat* in $S$ is a complete, connected, totally geodesic submanifold of $S$ with sectional curvature zero and of maximal possible dimension.

All flats are congruent, i.e. there is an isometry of $S$ which maps a given flat onto an arbitrary one. The common dimension of all flats is called the *rank* of $S$, cf. [He1] Ch. V.6. If $S$ has rank 1, then flats are just geodesics and the sectional curvature is bounded away from zero.

Every geodesic $\gamma$ in $S$ is contained in at least one flat. A geodesic $\gamma$ is called *regular* if it is contained in exactly one flat. Otherwise $\gamma$ is called *singular*. We fix a point $x$ in $S$ and a flat $F$ through $x$. The singular geodesics through $x$ in $F$ form a union of finitely many isometrically embedded hyperplanes. We call a connected component of the complement of these hyperplanes a *Weyl chamber in* $F \subset S$. The point $x$ is called the *apex* of the Weyl chamber.

We may write $S = G/K$. Then $G = I_0(S)$, the connected component of the identity in the group of isometries of $S$, is a non-compact, connected, semisimple, real Lie group with trivial centre. The isotropy group $K$ of $x_0$ is a maximal compact subgroup of $G$.

The geodesic symmetry $s_0$ at the base point $x_0$ of $S$ defines an involutive automorphism $\sigma$ of $G$. The eigenspace decomposition of its differential is a Cartan decomposition of the Lie algebra $\mathbf{g}$ of $G$, $\mathbf{g} = \mathbf{k} \oplus \mathbf{p}$, where $\mathbf{k}$ is the Lie algebra of $K$ and $\mathbf{p} \cong T_{x_0}S$.

We choose a maximal abelian subspace $\mathbf{a}$ in $\mathbf{p}$. Let $\Omega$ denote the set of restricted roots of $(\mathbf{g}, \mathbf{a})$ (cf. [Hel], Ch. VI.3). The kernel of any one of the restricted roots $\alpha \in \Omega$ defines a hyperplane $\mathscr{H}_\alpha := \{H \in \mathbf{a} \mid \alpha(H) = 0\}$ in the vector space $\mathbf{a}$. The set $\mathbf{a}_s := \{H \in \mathbf{a} \mid \exists \alpha \in \Omega, \alpha(H) = 0\}$ is a union of hyperplanes, decomposing $\mathbf{a}$ into finitely many connected components, called the *Weyl chambers of* $\mathbf{a}$. These are open, convex cones in $\mathbf{a}$. We fix one of these chambers and denote it by $\mathbf{a}^+$. Each $H \in \mathbf{a}^+$ is contained in precisely one maximal abelian subspace of $\mathbf{p}$, namely in $\mathbf{a}$. Equivalently, the centralizer of $H$ in $\mathbf{p}$ coincides with $\mathbf{a}$. The following proposition is of fundamental importance:

PROPOSITION (É. Cartan). *Let* $\overline{\mathbf{a}^+}$ *denote the closure of* $\mathbf{a}^+$ *in* $\mathbf{a}$, *then*

$$\mathbf{p} = \bigcup_{k \in K} \mathrm{Ad}\,(k)\overline{\mathbf{a}^+}.$$

For the proof see [Hel], Ch. V, Thm. 6.7 and Ch. VII, Prop. 2.12.

Geometrically this proposition means that the isotropy group $K$ operates transitively on the set of Weyl chambers in $\mathbf{p}$. Moreover, if the rank of $S$ is 1 (i.e. $\dim \mathbf{a} = 1$) it says that rank 1 spaces are *isotropic*. Thus a symmetric space of any rank can be called "Weyl chamber isotropic".

A flat $F_0$ through $x_0$ corresponds to a maximal abelian subspace $\mathbf{a}$ of $\mathbf{p}$. More precisely, $F_0 = Ax_0$ is the orbit under the abelian group $A := \exp \mathbf{a} \subset G$.

The Weyl chambers defined above (as subsets of the manifold $S$) are images of Weyl chambers defined by restricted roots under the exponential map $\mathrm{Exp}_{x_0}$ and under left translations $\tau(g)$ in $S$. Since for symmetric spaces of non-compact type $\mathrm{Exp}_{x_0}$ is a diffeomorphism, we can identify Weyl chambers in $\mathbf{a}$ with those Weyl chambers in the flat $F_0$ which have apex $x_0$.

## 2.2. *Decompositions of semisimple Lie groups*

The connected, non-compact, semisimple Lie group $G = I_0(S)$ can be written as

$$G = K\overline{A^+}K,$$

i.e. each $g \in G$ can be written as $g = k_1 a k_2$ with $k_1, k_2 \in K$ and a unique $a \in \overline{A^+}$.

This decomposition of $G$ is called *a Cartan decomposition*, cf. [Hel] Ch.IX.1.

Later we shall need more precise information on the factors $k_1$ and $k_2$ in this decomposition. For any $H \in \mathbf{a}$ let $Z_K(H) := \{k \in K \mid \mathrm{Ad}(k)H = H\}$ be the centralizer of $H$ in $K$.

LEMMA. *Suppose that* $a = \exp H \in \overline{A^+}$.

(a) *If* $g = k_1 a k_2 = k'_1 a k'_2$, *then* $k'_1 \equiv k_1 \bmod Z_K(H)$.

(b) *If* $k'_1 = k_1 m$ *for some* $m \in Z_K(H)$, *then* $k'_2 = m^{-1} k_2$ *for the same* $m \in Z_K(H)$.

(c) *Set* $\bar{k} := k_1 k_2$ *and* $\bar{k}' := k'_1 k'_2$, *then we also have* $\bar{k}' = \bar{k}$. *This means that in the (polar-) decomposition* $g = p\bar{k} := k_1 a k_1^{-1} k$ *the factors* $p$ *and* $\bar{k}$ *are unique.*

*Proof.* We first note that for the point $gK$ in the symmetric space $S = G/K$

$$\mathrm{Exp}_{x_0} \mathrm{Ad}(k_1)H = k_1 aK = gK = k'_1 aK = \mathrm{Exp}_{x_0} \mathrm{Ad}(k'_1)H.$$

As $\mathrm{Exp}_{x_0}$ is a diffeomorphism we get $\mathrm{Ad}(k_1)H = \mathrm{Ad}(k'_1)H$ or $\mathrm{Ad}(k_1^{-1}k'_1)H = H$. Thus $k_1^{-1}k'_1 \in Z_K(H)$ which proves (a). Statements (b) and (c) easily follow: $k_1 a k_2 = k'_1 a k'_2$ and $k'_1 = k_1 m$ for $m \in Z_K(H)$ implies $k'_1 a k'_2 = k_1 m a k'_2 = k_1 a m k'_2 = k_1 a k_2$ thus $k'_2 = m^{-1} k_2$ and $\bar{k}' = k'_1 k'_2 = k_1 m m^{-1} k_2 = k_1 k_2 = \bar{k}$. □

The choice of the Weyl chamber $\mathbf{a}^+$ in $\mathbf{a}$ defines an ordering in the set $\Omega \subset \mathrm{Hom}(\mathbf{a}, \mathbf{R})$ of all restricted roots. For $\alpha \in \Omega$ we denote the corresponding root space by $\mathbf{g}_\alpha$. We set $\Omega^+ := \{\alpha \in \Omega \mid \alpha(H) > 0 \text{ for all } H \in \mathbf{a}^+\}$, then $\mathbf{n} := \Sigma_{\alpha \in \Omega^+} \mathbf{g}_\alpha$ is a nilpotent subalgebra of $\mathbf{g}$. Let $K, A, N$ denote the analytic subgroups of $G$ corresponding to the subalgebras $\mathbf{k}, \mathbf{a}, \mathbf{n}$. The map

$$K \times A \times N \to G; (k, a, n) \mapsto kan$$

is an analytic diffeomorphism called the *Iwasawa decomposition of* $G$. A proof can be found in [Hel] Ch. IX.1.

The Weyl group $W$ of $S$ is generated by the reflections in the walls of the basic Weyl chamber $\mathbf{a}^+$. It operates simply transitively on the set of all Weyl chambers in $\mathbf{a}$ and is isomorphic to $M'/M$ where $M'$ is the normalizer and $M$ the centralizer of $\mathbf{a}$ in $K$. The group $P := MAN$ is a closed subgroup of $G$. For each element $w$ of the Weyl group $W$ we fix a representative $m_w \in M'$. Then $G$ is the disjoint union of double cosets of $P$,

$$G = \bigcup_{w \in W} P m_w P.$$

This is called the *Bruhat decomposition*. It is related to $P$-orbits in the homogeneous space $G/P$. There is one principal orbit whose dimension is the dimension of $G/P$.

It gives rise to the so-called "big cell" $Pm_{w^*}P$ in the Bruhat decomposition, which is an open and dense double coset in $G$. We have $Pm_{w^*}P = Nm_{w^*}P$ and if we fix a representative $m^*$ for $m_{w^*}$ the decomposition becomes unique: If $g \in Nm^*P$ with $g = nm^*p$, then $n \in N$ and $p \in P$ are uniquely determined by $g$.

For more details and proofs see [War] Ch. 1.2 or [Hel] Ch. IX.

REMARK. The geometric significance of the Lie group decompositions treated in this section is most clearly seen in terms of the set $CS$ of all Weyl chambers of $S$. We turn to the detailed study of this set in the subsequent paragraph.

## 2.3. *The trivial bundle structure of the Weyl chamber bundle*

If the rank of the symmetric space $S$ is 1, then the isotropy group $K$ acts transitively on the unit tangent sphere at $x_0$. For symmetric spaces of higher rank this is no longer true. However, $K$ still acts transitively on the set of Weyl chambers with apex $x_0$ (by the Proposition in Section 2.1). This observation leads to the study of the set $CS$ of all Weyl chambers of $S$ (first introduced and investigated by H.-C. Im Hof in [IH1]).

We fix a maximal abelian subspace $\mathbf{a}$ in $\mathbf{p} \cong T_{x_0}S$. In $\mathbf{a}$ we choose a Weyl chamber $\mathbf{a}^+$, which in turn determines a Weyl chamber $c_0$ in $S$ with apex $x_0$. Let $M$ denote the centralizer and $M'$ the normalizer of $\mathbf{a}$ in $K$. We recall two properties of Weyl chambers, which will be used repeatedly.

PROPOSITION. (a) *All Weyl chambers with apex $x_0$ in $S$ (or, equivalently, all Weyl chambers in $\mathbf{p}$) constitute a homogeneous space isomorphic to $K/M$ with base point $eM$ corresponding to $c_0$.* (b) *The Weyl group $W := M'/M$ operates simply transitively on the set of all Weyl chambers in $F_0 = Ax_0$ with apex $x_0$.*

For a proof see [Hel] Ch. VII, Theorem 2.12, Ch. V, Lemma 6.3.

DEFINITION. We denote by $CS$ the *set of all Weyl chambers* of $S$. To obtain it, we let $F$ run through all flats of $S$ and $x$, for each $F$, through all points of $F$.

In this way we can associate to each Weyl chamber on the one hand a point in $S$, namely its apex, and on the other hand a flat $F$, which completely contains it.

THEOREM 1. *Let $S$ be a symmetric space of non-compact type with base point $x_0$. If $S = G/K$, then $CS$ is a homogeneous space isomorphic to $G/M$.*

*Proof.* Let $F_0$ denote the flat which supports $c_0$. Choose $c \in CS$ with apex, say $x$, and supporting flat, say $F$. Then, since $G$ operates transitively, $gx = x_0$ for some suitable $g \in G$. The image of the flat $F$ under this $g$ is a flat $gF$ which contains $x_0$. An element $k \in K$ transforms $gF$ into $F_0$, $kgF = F_0$. The two Weyl chambers $kgc$ and $c_0$ have the same apex and the same supporting flat, so there is an $m' \in M'$ such that $m'kgc = c_0$. To determine the isotropy group of $c_0$ we suppose that $gc_0 = c_0$. Then we first have $gx_0 = x_0$ thus $g \in K$, moreover $gF_0 = F_0$ and thus $g \in M'$. As the Weyl group operates simply transitively on the set of Weyl chambers in $F_0$ with apex $x_0$, it follows that $g \in M$. □

DEFINITION. Let $k_0$ denote the projection onto the $K$-component of the Iwasawa decomposition: $k_0 : G = KAN \to K$; $g = kan \mapsto k_0(g) := k$. We call two Weyl chambers $g_1M$ and $g_2M$ *asymptotic* if they have the same $K$-component in their respective Iwasawa decompositions, i.e. if $k_0(g_1)M = k_0(g_2)M$.

Let $P = MAN$ be the minimal parabolic subgroup of $G$ associated to the Iwasawa decomposition $G = KAN$. The map $\beta : G/P \to K/M$; $gP \mapsto k_0(g)M$ is a diffeomorphism: From the definition of $P$ and $k_0$ it immediately follows that $\beta$ is one-to-one and onto. Moreover, $d\beta|_{eP} : \mathbf{k} \oplus \mathbf{a} \oplus \mathbf{n}/\mathbf{m} \oplus \mathbf{a} \oplus \mathbf{n} \cong \mathbf{k}/\mathbf{m} \to \mathbf{k}/\mathbf{m}$ is onto and therefore $d\beta|_{gP}$ is also onto for every $g \in G$. We interpret a point of $G/P$ as an equivalence class of asymptotic Weyl chambers using the diffeomorphism $\beta : G/P \to K/M$. Geometrically, the map $\beta$ associates to each point $gP$ the unique Weyl chamber $k_0(g)M$ whose apex is the base point $x_0 \in S$ and which is asymptotic to the chamber $gM$.

The map which associates to each Weyl chamber its apex is the projection map $\pi$ of the bundle $\pi : G/M \to G/K$; $gM \mapsto gK$. This bundle structure turns out to be particularly simple. Namely, both the Cartan and the Iwasawa decomposition give rise to a trivial bundle structure of $CS$ over $S$.

DEFINITION. Let $g \in G$ and take the Cartan decomposition $g = kak' = kak^{-1}\bar{k}$. We define $k(g) := \bar{k} \in K$; see the Lemma in Section 2.2.

THEOREM 2. *For $CS = G/M$ and $S = G/K$ the two maps*

$$\Phi_C : CS \to S \times K/M; gM \mapsto (gK, k(g)M) \quad \textit{and}$$

$$\Phi_I : CS \to S \times K/M; gM \mapsto (gK, k_0(g)M) \quad \textit{are diffeomorphisms.}$$

*Proof.* $\Phi_C$ is *onto*: Let $(hK, \bar{k}M)$ be given. There exist $k \in K$ and $H \in \overline{\mathbf{a}^+}$ such that $hK = \mathrm{Exp}_{x_0} \mathrm{Ad}\,(k)H$. Then for $a = \exp H$ we have $hK = kaK = kak^{-1}\bar{k}K$. Set $g := kak^{-1}\bar{k}$. Then $\Phi_C(gM) = (hK, \bar{k}M)$.

$\Phi_C$ is *one-to-one*: Let $g_1M = k_1a_1k_1^{-1}\bar{k}_1M$ and $g_2M = k_2a_2k_2^{-1}\bar{k}_2M$ be two Weyl chambers with $\Phi_C(g_1M) = \Phi_C(g_2M)$. This is equivalent to $k_1a_1k_1^{-1}\bar{k}_1K = k_2a_2k_2^{-1}\bar{k}_2K$ and $\bar{k}_1M = \bar{k}_2M$ i.e. there exist $k \in K$ and $m \in M$ such that $k_1a_1k_1^{-1}\bar{k}_1 = k_2a_2k_2^{-1}\bar{k}_2k$ and $\bar{k}_1 = \bar{k}_2m$. We now deduce from the uniqueness of the factors in the Cartan decomposition (see the Lemma in Section 2.2) that $\bar{k}_1 = \bar{k}_2k$, and therefore $\bar{k}_2m = \bar{k}_1 = \bar{k}_2k$. Thus $k = m \in M$, and $\Phi_C$ is one-to-one.

$\Phi_C$ is *regular*: Since $CS$ is homogeneous it suffices to show that $\Phi_C$ is regular at $eM$. Let $\mathbf{m}$ denote the Lie algebra of $M$. We have $\Phi_C(eM) = (eK, eM)$ and

$$T_{eM}CS \cong \mathbf{g}/\mathbf{m} \cong \mathbf{p} \oplus \mathbf{k}/\mathbf{m}, \qquad T_{(eK,eM)}(S \times K/M) \cong \mathbf{p} \oplus (\mathbf{k}/\mathbf{m}).$$

With these identifications we find that $d\Phi_C|_{eM}(X + L) = X + L$ for $X \in \mathbf{p}$ and $L \in \mathbf{k}/\mathbf{m}$. Hence $d\Phi_C|_{eM}$ is onto. As both tangent spaces have the same dimension $\Phi_C$ is regular at $eM$.

$\Phi_I$ is a *diffeomorphism*: In [IH1] it is proved that $\Phi : G/M \to G/K \times G/P$; $gM \mapsto (gK, gP)$ is a diffeomorphism. As we have seen above, there is a diffeomorphism $\beta : G/P \cong K/M$. Since $\Phi_I = (Id \times \beta) \circ \Phi$ the theorem is proved. □

COROLLARY. *The Weyl chamber bundle CS of an (irreducible) symmetric space S of non-compact type is diffeomorphic to a trivial bundle over S with canonical fibre K/M.*

REMARK. The trivial bundle structure of the Weyl chamber bundle generalizes the trivial bundle structure of the unit tangent bundle of a hyperbolic space, for in the rank 1 case Weyl chambers are unit tangent vectors resp. geodesic rays. We regard the Weyl chamber bundle as a generalization of the unit tangent bundle, a point of view that proves to be of fundamental importance.

### 2.4. *Geometric interpretations*

We turn to geometric interpretations of various Lie group decompositions that will be used later.

If we start from the basic Weyl chamber $c_0 = eM$, then we can reach an arbitrary Weyl chamber $gM$ in two ways (corresponding to the Iwasawa and to the Cartan decomposition, respectively). Starting with the Iwasawa decomposition $g = k_0bn$, we first map the basic Weyl chamber $c_0 = eM$ with apex $x_0$ to $k_0M$ (see Figure 1), then $k_0M$ is displaced in the flat $k_0F_0$ by means of $k_0bk_0^{-1}$ to the chamber $k_0bM$ and finally transformed with $(k_0b)n(k_0b)^{-1}$ to $gM$. Note, that $k_0M$, $k_0bM$ and $gM$ belong to the same asymptoticity class $gP$.

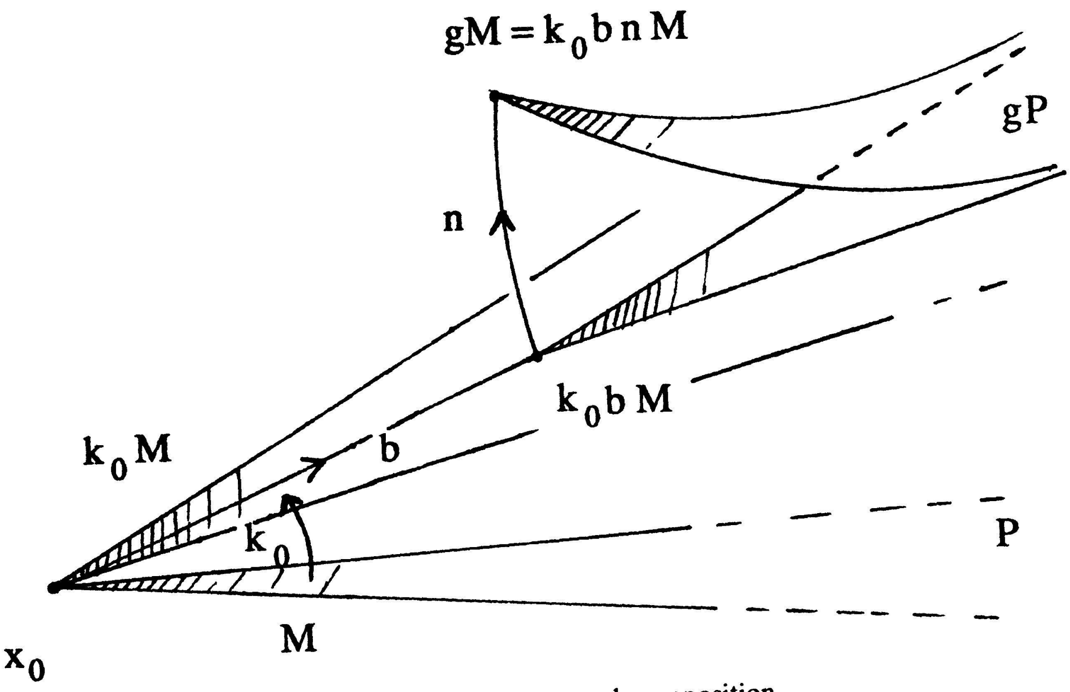


Figure 1. The Iwasawa decomposition.

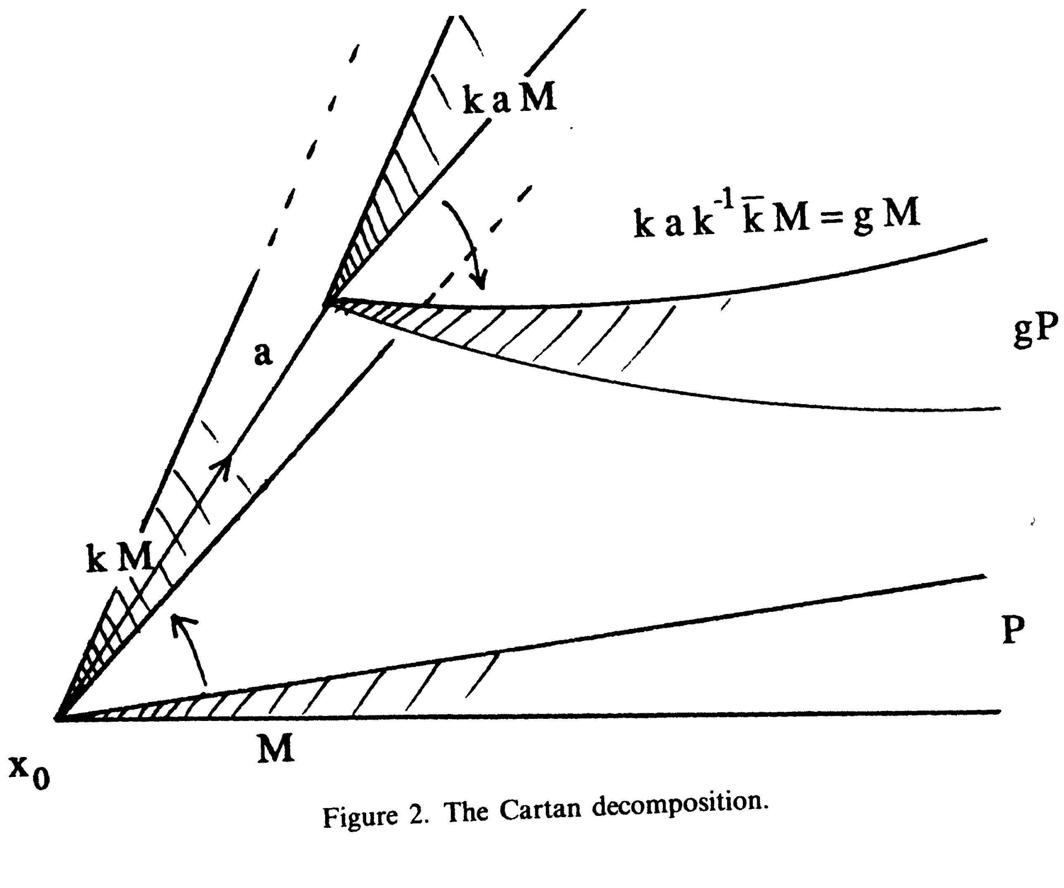


Figure 2. The Cartan decomposition.

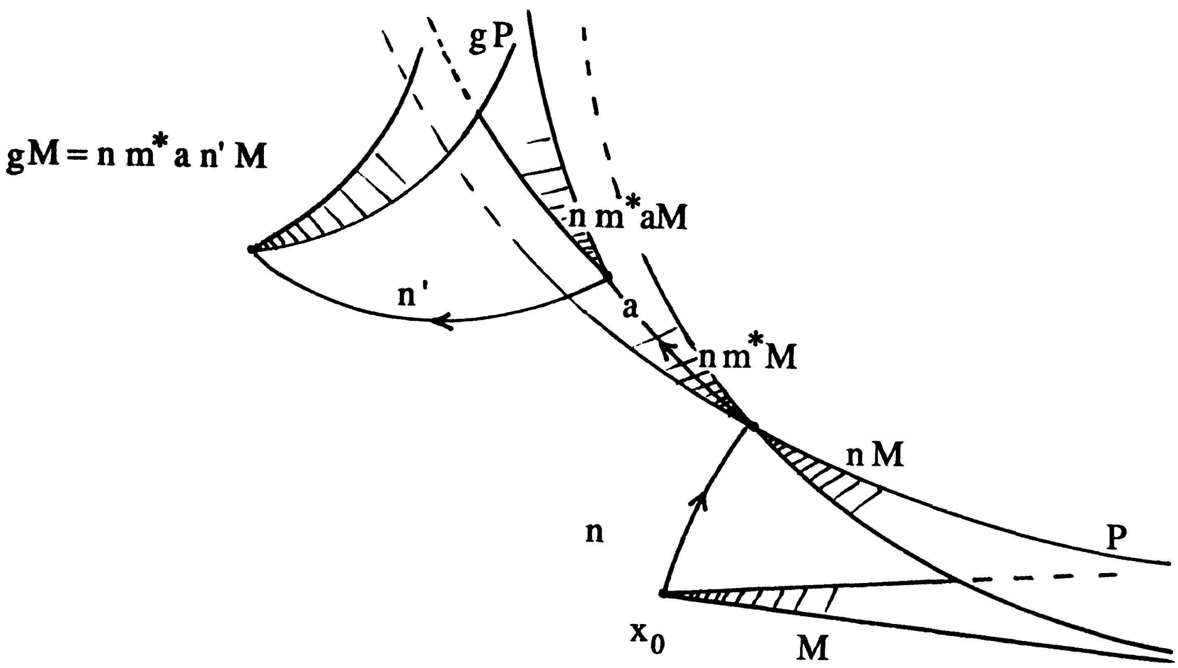


Figure 3. The Bruhat decomposition.

We can interpret the Cartan decomposition similarly (see Figure 2). We take a Weyl chamber $gM$. Let $g = kak^{-1}\bar{k}$ be the Cartan decomposition of $g$. Then the basic chamber $c_0$ is first rotated by means of $k$ into the flat $kF_0$. Next it is translated along this flat by means of $kak^{-1}$ to the chamber $kaM$. Finally this chamber is rotated by $ka(k^{-1}\bar{k})(ka)^{-1}$ to $gM$.

In Section 5 we shall also use the geometric interpretation of the big cell in the Bruhat decomposition (cf. Section 2.2). To describe it we define a *horocycle* in the symmetric space $S = G/K$ to be an orbit of a group conjugate to the nilpotent group $N$. Let $g$ be an element of $G$ in the big cell $Pm^*P \subset G$ and let $g = nm^*q$ be its Bruhat decomposition. The element $q \in P$ can be written as $q = an'm$ with $a \in A$, $n' \in N$ and $m \in M$. The Weyl chamber $gM = nm^*qM = nm^*an'M$ (see Figure 3) is obtained from the basic Weyl chamber $c_0 = M$ by first displacing $M$ by means of $n$ along the horocycle $N \cdot x_0$ to the chamber $nM$. This chamber is mapped to its opposite $nm^*M$, translated along the supporting flat by means of a conjugate of $a$ to $nm^*aM$, and finally displaced by means of a conjugate of $n'$ along the horocycle "centered at $gP$".

## 2.5. *The Weyl chamber flow on CS and a class of symplectic manifolds*

The following action of the abelian group $A = \exp \mathbf{a}$ on the Weyl chamber bundle $CS = G/M$ generalizes the geodesic flow.

DEFINITION. The map $\varphi : A \times G/M \to G/M$; $(a, gM) \mapsto gaM = (gag^{-1})gM$ is called the *Weyl chamber flow*. Note that for $m \in M$ we have $(gm)aM = gamM = gaM$ so that $\varphi$ is well defined.

Geometrically, $\varphi$ displaces a Weyl chamber along its supporting flat to an asymptotic Weyl chamber in the same flat. H.-C. Im Hof has shown in [IH2] that the Weyl chamber flow is an Anosov action, thus generalizing the same property of the geodesic flow of hyperbolic spaces. The $\varphi$-orbit of a Weyl chamber $gM \in G/M$ is the flat given by $\{gaM \in G/M \mid a \in A\} = gAM$.

DEFINITION. A *directed flat* is a flat with a distinguished class of asymptotic Weyl chambers.

REMARK. The set of $\varphi$-orbits, i.e. $G/AM$, coincides with the set of all directed flats in $S$. It is an $l$-fold covering space of the set $G/AM'$ of *all* flats in $S$; where $l$ is the order of the Weyl group $W = M'/M$, which in turn is isomorphic to the covering group of this covering.

There is another generalization of a property of the set of all geodesics of a hyperbolic space. Namely,

THEOREM 3. *The set $G/AM$ of all directed flats in an (irreducible) symmetric space of non-compact type $S = G/K$ is a homogeneous symplectic manifold of dimension* $2(\dim S - \operatorname{rank} S)$.

A detailed proof of this theorem can be found in [Leu].

REMARK. The theorem above is a special case of the following general situation. For an arbitrary Lie group $G$ *the co-adjoint action* $\mathrm{Ad}^*$ *of* $G$ *on* $\mathbf{g}^* = \mathrm{Hom}(\mathbf{g}, \mathbf{R})$ is given by $\mathrm{Ad}^*(g)\xi(X) = \xi(\mathrm{Ad}(g^{-1})X)$ with $\xi \in \mathbf{g}^*, X \in \mathbf{g}$. It is well-known that there exists a canonical symplectic structure on a co-adjoint orbit (Kirillov–Kostant–Souriau, cf. [GSt], Proposition 25.2). In our case, the group $G$ is semisimple so that the Killing-form $\kappa$ of $\mathbf{g}$ is non-degenerate and hence induces an isomorphism between $\mathbf{g}$ and $\mathbf{g}^*$. Thus, for $H \in \mathbf{a}_r$, $H^* := \kappa(H, \cdot) \in \mathbf{g}^*$ and for $X \in \mathbf{g}$ we have

$$\mathrm{Ad}^*(g)H^*(X) = H^*(\mathrm{Ad}(g^{-1})X) = \kappa(H, \mathrm{Ad}(g^{-1})X) = \kappa(\mathrm{Ad}(g)H, X).$$

The $G$-orbit of $H^*$ is given by $G \cdot H^* = \{\mathrm{Ad}^*(g)H^* \mid g \in G\} \cong G/AM$. Hence the manifold of flats is a co-adjoint orbit of $G$ and the symplectic structure we refer to in Theorem 3 is precisely the canonical symplectic structure on such an orbit.

## 3. Congruence classes of marked triangles

For many questions concerning symmetric spaces it is sufficient to work with $G = I_0(S)$, the connected component of the identity in the group of isometries of a Riemannian symmetric space $S$. But in studying geometric quantities of triangles it will be essential to work with $I(S)$, the *full* group of isometries of $S$.

Using the transitive operation of $I(S)$ on the Weyl chamber bundle $CS$ we obtain another model for this space: Let $\mathbf{M}^*$ denote the set of all elements in $I(S)$ which fix the basic Weyl chamber, then $CS = I(S)/\mathbf{M}^*$. Since the compact group $\mathbf{M}^*$ also plays an important rôle in connection with triangles, we first study some of its algebraic properties.

### 3.1. *Algebraic preliminaries*

DEFINITION. By $\mathbf{K}$ we denote the *full* isotropy group of the base point $x_0 \in S$.

As usual, we identify $T_{x_0}S$ with $\mathbf{p} \subset \mathbf{g}$. We recall that the linear isotropy representation of $\mathbf{K}$ on $\mathbf{p}$ is given by $\mathbf{K}_* = \mathrm{Ad}\,(\mathbf{K})|_{\mathbf{p}} \subset O(\mathbf{p})$. Since an isometry is completely defined by its value and its differential at a given point, the linear isotropy representation is faithful and $\mathbf{K}_* \cong \mathbf{K}$.

DEFINITION. For each element $v$ of $\mathbf{K}$ let $i_v$ denote the inner automorphism of $I(S)$ given by conjugation with $v$, i.e. $i_v(g) = vgv^{-1}$. We further denote by $\mathrm{Int}\,(K)$ the group of all inner automorphisms of $K$ and by $\mathrm{Aut}^G\,(K)$ the group of all automorphisms of $K$ which extend to $G$.

The structure of $\mathbf{K}$ was determined by Élie Cartan.

PROPOSITION (É. Cartan). *Let $s_0$ be the geodesic symmetry at $x_0$ and let $i_{v_j}|_K$ represent the cosets of* $\mathrm{Int}(K)$ *in* $\mathrm{Aut}^G\,(K)$ *for $j = 1, \ldots, r$. Then*

$$\mathbf{K} = \bigcup_{j=1}^{r} v_j \cdot (K \cup s_0 K).$$

For a proof see [Wol], Theorem 8.8.1 or [Ca2].

DEFINITION. There is a unique element $w^*$ in the Weyl group $W = M'/M$ of $S$, mapping the Weyl chamber $c_0$ corresponding to $M$ to its opposite chamber $c_0^*$. We have $w^* = m^*M$ for some $m^* \in M' = N_K(\mathbf{a})$.

REMARK. By our identifications, $m^*M$ may be considered both as an element of the Weyl group and as a Weyl chamber in the basic flat. $w^*$ has order 2, thus $(w^*)^2 = e$ or, equivalently, $(m^*)^2 \in M$. Details can be found in [Bou] Ch. V, Ex. 6.2.

DEFINITION. Let $\mathbf{a}$ be a fixed maximal abelian subspace of $\mathbf{p}$ and let $\mathbf{a}^+$ be a fixed Weyl chamber in $\mathbf{a}$. We denote the normalizer of $\mathbf{a}^+$ in $\mathbf{K}$ by $\mathbf{M}^*$.

REMARK. The quotient manifold $I(S)/\mathbf{M}^* \cong I(S) \cap G/\mathbf{M}^* \cap G = G/M = CS$ yields another model for the Weyl chamber bundle.

In the next Lemma we describe the structure of $\mathbf{M}^*$ in more detail:

LEMMA 1. *Using the notation of the Proposition above, we have*

$$\mathbf{M}^* = \bigcup_{j=1}^{r} v_j \cdot (M \cup s_0 m^* M).$$

*Proof.* Let $i_{v_j}|_K \in \mathrm{Aut}^G(K)$. Since $\mathrm{Ad}\,(v_j^{-1})\mathbf{a}^+$ is a Weyl chamber in $\mathbf{p}$, we have $\mathrm{Ad}\,(v_j^{-1})\mathbf{a}^+ = \mathrm{Ad}\,(k)\mathbf{a}^+$ for some $k \in K$, i.e. $\mathrm{Ad}\,(v_j k)\mathbf{a}^+ = \mathbf{a}^+$. Thus $v_j k \in \mathbf{M}^*$ and $i_{v_j k} \equiv i_{v_j} \bmod \mathrm{Int}\,(K)$. By choosing appropriate representatives we can therefore assume that $v_j \in \mathbf{M}^*$ for every $j$. By the above Proposition it is then enough to consider the following two cases: (1) $v_j h \in \mathbf{M}^*$ for $h \in K$. Then by our assumption $h \in \mathbf{M}^*$ and so $h \in M$ since $\mathbf{M}^* \cap K = M$. (2) $v_j s_0 h \in \mathbf{M}^*$ for $h \in K$. Then $s_0 h = h s_0 \in \mathbf{M}^*$. Since $\mathrm{Ad}\,(h)\mathbf{a}^+ = \mathrm{Ad}\,(s_0)\mathbf{a}^+ = \mathrm{Ad}\,(m^*)\mathbf{a}^+$ we conclude from the uniqueness of $m^*$ modulo $M$ that $h \in m^*M$. □

Let $k \in \mathbf{K}$. As both $G = I_0(S)$ and $K$ are connected subgroups of the group $I(S)$, $i_k|_K$ is an automorphism of $K$ and $i_k|_G$ is an automorphism of $G$. We use $i_k$ to denote also these restrictions. The following property of the elements of $\mathbf{M}^*$ will be used later.

LEMMA 2. *For an element $v \in \mathbf{M}^*$ consider $i_v \in \mathrm{Aut}\,(G)$. Then $i_v$ leaves the Iwasawa decomposition of $G$ invariant. More precisely: If $G = KAN$ is the Iwasawa decomposition of $G$ (determined by the choice of the basic Weyl chamber that defines $\mathbf{M}^*$), then $i_v(K) \subseteq K$, $i_v(A) \subseteq A$, $i_v(N) \subseteq N$.*
*Moreover $i_v(M) \subseteq M$, i.e. $M$ is a normal subgroup of finite index in $\mathbf{M}^*$.*

*Proof.* The claim on the invariance of $K$ is obvious. To see that $i_v(A) \subseteq A$, we only have to observe that $\mathbf{a}$ is spanned by vectors in $\mathbf{a}^+$, so that $\mathrm{Ad}\,(v)\mathbf{a} = \mathbf{a}$. Since $i_v \circ \exp|_{\mathbf{a}} = \exp|_{\mathbf{a}} \circ \mathrm{Ad}\,(v)$ the claim is proved. We show now that $i_v(N) \subseteq N$. Since $v$

is an isometry of $S$, Ad $(v)$ permutes the root hyperplanes and therefore the induced map $\mathrm{Ad}^*(v)$ in $\mathrm{Hom}(\mathbf{a}, \mathbf{R})$ permutes the roots. For a positive restricted root $\alpha \in \Omega^+$ and $H \in \mathbf{a}^+$ we have $\beta(H) := \mathrm{Ad}^*(v)\alpha(H) = \alpha(\mathrm{Ad}(v^{-1})H) > 0$ so that $\beta \in \Omega^+$. We conclude that $\mathrm{Ad}(v)$ permutes the root spaces $\mathbf{g}_\lambda$ for $\lambda \in \Omega^+$ and $\mathrm{Ad}(v)\mathbf{n} = \mathrm{Ad}(v)\Sigma_{\lambda \in \Omega^+} \mathbf{g}_\lambda = \mathbf{n}$. Combining this with $\exp \circ \mathrm{Ad}(v)|_{\mathbf{n}} = i_v \circ \exp|_{\mathbf{n}}$ shows that $i_v(N) \subseteq N$. In order to prove the last claim let $m \in M$ and $H \in \mathbf{a}^+$. If we set $H' := \mathrm{Ad}(v^{-1})H \in \mathbf{a}^+$, then we have $\mathrm{Ad}(vmv^{-1})H = \mathrm{Ad}(v)\,\mathrm{Ad}(m)H' = \mathrm{Ad}(v)H' = H$. As $i_v(K) \subseteq K$ and $M \cap K = M$, the claim follows. □

The following actions of $\mathbf{M}^*$ are needed below.

DEFINITION. The group $\mathbf{M}^*$ acts on $K/M$ by $(v, kM) \mapsto v \cdot kM := i_v(k)M = vkv^{-1}M$ and on $A^+$ by $(v, a) \mapsto v \cdot a := i_v(\exp H) = \exp \mathrm{Ad}(v)H$.

We use both the notations $\mathbf{M}^* \cdot kM$ and $\mathbf{k}$ for the $\mathbf{M}^*$-orbit $\{i_v(k)M \mid v \in \mathbf{M}^*\}$ of $kM$ in $K/M$. Similarly we use $\mathbf{M}^* \cdot a$ and $\mathbf{a}$ for $\{i_v(a) \mid v \in \mathbf{M}^*\}$, the (finite) orbit of $a \in A^+$ under $\mathbf{M}^*$.

### 3.2. *Intervals and angles in a symmetric space*

We consider a pair of points, say $(\mathscr{A}, \mathscr{B})$, in a Riemannian symmetric space $S$ of non-compact type. As $S$ is a Hadamard manifold there exists a unique geodesic $\gamma$ joining $\mathscr{A}$ and $\mathscr{B}$.

Let us assume that $\gamma$ is *regular* (cf. Section 2.1). Then $\gamma$ is contained in a unique Weyl chamber $c$ with apex $\mathscr{A}$. By the facts stated in Section 2.3 there is an isometry of $S$ which maps the chamber $c$ to a (chosen) basic Weyl chamber $c_0 = A^+ \cdot x_0$ with apex $x_0$. The pair $(\mathscr{A}, \mathscr{B})$ can thus be mapped isometrically to the pair $(\mathscr{A}_0, \mathscr{B}_0)$ with $\mathscr{A}_0 = x_0$ and $\mathscr{B}_0 = a \cdot x_0$ for a unique $a \in A^+$.

However, this "measurement" is unique only up to the action of the group $\mathbf{M}^*$ defined in the previous paragraph. An element $v \in \mathbf{M}^*$ maps the basic Weyl chamber $c_0$ to itself but does not necessarily leave it pointwise fixed.

In order to obtain a geometric quantity for a pair of points (i.e. a map that is constant on congruence classes) we make the

DEFINITION. We define the *interval* associated to the congruence class of the pair of points $(\mathscr{A}, \mathscr{B})$ to be the orbit $\mathbf{a} = \mathbf{M}^* \cdot a = \{i_v(a) \mid v \in \mathbf{M}^*\}$ in $A^+$.

Similarly we can associate a geometric quantity to an *ordered* pair of Weyl chambers $(c_1, c_2)$ with the *same* apex $x$.

First there is an isometry which maps the (ordered) pair $(c_1, c_2)$ to the pair $(c_0, c)$ with common apex $x_0$ (cf. Section 2.3). Again this isometry is unique only up to the left action of $\mathbf{M}^*$. Then, as the isotropy group $K$ of $x_0$ acts transitively on the set of Weyl chambers with apex $x_0$, there is a $k \in K$ such that $c = kc_0$. Here $k$ is unique modulo $M$ (cf. Section 2.3).

DEFINITION. We define the *oriented angle* associated to the congruence class of the ordered pair of Weyl chambers $(c_1, c_2)$ to be the orbit

$$\mathbf{k} = \mathbf{M}^* \cdot kM = \{i_v(k)M \mid v \in \mathbf{M}^*\} \quad \text{in } K/M.$$

We collect the previous remarks in the

PROPOSITION. *Let $S$ be a Riemannian symmetric space of non-compact type. Then*:

(i) *The space of invariants for the relative position of pairs of points (both situated on a regular geodesic) is isomorphic to the space of orbits $\mathbf{M}^*\backslash A^+$.*
(ii) *The space of invariants for the relative position of ordered pairs of Weyl chambers (with common apex) is isomorphic to the space of orbits $\mathbf{M}^*\backslash K/M$.*

### 3.3. *A congruence theorem for marked triangles*

DEFINITION. Three points $\mathscr{A}, \mathscr{B}, \mathscr{C}$ in a Riemannian symmetric space $S$ of non-compact type define a *geodesic triangle* $\mathscr{T} = \{\mathscr{A}, \mathscr{B}, \mathscr{C}\}$. The triangle $\mathscr{T}$ is called *regular* if the geodesic segments $\mathscr{A}\mathscr{B}$, $\mathscr{B}\mathscr{C}$ and $\mathscr{C}\mathscr{A}$ lie on regular geodesics.

A geodesic triangle $\{\mathscr{A}, \mathscr{B}, \mathscr{C}\}$ determines six *directed* geodesic segments: $\mathscr{A}\mathscr{B}$, $\mathscr{B}\mathscr{A}$, $\mathscr{B}\mathscr{C}$, $\mathscr{C}\mathscr{B}$, $\mathscr{C}\mathscr{A}$, $\mathscr{A}\mathscr{C}$. In order to associate well defined geometric quantities to a geodesic triangle we make the following

DEFINITION. A (regular) geodesic triangle $\{\mathscr{A}, \mathscr{B}, \mathscr{C}\}$ is called *marked* if one of its segments is distinguished, say $\mathscr{A}\mathscr{B}$. The so marked geodesic triangle is denoted by $\mathscr{A}\mathscr{B}\mathscr{C}$. Two marked geodesic triangles $\mathscr{A}\mathscr{B}\mathscr{C}$ and $\mathscr{D}\mathscr{E}\mathscr{F}$ in $S$ are *congruent* if there exists an isometry of $S$ which maps $\mathscr{A}\mathscr{B}\mathscr{C}$ to $\mathscr{D}\mathscr{E}\mathscr{F}$, i.e. the isometry must respect the marking. A congruence class of marked, regular geodesic triangles is called a *marked, regular triangle* of $S$ and the set of all marked, regular triangles in $S$ is denoted by $\triangle(S)$. A *geometric quantity of a marked, regular triangle* of $S$ is a map from $\triangle(S)$ into an arbitrary set $X$.

We first look for conditions of congruence for two marked, regular geodesic triangles. To fix ideas, we assume that the regular triangle $\{\mathscr{A}, \mathscr{B}, \mathscr{C}\}$ is marked by

$\mathscr{A}\mathscr{B}$. As the group of isometries of $S$ operates transitively on the set $CS$ of all Weyl chambers there is an isometry $g \in I(S)$ which maps $\mathscr{A}\mathscr{B}\mathscr{C}$ to a geodesic triangle $\mathscr{A}_0\mathscr{B}_0\mathscr{C}_0$ with $\mathscr{A}_0 = g\mathscr{A} = x_0$ and $\mathscr{A}_0\mathscr{B}_0 = g(\mathscr{A}\mathscr{B}) \in c_0$, where $c_0$ denotes the basic Weyl chamber with apex $x_0$, identified with $eM$ (cf. Section 2). Thus, $\mathscr{A}_0 = x_0$, $\mathscr{B}_0 = a_1 x_0$ and $a_1^{-1}\mathscr{C}_0 = \operatorname{Exp}_{x_0} \operatorname{Ad}(k_1)H_2 = k_1 a_2 x_0$, where $a_1$ and $a_2 = \exp H_2$ are in $A^+$ and $k_1 \in K$ is unique modulo $M$.

Note that the triangle $\mathscr{A}_0\mathscr{B}_0\mathscr{C}_0$ is not uniquely determined. If we replace the isometry $g$ above by $vg$ for some isometry $v \in \mathbf{M}^*$, the normalizer of $\mathbf{a}^+$ in the full isotropy group $\mathbf{K}$, then we obtain a congruent marked, geodesic triangle whose marked side is also contained in $c_0$.

DEFINITION. To the $\mathscr{A}\mathscr{B}$-marked geodesic triangle $\mathscr{A}\mathscr{B}\mathscr{C}$ we associate the *marking-data*

$$\mathbf{M}^* \cdot (a_1, k_1 M, a_2) := \{(i_v(a_1), i_v(k_1)M, i_v(a_2)) \mid v \in \mathbf{M}^*\}.$$

Note the *simultaneous* action of $\mathbf{M}^*$ on the three components.

The following theorem states that these data actually characterize a congruence class.

THEOREM 1. *Two marked, regular geodesic triangles $\mathscr{T}_1$ and $\mathscr{T}_2$ are congruent if and only if the corresponding marking-data are the same*:

$$\mathbf{M}^* \cdot (a_1, k_1 M, a_2) = \mathbf{M}^* \cdot (b_1, h_1 M, b_2).$$

*Proof.* Suppose first that $\mathscr{T}_1$ and $\mathscr{T}_2$ are congruent, then the corresponding data are equal by the isometry invariance of the definition. For the converse suppose, that $\mathscr{T}_1 = \mathscr{A}\mathscr{B}\mathscr{C}$ is $\mathscr{A}\mathscr{B}$-marked and $\mathscr{T}_2 = \mathscr{D}\mathscr{E}\mathscr{F}$ is $\mathscr{D}\mathscr{E}$-marked with $b_1 = va_1v^{-1}$, $h_1 = vk_1v^{-1}m$, $b_2 = va_2v^{-1}$ for some $v \in \mathbf{M}^*$ and $m \in M$. Without loss of generality we have $\mathscr{D} = x_0 = vx_0 = v\mathscr{A}$, $\mathscr{E} = b_1 x_0 = va_1 v^{-1} x_0 = va_1 x_0 = v\mathscr{B}$. By Lemma 2 in Section 3.1 there is an $m' \in M$ with $\mathscr{F} = b_1 h_1 b_2 x_0 = va_1 v^{-1} vk_1 v^{-1} mva_2 v^{-1} x_0 = va_1 k_1 a_2 v^{-1} m' x_0 = va_1 k_1 a_2 x_0 = v\mathscr{C}$. Hence $\mathscr{D}\mathscr{E}\mathscr{F} = v\mathscr{A}\mathscr{B}\mathscr{C}$ and both triangles are congruent. $\square$

If the directed segments $\mathscr{A}\mathscr{B}, \mathscr{B}\mathscr{C}, \mathscr{C}\mathscr{A}$ are considered as pairs of points, then they define three intervals $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in the sense of the previous paragraph. In order to define also angles for the congruence class $\Delta$ of the marked, regular geodesic triangle $\mathscr{A}\mathscr{B}\mathscr{C}$ we choose two Weyl chambers for each vertex. To do this in coincidence with the marking data we take the chambers $(a_1 \cdot c_0, a_1 k_1 \cdot c_0)$ at $\mathscr{B}$. Thus we get $\mathbf{k}_1 = \mathbf{M}^* \cdot k_1 M$ as the (exterior) angle at $\mathscr{B}$.

The marked, geodesic triangle $\mathscr{A}\mathscr{B}\mathscr{C}$ is naturally oriented: $\mathscr{A} \to \mathscr{B} \to \mathscr{C} \to \mathscr{A}$. We next determine the angles at $\mathscr{C}$ and $\mathscr{A}$ with respect to this orientation. Suppose that $\mathbf{M}^* \cdot (a_1, k_1 M, a_2)$ arises from the marking by means of $\mathscr{A}\mathscr{B}$. Then we may assume (by Theorem 1) that $\mathscr{A} = x_0$, $\mathscr{B} = a_1 x_0$, $\mathscr{C} = a_1 k_1 a_2 x_0$. Now marking the triangle with $\mathscr{B}\mathscr{C}$, we observe that $k_1^{-1} a_1^{-1} \mathscr{B} = k^{-1} x_0 = x_0$, $k_1^{-1} a_1^{-1} \mathscr{C} = a_2 x_0$, $k_1^{-1} a_1^{-1} \mathscr{A} = \mathrm{Exp}_{x_0} \mathrm{Ad}\,(k_2) H_3$ for $a_3 = \exp H_3 \in A^+$ and $k_2 \in K \bmod M$. From this we conclude that marking $\mathscr{A}\mathscr{B}\mathscr{C}$ with $\mathscr{B}\mathscr{C}$ leads to the marking-data $\mathbf{M}^* \cdot (a_2, k_2 M, a_3)$. In the same way we get $\mathbf{M}^* \cdot (a_3, k_3 M, a_1)$ for the third segment $\mathscr{C}\mathscr{A}$ compatible with the orientation.

REMARK. The moduli space $\triangle(S) \hookrightarrow \mathbf{M}^* \backslash (A^+ \times K/M \times A^+)$ of marked triangles has quite a complicated structure in general. For real hyperbolic spaces, however, we can identify it with $\mathbf{R}^+ \times [0, \pi/2] \times \mathbf{R}^+$. In that particular case the above theorem becomes a classical congruence theorem in hyperbolic geometry:

*Two geodesic triangles $\Delta$ and $\Delta'$ in $H^n\mathbf{R}$ are congruent if and only if two sides and the enclosed angle of one triangle coincide with the corresponding elements of the other.*

DEFINITION. In analogy with this classical case we define the sides of the triangle $\Delta \in \triangle(S)$ to be the $\mathbf{M}^*$-orbits $\mathbf{a}_1 = \mathbf{M}^* \cdot a_1$,$\mathbf{a}_2 = \mathbf{M}^* \cdot a_2$, $\mathbf{a}_3 = \mathbf{M}^* \cdot a_3$ in $\mathbf{M}^* \backslash A^+$. And we define the *(exterior) angles of the triangle* $\Delta \in \triangle(S)$ to be the $\mathbf{M}^*$-orbits $\mathbf{k}_1 = \mathbf{M}^* \cdot k_1 M$, $\mathbf{k}_2 = \mathbf{M}^* \cdot k_2 M$, $\mathbf{k}_3 = \mathbf{M}^* \cdot k_3 M$ in $\mathbf{M}^* \backslash K/M$.

Summing up our discussion we can attach six geometric quantities to any marked, regular triangle $\Delta \in \triangle(S)$. Namely, the three sides $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and the three exterior angles $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$. However, it is obvious (already from classical geometry) that these six quantities cannot be arbitrary. In order to be the sides and angles of a triangle they must satisfy some kind of "closing condition". In the next paragraph we shall derive a basic relation among representatives of the six quantities, which is characteristic for triangles (in the sense that this relation is necessary and sufficient for the corresponding sides and angles to be those of a triangle).

From Theorem 1 we know that, for $\Delta \in \triangle(S)$, two sides and the enclosed angle, for example $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{k}_1$, are related to the *defining quantities*: $\mathbf{M}^* \cdot (a_1, k_1 M, a_2)$.

By *trigonometry of the symmetric space $S$* we mean a (minimal) set of functional relations which allow us to deduce from the three defining quantities of a triangle the third side and the two remaining angles. In classical (hyperbolic) geometry this is achieved in using the functional relations usually called "law of cosines" and "law of sines". We shall extract such functional relations from a fundamental formula (in the Weyl chamber bundle of $S$) to which we now turn.

### 3.4. *A fundamental relation in the Weyl chamber bundle*

Let $\mathscr{A}\mathscr{B}\mathscr{C}$ be a marked, regular geodesic triangle in a symmetric space $S$ of non-compact type. Without loss of generality let $\mathscr{A}$ be the base point of $S$ and suppose that $\mathscr{A}\mathscr{B}$ lies in the basic Weyl chamber $c_0$ with apex $x_0$.

Furthermore, let $\Delta$ be the congruence class of $\mathscr{A}\mathscr{B}\mathscr{C}$ and let $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$, $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$ be the sides and angles of $\Delta$. We shall lift the triangle $\mathscr{A}\mathscr{B}\mathscr{C}$ to a "hexagon" in the Weyl chamber bundle $\pi : CS = G/M \to S = G/K$. To do this we trace the path taken by the Weyl chamber $c_0$ moving along $\mathscr{A}\mathscr{B}\mathscr{C}$ (cf. Figure 4).

First we identify $c_0$ with the base point $M$ of $G/M$. Then we displace $c_0$ in its supporting flat from $\mathscr{A}$ to the point $\mathscr{B} \in S$. In $CS$ this operation is described by the Weyl chamber flow for a particular element: $M \rightsquigarrow a_1 M$ for a unique $a_1 \in A^+$. The isotropy group of $\mathscr{B}$, $a_1 K a_1^{-1}$, operates transitively on the set of Weyl chambers with apex $\mathscr{B}$, i.e. on the fibre $\pi^{-1}(\mathscr{B}) = \pi^{-1}(a_1 K) = \{a_1 k M \mid k \in K\}$. As there is a unique Weyl chamber $c_1$ containing the regular geodesic $\mathscr{B}\mathscr{C}$, there is a $k_1 \in K$ such that $c_1 = a_1 k_1 M = (a_1 k_1 a_1^{-1}) a_1 M$, where $k_1$ is unique modulo $M$.

We next take the unique element $a_2 \in A^+$ which translates $c_1$ along the unique flat that contains the geodesic segment $\mathscr{B}\mathscr{C}$ to a chamber with apex $\mathscr{C}$ : $a_1 k_1 M \rightsquigarrow a_1 k_1 a_2 M$ and $\pi(a_1 k_1 a_2 M) = a_1 k_1 a_2 K = \mathscr{C}$. Note that this representation is independent of the representative chosen for the coset $k_1 M$, namely, if we take $k_1' = k_1 m$ for some $m \in M$ we have $a_1 k_1' a_2 M = a_1 k_1 m a_2 M = a_1 k_1 a_2 m M = a_1 k_1 a_2 M$. An element

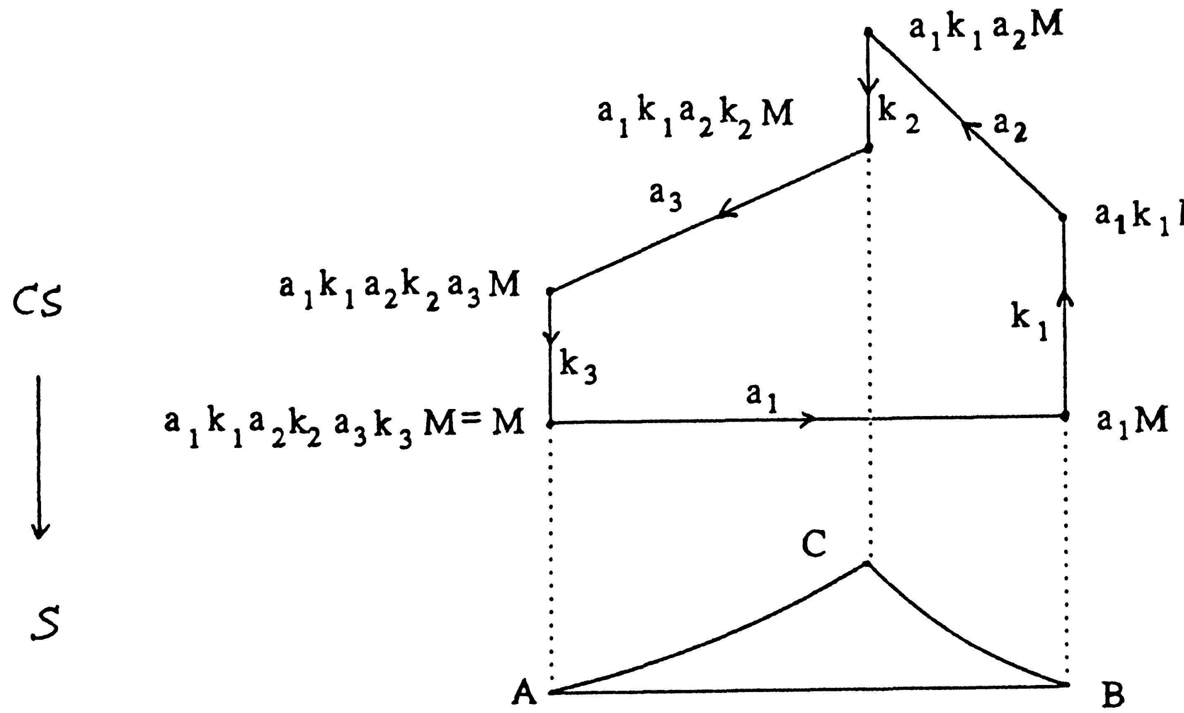


Figure 4.

of the isotropy group of $\mathscr{C}$ rotates our Weyl chamber to the new Weyl chamber $c_2$ which contains the geodesic $\mathscr{C}\mathscr{A}$, i.e. $a_1k_1a_2M \rightsquigarrow c_2 = a_1k_1a_2k_2M$. This procedure is repeated a final time. We first displace $c_2$ to $\mathscr{A}$ and then rotate it back to the basic Weyl chamber $c_0: a_1k_1a_2k_2M \rightsquigarrow a_1k_1a_2k_2a_3M \rightsquigarrow a_1k_1a_2k_2a_3k_3M = M$. Figure 4 summarizes the whole procedure.

REMARK. Every Weyl chamber is of the form $gM = gmM = g'M$. Suppose that $ghM = g'h'M = gmh'M$. Then $ghm' = gmh'$ for some $m' \in M$ and therefore $h' = m^{-1}hm'$.

THEOREM 2. (a) *Consider three intervals* $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ *and three angles* $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$. *In order that these six geometric quantities are the sides and angles of a triangle* $\Delta \in \triangle(S)$ *it is necessary and sufficient that there exist representatives* $a_i \in \mathbf{a}_i = \mathbf{M}^* \cdot a_i$ *and* $k_iM \in \mathbf{k}_i = \mathbf{M}^* \cdot k_iM$, *for* $i = 1, 2, 3$, *which satisfy the relation*

$$a_1k_1a_2k_2a_3k_3M = M.$$

(b) *If, moreover,* $a_1'k_1'a_2'k_2'a_3'k_3'M = M$ *holds for some other representatives of the sides and angles of* $\Delta$, *then* $a_i' = va_iv^{-1}$, $k_i' = m_{i-1}^{-1}vk_iv^{-1}m_i$ *for* $i = 1, 2, 3$, *where* $v \in \mathbf{M}^*$ *and* $m_j \in M$.

(c) *In particular,* $a_1'k_1'a_2'k_2'a_3'k_3'M = v(a_1k_1a_2k_2a_3k_3)v^{-1}M = M$ *and*

$$\mathbf{M}^* \cdot k_1'k_2'k_3'M = \mathbf{M}^* \cdot k_1k_2k_3M.$$

*Proof.* If the $\mathbf{a}_i$ and $\mathbf{k}_i$ are sides and angles of some $\Delta$, then we have already verified the relation $a_1k_1a_2k_2a_3k_3M = M$ for a chosen representative $\mathscr{A}\mathscr{B}\mathscr{C} \in \Delta$. If $\mathscr{A}\mathscr{B}\mathscr{C}$ is replaced by another representative, we know from Theorem 1 that there is a $v \in \mathbf{M}^*$ with $a_1' = va_1v^{-1}$, $a_2' = va_2v^{-1}$, $k_1' = vk_1v^{-1}m_1$ with $m_1 \in M$. We thus have $(k_2'a_3'k_3'm')^{-1} = a_1'k_1'a_2' = i_v(a_1k_1a_2m'') = i_v(k_2a_3k_3m)^{-1}$ for some $m'', m', m \in M$. Equivalently $k_2'a_3'k_3'm' = i_v(k_2a_3k_3m)$. The claim now follows from Lemma 2 in Section 3.1 and the uniqueness properties of the factors in the Cartan decomposition of $G$.

If, on the other hand, the relation $a_1k_1a_2k_2a_3k_3M = M$ holds, then we can take for $\Delta$ the congruence class of the geodesic triangle given by the three points $\mathscr{A} = x_0$, $\mathscr{B} = a_1x_0$ and $\mathscr{C} = a_1k_1a_2x_0$. □

DEFINITION. We call the equation $a_1k_1a_2k_2a_3k_3M = M$ in $CS$ the *fundamental relation* for the triangle $\mathscr{A}\mathscr{B}\mathscr{C} \in \Delta \in \triangle(S)$.

REMARKS. 1. Note the remarkable symmetry in the fundamental relation. The base point appears only in the form of its isotropy subgroup $K$, but neither $\mathscr{A}$

nor $\mathscr{B}$ nor $\mathscr{C}$ is distinguished: $a_1k_1a_2k_2a_3k_3M = M \Leftrightarrow a_2k_2'a_3k_3'a_1k_1'M = M \Leftrightarrow a_3\bar{k}_3a_1\bar{k}_1a_2\bar{k}_2M = M$ for $k_iM = k_i'M = \bar{k}_iM$.

2. Choosing suitable representatives in $K/M$ allows us to write the fundamental relation in the form: $e = a_1k_1a_2k_2a_3k_3m = a_1k_1a_2k_2a_3k_3'$.

3. We can transvect the basic Weyl chamber along the sides of a geodesic triangle *without* rotating at the vertices and thus imitating parallel translation of tangent vectors along the geodesic loop formed by the triangle. The angle $\mathbf{M}^* \cdot k_1k_2k_3M$ in Theorem 2 can then be interpreted as the "holonomy-angle" of the marked triangle $\Delta \in \triangle(S)$. In particular, for a geodesic triangle in the hyperbolic plane with angles $\alpha$, $\beta$, $\gamma$ we obtain the angular defect of the triangle: $\delta = \pi - (\alpha + \beta + \gamma)$.

## 4. Laws of cosines for symmetric spaces of non-compact type

Let $S = G/K$ be an irreducible Riemannian symmetric space of non-compact type with base point $x_0$. We choose a basic Weyl chamber $c_0 = A^+x_0$. To a marked, regular triangle $\Delta \in \triangle(S)$ we can associate sides $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbf{M}^*\backslash A^+$ and angles $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3 \in \mathbf{M}^*\backslash K/M$. These six quantities are linked by the fundamental relation $a_1k_1a_2k_2a_3k_3M = M$. It is our aim in this section to compute $\mathbf{a}_3 \in \mathbf{M}^*\backslash A^+$, the third side of $\Delta$, from the defining quantities $\mathbf{M}^* \cdot (a_1, k_1M, a_2)$ given by two sides $\mathbf{a}_1, \mathbf{a}_2$ and the enclosed angle $\mathbf{k}_1$.

DEFINITION. We call functional relations governing this *laws of cosines.*

### 4.1. *Quadratic representations and Invariant Theory*

By Theorem 2 and Remark 2 in Section 3.4 we may assume that the representatives for the sides and the angles of $\Delta$ are chosen in such a way that the fundamental relation becomes $a_1k_1a_2k_2a_3k_3 = e$. In a first step we would like to eliminate one of the three angles appearing in the fundamental relation. To that end we embed $S$ into $G = I_0(S)$.

DEFINITION. Let $\sigma = i_{s_0}$ denote the involution of $G$ induced by the geodesic symmetry at the base point $x_0$ of $S$ and set $Q : S \hookrightarrow G$; $gK \mapsto g\sigma(g^{-1})$. This is an embedding of $S$ as a submanifold of $G = I_0(S)$. The map $Q$ is called the *quadratic representation* of $S$ (see [Hel], p. 276 for the details).

Note that $Q(G/K) \subseteq \exp \mathbf{p}$. For $g = a_1k_1a_2 = k_3^{-1}a_3^{-1}k_2^{-1}$ we obtain $Q(gK) = g\sigma(g^{-1}) = a_1k_1a_2^2k_1^{-1}a_1 = k_3^{-1}a_3^{-2}k_3 = \exp(\mathrm{Ad}(k_3^{-1})(-2H_3))$. We see from these formulae that we should look for a set of $K$-invariant functions on $\mathbf{p}$ which uniquely determine the element $a_3 = \exp(H_3)$.

By Invariant Theory for symmetric spaces there are precisely $r = \operatorname{rank}(S)$ homogeneous polynomials on $\mathbf{p} \cong T_{x_0}S$ which are $K$-invariant and whose gradients at every regular $X \in \mathbf{p}$ are linearly independent. More precisely, for a maximal abelian subalgebra $\mathbf{a} \subset \mathbf{p}$ and the corresponding Weyl group $W$ of $S$ we have

PROPOSITION 1 (Chevalley restriction theorem). *The* $\mathbf{R}$*-algebra* $C_K^\infty(\mathbf{p})$ *of* $K$*-invariant* $C^\infty$*-functions on* $\mathbf{p}$ *is isomorphic to the* $\mathbf{R}$*-algebra* $C_W^\infty(\mathbf{a})$ *of* $W$*-invariant* $C^\infty$*-functions on* $\mathbf{a}$.

PROPOSITION 2 (C. Chevalley). *The* $\mathbf{R}$*-algebra* $P_W(\mathbf{a})$ *of* $W$*-invariant polynomials on* $\mathbf{a}$ *is generated by* $r = \operatorname{rank}(S)$ *algebraically independent homogeneous polynomials* $p_1, \ldots, p_r$ *and* 1.

PROPOSITION 3 (R. Steinberg). *Let* $J$ *be the Jacobian matrix of a basic set of invariants of* $W$ *(computed relative to any basis of* $\mathbf{a}$*). Let* $H$ *be any point of* $\mathbf{a}$*. Then the maximal number of linearly independent reflection hyperplanes containing* $H$ *coincides with the nullity of* $J$ *at* $H$.

The proofs can be found in [He2], Ch. II, Cor. 5.11., [War], Theorem 2.1.3.1. or [Che] and [Ste], respectively.

We use the polynomials $p_1, \ldots, p_r$ from Proposition 2 to define a map

$$p : \mathbf{a} \to \mathbf{R}^r; \qquad H \mapsto p(H) := (p_1(H), \ldots, p_r(H)).$$

Note that since the polynomials $p_1, \ldots, p_r$ are not unique, neither is the map $p$.

Let $\mathbf{a}_r := \mathbf{a} \backslash \mathbf{a}_s$ denote the set of regular elements in $\mathbf{a}$ (cf. Section 2.1). We know from Geometric Invariant Theory that $p$ separates the Weyl group orbits, i.e. if $p(H_1) = p(H_2)$ for $H_1, H_2 \in \mathbf{a}_r$, then there exists $w \in W$ such that $H_2 = w \cdot H_1$ (cf. [Spr], 2.4.8).

The map $p$, being constant on $W$-orbits, restricts to an injective map on $\mathbf{a}^+$. Using Proposition 3, we conclude that $p|_{\mathbf{a}^+}$ is a diffeomorphism onto its image in $\mathbf{R}^r$. Let $q$ denote the inverse of this diffeomorphism $q := (p|_{\mathbf{a}^+})^{-1} : p(\mathbf{a}^+) \to \mathbf{a}^+$.

By Proposition 1 the map $p$ has a unique $K$-invariant extension to $\mathbf{p}$ which we again denote by $p$.

We set $\mathbf{p}_r := \bigcup_{k \in K} \operatorname{Ad}(k)\mathbf{a}^+$. Then $S_r := \operatorname{Exp}_{x_0} \mathbf{p}_r$ is the set of those points in $S$ which lie on only one flat through $x_0$. Note that $p(\mathbf{p}_r) = p(\mathbf{a}^+)$.

DEFINITION. We set $C_1 := \exp \circ q \circ p \circ (\exp|_{\mathbf{p}_r})^{-1} : Q(S_r) \to A^+$.

THEOREM 1 (Laws of Cosines; first version). *Let $S$ be a Riemannian symmetric space of non-compact type and let $\mathbf{a}$ be a maximal abelian subspace of $\mathbf{p} \cong T_{x_0}S$ with Weyl group $W$. For algebraically independent, $W$-invariant polynomials $p_1, \ldots, p_r$ on $\mathbf{a}$ let $C_1$ be the composite just defined.*

*Then the map*

$$\Gamma_1 : \triangle(S) \to \mathbf{M}^*\backslash A^+; \mathbf{M}^* \cdot (a_1, k_1 M, a_2) \mapsto \mathbf{M}^* \cdot C_1(Q(a_1 k_1 a_2 K)) = \mathbf{M}^* \cdot a_3^2$$

*determines the third side $\mathbf{a}_3 = \mathbf{M}^* \cdot a_3 \in \mathbf{M}^*\backslash A^+$ of a given marked, regular triangle $\Delta \in \triangle(S)$ from the defining quantities $\mathbf{M}^* \cdot (a_1, k_1 M, a_2)$.*

*Proof.* As above we set $g = a_1 k_1 a_2 = k_3^{-1} a_3^{-1} k_2^{-1}$. The assumption that $\Delta$ is regular implies that the vertex $\mathscr{C} = gK$ of the representative $\mathscr{A}\mathscr{B}\mathscr{C} \in \Delta$ lies in $S_r$. We use the fundamental relation and the $K$-invariance of $p$ to compute

$$\begin{aligned} C_1(Q(gK)) &= C_1(Q(a_1 k_1 a_2 K)) = C_1(Q(k_3^{-1} a_3^{-1} k_2^{-1} K)) = C_1(k_3^{-1} a_3^{-2} k_3) \\ &= \exp \circ q \circ p(\mathrm{Ad}\,(k_3^{-1})(-2H_3)) = \exp \circ q \circ p(-2H_3). \end{aligned}$$

Let $s_0$ denote the geodesic symmetry at $x_0$ and let $m^*M$ denote the Weyl group element that maps the basic Weyl chamber to its opposite (cf. Section 3.1). Then $\mathrm{Ad}\,(s_0)H_3 = -H_3$ is in the opposite Weyl chamber and $\mathrm{Ad}\,(m^*s_0)H_3$ is again in $\mathbf{a}^+$. In particular $p(-H_3) = p(\mathrm{Ad}\,(m^*)(-H_3)) = p(\mathrm{Ad}\,(m^*s_0)H_3)$. We insert this in the above equation and get

$$\begin{aligned} C_1(Q(gK)) &= \exp \circ q \circ p(-2H_3) = \exp \circ q \circ p(\mathrm{Ad}\,(m^*s_0)2H_3) \\ &= m^*s_0 a_3^2 (m^*s_0)^{-1} = i_{m^*s_0}(a_3^2). \end{aligned}$$

It remains to check that the map $C_1$ is constant on $\mathbf{M}^* \cdot (a_1, k_1, a_2)$, i.e. that the map $\Gamma_1$ on $\triangle(S)$ is well defined. We used $g = a_1 k_1 a_2 = k_3^{-1} a_3^{-1} k_2^{-1}$. Another representative for $\mathbf{M}^* \cdot (a_1, k_1, a_2)$ is of the form $(i_v(a_1), i_v(k_1)m, i_v(a_2))$ for suitable $v \in \mathbf{M}^*$ and $m \in M$. Using Lemma 2 of Section 3.1, we then get

$$g' = va_1 v^{-1} vk_1 v^{-1} mva_2 v^{-1} = va_1 k_1 a_2 v^{-1} m' \qquad \text{for some } m' \in M.$$

Thus

$$Q(g'K) = i_v(a_1 k_1 a_2^2 k_1^{-1} a_1) = i_v(k_3^{-1} a_3^{-2} k_3) = i_v(k_3^{-1}) i_v(a_3^{-2}) i_v(k_3)$$

$$\text{where } i_v(k_3) \in K.$$

Similarly to the previous computation we obtain

$$C_1(Q(g'K)) = C_1(Q(i_v(g)m'K)) = i_{m^* s_0 v}(a_3^2) \in \mathbf{M}^* \cdot a_3^2,$$

and in conclusion $\mathbf{M}^* \cdot C_1(Q(g'K)) = \mathbf{M}^* \cdot C_1(Q(gK))$ so that $\Gamma_1$ is actually a map defined on $\triangle(S)$ as we claimed. Clearly $\mathbf{M}^* \cdot a_3^2$ determines $\mathbf{M}^* \cdot a_3$. $\square$

In this first version of the laws of cosines the inverse of the exponential map is involved rendering it inappropriate for most computations in explicit examples. We therefore provide an alternative approach which is based on linear representations of the semisimple group $G$.

### 4.2. *Isometric embeddings in a fundamental symmetric space*

We discuss the *fundamental symmetric space* $\mathscr{P}(n, \mathbf{C}) := SL(n, \mathbf{C})/SU(n)$ in some detail. We retain the general notation but use the subscript 0 to emphasize that we are dealing with the special case $\mathscr{P}(n, \mathbf{C})$.

Let $I_n$ denote the unit matrix of rank $n$. The canonical involution $\sigma_0$ induced by the geodesic symmetry at the base point $I_n \cdot SU(n)$ is given by $\sigma_0 : PSL(n, \mathbf{C}) \to PSL(n, \mathbf{C})$; $A \mapsto (\overline{A^t})^{-1}$. It follows that the quadratic representation is of the form $Q_0 : \mathscr{P}(n, \mathbf{C}) \to PSL(n, \mathbf{C})$; $Q_0(X \cdot SU(n)) = X\overline{X^t}$.

DEFINITION. Let $\sigma$ denote the canonical involution of $G$ induced by the geodesic symmetry at the base point of $S$. A representation $\rho : G \to PSL(n, \mathbf{C})$ is called *compatible with* $\sigma$ if $\rho \circ \sigma = \sigma_0 \circ \rho$ modulo scalars.

We shall make use of the following

PROPOSITION 4 (I. Satake). *Let $S = G/K$ be an irreducible symmetric space $S$ of non-compact type and $\rho : G \to PSL(n, \mathbf{C})$ an irreducible faithful representation of $G$ compatible with $\sigma$. Then $\mathscr{R}_\rho : S = G/K \to \mathscr{P}(n, \mathbf{C})$; $gK \mapsto \rho(g)\overline{\rho(g)^t}$ is an isometry and $\mathscr{R}_\rho(S)$ is a totally geodesic submanifold of the fundamental symmetric space $\mathscr{P}(n, \mathbf{C})$.*

The proof can be found in [Sat].

DEFINITION. We call $\mathscr{R}_\rho$ the *irreducible representation of $S$ determined by $\rho$.*

REMARK. There always exist faithful linear representations for $G = I_0(S)$: Since $G$ acts effectively on $S$ and since the centre of $G$ is contained in $K$, $G$ has trivial centre. Therefore $G$ can be identified (via the adjoint representation) with the group of inner automorphisms of its Lie algebra, $G \cong \mathrm{Ad}\, G \cong \mathrm{Int}\,(\mathbf{g})$.

Using the quadratic representation $Q_0$, the set $D$ of all positive definite diagonal matrices with determinant 1 can be chosen as the basic flat in $\mathscr{P}(n, \mathbf{C})$. Further, we can identify the basic Weyl chamber in $\mathscr{P}(n, \mathbf{C})$ with $D^+$, the set of real diagonal matrices $\mathrm{Diag}\,(d_1, d_2, \ldots, d_n) \in D$ with $d_1 > d_2 > \cdots > d_n > 0$.

DEFINITION. Let $m_0^*$ represent the element in the Weyl group of $\mathscr{P}(n, \mathbf{C})$ which maps the basic Weyl chamber $D^+$ to its opposite $D^-$, i.e.

$$i_{m_0^*}(\mathrm{Diag}\,(d_1, d_2, \ldots, d_n)) = \mathrm{Diag}\,(d_n, d_{n-1}, \ldots, d_1).$$

We denote by $\tau$ the composite map

$$\tau := \sigma_0 \circ i_{m_0^*} : D^+ \to D^+, \qquad \mathrm{Diag}\,(d_1, \ldots, d_n) \mapsto \mathrm{Diag}\,(d_n^{-1}, d_{n-1}^{-1}, \ldots, d_1^{-1}).$$

As $\mathscr{R}_\rho$ is an isometry and as $\mathscr{R}_\rho(S)$ is totally geodesic in $\mathscr{P}(n, \mathbf{C})$ we can choose the basic flat in the symmetric space $S$ in such a way that it is embedded by $\mathscr{R}_\rho$ into the basic flat $D$ of $\mathscr{P}(n, \mathbf{C})$.

LEMMA. *Let $m^*$ and $m_0^*$ denote representatives of the respective unique elements in the Weyl groups of $S$ and $\mathscr{P}(n, \mathbf{C})$ which map the basic Weyl chambers $A^+$ resp. $D^+$ to their opposites. If $\rho$ is a representation compatible with $\sigma$, then $\rho \circ i_{m^*}|_{A^+} = i_{m_0^*} \circ \rho|_{A^+}$ and $\rho \circ i_{m^*s_0}|_{A^+} = \tau \circ \rho|_{A^+}$.*

*Proof.* We denote by $A^-$ the Weyl chamber in $A \subset G$ opposite to $A^+$. As $\rho$ is an isometric embedding (cf. Proposition 4) with $\rho(A) \subseteq D$, we have $\rho(A^+) \subseteq D^+$ and $\rho(A^-) \subseteq D^-$. Moreover, $\rho(m^*)\rho(A^+)\rho(m^*)^{-1} = \rho(i_{m^*}(A^+)) = \rho(A^-)$. We see from this that $i_{\rho(m^*)}$ maps the subset $\rho(A^+)$ into $D^-$. Thus there is an element $w$ in the Weyl group $W_0$ of $\mathscr{P}(n, \mathbf{C})$ such that $i_{\rho(m^*)}(\rho(a)) = w \cdot \rho(a)$ for all $\rho(a) \in \rho(A^+)$ (cf. [Hel], Ch. VII, Proposition 2.2). Hence $i_{\rho(m^*)}$ is the restriction of $w$ and since $W_0$ operates simply transitively on the set of Weyl chambers in $D$, $w$ is represented by $m_0^*$.

For the second claim of the lemma just observe that $\rho \circ i_{m^*s_0}|_{A^+} = \rho \circ i_{m^*} \circ i_{s_0}|_{A^+} = \rho \circ i_{s_0} \circ i_{m^*}|_{A^+} = \rho \circ \sigma \circ i_{m^*}|_{A^+} = \sigma_0 \circ \rho \circ i_{m^*}|_{A^+} = \sigma_0 \circ i_{m_0^*} \circ \rho|_{A^+} = \tau \circ \rho|_{A^+}$. □

Every matrix in $\mathscr{P}(n, \mathbf{C})$ or, more precisely, in $Q_0(\mathscr{P}(n, \mathbf{C}))$ can be diagonalized by an element of $SU(n)$. Thus a set of functionally independent $SU(n)$-invariant functions on $\mathcal{P}(n, \mathbf{C})$ is given, for example, by the elementary symmetric polynomials $\sigma_j$ for $j = 1, \ldots, n$ in the eigenvalues $\lambda_j$ of elements of $\mathcal{P}(n, \mathbf{C})$.

DEFINITION. We define maps $p_0 : \mathcal{P}(n, \mathbf{C}) \to \mathbf{R}^{n-1}$; $X \mapsto (\sigma_1(X), \ldots, \sigma_{n-1}(X))$, and $q_0 : p_0(\mathcal{P}(n, \mathbf{C})) \to D^+$; $(\sigma_1(X), \ldots, \sigma_{n-1}(X)) \mapsto \operatorname{Diag}(\lambda_1(X), \ldots, \lambda_n(X))$, with $\lambda_1(X) > \cdots > \lambda_n(X) > 0$ for $X \in \mathcal{P}(n, \mathbf{C})$.

The elementary symmetric polynomials of a matrix with different eigenvalues determine these eigenvalues up to permutation. Thus the map $q_0$ is one-to-one. Note that $q_0 \circ p_0 : \mathcal{P}(n, \mathbf{C}) \to D^+$ associates to a positive definite hermitian matrix the set of its *ordered* eigenvalues. In particular $q_0 \circ p_0$ is $SU(n)$-invariant. The map $\pi : A \to A \cdot K$; $a \mapsto aK$ is clearly one-to-one and we define the map:

DEFINITION. $C_2 : S_r \subset G/K \to A^+$; $C_2(gK) := \pi^{-1} \circ \mathcal{R}_\rho^{-1} \circ q_0 \circ p_0 \circ \mathcal{R}_\rho(gK)$.

After these preliminary observations we again consider a marked, regular triangle $\Delta \in \triangle(S)$ with sides $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and angles $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and with associated fundamental relation $a_1 k_1 a_2 k_2 a_3 k_3 = e$.

THEOREM 2 (Laws of Cosines; second version). *Let $S$ be an irreducible Riemannian symmetric space of non-compact type and let $\mathcal{R}_\rho$ be an irreducible representation of $S$. If $C_2$ is the map defined above, then the map*

$$\Gamma_2 : \triangle(S) \to \mathbf{M}^* \backslash A^+;\ \mathbf{M}^* \cdot (a_1, k_1 M, a_2) \mapsto \mathbf{M}^* \cdot C_2(a_1 k_1 a_2 K) = \mathbf{a}_3$$

*determines the third side $\mathbf{a}_3 = \mathbf{M}^* \cdot a_3$ of a triangle $\Delta \in \triangle(S)$ uniquely from the defining quantities $\mathbf{M}^* \cdot (a_1, k_1 M, a_2)$.*

*Proof.* We first observe that for $g = a_1 k_1 a_2 = k_3^{-1} a_3^{-1} k_2^{-1}$, we have

$$\begin{aligned}\mathcal{R}_\rho(gK) &= \rho(g)\overline{\rho(g)}^t = \rho(k_3^{-1})\rho(a_3^{-1})\rho(k_2^{-1})\overline{\rho(k_2^{-1})}^t\overline{\rho(a_3^{-1})\rho(k_3^{-1})}^t \\ &= \rho(k_3)^{-1}\rho(a_3^{-2})\rho(k_3).\end{aligned}$$

Hence by the SU(n)-invariance of $q_0 \circ p_0$

$$\begin{aligned}C_2(gK) &= \pi^{-1} \circ \mathcal{R}_\rho^{-1} \circ q_0 \circ p_0(\rho(k_3)^{-1}\rho(a_3^{-2})\rho(k_3)) \\ &= \pi^{-1} \circ \mathcal{R}_\rho^{-1} \circ q_0 \circ p_0(\rho(a_3^{-2})).\end{aligned}$$

If $\rho(a_3) = \mathrm{Diag}\,(d_1, \ldots, d_n) \in D^+$ with $d_1 > \cdots > d_n > 0$, then by the definition of $\tau$ and the above Lemma

$$q_0 \circ p_0(\rho(a_3)^{-2}) = \mathrm{Diag}\,((d_n^{-2}, d_{n-1}^{-2}, \ldots, d_1^{-2}) = \tau(\mathrm{Diag}\,(d_1^2, \ldots, d_n^2))$$
$$= \tau \circ \rho(a_3^2) = \rho \circ i_{m^* s_0}(a_3^2) = \mathscr{R}_\rho(i_{m^* s_0}(a_3)K).$$

We substitute this into the above equation to obtain

$$C_2(gK) = \pi^{-1} \circ \mathscr{R}_\rho^{-1} \circ \mathscr{R}_\rho(i_{m^* s_0}(a_3)K) = \pi^{-1}(i_{m^* s_0}(a_3)K) = i_{m^* s_0}(a_3).$$

As in Theorem 1 the result is independent of the choice of the representative. □

## 5. Laws of sines for symmetric spaces of non-compact type

The aim of this Section is to generalize the laws of sines of hyperbolic geometry to arbitrary symmetric spaces $S$ of non-compact type. Recall from Section 3 that the sides of a marked, regular triangle $\Delta \in \triangle(S)$ are $\mathbf{M}^*$-orbits in $A^+$ and that the angles are $\mathbf{M}^*$-orbits in $K/M$.

For a triangle $\Delta \in \triangle(S)$ we consider two sides and two angles which are adjacent to the third side, e.g. $\mathbf{a}_1, \mathbf{a}_3$ and $\mathbf{k}_1, \mathbf{k}_2$.

We are looking for relations between these sides and these angles which allow us, at least implicitly, to determine from the two sides and one angle, say $\mathbf{k}_1$, adjacent to the third side $\mathbf{a}_2$, the second adjacent angle $\mathbf{k}_2$.

DEFINITION. We call functional relations by which this can be done *laws of sines.*

### 5.1. *Integrals for the Weyl chamber flow*

For triangles in the hyperbolic plane we have the well-known equalities

$$\frac{\sin\alpha}{\sinh a} = \frac{\sin\beta}{\sinh b} = \frac{\sin\gamma}{\sinh c}.$$

These formulae can be derived from an integral of the geodesic flow of the hyperbolic plane (cf. e.g. [Hsi] or [Leu]).

DEFINITION. Given a group $G$ operating on a set $X$ and a map $f: X \to Y$ from $X$ into an arbitrary set $Y$, we call $f$ an *integral* for the given $G$-action if $f$ is constant on the $G$-orbits.

In what follows we shall construct such integrals for the Weyl chamber flow, which we defined in Section 2.5. They are integrals for the geodesic flow if the rank of the symmetric space is 1.

Let $gM \in CS = G/M$ be a Weyl chamber and let $g = hbn$ be the Iwasawa decomposition of $g$ with $h \in K$, $b \in A$ and $n \in N$. For $g' = gM, m \in M$, we have $g' = h'b'n' = hbnm = hmb\bar{n}$ because $M$ normalizes $N$ and centralizes $A$. We thus have a well-defined map $\Theta : G/M \to K/M$; $gM \mapsto k_0(g)M$ where $k_0(g)$ denotes the $K$-component in the Iwasawa decomposition of $g \in G = KAN$. Geometrically $\Theta$ associates to a Weyl chamber its corresponding class of asymptotic Weyl chambers (cf. Section 2.3).

Recall from Section 3.1 that there is a unique element $w^*$ in the Weyl group $W = M'/M$ of $S$ which maps the Weyl chamber $c_0$ corresponding to $M$ to its opposite chamber $c_0^*$. We have $w^* = m^*M$ for some $m^* \in M' = N_K(\mathbf{a})$ and $(m^*)^2 \in M$.

DEFINITION. For a Weyl chamber $c = gM \in G/M$ we set $c^* := gm^*M$. This definition is independent of representatives, for $M$ is a normal subgroup of $M'$, so that $gmm^*M = gm^*m'M = gm^*M$ for some $m, m' \in M$.

Further, we define for the corresponding Weyl chambers $gM$ and $gm^*M$

$$hM := \Theta(gM), \qquad h_*M := \Theta(gm^*M), \quad \text{and} \quad \Pi(gM) := \mathbf{M}^* \cdot h_*^{-1}hM.$$

Again it is easy to see that these orbits are well defined.

We call $\Pi(gM) \in \mathbf{M}^*\backslash K/M$ the *subtended angle* of the flat $gAM$ with respect to the base point $x_0$ of $S$ (cf. Figure 5).

REMARK. Observe that the map $\hat{\Pi} : G/M \to M\backslash K/M$; $\hat{\Pi}(gM) := Mh_*^{-1}hM$ is also well-defined. We shall make use of this observation in the formulation of the laws of sines in the next paragraph. In hyperbolic geometry the subtended angle of a geodesic not containing the base point $x_0$ is *twice the angle of parallelism* with respect to $x_0$.

THEOREM 1. *The two maps $\Theta : G/M \to K/M$ and $\Pi : G/M \to \mathbf{M}^*\backslash K/M$ defined above are integrals of the Weyl chamber flow $\phi$.*

*Moreover, $\Theta(kgM) = k\Theta(gM)$ and $\Pi(kgM) = \Pi(gM)$.*

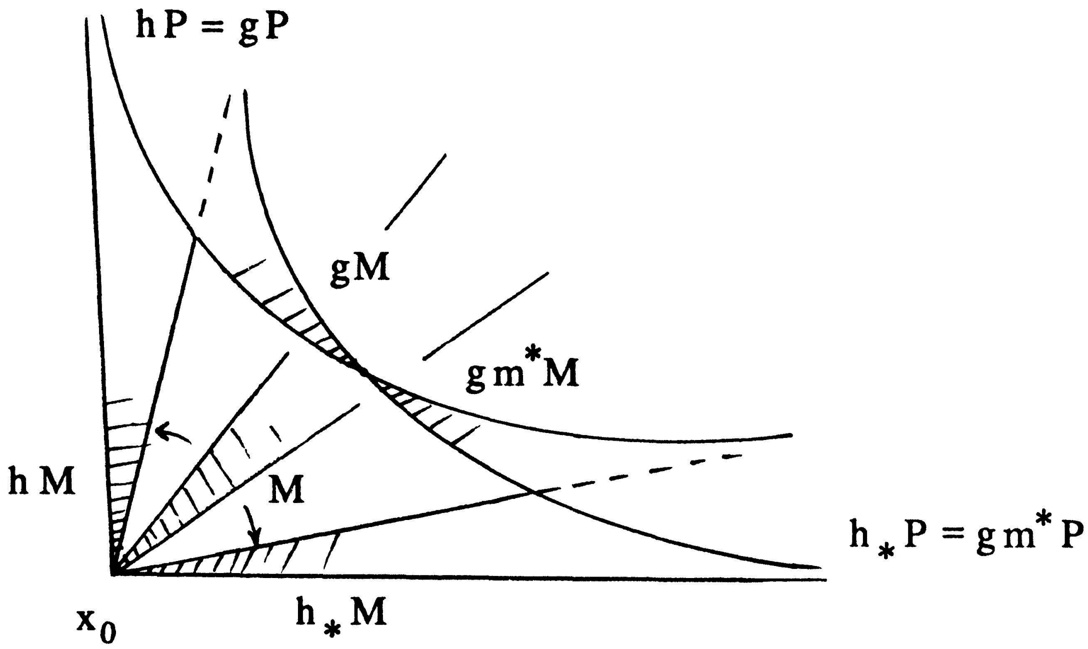


Figure 5.

*Proof.* We must show that, for $gM \in G/M$ and $a \in A$, $\Theta(gM) = \Theta(gaM)$ and $\Pi(gM) = \Pi(gaM)$. To prove this, we consider the Iwasawa decomposition $g = hbn$. As $A$ normalizes $N$, we have $ga = hbna = hban'$ and $\Theta(gM) = hM = \Theta(gaM)$. Clearly $\Theta(kgM) = khM = k\Theta(gM)$. As $am^* = m^*a'$ for some $a' \in A$, we also have $\Theta(gam^*M) = \Theta(gm^*a'M) = \Theta(gm^*M)$ and $\Pi(gaM) = \mathbf{M}^* \cdot h_*^{-1}hM = \Pi(gM)$.

Finally it is clear from the definitions that $\Theta(kgM) = khM$ and $\Theta(kgm^*M) = kh_*M$. This gives $\Pi(kgM) = \mathbf{M}^* \cdot (kh_*)^{-1}khM = \Pi(gM)$. □

REMARK. The subtended angles are precisely those orbits $\mathbf{M}^* \cdot kM$ for which $k \in K^* := K \cap Pm^*P$. To see this, simply note that, for $g = hbn \in G$, $gP = hP$, $gm^*P = h_*P$ and we can therefore write $h_* = gm^*p$, $h = gq$ for $p, q \in P$. Now $h_*^{-1}h \in K \cap Pm^*P$.

The set $K^*/M := \{kM \in K/M \mid k \in K^*\}$ is a connected, open and dense submanifold of $K/M$. This can be shown by using Proposition 1.2.3.5 in [War].

## 5.2. *The laws of sines*

We consider a marked, regular triangle $\Delta \in \triangle(S)$ with associated fundamental relation written in the form $a_1k_1a_2M = k_3^{-1}a_3^{-1}k_2^{-1}M$ (cf. Remark 1, Section 3.4). We now use the map $\hat{\Pi}$ introduced in Section 5.1.

THEOREM 2 (Laws of Sines). *Let $S$ be an irreducible Riemannian symmetric space of non-compact type. For a representative of a marked triangle $\Delta \in \triangle(S)$ with sides $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ and angles $\mathbf{k}_1, \mathbf{k}_2, \mathbf{k}_3$ and fundamental relation as above the map*

$$\Sigma : A \times K/M \to M\backslash K/M; (a, kM) \mapsto \hat{\Pi}(akM)$$

*satisfies*

$$\Sigma(a_1, k_1 M) = \Sigma(a_3^{-1}, k_2^{-1} M),$$
$$\Sigma(a_2, k_2 M) = \Sigma(a_1^{-1}, k_3^{-1} M),$$
$$\Sigma(a_3, k_3 M) = \Sigma(a_2^{-1}, k_1^{-1} M).$$

*Proof.* By the Remark preceeding Theorem 1 the map $\Sigma$ is well-defined.

We next verify the first equation in Theorem 2. Using Theorem 1 and the fundamental relation we find $\Theta(a_1 k_1 M) = \Theta(a_1 k_1 a_2 M) = \Theta(k_3^{-1} a_3^{-1} k_2^{-1} M)$; moreover, there is an $a_2' \in A$ such that $a_1 k_1 m^* a_2' M = a_1 k_1 a_2 m^* M = k_3^{-1} a_3^{-1} k_2^{-1} m^* M$ and therefore $\Theta(a_1 k_1 m^* M) = \Theta(a_1 k_1 m^* a_2' M) = \Theta(k_3^{-1} a_3^{-1} k_2^{-1} m^* M)$. As also $\Theta(kgM) = k\Theta(gM)$ for any $k \in K$, we obtain

$$\Theta(k_3^{-1} a_3^{-1} k_2^{-1} M) = k_3^{-1} \Theta(a_3^{-1} k_2^{-1} M),$$
$$\Theta(k_3^{-1} a_3^{-1} k_2^{-1} m^* M) = k_3^{-1} \Theta(a_3^{-1} k_2^{-1} m^* M).$$

If we set $h_3 M := \Theta(a_3^{-1} k_2^{-1} M)$ and $(h_3)_* M := \Theta(a_3^{-1} k_2^{-1} m^* M)$ then

$$\begin{aligned}\hat{\Pi}(a_3^{-1} k_2^{-1} M) &= M (h_3)_*^{-1} h_3 M = \hat{\Pi}(k_3^{-1} a_3^{-1} k_2^{-1} M) = \hat{\Pi}(a_1 k_1 a_2 M) \\ &= \hat{\Pi}(a_1 k_1 M).\end{aligned}$$

Cyclic permutation of the representatives of sides and angles in the fundamental relation completes the proof. □

REMARK. If we work with another representative of $\Delta$, i.e. if we replace $a_1 k_1 a_2 M$ by $i_v(a_1 k_1 a_2) M$ for some $v \in \mathbf{M}^*$, then the above computations lead to $\hat{\Pi}(i_v(a_3^{-1} k_2^{-1}) M) = M i_v((h_3)_*^{-1} h_3) M = \hat{\Pi}(i_v(a_1 k_1) M)$. Note that the associated subtended angle is the same for both representatives and thus is a geometric quantity associated to $\Delta$.

We next investigate the extent to which the first equation in Theorem 2 actually determines the second adjacent angle $\mathbf{k}_2$ of the considered triangle $\Delta \in \triangle(S)$.

Let us first look at the classical hyperbolic case. Given a geodesic triangle in the hyperbolic plane $H^2\mathbf{R}$ with sides $b$ and $c$ and angles $\beta$ and $\gamma$, we have the law of sines: $\sinh c \sin \beta = \sinh b \sin \gamma$. If $b$, $c$ and $\beta$ are given, then this equation in general has two different solutions, $\gamma$ and $\pi - \gamma$.

The actual solution (for the considered triangle) is only determined by taking the third side $a$ – which is part of the defining quantities $c, \beta, a$ – into account. We prove a generalization of this fact.

DEFINITION. For $g \in G$ let $g = kna$ be the Iwasawa decomposition with $G = KNA$ (in this order!). We set $A(g) := a$.

THOEREM 3. *Suppose that $\hat{\Pi}(akM) = \hat{\Pi}(ak'M)$ and $A(ak) = A(ak')$ hold for $a^{-1} \in A^+$ and $kM, k'M \in K/M$. Then $MkM = Mk'M$.*

*Proof.* We use the decompositions $ak = hnb$ and $ak' = h'n'b'$. Then $b = A(ak) = A(ak') = b'$. Moreover let $akm^* = h_* n_1 b_1$ and $ak'm^* = h'_* n'_1 b'_1$. Put $\hat{\Pi}(akM) = Mh_0M = \hat{\Pi}(ak'M)$. We then obtain $m'h_0m = h_*^{-1}h = n_1 b_1 m^* b^{-1} n^{-1}$, $h_0 = (h'_*)^{-1}h' = n'_1 b'_1 m^*(b')^{-1}(n')^{-1}$, for some $m, m' \in M$. The uniqueness of the factors in the Bruhat decomposition (cf. Section 2.2) implies that $n = m_1 n' m_2$ for $m_1, m_2 \in M$. Thus $h^{-1}akM = nbM = m_1 n' m_2 b'M = m_1 n' b'M = m_1 (h')^{-1} ak'M$. By the uniqueness properties of the factors in the Cartan decomposition (cf. Section 2.2) we conclude that $MkM = Mk'M$. □

We now apply Theorem 3 to a marked, regular geodesic triangle $\mathscr{A}\mathscr{B}\mathscr{C} \in \Delta$ with fundamental relation $a_1 k_1 a_2 k_2 a_3 k_3 M = M$.

Observe that $A(a_3^{-1} k_2^{-1}) = A(k_3^{-1} a_3^{-1} k_2^{-1}) = A(a_1 k_1 a_2) = A(a_1 k_1) a_2$. Thus, given the defining quantities $\mathbf{M}^* \cdot (a_1, k_1 M, a_2)$ for $\Delta$, the element $A(a_3^{-1} k_2^{-1}) \in A$ is fully determined. From this observation we deduce the following Corollary of Theorem 3 to supplement Theorem 2:

COROLLARY. *For a marked, regular geodesic triangle that represents a marked triangle $\Delta \in \triangle(S)$ the equation $\Sigma(a_1, k_1 M) = \Sigma(a_3^{-1}, k_2^{-1} M)$ together with the defining quantities $\mathbf{M}^* \cdot (a_1, k_1 M, a_2)$ of $\Delta$ uniquely determines the angle $\mathbf{k}_2 = \mathbf{M}^* \cdot k_2 M$.*

## 6. The space of Euclidean structures on $\mathbf{R}^n$ as an example

We illustrate our methods for $\mathscr{P}(n, \mathbf{R}) := SL(n, \mathbf{R})/SO(n)$, the space of scalar products (with determinant 1) on $\mathbf{R}^n$, for any $n \geq 2$. For the trigonometry of the spaces $SL(n, \mathbf{C})/SU(n)$ the interested reader is refered to [Leu].

### 6.1. *The orbit spaces* $\mathbf{M}^*\backslash K/M$ *and* $\mathbf{M}^*\backslash A^+$

The Riemannian symmetric space $\mathscr{P}(n, \mathbf{R})$ can be identified with the set of the positive definite $n \times n$ matrices with determinant one. $SL(n, \mathbf{R})$ acts on $\mathscr{P}(n, \mathbf{R})$ by

$$SL(n, \mathbf{R}) \times \mathscr{P}(n, \mathbf{R}) \to \mathscr{P}(n, \mathbf{R}); \qquad (A, X) \mapsto AXA^t.$$

The geodesic symmetry $s_0$ at the base point $I_n := \mathrm{Diag}(1, \dots, 1)$, the $n \times n$ unit matrix in $\mathscr{P}(n, \mathbf{R})$, induces the involution $i_{s_0}(X) = \sigma(X) = (X^{-1})^t$, for $X \in SL(n, \mathbf{R})$.

The abelian group $A$ in $SL(n, \mathbf{R})$ can be chosen as the set of all diagonal matrices with determinant 1, i.e. $a = \mathrm{Diag}(a_1, \dots, a_n)$; $\Pi_{i=1}^n a_i = 1$. A Weyl chamber $A^+ \subset A$ is given by those $a \in A$ for which $a_1 > \cdots > a_n > 0$.

The matrix $\omega = \mathrm{Diag}(-1, 1, \dots, 1)$ defines an isometry that reverses orientation. If the rank of $\mathscr{P}(n, \mathbf{R})$ is odd (i.e. for $n$ even) then

$$\mathbf{K} = SO(n)/\{\pm I_n\} \cup s_0 SO(n)/\{\pm I_n\} \cup \omega(SO(n)/\{\pm I_n\} \cup s_0 SO(n)/\{\pm I_n\})$$

and if the rank of $\mathscr{P}(n, \mathbf{R})$ is even $\mathbf{K} = SO(n) \cup s_0 SO(n)$ (cf. [Ca2], p. 389).

We have $M = \{\mathrm{Diag}(\varepsilon_1, \dots, \varepsilon_n) \mid \Pi_{i=1}^n \varepsilon_i = 1, \varepsilon_i \in \{1, -1\}\}/\{\pm I_n\}$ for $n$ even resp. $M = \{\mathrm{Diag}(\varepsilon_1, \dots, \varepsilon_n) \mid \Pi_{i=1}^n \varepsilon_i = 1, \varepsilon_i \in \{1, -1\}\}$ for $n$ odd. From Lemma 1, Section 3.1, we see that for $n$ even

$$\mathbf{M}^* = M \cup s_0 m^* M \cup \omega(M \cup s_0 m^* M)$$

and for $n$ odd

$$\mathbf{M}^* = M \cup s_0 m^* M.$$

Moreover, for $\mathrm{Diag}(a_1, \dots, a_n) \in A^+$ we compute

$$\mathbf{M}^* \cdot \mathrm{Diag}(a_1, \dots, a_n) = \{\mathrm{Diag}(a_1, \dots, a_n), \mathrm{Diag}(a_n^{-1}, \dots, a_1^{-1})\}.$$

Finally, we note that the group $M$ is discrete; thus the connected manifold of principal $M$-orbits in $K/M$ has the same dimension as $K = SO(n)$, i.e. $\frac{1}{2}n(n-1)$.

#### 6.1.1. *The laws of cosines for* $\mathscr{P}(n, \mathbf{R})$

Let $\mathscr{A}\mathscr{B}\mathscr{C}$ be a regular, geodesic triangle in $\mathscr{P}(n, \mathbf{R})$. Then the two sides $\mathbf{M}^* \cdot a$, $\mathbf{M}^* \cdot b$ and the angle $\mathbf{M}^* \cdot k_1 \mathbf{M}$ at $\mathscr{B}$ are represented by the matrices

$$a = \mathrm{Diag}(a_1, \dots, a_n), \qquad b = \mathrm{Diag}(b_1, \dots, b_n) \in A^+$$

and $k_1 = [v_i^{\,j}] \in SO(n)$. The other representatives of $Mk_1M$ are all of the form $k_1' = [\varepsilon_i \delta_j v_i^{\,j}]$ with $\varepsilon_i, \delta_j \in \{1, -1\}$.

Let $v_i$ denote the $i$-th row of $k_1$ and set

$$(v_i, v_j)_b := \sum_{k=1}^{n} b_k^2 v_i^{\,k} v_j^{\,k}.$$

We use Theorem 2 in Section 4 to compute the laws of cosines.

With the canonical matrix representation $\rho = id$ of $SL(n, \mathbf{R})$ and $g = ak_1 b$ we get

$$\mathscr{R}_\rho(gK) = \mathscr{R}(gK) = (ak_1)b^2(ak_1)',$$

$$\mathscr{R}(gK)_{ij} = a_i a_j (v_i, v_j)_b.$$

Let $c = \mathrm{Diag}\,(c_1, \ldots, c_n) \in A^+$ represent the side $\mathscr{AC}$ of the triangle $\mathscr{ABC}$.

The *laws of cosines of* $P(n, \mathbf{R})$ are given by the $(n-1) = \mathrm{rank}\,\mathscr{P}(n, \mathbf{R})$ equations

$$\sum_{i=1}^{n} c_i^{2m} = \mathrm{trace}\,(\mathscr{R}(gK))^m$$

$$= \sum_{i_1, \ldots, i_m = 1}^{n} a_{i_1}^2 \cdots a_{i_m}^2 (v_{i_1}, v_{i_2})_b (v_{i_2}, v_{i_3})_b \cdots (v_{i_{m-1}}, v_{i_m})_b (v_{i_m}, v_{i_1})_b$$

where $m = 1, \ldots, n-1$. The proof is by induction and straightforward.

Note that the additional equation $\Pi_{k=1}^{n}(c_k)^2 = 1$ must be taken into account.

REMARK. The space $\mathscr{P}(2, \mathbf{R})$ is the "hyperboloid"-model for the real hyperbolic plane $H^2\mathbf{R}$ and there is only one law of cosines, namely,

$$t_1 = \mathrm{trace}\,\mathscr{R}(gK) = \sum_{i=1}^{2} a_i^2 (v_i, v_i)_b = \sum_{i=1}^{2} c_i^2.$$

If we set $a_1 = e^{t_1/2}$, $a_2 = e^{-t_1/2}$, $b_1 = e^{t_2/2}$, $b_2 = e^{-t_2/2}$, $c_1 = e^{t_3/2}$, $c_2 = e^{-t_3/2}$ and $v_1 = (\cos(\phi/2), -\sin(\phi/2))$, $v_2 = (\sin(\phi/2), \cos(\phi/2))$. Then the above equation takes the form

$$e^{t_3} + e^{-t_3} = e^{t_1}\left(\cos^2\frac{\phi}{2} e^{t_2} + \sin^2\frac{\phi}{2} e^{-t_2}\right) + e^{-t_1}\left(\sin^2\frac{\phi}{2} e^{t_2} + \cos^2\frac{\phi}{2} e^{-t_2}\right)$$

$$= \cos^2\frac{\phi}{2} e^{t_2}(e^{t_1} - e^{-t_1}) - \cos^2\frac{\phi}{2} e^{-t_2}(e^{t_1} - e^{-t_1}) + e^{t_1 - t_2} + e^{-t_1 + t_2}$$

and we get the well-known formula (note that $\phi$ is the *exterior* angle at $\mathscr{B}$)

$$\begin{aligned}\cosh t_3 &= \sinh t_1 \sinh t_2 2 \cos^2 \frac{\phi}{2} + \cosh (t_1 - t_2)\\ &= \cosh t_1 \cosh t_2 - \cos (\pi - \phi) \sinh t_1 \sinh t_2.\end{aligned}$$

6.1.2. *The laws of sines for $\mathscr{P}(n, \mathbf{R})$*

We wish to compute the angle $\mathbf{k}_2$ from the sides $\mathbf{a}, \mathbf{c}$ and the angle $\mathbf{k}_1$ for the congruence class of the triangle $\mathscr{A}\mathscr{B}\mathscr{C}$ in $\mathscr{P}(n, \mathbf{R})$. As we already remarked above the set of (principal) $M$-orbits in $K/M$ has dimension $\frac{1}{2}n(n-1)$. This is precisely the number of laws of sines that we need to characterize the angle $\mathbf{k}_2$, represented by the matrix $k_2 = [w_i^j] \in SO(n)$, at the vertex $\mathscr{C}$.

Let $(\ ,\ ) = (\ ,\ )_E$ denote the standard scalar product on $\mathbf{R}^n$ and $\|\ \|$ the norm induced by it. For the diagonal matrix $a$ which represents $\mathbf{a}$ and for the column vectors $v^j$ of the orthogonal matrix $k_1 = [v_i^j]$ the matrix product $av^j$ is defined.

We claim that

$$\frac{|(av^i, av^j)|}{\|av^i\| \, \|av^j\|} = \frac{|(cw^i, cw^j)|}{\|cw^i\| \, \|cw^j\|}$$

with $1 \leq i < j \leq n$ are the *sine laws* for $\mathscr{P}(n, \mathbf{R})$ we are looking for.

In order to prove this claim let $gM$ be a Weyl chamber in $C\mathscr{P}(n, \mathbf{R}) = SL(n, \mathbf{R})/M$. For $1 \leq i < j \leq n$ we define the functions

$$\mathscr{S}_{ij} : C\mathscr{P}(n, \mathbf{R}) \to \mathbf{R}; \qquad \mathscr{S}_{ij}(gM) := \frac{|(x^i, x^j)|}{\|x^i\| \, \|x^j\|}$$

where $x^i$ is the $i$-th column of a representative of the Weyl chamber $gM$.

Note that these functions are well defined, since for $g' = gm$, the columns of the matrix $g'$ are those of $g$ multiplied with a factor $\pm 1$.

The functions $\mathscr{S}_{ij}$ are functionally independent as $SO(n)$-invariants of $n$ vectors in $\mathbf{R}^n$ (cf. [Wey], Theorems 2.9.A and 2.17.A).

We next show that the functions $\mathscr{S}_{ij}$ are $SO(n)$-invariant integrals of the Weyl chamber flow.

Let $g = hak$ be the Cartan decomposition with $h, k \in SO(n)$ and $a \in A^+$. We write $hak = [h \cdot av^1, h \cdot av^2, \ldots, h \cdot av^n]$ and denote by $x^i$ the $i$-th column of the matrix $hak$ representing $gM$.

First we have $(x^i, x^j) = (h \cdot av^i, h \cdot av^j) = (av^i, av^j)$, $h$ being an orthogonal matrix.

Furthermore the matrix $akd$ with $d \in A^+$ has the $(i, j)$-component $(akd)_{ij} = a_i d_j v_i{}^j$. Thus denoting the $i$-th column of $akd$ by $y^i$, we find

$$(y^i, y^j) = d_i d_j \sum_{k=1}^{n} a_k^2 v_k{}^i v_k{}^j = d_i d_j (av^i, av^j).$$

In conclusion we get:

$$\frac{|(x^i, x^j)|}{\|x^i\| \, \|x^j\|} = \frac{|(av^i, av^j)|}{\|av^i\| \, \|av^j\|} = \frac{|(y^i, y^j)|}{\|y^i\| \, \|y^j\|}.$$

This proves that the $\frac{1}{2}n(n-1)$ functions are $SO(n)$-invariant functionally independent integrals of the Weyl chamber flow and we obtain the laws of sines mentioned above.

REMARK. For $n = 2$, i.e. $\mathscr{P}(2, \mathbf{R}) \cong H^2\mathbf{R}$, we can write $a = \mathrm{Diag}\,(e^{t_1/2}, e^{-t_1/2})$, $c = \mathrm{Diag}\,(e^{t_3/2}, e^{-t_3/2})$ and $k_1 = D_{\beta/2}$, $k_2 = D_{\gamma/2} \in SO(2)$. Then $(av^1, av^2) = -\sin\beta \sinh t_1$ and

$$\|av^1\|^2 = \cos^2\frac{\beta}{2} e^{t_1} + \sin^2\frac{\beta}{2} e^{-t_1}, \qquad \|av^2\|^2 = \sin^2\frac{\beta}{2} e^{t_1} + \cos^2\frac{\beta}{2} e^{-t_1};$$

thus

$$\frac{(av^1, av^2)}{\|av^1\| \, \|av^2\|} = -\frac{\sinh t_1 \sin\beta}{(1 + \sinh^2 t_1 \sin^2\beta)^{1/2}}.$$

And finally we obtain the laws of sines for the hyperbolic plane in the well-known form

$$\pm\frac{\sinh t_1 \sin\beta}{(1 + \sinh^2 t_1 \sin^2\beta)^{1/2}} = \frac{\sinh t_3 \sin\gamma}{(1 + \sinh^2 t_3 \sin^2\gamma)^{1/2}}$$

$$\Leftrightarrow \pm\sinh t_1 \sin\beta = \sinh t_3 \sin\gamma \Leftrightarrow \pm\frac{\sin\gamma}{\sinh t_1} = \frac{\sin\beta}{\sinh t_3}.$$

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