Comment. Math. Helvetici 67 (1992) 214–225 0010-2571/92/020214-12\$1.50 + 0.20/0

Relative cyclic homology and the Bass conjecture

JAMES A. SCHAFER

0. Introduction

Let A be an arbitrary ring with unit. If P is a finitely generated projective A-module one would like to associate to P a rank function generalizing the function which assigns to the free A-module A^n the integer n. Since in the commutative case n is the trace of the identity endomorphism of A^n, one wishes to define a trace for endomorphisms of finitely generated projective A-modules in the case of non-commutative A. This was achieved independently by Hattori and Stallings [6, 11]. Unfortunately in order for the "trace" to have the natural property of a trace function i.e., for the trace of ab to be equal to the trace of ba, one is forced to have the trace take values not in A but in $A/[A, A]$ where $[A, A]$ is the subgroup of A generated by all commutators ab-ba. The resulting trace function $\mathrm{tr}_P : \mathrm{End}_A (P) \to A/[A, A]$ has many of the properties of the trace function in the commutative case, including additivity, commutativity, and linearity. For details, see [2]. We note in particular,

1. *Functoriality*: If $\alpha : A \to B$, then α induces a map

$$\alpha : A/[A, A] \to B/[B, B]$$

and if $u \in \mathrm{End}_A P)$ then $\mathrm{tr}_{P \otimes_A B} (u \otimes id) = \alpha_*(\mathrm{tr}_P (u))$.

2. *Linearity*: Suppose $P = P_1 \oplus P_2$ and $u \in \mathrm{End}_A (P)$ restricts to $u_1 \in \mathrm{End}_A (P_1)$ and to $u_2 \in \mathrm{End}_A (P_2)$ then

$$\mathrm{tr}_P(u) = \mathrm{tr}_{P_1}(u_1) + \mathrm{tr}_{P_2}(u_2).$$

This last property allows one to note that if the P is a finitely generated projective A-module and one defines the rank r_P of P to be $\mathrm{tr}_P (id_P)$, then if P is a direct summand of the free A-module F and $e : F \to F$ is the idempotent defining P, that is $P = e(F)$, then $r_P = \mathrm{tr}_F (e)$. Also since $e \in M_d(A)$ for some d and the matrix defining e only involves finitely many elements of A, we see from property 1 there exists a finitely generated subring A' of A and a finitely generated projective A'-module so that $r_p = \alpha_*(r_{P'})$, where α is the natural map of A' into A.

If R is any commutative ring and G an arbitrary group, then it is easy to see for the group ring $R(G)$, $T(G) = RG/[RG, RG]$ is the free R-module with one generator for each conjugacy class of G. (For $\hat{g}, h \in G$, $gh - hg = h^{-1}g'h - g'$ where $g = h^{-1}g'$). One denotes the component of r_P on the conjugacy class s by $r_P(s)$. The *Bass Conjecture* [3] is then as follows:

Let G be an arbitrary group and R any subring of the complex numbers $\mathbb{C}$. Let $D = \{d \in \mathbb{Z} \mid \exists a \in \mathbb{Z}$ with $a/d \in \mathbb{Q} \cap R\}$. If $1 \neq s \in G$ is such that order $(s) \notin D$, *then $r_P(s) = 0$ for any finitely generated projective RG-module P.*

This paper is concerned with showing how relative cyclic homology can be used to obtain results on this conjecture. In the first section we describe the relative cyclic homology of a pair of k-algebras (A, S) as defined by L. Kadison [7]. If $S = k1_A$, this is nothing more than ordinary cyclic homology of A. The second and third sections are concerned with the case $(A, S) = (kG, kH)$ where k is a field of characteristic zero and H is a normal subgroup of the group G. Here we generalize to the case $HC_*(kG, kH)$ a result of Burghelea [4] calculating $HC_*(kG)$. We are much indebted to Marciniak's [9] algebraic proof of this result of Burghelea. Finally we use this calculation to obtain results on the Bass conjecture. In particular we show

THEOREM. *If s has an infinite order in G/G_n where G_n is the nth term of the lower central series for G, then $r_P(s) = 0$ for any finitely generated projective QG-module P.*

1. Relative tensor products and relative cyclic homology

Let A be a k-algebra and S a subalgebra containing $k \cdot 1_A$. The n-fold circular tensor product of A over S, $\hat{\otimes}_S^n A$ or $A \hat{\otimes}_S A \cdots \hat{\otimes}_S A$ (n factors), is defined to the the ordinary n-fold tensor product $\hat{\otimes}_S^n A$ modulo the k-submodule generated by $\{(sa_0) \otimes \cdots \otimes a_n - a_0 \otimes \cdots \otimes (a_n s)\}$ for all $s \in S$ and $a_i \in A$. This of course can be more generally defined for S bimodules and seen in the case $n = 2$ is easily seen to be the ordinary tensor product over $S \otimes_k S^{0p}$. Note the 1-fold tensor product of A over S is nothing more than $A/[A, S]$. For details see [7].

Given a k-algebra A and a subalgebra S containing S containing $k \cdot \mathbf{1}_A$ define a cyclic set $Z_*(A, S)$ as follows. $Z_n(A, S) = \hat{\otimes}_S^n A$ and the maps are

$$d_i(a_0 \hat{\otimes} \cdots \hat{\otimes} a_n) = a_0 \hat{\otimes} \cdots \hat{\otimes} a_i \hat{\otimes} a_i a_{i+1} \hat{\otimes} \cdots \hat{\otimes} a_n, \qquad 0 \le i < n,$$
$$= a_n a_0 \hat{\otimes} \cdots \hat{\otimes} a_{n-1}, \qquad i = n,$$
$$s_i(a_0 \hat{\otimes} \cdots \hat{\otimes} a_n) = a_0 \hat{\otimes} \cdots \hat{\otimes} a_i \hat{\otimes} 1 \hat{\otimes} a_{i+1} \hat{\otimes} \cdots \hat{\otimes} a_n, \qquad 0 \le i \le n,$$

and $t_n(a_0 \hat{\otimes} \cdots \hat{\otimes} a_n) = a_n \hat{\otimes} \cdots \hat{\otimes} a_0$.

The first two maps clearly form a simplicial k-module for $Z_*(A, S)$ and the proof the addition of t_n gives a cyclic set is exactly as in the non-relative case if one recalls the tensor product is a circular tensor product.

One now defines the cyclic homology of the pair (A, S) in the usual way when one has a cyclic set, i.e., one forms the Tsygan double complex from the cyclic set and then defines cyclic homolgy as the homology of this double complex. Recall the Tsygan complex $W_{*,*}$ is formed as follows. Let $T_p = (-1)t_p$, $W_{p,q} = Z_p$, $p, q \ge 0$,

$$d'_{p,q} = \sum_{i=0}^{q} (-1)^i d_i : Z_q \to Z_{q-1}, \qquad p \text{ even},$$

$$d'_{p,q} = \sum_{i=0}^{q-1} (-1)^i d_i : Z_q \to Z_{q-1}, \qquad p \text{ odd},$$

$$d''_{p,q} = 1 - T_p : Z_p \to Z_p, \qquad q \text{ odd},$$

$$d''_{p,q} = 1 + T_p + T_p^2 + \cdots + T_p^p : Z_p \to Z_p, \qquad q \text{ even}.$$

The odd columns are acyclic and the even columns give the Hochschild homology of the complex. Note that everything is functorial in the pair (A, S).

2. The complex $Z_*(kG, kH)$

Let G be a group and H be an arbitrary subgroup. Let H act on G by conjugation (${}^{\rho}g = \rho^{-1}g\rho$) and let $(G)_H$ be the orbit space. Denote by Λ_H the free k-module with one generator for each element of $(G)_H$ i.e., for every H-conjugacy class of G.

PROPOSITION 1. *Suppose H is normal in G, then G/H acts (on the right) of Λ_H by conjugation.*

Proof. If $g \equiv_H g'$ i.e. $g' = h^{-1}gh$ for some $h \in H$ then ${}^{\rho}g' \equiv_H {}^{\rho}g$ since by normality $h\rho = \rho h'$, $h' \in H$ and so

$${}^{\rho}g' = (h')^{-1}\rho^{-1}g\rho h' = (h'^{-1})^{\rho}gh'.$$

Therefore G acts on Λ_H and clearly H acts trivially. □

We will continue with the assumption that H is normal in G. For $g \in G$, let $\bar{g}$ denote the image of g in Λ_H and $\tilde{g}$ denote the image of g in G/H.

Let $S_*(G/H)$ denote the homogeneous Bar construction for G/H. Define a simplicial set, $S_*(kG, kH)$ as $S_n(kG, kH) = \Lambda_H \otimes_{G/H} S_n(G/H)$,

$$d_i(\bar{g} \otimes (\tilde{x}_0, \ldots, \tilde{x}_n) = \bar{g} \otimes (\tilde{x}_0, \ldots, \hat{\tilde{x}}_i, \ldots, \tilde{x}_n), \qquad 0 \leq i \leq n,$$

$$s_i(\bar{g} \otimes (\tilde{x}_0, \ldots, \tilde{x}_n) = \bar{g} \otimes (\tilde{x}_0, \ldots, \tilde{x}_i, \tilde{x}_i, \ldots, \tilde{x}_n), \qquad 0 \leq i \leq n.$$

Define a map $\tau_n : S_n(kG, kH) \to S_n(kG, kH)$ by

$$\tau_n(\bar{g} \otimes (\tilde{x}_0, \ldots, \tilde{x}_n) = \bar{g} \otimes (\tilde{g}^{-1}\tilde{x}_n, \tilde{x}_0, \ldots, \tilde{x}_{n-1}), \qquad n \geq 0.$$

PROPOSITION 2. *With respect to the above maps, $S_*(kG, kH)$ forms a cyclic k-module.*

Proof. It is immediate that these maps form a cyclic k-module as soon as one sees that τ_n is well-defined. As for this it is clear that changing x_i in its coset modulo H affects nothing, while if $\bar{g} = \bar{g}'$, then there exists $h \in H$ with $g' = h^{-1}gh$, and hence $g' = h^{-1}gh = h^{-1}(ghg^{-1})g = h'g$. Therefore $\tilde{g} = \tilde{g}'$ in G/H. □

Define a map

$$\alpha_n : \Lambda_H \otimes_{G/H} S_n(G/H) \to Z_n(kG, kH)$$

as follows.

$$\alpha_0(\bar{g} \otimes_{G/H} (\tilde{x}_0)) = x_0^{-1}gx_0 \text{ modulo } [kG, kH],$$

$$\alpha_n(\bar{g} \otimes_{G/H} (\tilde{x}_0, \ldots, \tilde{x}_n)) = x_n^{-1}gx_0 \mathbin{\hat{\otimes}} x_0^{-1}x_1 \mathbin{\hat{\otimes}} \cdots \mathbin{\hat{\otimes}} x_{n-1}^{-1}x_n.$$

THEOREM 1. *α_* is an isomorphism of cyclic k-modules.*

Proof. (i) α_* is well defined.

(a) α_* is independent of the H-conjugacy class of g. Let $g' = h^{-1}gh$.
For α_0, we have

$$\begin{aligned} x_0^{-1}g'x_0 &= x_0^{-1}h^{-1}ghx_0 \\ &= x_0^{-1}h^{-1}x_0x_0^{-1}gx_0x_0^{-1}hx_0 \\ &= x_0^{-1}gx_0(x_0^{-1}hx_0)x_0^{-1}h^{-1}x_0 \bmod [kG, kH] \\ &= x_0^{-1}gx_0 \bmod [kG, kH]. \end{aligned}$$

For α_n, $n \geq 1$, recall the right hand side in the definition for α_n is the circular tensor product of kG over kH. Hence

$$\begin{aligned}
&x_n^{-1}g'x_0 \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n \\
&\quad = x_n^{-1}h^{-1}ghx_0 \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n \\
&\quad = x_n^{-1}h^{-1}x_n(x_n^{-1}gx_0)x_0^{-1}hx_0 \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n \\
&\quad = (x_n^{-1}gx_0)(x_0^{-1}hx_0) \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n(x_n^{-1}h^{-1}x_n) \\
&\quad = (x_n^{-1}gx_0)(x_0^{-1}hx_0) \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-2}^{-1}x_{n-1}x_{n-1}^{-1}h^{-1}x_{n-1} \hat{\otimes} x_{n-1}^{-1}x_n \\
&\quad = (x_n^{-1}gx_0)(x_0^{-1}hx_0)(x_0^{-1}h^{-1}x_0) \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n \\
&\quad = x_n^{-1}gx_0 \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n.
\end{aligned}$$

(b) α_* is independent of the coset representatives of G/H. For α_0, if $x_0' = hx_0$ then

$$\begin{aligned}
x_0'^{-1}gx_0' &= x_0^{-1}h^{-1}ghx_0 \\
&= x_0^{-1}h^{-1}x_0x_0^{-1}gx_0x_0^{-1}hx_0 \\
&= x_0^{-1}gx_0 \bmod [kG, kH].
\end{aligned}$$

For α_n, $n \geq 1$ if $x_i' = hx_i$, $i = 0, \ldots, n-1$ then

$$\begin{aligned}
&x_n^{-1}gx_0 \hat{\otimes} \cdots \hat{\otimes} x_{i-1}^{-1}hx_i \hat{\otimes} x_i^{-1}h^{-1}x_{i+1} \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n \\
&\quad = x_n^{-1}gx_0 \hat{\otimes} \cdots \hat{\otimes} x_{i-1}^{-1}hx_ix_i^{-1}h^{-1}x_i \hat{\otimes} x_i^{-1}x_{i+1} \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n
\end{aligned}$$

with an obvious modification for $i = 0$. For $i = n$, we have

$$\begin{aligned}
&x_n^{-1}h^{-1}gx_0 \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}hx_n \\
&\quad = x_n^{-1}h^{-1}gx_0 \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n(x_n^{-1}hx_n) \\
&\quad = (x_n^{-1}hx_n)x_n^{-1}h^{-1}gx_0 \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n \\
&\quad = x_n^{-1}gx_0 \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n.
\end{aligned}$$

(c) The map is linear in both variables and since

$$(\bar{g}\rho, (\tilde{x}_0, \ldots, \tilde{x}_n)) \to x_n^{-1}\rho^{-1}g\rho x_0 \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1}x_n$$

as does $(\bar{g}, \rho(\tilde{x}_0, \ldots, \tilde{x}_n))$ we obtain a well defined k-linear map

$$\alpha_* : \Lambda_H \otimes_{G/H} S_*(G/H) \to Z_*(kG, kH).$$

(ii) To show α_* is an isomorphism of k-modules one constructs an inverse as follows. Define a map

$$\beta_* : Z_*(kG, kH) \to \Lambda_H \otimes_{G/H} S_*(G/H)$$

by

$$\beta_0(g \bmod [kG, kH]) = \bar{g} \otimes_{G/H} (\tilde{1}) \qquad \text{for } g \in G,$$

$$\beta_n(y_0 \hat{\otimes} \cdots \hat{\otimes} y_n) = \overline{y_1 \cdots y_n y_0} \otimes_{G/H} (\tilde{1}, \tilde{y}_1, \tilde{y}_1\tilde{y}_2, \ldots, \tilde{y}_1 \cdots \tilde{y}_n), \qquad n \geq 1.$$

This is well defined. For β_0 because $gh - hg \to (\overline{gh} - \overline{hg}) \otimes (\tilde{1}) = \overline{gh} \otimes (\tilde{1}) - \overline{hg} \otimes (\tilde{1}) = 0$ since $gh \equiv_H hg$. For β_n, let y_i be replaced by hy_i for $i = 0, \ldots, n$. Then

$$\beta_n(y_0 \hat{\otimes} \cdots \hat{\otimes} hy_i \hat{\otimes} \cdots \hat{\otimes} y_n) =$$

$$\overline{y_1 \cdots (hy_i) \cdots y_n y_0} \otimes_{G/H} (\tilde{1}, \tilde{y}_1, \tilde{y}_1\tilde{y}_2, \ldots, \tilde{y}_1 \cdots \tilde{h}\tilde{y}_i, \ldots, \tilde{y}_1 \cdots \tilde{h}\tilde{y}_i \cdots \tilde{y}_n),$$

$$\beta_n(y_0 \hat{\otimes} \cdots \hat{\otimes} y_{i-1} h \hat{\otimes} \cdots \hat{\otimes} y_n) =$$

$$\overline{y_1 \cdots (y_{i-1} h) \cdots y_n y_0} \otimes_{G/H} (\tilde{1}, \tilde{y}_1, \tilde{y}_1\tilde{y}_2, \ldots, \tilde{y}_1 \cdots \tilde{y}_{i-1}\tilde{h} \cdots, \ldots, \tilde{y}_1 \cdots \tilde{y}_n).$$

The terms to the right of the tensor product sign are equal since these are elements of the coset space G/H and not G, while the terms to the left of the tensor product are equal for $i \neq 1$ by associativity of the product in G and for $i = 1$ since $hy_1 \cdots y_0 \equiv_H y_1 \cdots y_0 h$.

Both compositions are the identity. For $n = 0$, $\beta_0\alpha_0(\bar{g} \otimes (\tilde{x}_0)) = \beta_0(x_0^{-1} g x_0 \bmod [kG, kH]) = \overline{x_0^{-1} g x_0} \otimes (\tilde{1}) = \bar{g} \otimes (\tilde{x}_0)$, while $\alpha_0\beta_0(g \bmod [kG, kH]) = \alpha_0(\bar{g} \otimes (\tilde{1})) = g \bmod [kG, kH]$. For $n > 0$,

$$\begin{aligned}
\beta_n\alpha_n(\bar{g} \otimes (\tilde{x}_0, \ldots, \tilde{x}_n)) &= \beta_n(x_n^{-1} g x_0 \hat{\otimes} x_0^{-1} x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1} x_n) \\
&= \overline{x_0^{-1} g x_0} \otimes (\tilde{1}, \tilde{x}_0^{-1}\tilde{x}_1, \tilde{x}_0^{-1}\tilde{x}_2, \ldots, \tilde{x}_0^{-1}\tilde{x}_n) \\
&= \bar{g} \otimes (\tilde{x}_0, \ldots, \tilde{x}_n),
\end{aligned}$$

$$\begin{aligned}
\alpha_n\beta_n(y_0 \hat{\otimes} \cdots \hat{\otimes} y_n) &= \alpha_n(\overline{y_1 \cdots y_n y_0} \otimes_{G/H} (\tilde{1}, \tilde{y}_1, \tilde{y}_1\tilde{y}_2, \ldots, \tilde{y}_1 \cdots \tilde{y}_n) \\
&= (y_1 \cdots y_n)^{-1} y_1 \cdots y_n y_0 \hat{\otimes} y_1 \hat{\otimes} \cdots \hat{\otimes} y_n \\
&= y_0 \hat{\otimes} \cdots \hat{\otimes} y_n.
\end{aligned}$$

(iii) α_* is an isomorphism of cyclic k-modules. The calculation that α_* commutes with d_i and s_i is immediate as is the calculation that α_* commutes with t_n which we give anyway.

$$\begin{aligned} t_n\alpha_n(\bar{g} \otimes (\tilde{x}_0, \ldots, \tilde{x}_n) &= t_n(x_n^{-1}gx_0 \hat{\otimes} x_0^{-1}x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1} x_n) \\ &= x_{n-1}^{-1}x_n \hat{\otimes} x_n^{-1} gx_0 \hat{\otimes} x_0^{-1} x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-2}^{-1} x_{n-1}, \end{aligned}$$

while

$$\begin{aligned} \alpha_n\tau_n(\bar{g} \otimes (\tilde{x}_0, \ldots, \tilde{x}_n) &= \alpha_n(\bar{g} \otimes (\tilde{g}^{-1}\tilde{x}_n, \tilde{x}_0, \ldots, \tilde{x}_{n-1}) \\ &= x_{n-1}^{-1}x_n \hat{\otimes} x_n^{-1} gx_0 \hat{\otimes} x_0^{-1} x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-2}^{-1} x_{n-1}. \quad \square \end{aligned}$$

$S_*(kG, kH)$ is clearly functorial in mappings of pairs of groups (G, H) with H normal in G and it is obvious α_* is a natural equivalence of the functors $S_*(kG, kH)$ and $Z_*(kG, kH)$.

3. The cyclic homology of the complex $S_*(kG, kH)$

In this section we more or less follow Marciniak's algebraic proof [9] of Burghelea's calculation [4] of the cyclic homology of kG keeping track of the modifications demanded in the relative case.

Let $T(G)$ denote the G-conjugacy classes of G. For $c \in T(G)$, let Λ_c denote the k-submodule of Λ_H generated by $[g]$ for $g \in c$. It is obvious Λ_c is a G/H submodule of Λ_H and $\Lambda_H \cong \Sigma\, \Lambda_c$ as $k(G/H)$-modules. Moreover it is clear from the definitions that d_i, s_i and τ_n respect this decomposition, i.e., as cyclic k-modules

$$S_*(kG, kH) \cong \sum_{T(G)} \Lambda_c \hat{\otimes}_{G/H} S_*(G/H)$$

where the maps defining the cyclic structure on the right hand side of the equation are given by the same formulas as on the left. Again this isomorphism is functorial in the pair (G, H).

Let Γ be an arbitrary group and $\gamma \in \Gamma$ a central element. Define a cyclic k-module, $\mathscr{X}_*(\Gamma, \gamma)$ by $\mathscr{X}_n(\Gamma, \gamma) = k \otimes_\Gamma S_n(\Gamma)$ with d_i and s_i induced from $S_*(\Gamma)$ and

$$\tau_n(1 \otimes (\gamma_0, \gamma_1, \ldots, \gamma_n)) = 1 \otimes (\gamma^{-1}\gamma_n, \gamma_0, \ldots, \gamma_{n-1}) .$$

The proof this is a cyclic k-module is immediate and clearly this construction is functorial in pairs (Γ, γ) with γ central in Γ.

For each $c \in T(G)$ choose $z \in c$ and let Stab $(\bar{z})$ denote the isotropy group of $\bar{z} \in (G)_H$ contained in $\tilde{G} = G/H$. Since there is a bijection of the right coset space Stab $(\bar{z})\backslash\tilde{G} \to c$ induced by $\tilde{g} \to \bar{z}^g$, it follows immediately that this map induces an isomorphism

$$k(\mathrm{Stab}\,(\bar{z})\backslash\tilde{G}) \to \Lambda_c$$

of right G/H-modules. Since $k \,\hat{\otimes}_{\mathrm{Stab}\,(z)}\, \mathrm{k}(\tilde{\mathrm{G}}) \cong \mathrm{k}(\mathrm{Stab}\,(\bar{z})\backslash\tilde{G})$ as right $\tilde{G}$-modules via the map $1 \otimes \tilde{g} \to (\mathrm{Stab}\,(\bar{z}))\tilde{g}$, we obtain for each n an isomorphism

$$(k \otimes_{\mathrm{Stab}\,(z)} k(G/H)) \otimes_{G/H} S_n(G/H) \to \Lambda_c \otimes_{G/H} S_n(G/H).$$

Since the left hand side is isomorphic to $k \,\hat{\otimes}_{\mathrm{Stab}\,(z)} S_n(G/H)$ we have an isomorphism of k-modules

$$k \otimes_{\mathrm{Stab}\,(z)} S_n(G/H) \to \Lambda_c \otimes_{G/H} S_n(G/H)$$

given by $1 \otimes_{\mathrm{Stab}\,(z)} (\tilde{x}_0, \ldots, \tilde{x}_n) \to \bar{z} \otimes_{G/H} (\tilde{x}_0, \ldots, \tilde{x}_n)$. If one defines a cyclic structure on $k \otimes_{\mathrm{Stab}\,(z)} S_n(G/H)$ by inducing the simplicial structure from $S_n(G/H)$ and the cyclic map being defined to be

$$\tau_n(1 \otimes_{\mathrm{Stab}\,(z)} (\tilde{x}_0, \ldots, \tilde{x}_n)) = 1 \otimes (\bar{z}^{-1}\tilde{x}_n, \tilde{x}_0, \ldots, \tilde{x}_{n-1}),$$

one sees immediately that the above map is an isomorphism of cyclic k-modules.

Consider the map of cyclic k-modules

$$\rho : \mathscr{X}_n(\mathrm{Stab}\,(\bar{z}), \bar{z}) = k \otimes_{\mathrm{Stab}\,(z)} S_n(\mathrm{Stab}\,(\bar{z})) \to k \otimes_{\mathrm{Stab}(z)} S_n(G/H)$$

induced by the inclusion of Stab $(\bar{z}) \to G/H$. We wish to show ρ induces an isomorphism on cyclic homology and this will follow from the following observation.

Observation. One knows that if one has a map of filtered differential complexes which incudes an isomorphism on any level of the associated spectral sequences then it induces an isomorphism on homology. In particular by applying this remark to the vertical filtration on the associated Tsygan complexes of two cyclic sets one concludes that a map of cyclic sets inducing an isomorphism in Hochschild homology induces an isomorphism in cyclic homology. (This follows immediately from the natural Connes sequence relating Hochshild and cyclic homology. However it is more useful in this follow as one can apply it also to the associated horizontal filtration of the Tsygan complex as in [9].)

PROPOSITION 1. $\rho : \mathscr{X}_n(\text{Stab}(\bar{z}), \bar{z}) \to k \otimes_{\text{Stab}(z)} S_n(G/H)$ *induces an isomorphism in cyclic homology.*

Proof. Since both $S_*(\text{Stab}(\bar{z}))$ and $S_*(G/H)$ are $k(\text{Stab}(\bar{z}))$-projective resolutions of k, the Hochschild homology of both sides is $H_*(\text{Stab}(\bar{z}), k)$, and ρ induces an isomorphism on Hochschild homology since the inclusion map induces a chain lift of the id_k. □

PROPOSITION 2. *Let* $z \in G$, *then* $\text{Stab}(\bar{z}) = C_G(z)H/H \subseteq G/H$, *where* $C_G(z)$ *denotes the centralizer of* z *in* G.

Proof. Immediate. □

Combining the above maps we obtain the following. Let $\{z\}$ be a set of representatives of the G-conjugacy classes of G. For each $z \in \{z\}$ we have a map of cyclic k-modules

$$\rho_z : \mathscr{X}_*(C_G(z)H/H, Hz) \to Z_*(kG, kH)$$

given by $1 \otimes (\tilde{x}_0, \dots, \tilde{x}_n) \to x_n^{-1} z x_0 \hat{\otimes} x_0^{-1} x_1 \hat{\otimes} \cdots \hat{\otimes} x_{n-1}^{-1} x_n$.

The above results give

PROPOSITION 3. *Let H be normal in G, then the map*

$$\bigoplus \rho_z : \sum \mathscr{X}_*(C_G(z)H/H, Hz) \to Z_*(kG, kH)$$

induces an isomorphism on cyclic homology.

Remarks on functoriality. It is clear that if $f : (G, H) \to (G', H')$, then

$$\begin{array}{ccc}
\mathscr{X}_*(C_G(z)H/H, Hz) & \xrightarrow{\rho_z} & Z_*(kG, kH) \\
\downarrow \bar{f} & & \downarrow f \\
\mathscr{X}_*(C_G(fz)H'/H, H'fz) & \xrightarrow{\rho_{fz}} & Z_*(kG', kH')
\end{array}$$ commutes,

where the map f naturally induces both a map $\bar{f} : G/H \to G'/H'$ and a map $\bar{f} : (G)_H \to (G')_{H'}$ which is $\bar{f}$ equivariant. Hence we obtain a map $\bar{f} : \text{Stab}(\bar{z}) = C_G(z)H/H \to \text{Stab}(\overline{fz}) = C_{G'}(fz)H'/H'$ inducing the $\bar{f}$ above. Unfortunately summing over the G-conjugacy classes is not possible as the map $\bar{f}$ is in general not one-to-one. However if we fix the group G *and* a set of representatives $\{z\}$ of the conjugacy classes of G we obtain a well defined natural transformation $\bigoplus \rho_z$ of the functors $\mathscr{X}_*(N) = \Sigma\, \mathscr{X}_*(C_G(z)N/N, Nz)$ and $Z_*(N) = Z_*(kG, kN)$ defined on the category of normal subgroups of G and inclusion maps and which induces isomorphisms in cyclic homology.

We now have all the ingredients for the following

THEOREM. *Let N be normal in G and let k be a field of characteristic zero, then*

$$HC_*(kG, kN) \cong \bigoplus_{c \in T^0(N)} H_*(G_c, k) \otimes HC_*(k) \oplus \bigoplus_{c \in T^\infty(N)} H_*(G_c, k)$$

where $T^0(N)$ (resp. $T^\infty(N)$) $=$ G-conjugacy classes $[z]$ of G such that Nz is of finite (resp. infinite) order in G/N, and $G_c = C_G(z)N/(z)N \cong$ Stab $(\bar{z})/(Nz)$.

Proof. We have seen we can calculate $HC_*(kG, kN)$ from the direct sum over the G-conjucacy classes of the cyclic k-modules $\mathscr{X}_*(C_G(z)N/N, Nz)$. We can now compute the cyclic homology of these cyclic sets as in Marciniak [9] or Burghelea [4]. □

4. Applications to the traces of projective modules

Let A be a k-algebra. The Chern character is a natural transformation $ch_n : K_n(A) \to HC_n(A)$ which in dimension zero coincides with the Stong-Hattori trace for finitely generated projective A-modules. Karoubi [8] has produced a lifting of ch_n, $ch_n^i : K_n(A) \to HC_{n+2i}(A)$ commuting with the natural map $S : HC_{n+2}(A) \to HC_n(A)$. Let S be a k-subalgebra of A containing $k1_A$. We have a natural map $HC_*(A) \to HC_*(A, S)$ and we have the

PROPOSITION. *The natural map $HC_0(A) \to HC_0(A, S)$ is an isomorphism.*

Proof. It is immediate from the definitions that both sides are $A/[A, A]$ and the induced map is induced from the identity of A. □

Consider the following commutative diagram.

$$\begin{tikzcd} & HC_{2n}(A) \arrow[r] \arrow[d, "S^n"] & HC_{2n}(A, S) \arrow[d, "S^n"] \\ K_0(A) \arrow[ur, "ch_0^n"] \arrow[r, "ch_0"] & HC_0(A) \arrow[r, "\cong"] & HC_0(A, S). \end{tikzcd}$$

Letting (A, S) be (kG, kN) for N a normal subgroup of G and using the above theorem computing the cyclic homology of the pair (kG, kN) one sees if one has vanishing theorems for some components of $HC_*(kG, kN)$ one would obtain vanishing theorems for traces of finitely generated projective kG-modules on certain conjugacy classes. To this end we recall a theorem of Eckmann, used for the same purpose in the non relative case. Recall $hd_Q(G) = \sup \{k \mid H_k(G, A) \neq 0$ for some QG-module $A\}$.

THEOREM (Eckmann [5]). *Let G have $hd_Q(G) = n < \infty$, and suppose G belongs to one of the following classes*: (a) *nilpotent groups*; (b) *torsion free solvable groups*; (c) *linear groups, i.e., subgroups of $GL_n(F)$ where F is a field of characteristic zero*; (d) *groups of cohomological dimension ≤ 2. Then if x is a central element of infinite order, $H_j(G/(x), Q) = 0$ for $j \geq n$ ($n = 2$ in case d). Using this result we have the immediate*

THEOREM. *Let $z \in G$. Suppose $\exists N$ normal in G such that* (i) *Nz is of infinite order in G/N,* (ii) *$hd_Q(G/N) < \infty$,* (iii) *G/N is one of the types* (a)–(d), *then if P is any finitely generated projective QG-module $r(P)_z = 0$.*

Remark. By applying Eckmann's result directly, i.e., the above result in the case $N = (1)$, one only obtains $\Sigma_{Ny = Nz}\, r(P)_y = 0$.

COROLLARY 1. *Let G_n be the nth term of the lower central series for G. Suppose z has an infinite order in G/G_n, then for any finitely generated projective QG-module P, $r(P)_z = 0$.*

Proof. Since G/G_n is nilpotent, any finitely generated subgroup is polycylic and hence has finite Hirsch number. But, on the class of solvable groups, $hd_Q(G)$ equals the Hirsch number [11] and therefore since homology commutes with direct limits and any group is the direct limit of its finitely generated subgroups we have $hd_Q(G/G_n) < \infty$. Hence the result. □

It is amusing that while the corollary to the last theorem says something about nilpotent quotients of G, the following result says something about nilpotent subgroups of G.

COROLLARY 2. *Suppose G is a split extension of a finitely generated torsion free nilpotent group N and an arbitrary group A. If $1 \neq_t z \in N$ and P is any finitely generated projective QG-module, then $r(P)_z = 0$.*

Proof. N has an embedding in $GL_n(Z)$ as a group of unipotent matrices ([13], p. 23). By a result of Swan [12] (see [13], p. 22), there exists $\varphi : G \to GL_m(Z)$ such that $\ker(\varphi) \cap N = (1)$. Hence z has infinite order in $G/\ker\varphi \subseteq GL_m(Z)$. Any unipotent subgroup of $GL_m(Z)$ is conjugate in $GL_m(Q)$ to a subgroup of the group of upper triangular matrices with diagonal entries equal to one and such a group has cohomological dimension $\leq n(n-1)$. By a result of Alperin and Shalen [1] $G/\ker\varphi$ has finite virtual cohomological dimension and hence $hd_Q(G/\ker\varphi) < \infty$. Hence $G/\ker\varphi$ is a linear group with $hd_Q < \infty$ and the result follows from the theorem.

□

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Department of Mathematics
University of Maryland 20740
College Park, Maryland
USA

Received July 19, 1990