Locally toroidal regular polytopes of rank 4.

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Locally toroidal regular polytopes of rank 4

PETER MCMULLEN AND EGON SCHULTE*

Abstract. The paper studies various relationships between locally toroidal regular 4-polytopes of types $\{6, 3, p\}$ and $\{3, 6, 3\}$. These relationships are based on corresponding relationships between the regular honeycombs with the same Schläfli-symbol in hyperbolic 3-space. Also the paper discusses regular tessellations (sections of rank 3) which are locally inscribed into regular 4-polytopes. In particular, this leads to local criteria for the finiteness of the polytopes.

1. Introduction

The study of regular polytopes has a long history (cf. Coxeter [7]). The classical regular polytopes make their appearance in many different branches of mathematics, ranging from Lie algebras to Tits buildings. Traditionally, a polytope is topologically a sphere, and also has spherical facets and spherical vertex-figures; in other words, it is locally and globally of spherical type.

In recent years the classical notion of a regular polytope has been generalized to abstract regular polytopes. Abstract regular polytopes are combinatorial and geometrical structures which resemble the classical regular polytopes but are not necessarily of spherical type, neither locally nor globally (Danzer–Schulte [12], McMullen–Schulte [19–23, 26]). For related notions see also McMullen [17], Grünbaum [16], Dress [13], Buekenhout [1] and Tits [34, 35].

For a locally toroidal regular polytope $\mathscr{P}$ of rank 4, the facets and vertex-figures are spherical or toroidal regular maps (tessellations), but are not all spherical. The corresponding Schläfli-types are $\{4, 4, 3\}$, $\{4, 4, 4\}$, $\{3, 6, 3\}$, $\{6, 3, p\}$ with $3 \leq p \leq 6$, and their duals. For the types $\{4, 4, 3\}$ and $\{6, 3, p\}$ a complete classification of the finite universal polytopes $\mathscr{P}$ was obtained in [23] and [22], respectively. An almost complete classification is known for the type $\{4, 4, 4\}$ [23]. For the type $\{3, 6, 3\}$ only a few classes have been settled so far ([22], Weiss [36]).

The situation is particularly satisfactory for the types $\{6, 3, p\}$ (and the known cases of $\{3, 6, 3\}$). Here, the structure of the polytopes $\mathscr{P}$ is governed by a complex hermitian form. In particular, the polytope is finite if and only if the corresponding form is positive definite. This generalizes the well-known classical situation where a real quadratic form (associated with the underlying Coxeter group) determines the structure of the polytope (cf. [7]).

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In this paper we discuss various relationships between the locally toroidal regular 4-polytopes of types $\{6, 3, p\}$ and $\{3, 6, 3\}$. As a by-product, this also leads to some new classification results. The relationships between the polytopes are based on corresponding relationships between the regular honeycombs with the same Schläfli-symbol in hyperbolic 3-space H^3. These results are described in Sections 3 and 4; basic definitions are given in Section 2.

In Section 5 we associate with certain polytopes $\mathscr{P}$ a regular tessellation $\mathscr{H}$ which in a sense cuts right through the polytope. In particular, this gives local criteria for the finiteness of the polytopes. In some cases the geometrical representation of [22] for the group of $\mathscr{P}$ can be used to find the structure of $\mathscr{H}$ explicitly.

Finally, Section 6 deals with flat polytopes. For the discussion of locally and globally toroidal regular polytopes of higher rank (≥ 5) the reader is referred to [25].

2. Basic notions

For a detailed introduction the reader is referred to [12, 20–22]. An (*abstract*) *polytope $\mathscr{P}$ of rank n*, or briefly an *n-polytope*, is a partially ordered set with a strictly monotone rank function with range $\{-1, 0, \ldots, n\}$. The elements of rank i are called the *i-faces* of $\mathscr{P}$, or *vertices* or *facets* of $\mathscr{P}$ if $i = 0$ or $n-1$, respectively. The *flags* (maximal totally ordered subsets) of $\mathscr{P}$ all contain exactly $n+2$ faces, including the unique (least) (-1)-face F_{-1} and the unique (greatest) n-face F_n of $\mathscr{P}$. Further defining properties of $\mathscr{P}$ are the (*global* and *local*) *flag-connectedness* as well as the homogeneity property that, for any $(i-1)$-face F and any $(i+1)$-face G with $F < G$, there are exactly *two* i-faces H of $\mathscr{P}$ such that $F < H < G$ $(i = 0, \ldots, n-1)$.

For two faces F and G with $F \leq G$ we call $G/F := \{H \mid F \leq H \leq G\}$ a *section* of $\mathscr{P}$. There is little possibility of confusion if we identify a face F with the section F/F_{-1}. We call F_n/F the *co-face* (*of F_n*) *at* F, or the *vertex-figure at* F if F is a vertex.

An n-polytope $\mathscr{P}$ is *regular* if its automorphism group $A(\mathscr{P})$ is flag-transitive. The group of a regular polytope $\mathscr{P}$ is generated by *distinguished generators* $\rho_0, \ldots, \rho_{n-1}$, where ρ_i is the unique automorphism which keeps fixed all but the i-face of some base flag $\Phi = \{F_{-1}, F_0, \ldots, F_n\}$ of $\mathscr{P}$. If $\{p_1, \ldots, p_{n-1}\}$ is the (Schläfli-)*type* of $\mathscr{P}$, then these generators satisfy the relations

$$(\rho_i\rho_j)^{p_{ij}} = 1 \qquad (i, j = 0, \ldots, n-1) \tag{1}$$

where $p_{ii} = 1$, $p_{ij} = p_{ji} =: p_{i+1}$ if $j = i+1$, and $p_{ij} = 2$ if $|i-j| \geq 2$. Further, $A(\mathscr{P})$

has the *intersection property*

$$\langle \rho_i \mid i \in I \rangle \cap \langle \rho_i \mid i \in J \rangle = \langle \rho_i \mid i \in I \cap J \rangle \qquad \text{for } I, J \subset \{0, \ldots, n-1\}. \tag{2}$$

By a *C-group* we mean a group which is generated by involutions such that (1) and (2) hold. The C-groups are precisely the groups of abstract regular polytopes [19, 28].

Given regular n-polytopes $\mathscr{P}_1$ and $\mathscr{P}_2$ such that the vertex-figures of $\mathscr{P}_1$ are isomorphic to the facets of $\mathscr{P}_2$, we denote by $\langle \mathscr{P}_1, \mathscr{P}_2 \rangle$ the class of all regular $(n+1)$-polytopes $\mathscr{P}$ with facets isomorphic to $\mathscr{P}_1$ and vertex-figures isomorphic to $\mathscr{P}_2$. Each non-empty class $\langle \mathscr{P}_1, \mathscr{P}_2 \rangle$ contains a *universal polytope* denoted by $\{\mathscr{P}_1, \mathscr{P}_2\}$; all polytopes in $\langle \mathscr{P}_1, \mathscr{P}_2 \rangle$ are obtained from $\{\mathscr{P}_1, \mathscr{P}_2\}$ by identifications [30].

A regular polytope $\mathscr{P}$ of rank 4 is said to be *locally of genus* (*at most*) g if its facets and its vertex-figures are regular maps on orientable surfaces of genus at most g, and either its facets or its vertex-figures are actually maps of genus g. We call $\mathscr{P}$ *locally toroidal* if it is locally of genus 1. Note that locally toroidal regular 4-polytopes are necessarily of type $\{3, 4, 4\}$, $\{4, 4, 3\}$, $\{4, 4, 4\}$, $\{3, 6, 3\}$, $\{6, 3, p\}$ or $\{p, 3, 6\}$ with $p = 3$, 4, 5 or 6 (provided all entries in the symbol are at least 3). See [29, 37] for early examples of such polytopes. For examples of higher genus see also [27].

Toroidal regular maps are discussed in [9]. However, in this paper we shall change the notation of [9] and denote the torus maps $\{4, 4\}_{s,t}$, $\{3, 6\}_{s,t}$ and $\{6, 3\}_{s,t}$ by the slightly more complicated symbols $\{4, 4\}_{(s,t)}$, $\{3, 6\}_{(s,t)}$ and $\{6, 3\}_{(s,t)}$, respectively. This change of notation is explained in [25] and is motivated by results on the classification of the globally toroidal (and locally spherical) regular polytopes of higher ranks. We shall often use the fact that for the maps $\{3, 6\}_{(s,t)}$ the extra relation for the generators ρ_i is

$$\left.\begin{array}{ll} (\rho_0(\rho_1\rho_2)^2)^{2s} = 1 & \text{if } s = t \geq 1; \\ (\rho_0\rho_1\rho_2)^{2s} = 1 & \text{if } s \geq 1, t = 0 \end{array}\right\} \tag{3}$$

(cf. [9], p. 108).

Recall that a *Petrie polygon* of a regular map $\mathscr{M}$ is a zig-zag along the edges such that each 2, but no 3, consecutive edges lie in a face, that is, 2-face [9]. A k-*chain* of $\mathscr{M}$ is a path along edges which leaves, at each vertex, k faces to the right [3]. The lengths of the Petrie polygons and k-chains of $\mathscr{M}$ are the periods of the elements $\rho_0\rho_1\rho_2$ and $\rho_0\rho_1(\rho_2\rho_1)^{k-1}$ in $A(\mathscr{P})$, respectively.

In some instances we shall construct polytopes from *mixing operations* on groups W which are generated by involutions $\sigma_0, \ldots, \sigma_{m-1}$ [19]. Then we derive new groups A by taking as generators $\rho_0, \ldots, \rho_{n-1}$ suitably chosen products of the σ_i's, so that A becomes a subgroup of W. We denote such a mixing operation by

$$\mu : (\sigma_0, \ldots, \sigma_{m-1}) \mapsto (\rho_0, \ldots, \rho_{n-1}).$$

If W is the group of a regular polytope $\mathscr{P}$ and $A = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is a C-group, then we write $\mathscr{P}^\mu$ for the regular polytope whose group is A.

An example of a mixing operation is the *facetting operation*

$$\mu = \varphi_2 : (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0, \sigma_1\sigma_2\sigma_1, \sigma_2) =: (\rho_0, \rho_1, \rho_2) \tag{4}$$

on the group $A(\mathscr{P}) = \langle \sigma_0, \sigma_1, \sigma_2 \rangle$ of a regular map $\mathscr{P}$ of type $\{p, q\}$ ([19], Section 4.2). The effect of φ_2 is best studied by employing Wythoff's construction to find $\mathscr{P}^{\varphi_2}$ [7]. If q is odd, then $A = A(\mathscr{P})$ and A is the group of a regular map $\mathscr{P}^{\varphi_2}$ of type $\{l, q\}$, with l equal to the length of the 2-chains of $\mathscr{P}$. If q is even, then in general A is a proper subgroup of $A(\mathscr{P})$.

If $\mathscr{P} = \{3, 6\}$ then $\mathscr{P}^{\varphi_2} = \{6, 3\}$ and $\mathscr{P}^{\varphi_2}$ takes only $\frac{2}{3}$ of the vertices of $\mathscr{P}$; see Figure 2 in Section 3. If $\mathscr{P} = \{3, 6\}_{(3r,0)}$, then $\mathscr{P}^{\varphi_2} = \{6, 3\}_{(r,r)}$ and $\mathscr{P}^{\varphi_2}$ takes again only $\frac{2}{3}$ of the vertices of $\mathscr{P}$. For $\mathscr{P} = \{3, 6\}_{(s,0)}$ with $3 \nmid s$ we have $\mathscr{P}^{\varphi_2} = \{6, 3\}_{(s,0)} = \mathscr{P}^*$ (the dual of $\mathscr{P}$) and $\mathscr{P}^{\varphi_2}$ covers the vertices of $\mathscr{P}$ twice. (Note that in [19] it was incorrectly remarked that for $(s, t) = (2, 0)$ the operation φ_2 does not give a map $\mathscr{P}^{\varphi_2}$.) Finally, if $\mathscr{P} = \{3, 6\}_{(s,s)}$, then $\mathscr{P}^{\varphi_2} = \{6, 3\}_{(s,0)}$ and $\mathscr{P}^{\varphi_2}$ takes only $\frac{2}{3}$ of the vertices of $\mathscr{P}$.

In Section 4 we shall also use the operation

$$\mu : (\sigma_0, \sigma_1, \sigma_2) \mapsto (\sigma_0\sigma_1\sigma_2\sigma_1\sigma_0, \sigma_2, \sigma_1) =: (\rho_0, \rho_1, \rho_2) \tag{5}$$

on the group $A(\mathscr{P}) = \langle \sigma_0, \sigma_1, \sigma_2 \rangle$ of a regular map $\mathscr{P}$ of type $\{3, 6\}$; see Figure 1. Again we can apply Wythoff's construction to find $\mathscr{P}^\mu$.

In particular, for $\mathscr{P} = \{3, 6\}$ we find $\mathscr{P}^\mu = \{3, 6\}$, with $A(\mathscr{P}^\mu) = \langle \rho_0, \rho_1, \rho_2 \rangle$ a subgroup of index 3 in $A(\mathscr{P})$; see Figure 4 in Section 3. It follows that $|A(\mathscr{P}) : A(\mathscr{P}^\mu)| = 1$ or 3 for any map $\mathscr{P}$ of type $\{3, 6\}$. If $s \geq 2$ and $\mathscr{P} = \{3, 6\}_{(s,s)}$, then $\mathscr{P}^\mu = \{3, 6\}_{(s,0)}$ and $\mathscr{P}^\mu$ takes only $\frac{1}{3}$ of the vertices of $\mathscr{P}$. For $\mathscr{P} = \{3, 6\}_{(1,1)}$ the operation (5) gives the map $\{3, 6\}_{(1,0)}$ which is not an abstract polytope. If $\mathscr{P} = \{3, 6\}_{(3r,0)}$, then $\mathscr{P}^\mu = \{3, 6\}_{(r,r)}$ and $\mathscr{P}^\mu$ takes again only $\frac{1}{3}$ of the vertices of $\mathscr{P}$. Finally, if $3 \nmid s$ and $\mathscr{P} = \{3, 6\}_{(s,0)}$, then $\mathscr{P}^\mu = \{3, 6\}_{(s,0)}$ and $\mathscr{P}^\mu$ is isomorphic to $\mathscr{P}$.

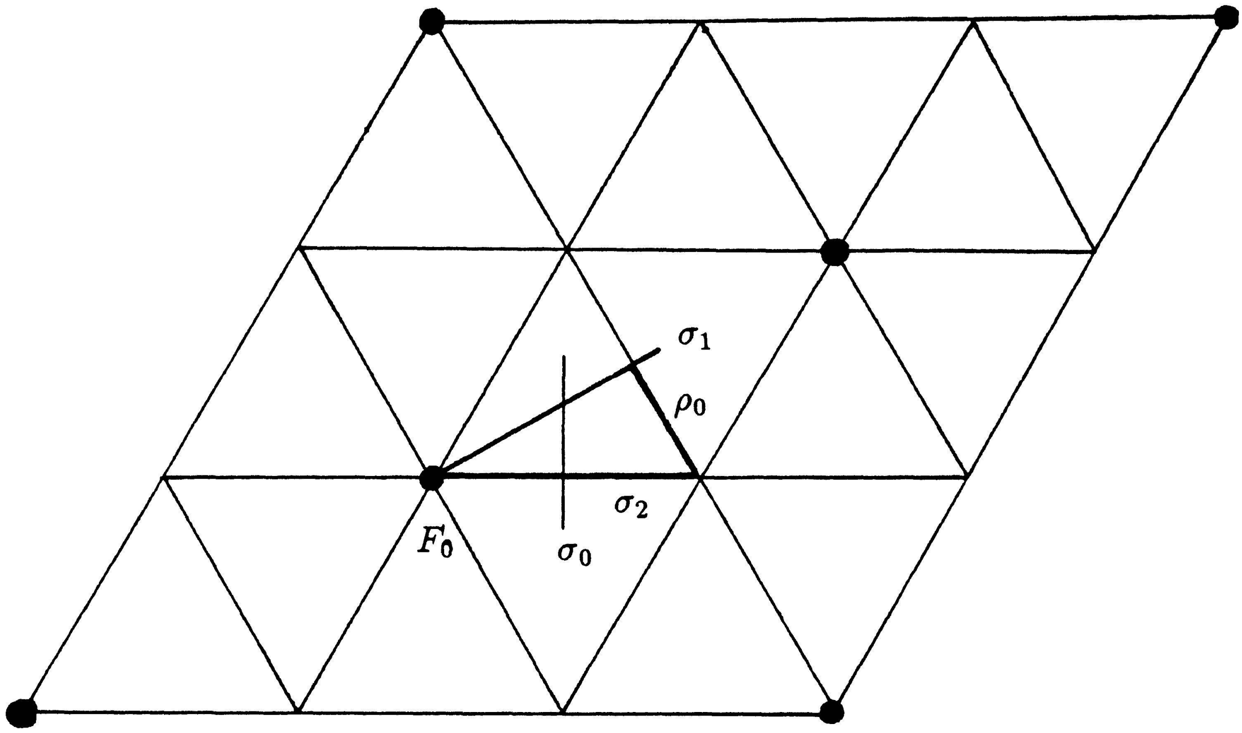


Figure 1

These facts follow easily from (3) and

$$\rho_0\rho_1\rho_2 = \sigma_0(\sigma_1\sigma_2)^2(\sigma_0\sigma_1) = \sigma_0(\sigma_1\sigma_2)^2\sigma_1\sigma_0\sigma_1\sigma_0 = \sigma_0\sigma_1(\sigma_2\sigma_1)^2\sigma_0\sigma_1\sigma_0 \sim (\sigma_2\sigma_1)^2\sigma_0$$

and

$$\rho_0(\rho_1\rho_2)^2 = (\sigma_0\sigma_1)\sigma_2\sigma_1\sigma_2(\sigma_0\sigma_1\sigma_2)\sigma_1 = \sigma_1(\sigma_0\sigma_1\sigma_0)\sigma_2\sigma_1\sigma_2(\sigma_0\sigma_1\sigma_2)\sigma_1 \sim (\sigma_0\sigma_1\sigma_2)^3.$$

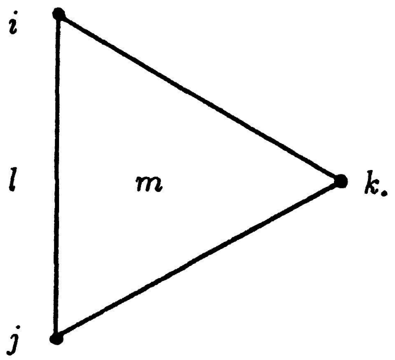
Here $\sim$ indicates conjugacy.

We shall also use *twisting operations* on groups $W = \langle\sigma_0, \ldots, \sigma_{m-1}\rangle$. If W admits involutory group automorphisms τ permuting the generators σ_i, then we can augment W by their addition to construct a semi-direct product A of W by the group generated by the τ's. In suitable cases A will be the group of a regular polytope. If $\tau_1, \ldots, \tau_k$ are the corresponding group automorphisms, we shall denote the twisting operation by

$$\kappa : (\sigma_0, \ldots, \sigma_{m-1}; \tau_1, \ldots, \tau_k) \mapsto (\rho_0, \ldots, \rho_{n-1}).$$

In our applications the groups W will be defined by diagrams $\mathscr{D}$ and the group automorphisms τ will correspond to symmetries of $\mathscr{D}$.

Following [4, 5] we denote by $[1\ 1\ 1^l]^m$ the group abstractly defined by the diagram



(6)

Here the mark m inside the triangle indicates that a set of defining relations is obtained by adding to the *standard relations* for σ_i, σ_j, σ_k (given by the underlying Coxeter diagram) the one extra relation

$$(\sigma_i \sigma_j \sigma_k \sigma_j)^m = 1, \tag{7}$$

or any of the six equivalent relations obtained by relabelling the nodes.

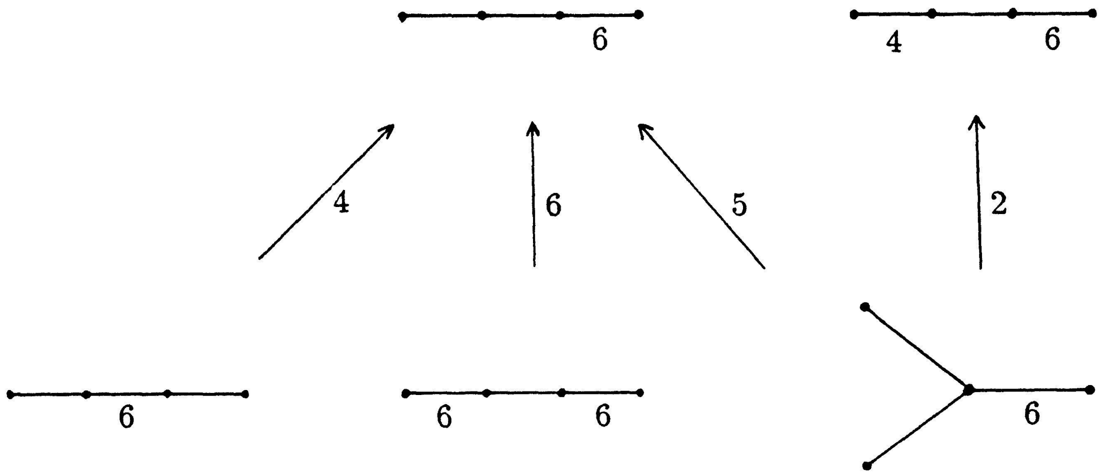
We shall also consider more general diagrams $\mathscr{D}$ consisting of a labelled simplicial 2-complex whose edge graph is a Coxeter diagram and all of whose triangles are marked by a number m indicating an extra defining relation of type (7). The corresponding (abstract) group is denoted by $W(\mathscr{D})$. Examples of such groups are the finite unitary reflexion groups $[p\,q\,r^l]^m$; see [2, 4, 31, 32]. More generally, if a group U (say) and its generators satisfy all defining relations of $W(\mathscr{D})$ but possibly other independent relations too, then U is said to *belong to the diagram* $\mathscr{D}$.

3. Subgroup relations for hyperbolic honeycombs

In Section 4 we shall discuss various relationships between locally toroidal regular 4-polytopes of types $\{3, 3, 6\}$, $\{4, 3, 6\}$, $\{6, 3, 6\}$ and $\{3, 6, 3\}$. These are based on relations between the symmetry groups of the corresponding hyperbolic honeycombs.

It is well-known that $\{3, 3, 6\}$, $\{4, 3, 6\}$, $\{6, 3, 6\}$ and $\{3, 6, 3\}$ are (the Schläfli-symbols of) four of the fifteen regular honeycombs in hyperbolic 3-space H^3 (cf. Coxeter [6, 10]). These four honeycombs have all their vertices at infinity (that is, on the absolute quadric). The two self-dual honeycombs $\{6, 3, 6\}$ and $\{3, 6, 3\}$ also have their facets inscribed in horospheres.

The symmetry group $[p, q, r]$ of the honeycomb $\{p, q, r\}$ is the Coxeter group with diagram $\bullet\underset{p}{\text{——}}\bullet\underset{q}{\text{——}}\bullet\underset{r}{\text{——}}\bullet$. In particular we have the following subgroup relationships:

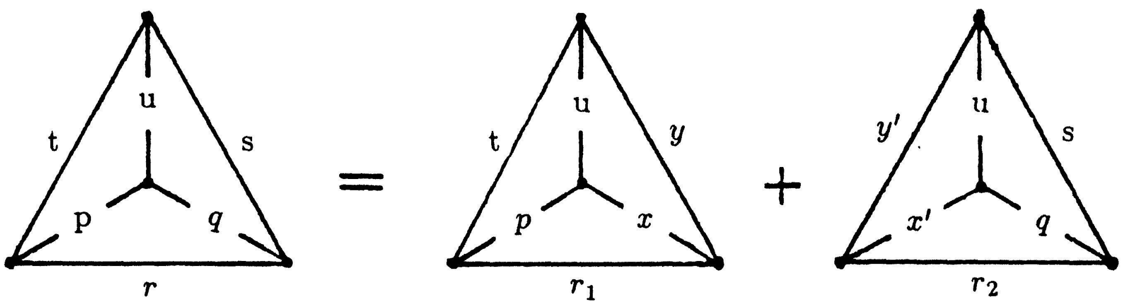


(8)

Here the subgroups are in the second row, with inclusion in the larger group of the first row as indicated; the extra mark indicates the index of the subgroup. Note that the diagram in the lower right corner does not belong to a regular honeycomb.

The subgroup relations in (8) can be obtained by simplex dissection of hyperbolic tetrahedra. If the index of the subgroup is k, then its fundamental tetrahedron is dissected into k congruent copies of the fundamental region for the larger group.

To describe these dissections we use (up to relabelling) the diagram notation of [7], p. 281. The tetrahedra are represented by graphs on 4 nodes. The nodes represent the 2-faces of the tetrahedron, and a branch marked (by an integer or a fraction) p indicates the dihedral angle π/p between two faces. As usual, an unmarked branch stands for a branch marked with $p = 3$, and branches with mark $p = 2$ are omitted. The rule for dissecting a tetrahedron (by dividing the angle π/r into two angles π/r_1 and π/r_2) is given by



(9)

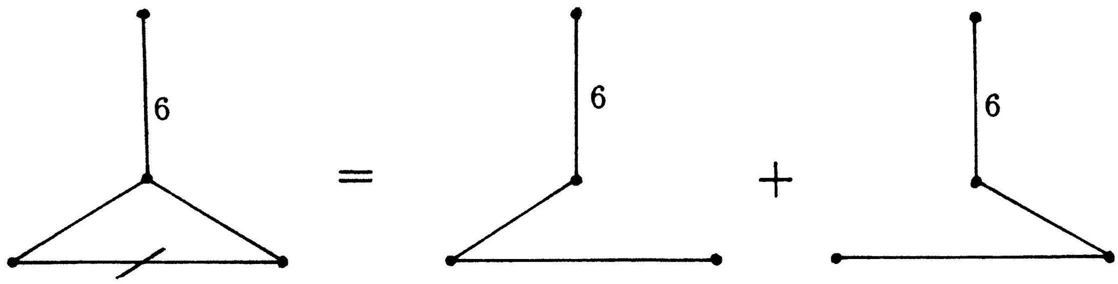
where

$$\frac{1}{r_1}+\frac{1}{r_2}=\frac{1}{r}, \qquad \frac{1}{x'}+\frac{1}{x}=1, \qquad \frac{1}{y'}+\frac{1}{y}=1,$$

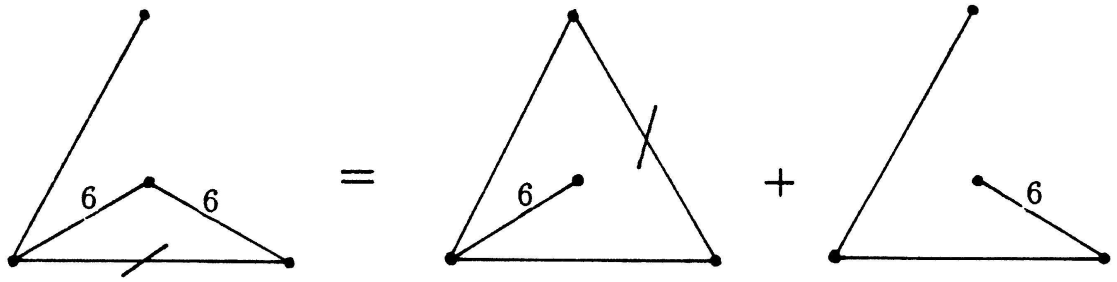
$$\cos\frac{\pi}{x}=\left(\cos\frac{\pi}{q}\sin\frac{\pi}{r_1}-\cos\frac{\pi}{p}\sin\frac{\pi}{r_2}\right)\Big/\sin\frac{\pi}{r}, \tag{10}$$

$$\cos\frac{\pi}{y}=\left(\cos\frac{\pi}{s}\sin\frac{\pi}{r_1}-\cos\frac{\pi}{t}\sin\frac{\pi}{r_2}\right)\Big/\sin\frac{\pi}{r}.$$

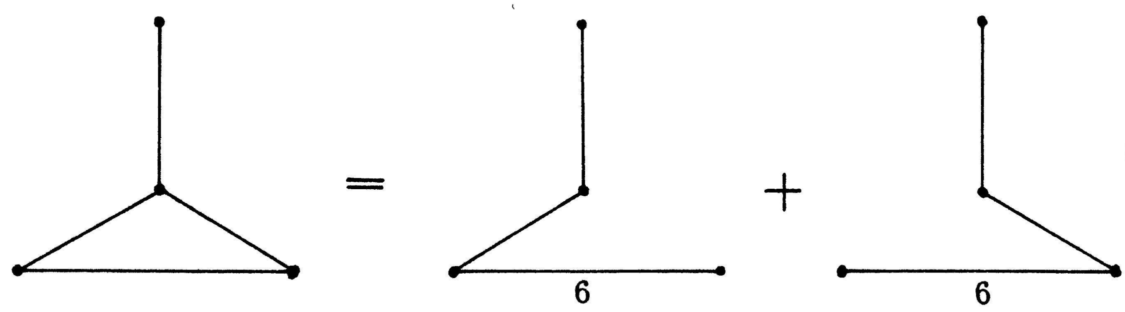
The subgroup relations in (8) are derived from the following applications of this rule. As above, the dihedral angle π/r which is dissected corresponds to the (possibly missing) horizontal branch in the diagram. For simplicity we write $\bullet\!\!-\!\!\!/\!\!\!-\!\!\bullet$ for $\bullet\underset{3/2}{\text{——}}\bullet$:



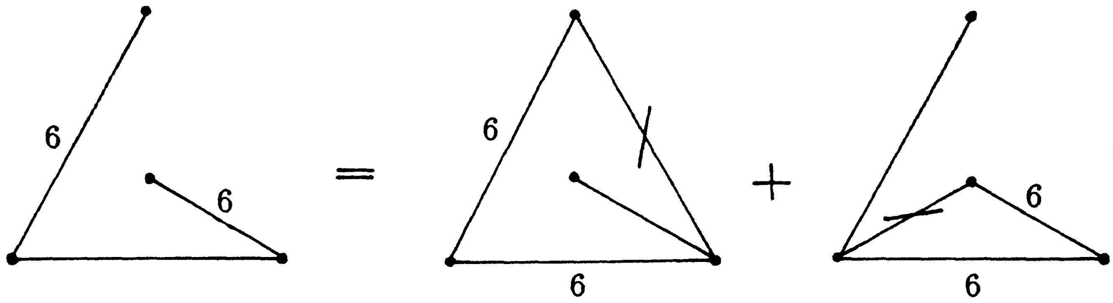
(11a)



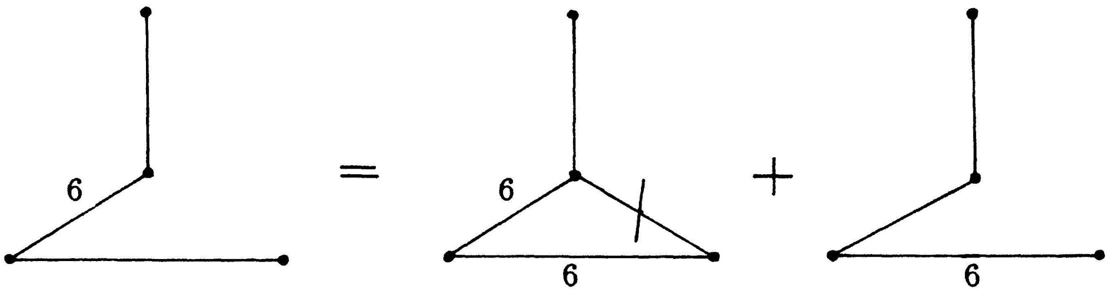
(11b)



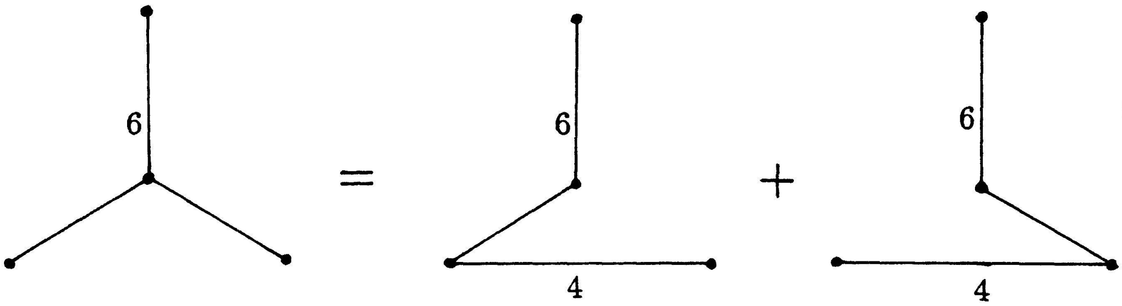
(11c)



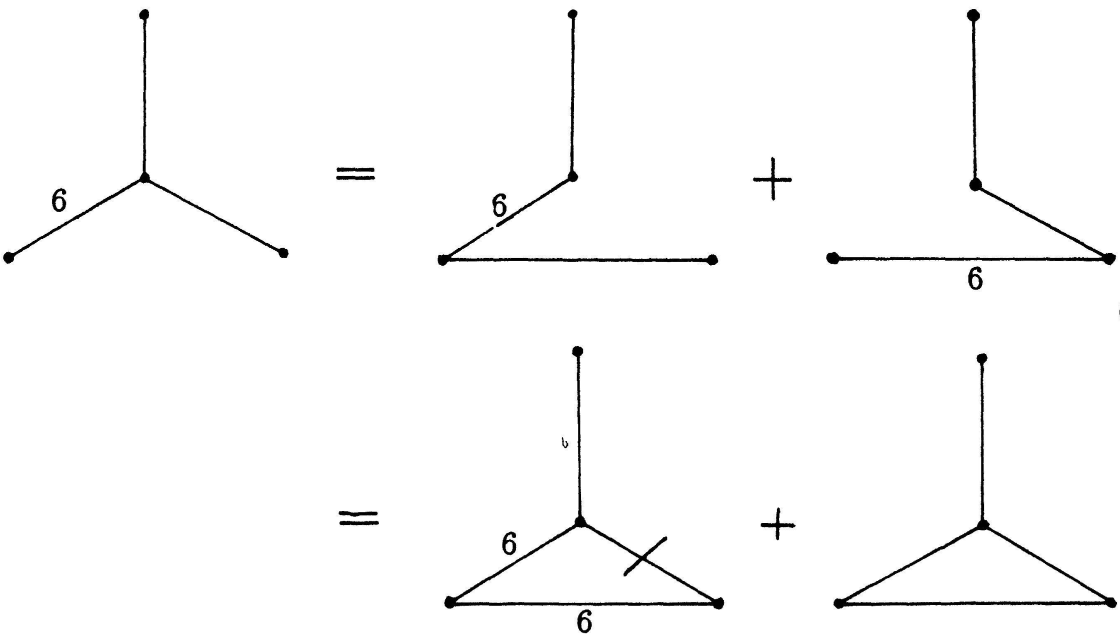
(11d)



(11e)



(11f)

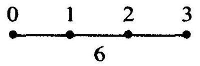


(11g)

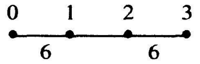
As an example, to find the index of [6, 3, 6] in [3, 3, 6] use (11d), (11b) and (11a), in this order.

Using these simplex dissections, it is easy to obtain generators for the appropriate subgroups. Let $[3, 3, 6] = \langle \sigma_0, \ldots, \sigma_3 \rangle$. Then we get generators $\rho_0, \ldots, \rho_3$ for the subgroups by the following operations:

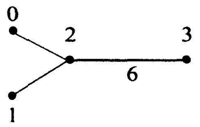
$$\alpha : (\sigma_0, \ldots, \sigma_3) \mapsto (\sigma_0, \sigma_1, \sigma_2\sigma_3\sigma_2, \sigma_3) =: (\rho_0, \ldots, \rho_3) \qquad (12)$$



$$\beta : (\sigma_0, \ldots, \sigma_3) \mapsto (\sigma_0, \sigma_1\sigma_2\sigma_3\sigma_2\sigma_1, \sigma_3, \sigma_2) =: (\rho_0, \ldots, \rho_3) \qquad (13)$$

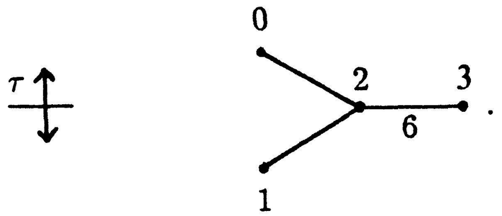


$$\gamma : (\sigma_0, \ldots, \sigma_3) \mapsto (\sigma_0, \sigma_2, \sigma_1, \sigma_3\sigma_2\sigma_3\sigma_2\sigma_3) =: (\rho_0, \ldots, \rho_3) \qquad (14)$$



For example, to prove (12) observe that the fundamental tetrahedron for [3, 6, 3] can be constructed from the fundamental region for [3, 3, 6] by preserving all but the third wall (corresponding to σ_2) while replacing the reflexion wall of σ_2 by the image under σ_2 of the reflexion wall of σ_3. See Figure 3 below.

The group [4, 3, 6] is derived from the group in (14) by the twisting operation indicated by



(15)

More exactly, the operation is given by

$$\kappa : (\rho_0, \ldots, \rho_3; \tau) \mapsto (\tau, \rho_1, \rho_2, \rho_3) =: (\psi_0, \ldots, \psi_3). \qquad (16)$$

Next we consider the geometrical counterparts of (8), (12) and (13). The following relationships between the honeycombs can be checked by using Wythoff's construction [7].

The facets of $\{3, 6, 3\}$ are (in one-to-one correspondence with) certain vertex-figures of $\{3, 3, 6\}$, while its vertex-figures are tessellations $\{6, 3\}$ whose vertices and edges occur among those of the original vertex-figure $\{3, 6\}$, as in Figure 2. To prove this consider the tetrahedral 3-face F_3 in the base flag $\Phi = \{F_{-1}, F_0, \ldots, F_4\}$ of $\{3, 3, 6\}$ and the fundamental tetrahedron T for [3, 3, 6] defined by Φ; see Figure 3. For simplicity we denote the vertices of T by $F_0, \ldots, F_3$. The operation in (12) is equivalent to a change from T to the (4 times)

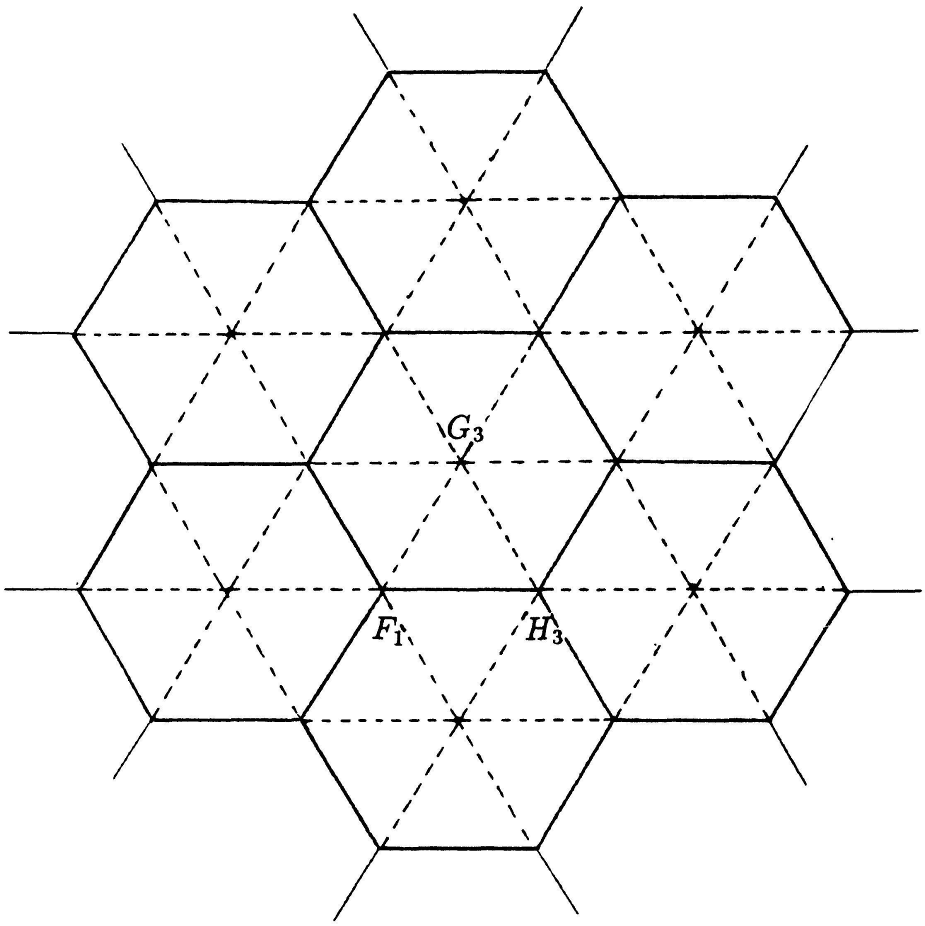


Figure 2

larger fundamental region T_α for [3, 6, 3] with vertices $G_0 = F_0$, $G_1 = F_1$, $G_2 = F_2$ and G_3. Our notation is such that G_i corresponds to (the "centre" of) the i-face in the base flag of $\{3, 6, 3\}$ belonging to $\rho_0, \ldots, \rho_3$. Now, the vertex G_3 of T_α is fixed by $\langle \rho_0, \rho_1, \rho_2 \rangle$, so that Wythoff's construction (with initial vertex $G_0 = F_0$) shows that the 3-face G_3 of $\{3, 6, 3\}$ corresponds to the vertex-figure of $\{3, 3, 6\}$ at its vertex $\sigma_2\sigma_1\sigma_0(F_0)(= G_3)$. Similarly, the vertex F_0 of T_α is fixed by $\langle \rho_1, \rho_2, \rho_3 \rangle$, so that the vertex-figure of $\{3, 6, 3\}$ at F_0 is obtained by clustering triangles in the vertex-figure of $\{3, 3, 6\}$ at F_0 as indicated in Figure 2. This clustering corresponds to an application of the facetting operation φ_2; see Section 2.

In a similar fashion, we find that for the honeycomb $\{6, 3, 6\}$ the facets are just some of the vertex-figures of $\{3, 6, 3\} = \{3, 3, 6\}^\alpha$, while its vertex-figures are tessellations $\{3, 6\}$ obtained from vertex-figures $\{6, 3\}$ of $\{3, 3, 6\}^\alpha$ by the operation indicated in Figure 4. The latter corresponds to an application of μ of (5) to the vertex-figures of $\{3, 3, 6\}$. In fact, (13) is equivalent to the change from T to the (6 times larger) fundamental tetrahedron T_β for [6, 3, 6] with vertices $H_0 = F_0$, $H_1 = F_1$, H_2 and $H_3 = \sigma_1\sigma_0(F_0)$ in Figure 3. Now, to find the structure of the facets

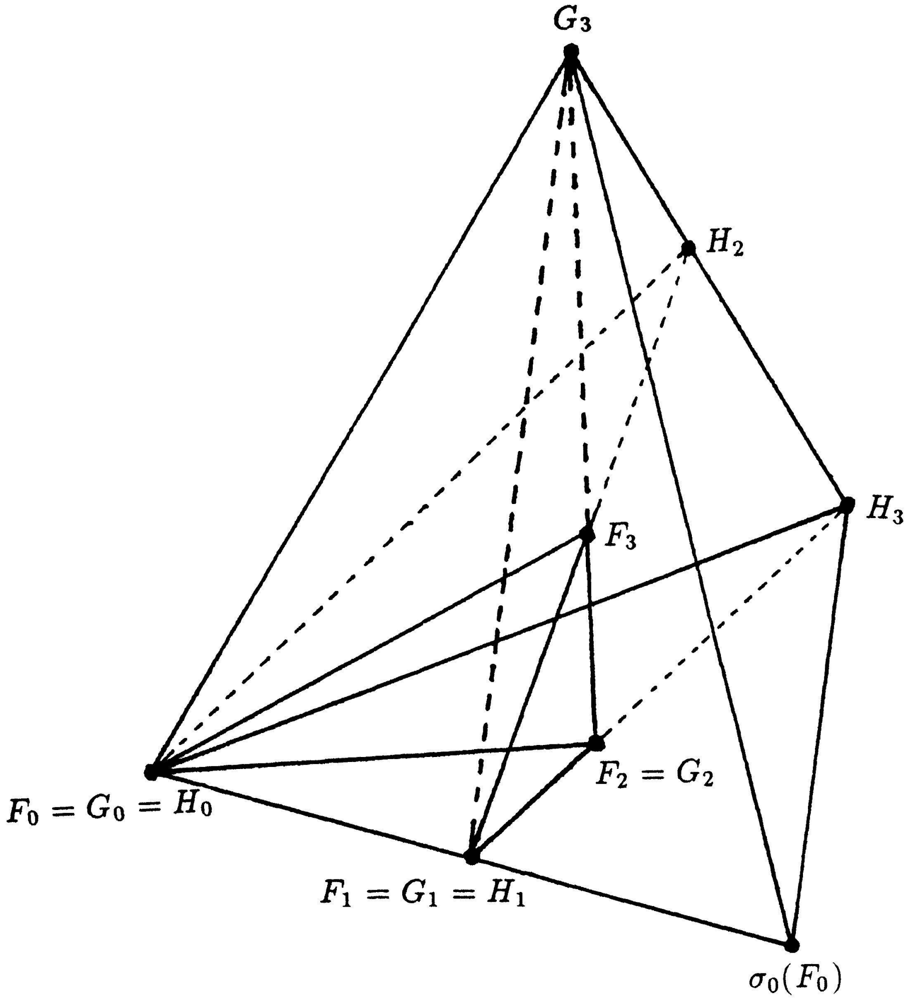


Figure 3

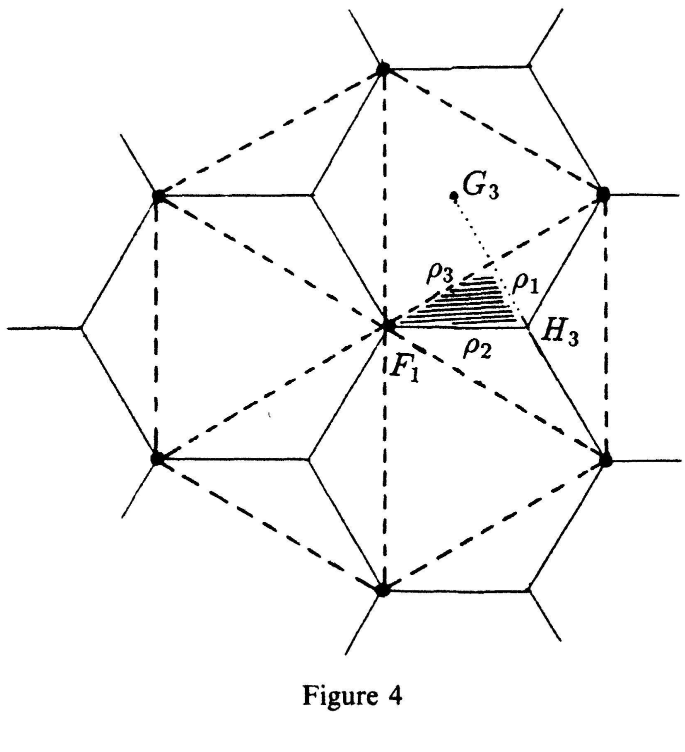


Figure 4

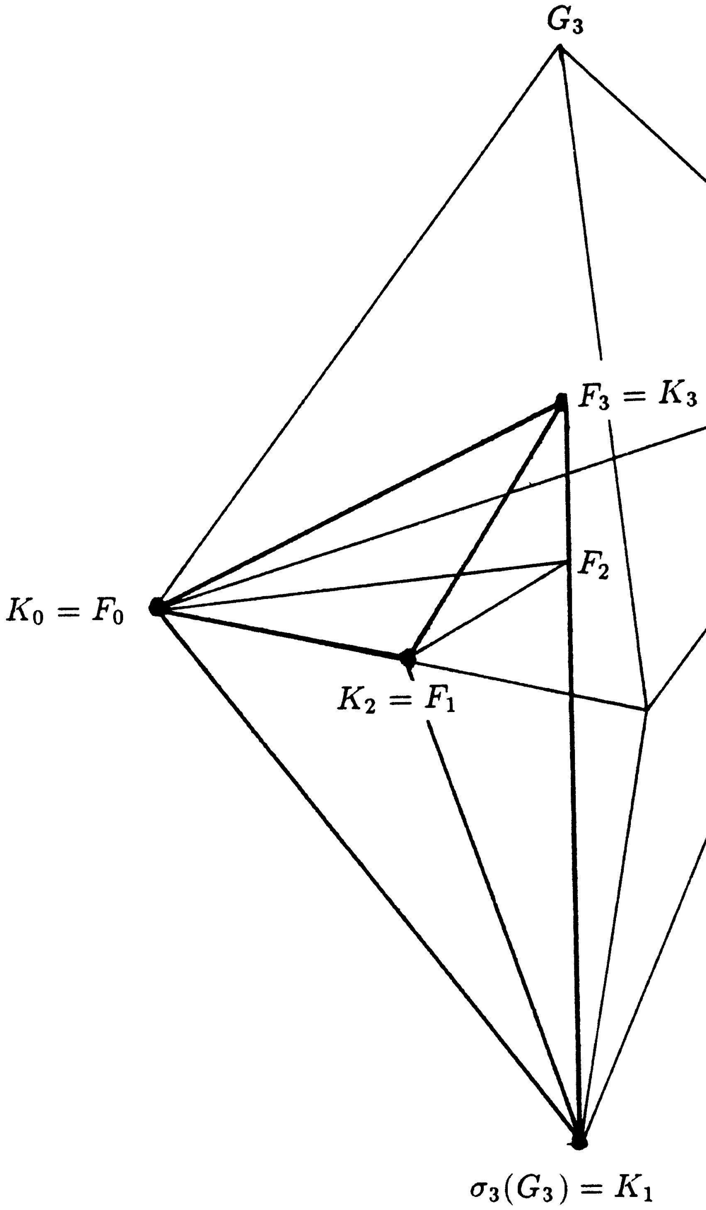


Figure 5

note that $\langle\rho_0, \rho_1, \rho_2\rangle$ fixes H_3 and that $H_3(=\sigma_1\sigma_0(F_0))$ is a vertex of $\{3, 3, 6\}^\alpha$. For the vertex-figures, note that $\langle\rho_1, \rho_2, \rho_3\rangle$ fixes F_0 while $\langle\rho_2, \rho_3\rangle$ fixes F_1.

The operation (14) also has a geometric counterpart; see Figure 5. If T_γ denotes the tetrahedron with vertices $K_0, \ldots, K_3$, then $\rho_0, \ldots, \rho_3$ are the generating reflexions in the walls of T_γ, with ρ_i the reflexion in the wall opposite to K_i. Now, since $\langle\rho_0, \rho_1, \rho_2\rangle = \langle\sigma_0, \sigma_1, \sigma_2\rangle$ is the group of the facet F_3 of $\{3, 3, 6\}$, the transforms of T_γ under $\langle\rho_0, \rho_1, \rho_2\rangle$ fit together to give five tetrahedral facets of $\{3, 3, 6\}$, namely F_3 and its four adjacent facets. These five tetrahedra form a cube, and this clustering of tetrahedra in fives extends to give a honeycomb $\{4, 3, 6\}$ inscribed in $\{3, 3, 6\}$. Note that in this context the cubical facets really have the symmetry of tetrahedra. The index of $\langle \rho_0, \ldots, \rho_3 \rangle$ in $\langle \sigma_0, \ldots, \sigma_3 \rangle$ is 5, since T_γ is a fundamental region for $\langle \rho_0, \ldots, \rho_3 \rangle$ and T_γ can be decomposed into 5 copies of T.

4. Relationships between locally toroidal polytopes

In this section we discuss relationships between locally toroidal regular 4-polytopes of types $\{3,3,6\}$, $\{4,3,6\}$, $\{6,3,6\}$ and $\{3,6,3\}$. Since their groups are quotients of the groups of the corresponding hyperbolic honeycombs, we can use the results of Section 3 to gain more insight into their structure. Note that by (8), (12) and (13) any quotient of $[3,3,6]$ contains as subgroups certain quotients of $[3,6,3]$ and $[6,3,6]$ whose indices divide 4 or 6, respectively. A similar remark applies to the other cases in (8).

First, recall that for any abstract n-polytope $\mathscr{P}$ (and more generally, any poset) the *order complex* $\Delta(\mathscr{P})$ is the simplicial $(n-1)$-complex whose simplices are the totally ordered subsets of $\mathscr{P}$ which do not contain the (-1)-face and n-face of $\mathscr{P}$ [33]. If $\mathscr{P}$ is a hyperbolic honeycomb $\{p,q,r\}$, then $\Delta(\mathscr{P})$ is isomorphic to the barycentric subdivision of $\mathscr{P}$.

To consider the operations (12), (13) and (14) on abstract polytopes let $\mathscr{P}$ be a regular polytope in the class $\langle \{3,3\}, \{3,6\}_{(s,t)} \rangle$ with group $A(\mathscr{P}) = \langle \sigma_0, \ldots, \sigma_3 \rangle$. The facets of $\mathscr{P}$ are 3-simplices, so that $\mathscr{P}$ is a "simplicial poset". In fact, in almost all cases $\mathscr{P}$ is actually a simplicial complex in the usual sense. For example, for $\mathscr{P} = \mathscr{P}_{(s,t)} := \{\{3,3\}, \{3,6\}_{(s,t)}\}$ this is true unless $(s,t) = (2,0)$. Now, for a general $\mathscr{P}$, subdividing each simplicial facet barycentrically gives (an isomorphic copy of) the order complex $\Delta(\mathscr{P})$; this is always a simplicial complex. The 3-simplex of $\Delta(\mathscr{P})$ which corresponds to the base flag $\Phi = \{F_{-1}, F_0, \ldots, F_4\}$ of $\mathscr{P}$ is a fundamental tetrahedron for the action of $\mathscr{P}$ on $\Delta(\mathscr{P})$ (or on the underlying topological space $|\Delta(\mathscr{P})|$). The complex $\Delta(\mathscr{P})$ ($|\Delta(\mathscr{P})|$, respectively) can be obtained from the barycentric subdivision of the hyperbolic honeycomb $\{3,3,6\}$ (H^3, respectively) by identifications corresponding to the extra relations for $A(\mathscr{P})$; see (3).

4.1. *The types* $\{3,3,6\}$ *and* $\{3,6,3\}$

We begin with the construction of the polytopes $\mathscr{P}^\alpha$ of type $\{3,6,3\}$ from those of type $\{3,3,6\}$, with

$$\alpha : (\sigma_0, \ldots, \sigma_3) \mapsto (\sigma_0, \sigma_1, \sigma_2\sigma_3\sigma_2, \sigma_3) =: (\rho_0, \ldots, \rho_3),$$

as in (12). On $\mathscr{P}$ we impose the (weak) condition that its *graph* (consisting of all vertices and edges of $\mathscr{P}$) has no loops; that is, any two vertices of $\mathscr{P}$ are joined by at most one edge of $\mathscr{P}$. This is satisfied if $\mathscr{P}$ is a simplicial complex. We do not know of a polytope in $\langle\{3,3\},\{3,6\}_{(s,t)}\rangle$ which violates the condition; even $\mathscr{P}_{(2,0)}$ does not.

Now, our above remarks on $\Delta(\mathscr{P})$ imply that we can construct $\mathscr{P}^{\alpha}$ from $\mathscr{P}$ in a similar fashion as $\{3,6,3\}$ from $\{3,3,6\}$. Again Figure 3 illustrates how the fundamental tetrahedra for $\mathscr{P}$ and $\mathscr{P}^{\alpha}$ (or for $\Delta(\mathscr{P})$ and $\Delta(\mathscr{P}^{\alpha})$) are related; we use the same notation as in Section 3.

Employing Wythoff's construction shows that the facets of $\mathscr{P}^{\alpha}$ are certain vertex-figures $\{3,6\}_{(s,t)}$ of $\mathscr{P}$, while its vertex-figures are transforms under the facetting operation φ_2 of the original vertex-figures $\{3,6\}_{(s,t)}$. To check that $A:=\langle\rho_0,\ldots,\rho_3\rangle$ is indeed a C-group with $\mathscr{P}^{\alpha}$ as the corresponding polytope, let $\psi\in\langle\rho_0,\rho_1,\rho_2\rangle\cap\langle\rho_1,\rho_2,\rho_3\rangle$. Then ψ fixes the vertices F_0 and G_3 of $\mathscr{P}$. Now, since the graph of $\mathscr{P}$ has no loops, F_0 and G_3 are joined by only one edge, namely the edge of F_3 connecting F_0 and G_3. It follows that $\psi\in\langle\rho_1,\rho_2\rangle$, as required for the intersection property to hold.

To find the index of A in $A(\mathscr{P})$ observe that in $\Delta(\mathscr{P})$ the simplex T_{α} with vertices G_0, G_1, G_2, G_3 is dissected into four copies of the fundamental region for $A(\mathscr{P})$ with vertices F_0, F_1, F_2, F_3, namely T, $\sigma_2(T)$, $\sigma_2\sigma_1(T)$ and $\sigma_2\sigma_1\sigma_0(T)$. If the index is not 4, then T_{α} is not a fundamental region for A, so that two of these copies must be equivalent under A. Since $\sigma_0,\sigma_1\in A$ it follows in this case that $\sigma_2\in A$ and thus $A=A(\mathscr{P})$. Hence $|A(\mathscr{P}):A|=1$ or 4. If $A=A(\mathscr{P})$, then $\sigma_2\in A$ and one is tempted to conclude $\sigma_2\in\langle\rho_1,\rho_2,\rho_3\rangle$ (since σ_2 stabilizes the base vertex $G_0=F_0$ of $\mathscr{P}^{\alpha}$). However we cannot be sure that Wythoff's construction of $\mathscr{P}^{\alpha}$ in $\Delta(\mathscr{P})$ gives a faithful realization.

Part (a) of the following theorem follows from our remarks in Section 2 on the facetting operation. The proof of (b) is given later. Note that there are other proofs of Theorem 1(b) (and Theorem 3(b)) which use coset enumeration.

THEOREM 1. *Let $\mathscr{P}$ be a regular 4-polytope in the class $\langle\{3,3\},\{3,6\}_{(s,t)}\rangle$, and let α be the operation (12). Assume that the graph of $\mathscr{P}$ has no loops.*

(a) *Then $\mathscr{P}^{\alpha}$ is a regular 4-polytope in the class*

(i) $\langle\{3,6\}_{(3r,0)},\{6,3\}_{(r,r)}\rangle$ *if* $s=3r\geq 3$, $t=0$;

(ii) $\langle\{3,6\}_{(s,0)},\{6,3\}_{(s,0)}\rangle$ *if* $3\nmid s$, $t=0$;

(iii) $\langle\{3,6\}_{(s,s)},\{6,3\}_{(s,0)}\rangle$ *if* $s=t\geq 2$.

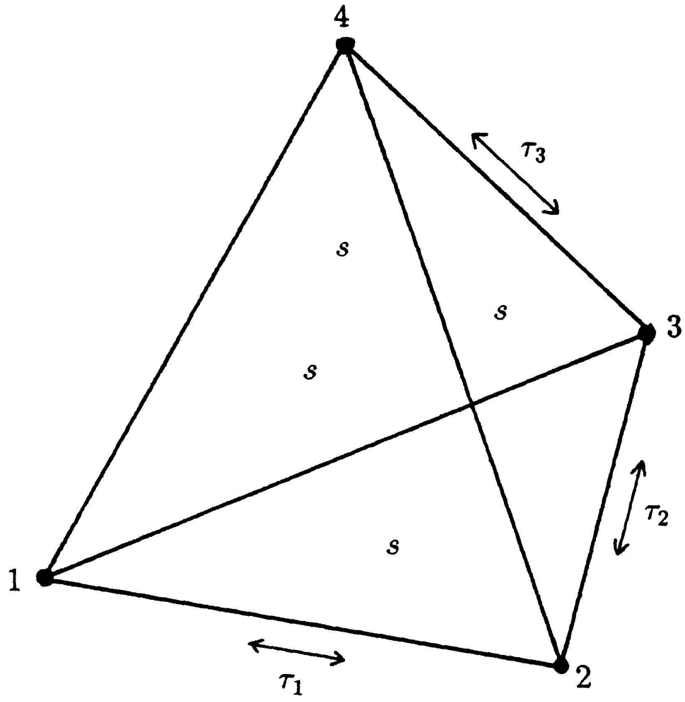
Furthermore, $|A(\mathscr{P}):A(\mathscr{P}^{\alpha})|=1$ *or* 4, *and* $A(\mathscr{P}^{\alpha})=A(\mathscr{P})$ *in case* (ii).

(b) *If* $\mathscr{P}=\mathscr{P}_{(s,t)}:=\{\{3,3\},\{3,6\}_{(s,t)}\}$, *then for* (i), (ii) *and* (iii) *the index is given by* $|A(\mathscr{P}):A(\mathscr{P}^{\alpha})|=4$, 1 *and* 4, *respectively. In particular,* $\mathscr{P}^{\alpha}_{(s,s)}=\{\{3,6\}_{(s,s)},\{6,3\}_{(s,0)}\}$ *for* $s\geq 2$.

It is clear from the construction that if $\mathcal{P}$ is universal in its class, then $\mathcal{P}^{\alpha}$ is universal among all polytopes obtained by α from polytopes of the same class as $\mathcal{P}$. Theorem 1(b) states that at least for $\mathcal{P} = \mathcal{P}_{(s,s)}$ the new polytope is indeed universal in its class.

The finite universal polytopes $\mathcal{P}_{(s,t)}$ were classified in [22]. The only finite instances are obtained for $(s, t) = (2, 0)$, $(3, 0)$, $(4, 0)$ and $(2, 2)$. Also known is the classification of the universal polytopes $\{\{3, 6\}_{(3r,0)}, \{6, 3\}_{(r,r)}\}$ and $\{\{3, 6\}_{(s,s)}, \{6, 3\}_{(s,0)}\}$; the only finite instances occur for $r = 1$ and $s = 2$, respectively [22]. Below we briefly recall some of the corresponding constructions.

First, $\mathcal{P}_{(s,s)}$ was constructed from the abstract group $W_1 = \langle \alpha_1, \ldots, \alpha_4 \rangle$ with tetrahedral diagram



(17)

by applying the twisting operation

$$\kappa : (\alpha_1, \ldots, \alpha_4; \tau_1, \tau_2, \tau_3) \mapsto (\tau_3, \tau_2, \tau_1, \alpha_1) := (\sigma_0, \ldots, \sigma_3). \tag{18}$$

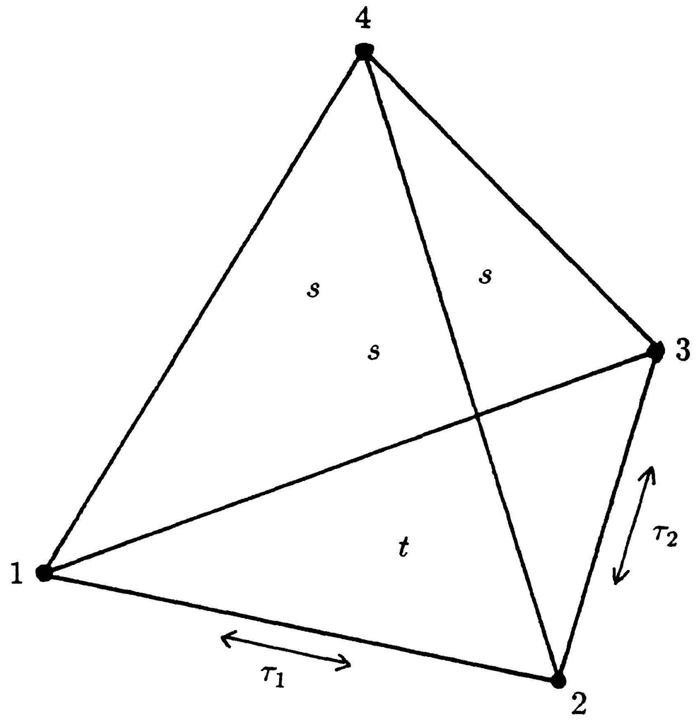
Then we have semi-direct products $A(\mathcal{P}_{(s,s)}) = \langle \alpha_1, \ldots, \alpha_4 \rangle \rtimes \langle \tau_1, \tau_2, \tau_3 \rangle = W_1 \rtimes S_4$. In particular, $\mathcal{P}_{(s,s)}$ is finite if and only if $s = 2$. If $s = 2$ then $W_1 \simeq S_5$ and $A(\mathcal{P}_{(2,2)}) = S_5 \times S_4$.

Now, the generators $\rho_0, \ldots, \rho_3$ of $A = A(\mathcal{P}^{\alpha}_{(s,s)})$ are given by

$$(\rho_0, \ldots, \rho_3) = (\sigma_0, \sigma_1, \sigma_2\sigma_3\sigma_2, \sigma_3) = (\tau_3, \tau_2, \alpha_2, \alpha_1), \tag{19}$$

so that $A = W_1 \rtimes S_3$. In particular, for the index we have $|A(\mathcal{P}_{(s,s)}) : A| = 4$.

Next, recall from [22] that the universal polytope $\{\{3,6\}_{(s,0)},\{6,3\}_{(t,t)}\}$ (with $s=t$ or $s=3t$) was constructed from the abstract group $W_2=\langle\beta_1,\ldots,\beta_4\rangle$ with tetrahedral diagram



(20)

(with the bottom 2-face marked t, the other 2-faces marked s) by means of

$$\kappa:(\beta_1,\ldots,\beta_4;\tau_1,\tau_2)\mapsto(\beta_4,\beta_3,\tau_2,\tau_1). \tag{21}$$

Its group $W_2\rtimes S_3$ is finite if and only if $s=t=2$. If $s=t=2$ the group is $S_5\times S_3$.

To find $\mathscr{P}^{\alpha}_{(s,s)}$ we only have to note that the generators of (19) correspond to those of (21) in reverse order (with $t=s$ in (20)). It follows that $\mathscr{P}^{\alpha}_{(s,s)}=\{\{3,6\}_{(s,s)},\{6,3\}_{(s,0)}\}$. This proves part of Theorem 1(b). The only finite instance is

$$\{\{3,3\},\{3,6\}_{(2,2)}\}\xrightarrow[4]{\alpha}\{\{3,6\}_{(2,2)},\{6,3\}_{(2,0)}\},$$

with the mark at the arrow indicating the index.

We continue our discussion with case (ii) of Theorem 1(b). The classification of the universal polytopes $\{\{3,6\}_{(s,0)},\{6,3\}_{(s,0)}\}$ is still open. In Weiss [36], it is proved that for $s=2,3$ the polytopes are finite. The polytopes $\mathscr{P}_{(2,0)}$ and $\{\{3,6\}_{(2,0)},\{6,3\}_{(2,0)}\}$ both have group $S_5\times C_2$, so that $\mathscr{P}^{\alpha}_{(2,0)}$ is indeed universal;

that is,

$$\{\{3,3\},\{3,6\}_{(2,0)}\} \xrightarrow[1]{\alpha} \{\{3,6\}_{(2,0)},\{6,3\}_{(2,0)}\}.$$

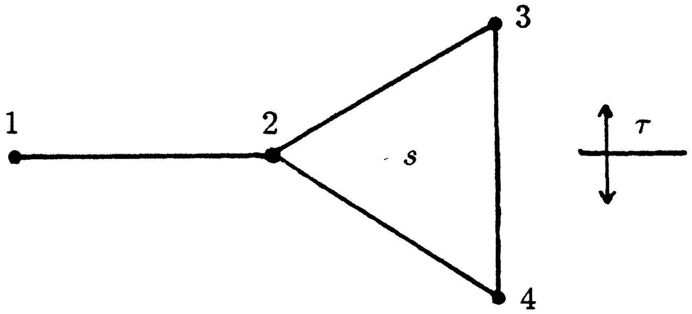
For $\mathscr{P}_{(4,0)}$ we are again in case (ii) of Theorem 1, but here $\{\{3,6\}_{(4,0)},\{6,3\}_{(4,0)}\}$ is likely to be infinite while $\mathscr{P}^{\alpha}_{(4,0)}$ is finite.

By Theorem 1 we have $\mathscr{P}^{\alpha}_{(s,0)} \in \langle\{3,6\}_{(s,0)},\{6,3\}_{(s,0)}\rangle$ if $3\nmid s$. This implies that the universal polytope $\{\{3,6\}_{(s,0)},\{6,3\}_{(s,0)}\}$ exists for all such s, but $\mathscr{P}^{\alpha}_{(s,0)}$ is finite if and only if $\mathscr{P}_{(s,0)}$ is finite.

THEOREM 2. *The universal regular polytope* $\{\{3,6\}_{(s,0)},\{6,3\}_{(s,0)}\}$ *exists for all* s *with* $3\nmid s$. *For* $s \geq 5$ (*and most likely also for* $s=4$) *it is infinite. For* $s=2$ *it is finite and its group is* $S_5 \times C_2$.

We remark that the results of [24] carry over to the situation of Theorem 2. Note also that Theorem 2 implies that for any s with $s \neq 2^k 3^l$ the universal $\{\{3,6\}_{(s,0)},\{6,3\}_{(s,0)}\}$ either does not exist or, if it exists, must be infinite. In fact, if $s \neq 2^k 3^l$, choose a prime p with $p \geq 5$ and $p \mid s$, and apply Theorem 2 with s replaced by p. Since the group for p is a quotient of the group for s, the above follows.

We proceed with the discussion of case (i) of Theorem 1. Recall from [22] that the universal polytope $\mathscr{P}_{(s,0)} = \{\{3,3\},\{3,6\}_{(s,0)}\}$ was constructed from the abstract group $W_3 = [1\ 1\ 2^3]^s$ with diagram



(22)

and generators $\alpha_1, \ldots, \alpha_4$ (say) by means of the twisting operation

$$\kappa : (\alpha_1, \ldots, \alpha_4; \tau) \mapsto (\alpha_1, \alpha_2, \alpha_3, \tau) =: (\sigma_0, \ldots, \sigma_3). \tag{23}$$

In particular, $A(\mathscr{P}_{(s,0)}) = W_3 \rtimes C_2$. This is a finite group if and only if $s \leq 4$.

First we complete the proof of Theorem 1(b). For each r the polytope $\mathscr{P}_{(3r,0)}$ projects onto $\mathscr{P}_{(3,0)}$, since $A(\mathscr{P}_{(3,0)})$ is a quotient of $A(\mathscr{P}_{(3r,0)})$. It follows that $\mathscr{P}^{\alpha}_{(3r,0)}$ projects onto $\mathscr{P}^{\alpha}_{(3,0)}$. Hence, to prove $A(\mathscr{P}^{\alpha}_{(3r,0)}) \neq A(\mathscr{P}_{(3r,0)})$ (and thus

$|A(\mathscr{P}_{(3r,0)}) : A(\mathscr{P}^{\alpha}_{(3r,0)})| = 4$), it suffices to check this for $r = 1$. But for $r = 1$ we have

$$\{\{3, 3\}, \{3, 6\}_{(3,0)}\} \xrightarrow[4]{\alpha} \{\{3, 6\}_{(3,0)}, \{6, 3\}_{(1,1)}\}.$$

In fact, while $A(\mathscr{P}_{(3,0)}) = W_3 \ltimes C_2$ has order 1296, the group of $\{\{3, 6\}_{(3,0)}, \{6, 3\}_{(1,1)}\}$ is $[1\ 1\ 1]^3 \ltimes S_3$ of order 324 (cf. [22], Section 8). This completes the proof of Theorem 1(b).

By Theorem 1 we know that $\mathscr{P}^{\alpha}_{(3r,0)} \in \langle\{3, 6\}_{(3r,0)}, \{6, 3\}_{(r,r)}\rangle$. Now, the universal $\{\{3, 6\}_{(3r,0)}, \{6, 3\}_{(r,r)}\}$ is obtained from $W_2 = \langle\beta_1, \ldots, \beta_4\rangle$ (with $s = 3r$, $t = r$) in (20) by operation (21). It follows that $A(\mathscr{P}^{\alpha}_{(3r,0)}) = \langle\rho_0, \ldots, \rho_3\rangle$ is a quotient of $W_2 \ltimes S_3$. A more instructive way (independent of Theorem 1) to see this is obtained as follows.

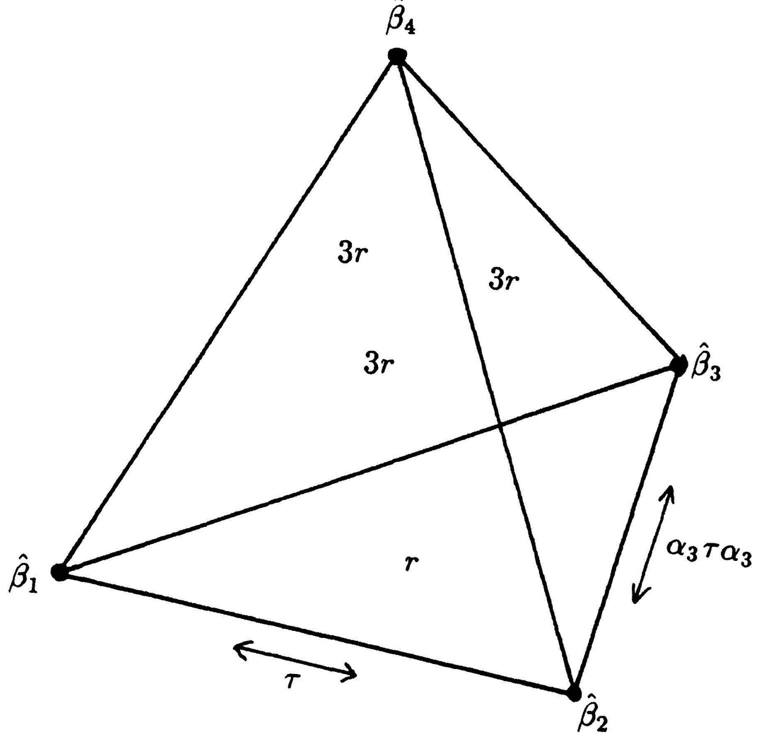
First, observe that by (12) and (23)

$$(\rho_0, \ldots, \rho_3) = (\alpha_1, \alpha_2, \alpha_3\tau\alpha_3, \tau).$$

Then

$$\begin{aligned} &\hat{\beta}_4 := \alpha_1, \quad \hat{\beta}_3 := \alpha_2, \quad \hat{\beta}_2 := \alpha_3\tau\alpha_3 \cdot \hat{\beta}_3 \cdot \alpha_3\tau\alpha_3 = \alpha_3\alpha_4\alpha_2\alpha_4\alpha_3, \\ &\hat{\beta}_1 := \tau \cdot \hat{\beta}_2 \cdot \tau = \alpha_4\alpha_3\alpha_2\alpha_3\alpha_4, \quad \hat{W}_2 := \langle\hat{\beta}_1, \ldots, \hat{\beta}_4\rangle. \end{aligned} \tag{24}$$

Note that $\hat{W}_2$ is a subgroup of W_3. The elements $\alpha_3\tau\alpha_3$ and τ act on W_2 as indicated in the diagram



(25)

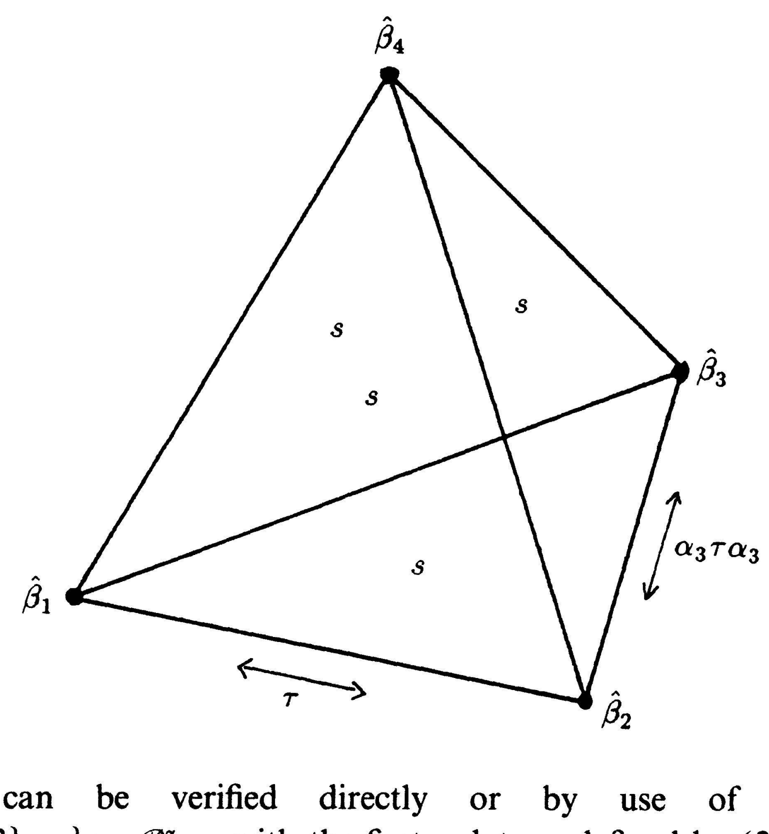
Note that products like $\hat{\beta}_2\hat{\beta}_3$, $\hat{\beta}_1\hat{\beta}_2\hat{\beta}_3\hat{\beta}_2$ and $\hat{\beta}_2\hat{\beta}_3\hat{\beta}_4\hat{\beta}_3$ indeed have the correct order, namely 3, r and $3r$, respectively. Hence $\beta_i \mapsto \hat{\beta}_i$ $(i = 1, \dots, 4)$ defines a homomorphism of W_2 onto $\hat{W}_2$. This extends in an obvious way to a homomorphism of $W_2 \ltimes S_3 = W_2 \ltimes \langle \tau_1, \tau_2 \rangle$ onto $A(\mathscr{P}^{\alpha}_{(3r,0)})$.

We cannot completely rule out here that the factorization

$$A(\mathscr{P}^{\alpha}_{(3r,0)}) = \hat{W}_2 \cdot \langle \tau, \alpha_3\tau\alpha_3 \rangle$$

is not a semi-direct product (though this is very unlikely), or equivalently, that $\alpha_3\alpha_4 = \alpha_3\tau\alpha_3\tau \in \hat{W}_2$.

We can similarly proceed for $\mathscr{P}_{(s,0)}$ with $3 \nmid s$. Then $\mathscr{P}^{\alpha}_{(s,0)} \in \langle \{3, 6\}_{(s,0)}, \{6, 3\}_{(s,0)} \rangle$. Defining $\hat{\beta}_i$ and $\hat{W}_2$ as above, we find that (25) has to be replaced by



(26)

Again this can be verified directly or by use of the projection $\{\{3, 6\}_{(s,0)}, \{6, 3\}_{(s,s)}\} \mapsto \mathscr{P}^{\alpha}_{(s,0)}$, with the first polytope defined by (20) and (21) with $s = t$. Now, since $\{3, 6\}^{\varphi_2}_{(s,0)} = \{6, 3\}_{(s,0)}$, the subgroup $\langle \rho_1, \rho_2, \rho_3 \rangle$ has order $12s^2$. On the other hand, $\langle \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3 \rangle \simeq [1\ 1\ 1]^s$ (of order $6s^2$) and $\langle \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_2 \rangle \langle \tau, \alpha_3\tau\alpha_3 \rangle \leq \langle \rho_1, \rho_2, \rho_3 \rangle$. It follows that $\alpha_3\alpha_4 = \alpha_3\tau\alpha_3\tau \in \hat{W}_2$. Hence $A(\mathscr{P}^{\alpha}_{(s,0)}) = \hat{W}_2 \cdot \langle \tau, \alpha_3\tau\alpha_3 \rangle$ is not a semi-direct product. However,

$$A(\mathscr{P}^{\alpha}_{(s,0)}) = \hat{W}_2 \ltimes \langle \tau \rangle = \hat{W}_2 \ltimes C_2.$$

By Theorem 1 we have $A(\mathscr{P}^{\alpha}_{(s,0)}) = A(\mathscr{P}_{(s,0)})$, so that $\hat{W}_2 = W_3$. In particular we have proved the following interesting fact:

For $3 \nmid s$ *the group* $[1\ 1\ 2]^s$ *of* (22) *belongs to the diagram* (26). (27)

However, in general, additional relations to those of (26) are needed to define W_3 (for example for $s = 4$). If $s = 2$ the relations do suffice, defining the group S_5 (cf. [22]).

It is worth remarking that for $3 \nmid s$ we can construct from $\mathscr{P}_{(s,0)}$ (or $\mathscr{P}^{\alpha}_{(s,0)}$) a polytope $\mathscr{L}$ in $\langle \{3, 6\}_{(s,0)}, \{6, 3\}_{(s,s)} \rangle$ with group $A(\mathscr{L}) = [1\ 1\ 2]^s \rtimes S_3$. In particular, $A(\mathscr{L})$ contains $A(\mathscr{P}_{(s,0)})$ as a subgroup of index 3. In fact, as (26) indicates, the element $\alpha_3 \tau \alpha_3$ acts on $\hat{W}_2$ in the same way as τ_2 in (20). It follows that we can adjoin to $A(\mathscr{P}_{(s,0)}) = A(\mathscr{P}^{\alpha}_{(s,0)})$ a suitable involutory element, so that $\mathscr{L}$ can be constructed as in (21). Note for this that the intersection property for $A(\mathscr{P}^{\alpha}_{(s,0)}) = \langle \rho_0, \ldots, \rho_3 \rangle$ implies the property $\langle \hat{\beta}_1, \hat{\beta}_2, \hat{\beta}_3 \rangle \cap \langle \hat{\beta}_2, \hat{\beta}_3, \hat{\beta}_4 \rangle = \langle \hat{\beta}_2, \hat{\beta}_3 \rangle$, and this in turn the intersection property for $A(\mathscr{L})$.

If $s = 2$ then $\mathscr{L} = \{\{3, 6\}_{(2,0)}, \{6, 3\}_{(2,2)}\}$ and $A(\mathscr{L}) = S_5 \times S_3$ [22]. For $s = 4$ the polytope $\mathscr{L}$ is finite but $\{\{3, 6\}_{(4,0)}, \{6, 3\}_{(4,4)}\}$ is infinite.

4.2. *The types* $\{3, 3, 6\}$ *and* $\{6, 3, 6\}$

We next discuss the polytopes $\mathscr{P}^{\beta}$ of type $\{6, 3, 6\}$, where $\mathscr{P}$ is a polytope in $\langle \{3, 3\}, \{3, 6\}_{(s,t)} \rangle$ and β is the operation on $A(\mathscr{P}) = \langle \sigma_0, \ldots, \sigma_3 \rangle$ given by

$$\beta : (\sigma_0, \ldots, \sigma_3) \mapsto (\sigma_0, \sigma_1\sigma_2\sigma_3\sigma_2\sigma_1, \sigma_3, \sigma_2) =: (\rho_0, \ldots, \rho_3),$$

as in (13). As in Section 4.1, we assume that the graph of $\mathscr{P}$ has no loops. Again, we can construct $\mathscr{P}^{\beta}$ from $\mathscr{P}$ as $\{6, 3, 6\}$ from $\{3, 3, 6\}$; see Figure 3.

Applying Wythoff's construction to the order complex $\Delta(\mathscr{P})$, we find that the facets of $\mathscr{P}^{\beta}$ are just some of the vertex-figures of $\mathscr{P}^{\alpha}$, while its vertex-figures are transforms of the original vertex-figure $\{3, 6\}_{(s,t)}$ under the operation μ of (5). To check that $A := \langle \rho_0, \ldots, \rho_3 \rangle$ is indeed a C-group with $\mathscr{P}^{\beta}$ as the corresponding polytope let $\psi \in \langle \rho_0, \rho_1, \rho_2 \rangle \cap \langle \rho_1, \rho_2, \rho_3 \rangle$. Then ψ fixes the vertices $F_0 (= H_0)$ and H_3 of $\mathscr{P}$ and thus the unique edge (of F_3) connecting F_0 and H_3; here we used our assumption that the graph of $\mathscr{P}$ has no loops. It follows that $\psi \in \langle \rho_1, \rho_2 \rangle$, as required.

Now, to find the index of A in $A(\mathscr{P})$, observe that in $\Delta(\mathscr{P})$ the simplex T_{β} with vertices H_0, H_1, H_2 and H_3 is dissected into six copies of the fundamental region T for $A(\mathscr{P})$ with vertices $F_0, \ldots, F_3$. These copies are T, $\sigma_1(T)$, $\sigma_1\sigma_0(T)$, $\sigma_1\sigma_2(T)$, $\sigma_1\sigma_2\sigma_0(T)$ and $(\sigma_2\sigma_1\sigma_0)^2(T)$. We shall see below that in the most interesting case we have $|A(\mathscr{P}) : A| = 1$ or 6, proving that either T or T_β is a fundamental region for A, respectively. In particular, if the index is 6, then $(\sigma_2\sigma_1\sigma_0)^2 \notin A$; note that in Figure 3 the element $(\sigma_2\sigma_1\sigma_0)^2$ corresponds to the "half-turn" of F_3 about the edge connecting F_0 and H_3, and thus maps T_β onto itself.

Note that A can only be normal in $A(\mathscr{P})$ if $A = A(\mathscr{P})$. In fact, if A is normal then $\sigma_0\sigma_1\sigma_0 = \sigma_1\sigma_0\sigma_1 \in \sigma_1 A\sigma_1 = A$ and thus $A = A(\mathscr{P})$. It follows that the index cannot be 2.

Further, note that by our remarks on (5) we have $A = A(\mathscr{P})$ if $3 \nmid s$ and $t = 0$. It is likely that this is the only case where $A = A(\mathscr{P})$. In fact, if $A = A(\mathscr{P})$ then $\sigma_1 \in A$ and we are tempted to conclude that $\sigma_1 \in \langle \rho_1, \rho_2, \rho_3 \rangle$ (since σ_1 stabilizes the base vertex $F_0 = H_0$ of $\mathscr{P}^\beta$). However, as in Section 4.1, we cannot be sure that the realization of $\mathscr{P}^\beta$ on $\Delta(\mathscr{P})$ is faithful. If indeed $\sigma_1 \in \langle \rho_1, \rho_2, \rho_3 \rangle$, then our remarks on (5) imply that $3 \nmid s$ and $t = 0$.

Now let $\mathscr{P} = \mathscr{P}_{(3r,0)} = \{\{3,3\}, \{3,6\}_{(3r,0)}\}$ with $r \geq 1$; see (22) and (23). Then $A(\mathscr{P}_{(3,0)})$ is a quotient of $A(\mathscr{P})$. Since the index of a subgroup can only get smaller under a homomorphism, it suffices to prove $|A(\mathscr{P}) : A| = 6$ if $r = 1$. Now, if $r = 1$ then $\mathscr{P}^\beta \in \langle \{6,3\}_{(1,1)}, \{3,6\}_{(1,1)} \rangle$. The universal $\{\{6,3\}_{(1,1)}, \{3,6\}_{(1,1)}\}$ is flat, and so it is the only member in its class; see Section 6. It follows that $\mathscr{P}^\beta = \{\{6,3\}_{(1,1)}, \{3,6\}_{(1,1)}\}$ and $A(\mathscr{P}^\beta) = S_3 \ltimes A(\{3,6\}_{(1,1)})$, of order 216 [22]. On the other hand, $A(\mathscr{P}) = [1\ 1\ 2]^3 \ltimes C_2$, of order 1296. Hence, $|A(\mathscr{P}) : A| = 6$, as required.

If $\mathscr{P}$ is a non-universal member of $\langle \{3,3\}, \{3,6\}_{(3r,0)} \rangle$, we do not know if $A(\mathscr{P}_{(3,0)})$ is always a quotient of $A(\mathscr{P})$. Here we cannot completely rule out that the index is not 6, though this is very likely.

Now let $\mathscr{P} = \mathscr{P}_{(s,s)} = \{\{3,3\}, \{3,6\}_{(s,s)}\}$ with $s \geq 2$; see (17) and (18). Then by (13) the generators of A are given by

$$(\rho_0, \ldots, \rho_3) = (\tau_3, \alpha_3, \alpha_1, \tau_1). \tag{28}$$

It follows that $A = W_1 \ltimes \langle \tau_1, \tau_3 \rangle = W_1 \ltimes (C_2 \times C_2)$. But $A(\mathscr{P}) = W \ltimes S_4$, so that $|A(\mathscr{P}) : A| = 6$, as required. Note further that (28) itself defines a twisting operation on $W_1 = \langle \alpha_1, \ldots, \alpha_4 \rangle$. This is in fact the same operation which was used in [22] to construct the universal $\{\{6,3\}_{(s,0)}, \{3,6\}_{(s,0)}\}$; see also (35) and (36) below. Hence, $\mathscr{P}^\beta = \{\{6,3\}_{(s,0)}, \{3,6\}_{(s,0)}\}$.

Finally, if $\mathscr{P}$ is any member of $\langle \{3,3\}, \{3,6\}_{(s,s)} \rangle$, then $A(\mathscr{P})$ is the image of $A(\mathscr{P}_{(s,s)}) = W_1 \ltimes S_4$ under a homomorphism f (say) mapping distinguished generators to distinguished generators. But then $A(\mathscr{P}) = f(W_1) \cdot f(\langle \tau_1, \tau_2, \tau_3 \rangle)$, $f(\langle \tau_1, \tau_2, \tau_3 \rangle) \simeq S_4$, and $A(\mathscr{P}^\beta) = f(W_1) \cdot f(\langle \tau_1, \tau_3 \rangle)$. If f is such that $f(W_1) \cap f(\langle \tau_1, \tau_2, \tau_3 \rangle) = \{1\}$, then the two products are semi-direct products and the index is 6. However we are not sure if this is always true.

THEOREM 3. *Let $\mathscr{P}$ be a regular polytope in the class* $\langle\{3,3\},\{3,6\}_{(s,t)}\rangle$, *and let β be the operation* (13). *Assume that the graph of $\mathscr{P}$ has no loops.*

(a) *Then $\mathscr{P}^\beta$ is a regular 4-polytope in the class*

(i) $\langle\{6,3\}_{(r,r)},\{3,6\}_{(r,r)}\rangle$ *if* $s=3r\geq 3$, $t=0$;

(ii) $\langle\{6,3\}_{(s,0)},\{3,6\}_{(s,0)}\rangle$ *if* $3\nmid s$, $t=0$;

(iii) $\langle\{6,3\}_{(s,0)},\{3,6\}_{(s,0)}\rangle$ *if* $s=t\geq 2$.

Furthermore, $|A(\mathscr{P}):A(\mathscr{P}^\beta)|=1$ *in case* (ii), *and* $|A(\mathscr{P}):A(\mathscr{P}^\beta)|=1$, 3 *or* 6 *in case* (i) *and* (iii).

(b) *If* $\mathscr{P}=\mathscr{P}_{(s,t)}=\{\{3,3\},\{3,6\}_{(s,t)}\}$, *then for* (i), (ii) *and* (iii) *the index is given by* $|A(\mathscr{P}):A(\mathscr{P}^\beta)|=6$, 1 *and* 6, *respectively. Also,* $\mathscr{P}^\beta_{(s,s)}=\{\{6,3\}_{(s,0)},\{3,6\}_{(s,0)}\}$ *for* $s\geq 2$.

The finite universal regular polytopes $\{\{6,3\}_{(q,r)},\{3,6\}_{(s,t)}\}$ have been classified in [22], Theorem 4. The only finite instances are: $\{\{6,3\}_{(2,2)},\{3,6\}_{(2,0)}\}$ with group $S_5\times S_4\times C_2$; $\{\{6,3\}_{(q,0)},\{3,6\}_{(2,0)}\}$ for $q=2,3,4$, with group $[1\ 1\ 2]^q\ltimes(C_2\times C_2)$ of order $4\cdot 5!$, $4\cdot 3^3\cdot 4!$ and $256\cdot 5!$, respectively; and the duals of these polytopes.

We briefly discuss the application of Theorem 3 to finite universal polytopes. In Theorem 3(a)(ii), if $s=2$ or 4, then $A(\mathscr{P}^\beta_{(s,0)})=S_5\times C_2$ or $[1\ 1\ 2]^4\ltimes C_2$, respectively. Hence $\mathscr{P}^\beta_{(s,0)}$ is not universal in its class and $A(\mathscr{P}^\beta)$ has index 2 or ∞ in the group of the universal polytope, respectively. In the remaining cases we have

$$\{\{3,3\},\{3,6\}_{(2,2)}\}\xrightarrow[6]{\beta}\{\{6,3\}_{(2,0)},\{3,6\}_{(2,0)}\},$$

$$\{\{3,3\},\{3,6\}_{(3,0)}\}\xrightarrow[6]{\beta}\{\{6,3\}_{(1,1)},\{3,6\}_{(1,1)}\}.$$

4.3. *The types* $\{3,3,6\}$ *and* $\{4,3,6\}$

In Section 3 the hyperbolic honeycomb $\{4,3,6\}$ was constructed from $\{3,3,6\}$ by clustering tetrahedra in fives. For the groups this implied an application of the operation γ in (14) followed by a twisting operation κ as in (16).

Now, let $\mathscr{P}$ be a regular polytope in $\langle\{3,3\},\{3,6\}_{(s,t)}\rangle$ with group $A(\mathscr{P})=\langle\sigma_0,\ldots,\sigma_3\rangle$. We cannot generally expect to obtain a new regular polytope of type $\{4,3,6\}$ from $\mathscr{P}$ by applying the corresponding operations γ and κ. Clearly γ can be defined as before by

$$\gamma:(\sigma_0,\ldots,\sigma_3)\mapsto(\sigma_0,\sigma_2,\sigma_1,\sigma_3\sigma_2\sigma_3\sigma_2\sigma_3)=:(\rho_0,\ldots,\rho_3),$$

but in general the resulting group will not admit a suitable group automorphism τ to allow a twisting operation. Equivalently, in general the clustering of the tetrahedral facets of $\mathscr{P}$ in fives will only give a polytope with cubical facets which is not regular. The following explains why the construction fails.

First, note that the construction gives new vertex-figures of two kinds: old vertex-figures $\{3, 6\}_{(s,t)}$, and new vertex-figures obtained from the old vertex-figure $\{3, 6\}_{(s,t)}$ by clustering faces $\{3\}$ in fours. If s is even (and $t = 0$ or $t = s$), this latter vertex-figure is of type $\{3, 6\}_{(s/2,\, t/2)}$; otherwise it is again of type $\{3, 6\}_{(s,t)}$ and takes every face of the old vertex-figure four times (but with roles switched). The two kinds of vertex-figures correspond to the subgroups $\langle \rho_0, \rho_2, \rho_3 \rangle = \sigma_3 \langle \sigma_0, \sigma_1, \sigma_2\sigma_3\sigma_2 \rangle \sigma_3$ and $\langle \rho_1, \rho_2, \rho_3 \rangle$ of $\langle \rho_0, \ldots, \rho_3 \rangle$, respectively; see also (12).

Hence, for the construction to give a polytope which is regular, s must be odd. But then $\sigma_3 \in \langle \rho_1, \rho_2, \rho_3 \rangle$ and hence $\langle \rho_0, \ldots, \rho_3 \rangle = A(\mathscr{P})$. But now, since σ_3 maps the facet F_3 of $\mathscr{P}$ to its neighbour, the clustering process covers each facet of $\mathscr{P}$ five times (but with roles switched). It follows that each edge of $\mathscr{P}$ is also a new edge. However, only in case $(s, t) = (1, 1)$ or $(s, t) = (3, 0)$ are the faces $\{3\}$ of the (second kind of) new vertex-figures equivalent under the group to the faces $\{3\}$ of the old vertex-figures, so only here can there be a twisting operation. It follows that the construction gives a regular polytope only if $(s, t) = (1, 1)$ or $(s, t) = (3, 0)$.

However, there are other interesting cases where the clustering process gives a non-regular polytope of type $\{4, 3, 6\}$. For example, if $\mathscr{P} = \{\{3, 3\}, \{3, 6\}_{(4,0)}\}$ and thus $A(\mathscr{P}) = [1\ 1\ 2]^4 \ltimes C_2$, we obtain a polytope with 80 vertices (at 16 of which the vertex-figure is $\{3, 6\}_{(4,0)}$ while at the remaining 64 it is $\{3, 6\}_{(2,0)}$), 256 edges, 384 square faces and 128 cubical facets.

5. Local regular tessellations in polytopes

In this section we shall associate with a regular polytope $\mathscr{P}$ of rank 4 a regular map $\mathscr{H} = \mathscr{H}(\mathscr{P})$ on a surface. In several cases this map is a regular tessellation on the 2-sphere or in the euclidean or hyperbolic plane. In a sense which we make precise below, $\mathscr{H}$ cuts right through the polytope. It is remarkable that for many (but not all) classes of polytopes the polytope is finite if and only if the map is finite (or the tessellation spherical, respectively). The exceptional cases indicate that the tessellations are (in a sense) only "locally inscribed" into the polytopes. Therefore the corresponding finiteness criteria (of Theorems 4, 5, 6) are only local criteria. It is worth pointing out that our considerations below will not imply the existence of the polytopes; in this respect we need to refer to earlier constructions.

Similar results have been studied in [23] for polytopes of type $\{3, 4, 4\}$, $\{4, 4, 3\}$ and $\{4, 4, 4\}$. Here we discuss the types $\{3, 3, 6\}$, $\{6, 3, 6\}$ and $\{3, 6, 3\}$.

5.1. *The types* $\{p, 3, 6\}$

Let $\mathscr{P}$ be a regular 4-polytope of type $\{p, q, r\}$ with group $A(\mathscr{P}) = \langle \rho_0, \ldots, \rho_3 \rangle$. Assume that the vertex-figures of $\mathscr{P}$ have 3-chains of lengths l (say). Then the operation

$$(\rho_0, \ldots, \rho_3) \mapsto (\rho_0, \rho_1, \rho_2\rho_3\rho_2\rho_3\rho_2) =: (\psi_0, \psi_1, \psi_2) \tag{29}$$

gives a subgroup $\langle \psi_0, \psi_1, \psi_2 \rangle$ of $A(\mathscr{P})$ which is the group of a regular map $\mathscr{H} = \mathscr{H}(\mathscr{P})$ of type $\{p, l\}$. Note for this that $\psi_1\psi_2 = \rho_1\rho_2(\rho_3\rho_2)^2$.

Let $F_0, \ldots, F_3$ be the (proper) faces in the base flag of $\mathscr{P}$. The map $\mathscr{H}$ can be constructed from $\mathscr{P}$ by Wythoff's construction with initial vertex F_0 (which is fixed by ψ_1 and ψ_2); see [7]. Then $\{F_0, F_1, F_2\}$ becomes the base flag of $\mathscr{H}$. The neighbouring vertices of F_0 in $\mathscr{H}$ are

$$(\psi_1\psi_2)^j(\psi_0(F_0)) = (\rho_1\rho_2(\rho_3\rho_2)^2)^j(\rho_0(F_0)) \qquad \text{for } j = 0, \ldots, l-1;$$

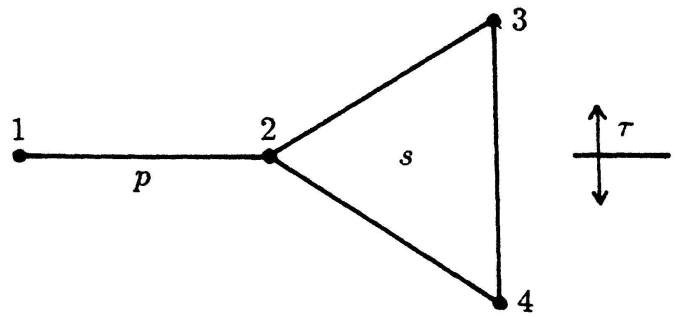
that is, as we go around F_0 in $\mathscr{H}$ we pick precisely the vertices of a 3-chain of the vertex-figure of $\mathscr{P}$ at F_0. If we span topological discs into the 2-faces of $\mathscr{H}$, we can think of $\mathscr{H}$ as a surface which in a sense cuts right through $\mathscr{P}$.

If $\mathscr{P}$ is of type $\{p, q, 6\}$, then ρ_3 commutes with ψ_0, ψ_1 and ψ_2, so that $\mathscr{H}$ is invariant under ρ_3. Hence, in a sense we can think of $\mathscr{H}$ as lying on the reflexion wall of ρ_3.

Now, let $\mathscr{P} := \{\{p, 3\}, \{3, 6\}_{(s,t)}\}$ with $p \geq 2$. We are particularly interested in the case $p = 3$, 4 or 5. The 3-chains of $\{3, 6\}_{(s,t)}$ have lengths $l = s$ or $3s$ if $t = 0$ or s, respectively. To find $\mathscr{H}$ we make use of the corresponding constructions of $\mathscr{P}$. We begin with the case $t = 0$.

5.2. *Polytopes with vertex-figures* $\{3, 6\}_{(s,0)}$

Recall from [22], Sections 4 and 5, that $\mathscr{P} = \{\{p, 3\}, \{3, 6\}_{(s,0)}\}$ can be constructed from the abstract group $W = \langle \alpha_1, \ldots, \alpha_4 \rangle$ with diagram



(30)

by the twisting operation

$$(\alpha_1, \ldots, \alpha_4; \tau) \mapsto (\alpha_1, \alpha_2, \alpha_3, \tau) =: (\rho_0, \ldots, \rho_3).$$

In particular, $A(\mathscr{P}) = W \rtimes C_2$. See (22) and (23) for the case $p = 3$.

But now

$$(\psi_0, \psi_1, \psi_2) = (\alpha_1, \alpha_2, \alpha_3\alpha_4\alpha_3), \tag{31}$$

so that the second entry in the type $\{p, s\}$ of $\mathscr{H}$ corresponds to the extra defining relation (7) of W. Equivalently, the period s of

$$\psi_1\psi_2 = \rho_1\rho_2(\rho_3\rho_2)^2 = \rho_2(\rho_1\rho_2\rho_3)^2\rho_2 \sim (\rho_1\rho_2\rho_3)^2$$

genuinely does specify the original polytope. We proceed with the following lemma which is of interest in its own right.

LEMMA 1. *Let $p, s \geq 2$ and $W = \langle \alpha_1, \ldots, \alpha_4 \rangle$ be the abstract group with diagram* (30). *Then the subgroup $\langle \alpha_1, \alpha_2, \alpha_3\alpha_4\alpha_3 \rangle$ of W is isomorphic to the Coxeter group with diagram* $\bullet\overset{}{\underset{p}{-\!-\!-}}\bullet\underset{s}{-\!-\!-}\bullet$.

Proof. The proof uses the geometric representation for W described in [22]; see also [4]. In [22] the representation was only considered for $p \leq 5$ but the methods extend generally.

Consider the sesquilinear form h on complex 4-space C^4,

$$h(x, y) = \sum_{i=1}^{4} x_i\bar{y}_i - \frac{1}{2}\sum_{i \neq j} c_{ij}x_i\bar{y}_j, \tag{32}$$

with

$$c_{13} = c_{14} = 0, \qquad c_{23} = c_{24} = 1, \qquad c_{12} = 2\cos\frac{\pi}{p}, \qquad c_{34} = e^{2\pi i/s} =: c_s. \tag{33}$$

(Note that the diagrams in [22] are mirror images of our diagram (30).) Let $a_1, \ldots, a_4$ be the canonical basis of C^4. For $k = 1, \ldots, 4$ define the linear map $R_k : C^4 \mapsto C^4$ by

$$R_k(x) = x - 2h(x, a_k)a_k. \tag{34}$$

Then $R_1, \ldots, R_4$ are hyperplane reflexions which preserve h, and the map $\alpha_i \mapsto R_i$ $(i = 0, \ldots, 3)$ defines a homomorphism f: $W \mapsto U := \langle R_1, \ldots, R_4 \rangle$. In particular, W is finite if and only if the corresponding hermitian form $h(x) := h(x, x)$ is positive definite. In this case (but probably also in other cases) the representation of W is faithful.

Define $S_1 := R_1$, $S_2 := R_2$ and $S_3 := R_3 R_4 R_3$, so that $\langle S_1, S_2, S_3 \rangle$ corresponds to $\langle \alpha_1, \alpha_2, \alpha_3\alpha_4\alpha_3 \rangle$ under f. Then S_1, S_2 and S_3 are the reflexions orthogonal to $b_1 := a_1$, $b_2 := a_2$ and $b_3 := R_3(a_4) = a_4 + \bar{c}_s a_3$, respectively. In particular, $S_k(x) = x - 2h(x, b_k)b_k$ for all k. It follows that $\langle S_1, S_2, S_3 \rangle$ leaves invariant the subspace $\langle b_1, b_2, b_3 \rangle$ whose orthogonal complement $\langle b_1, b_2, b_3 \rangle^\perp$ is the intersection of the reflexion hyperplanes of S_1, S_2 and S_3. Since $\langle S_1, S_2, S_3 \rangle$ acts trivially on $\langle b_1, b_2, b_3 \rangle^\perp$, it suffices to study its action on $\langle b_1, b_2, b_3 \rangle$.

Now, the Gram matrix of b_1, b_2, b_3 is given by

$$(h(b_i, b_j))_{i,j} = \begin{pmatrix} 1 & -\cos\dfrac{\pi}{p} & 0 \\ -\cos\dfrac{\pi}{p} & 1 & -\frac{1}{2} - \frac{1}{2}c_s \\ 0 & -\frac{1}{2} - \frac{1}{2}\bar{c}_s & 1 \end{pmatrix}.$$

But $(1 + c_s)(1 + \bar{c}_s) = 4\cos^2(\pi/s)$, so that $\lambda := 2(1 + \bar{c}_s)^{-1}\cos(\pi/s)$ has unit modulus. It follows that the Gram matrix for the new basis $b_1, b_2, \lambda b_3$ has the form

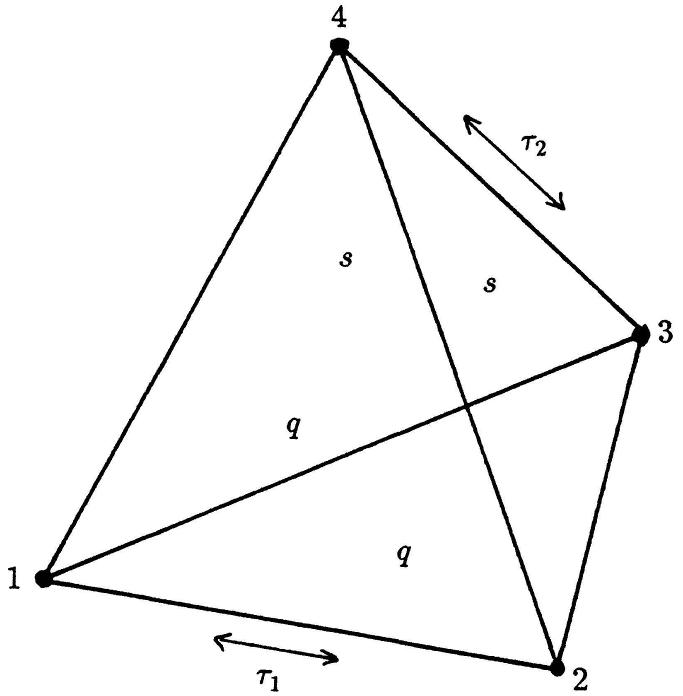
$$\begin{pmatrix} 1 & -\cos\dfrac{\pi}{p} & 0 \\ -\cos\dfrac{\pi}{p} & 1 & -\cos\dfrac{\pi}{s} \\ 0 & -\cos\dfrac{\pi}{s} & 1 \end{pmatrix}.$$

But this is the familiar matrix for the Coxeter group $[p, s]$. Hence both $\langle S_1, S_2, S_3 \rangle$ and $\langle \alpha_1, \alpha_2, \alpha_3\alpha_4\alpha_3 \rangle$ are isomporphic to this group.

THEOREM 4. *Let* $p, s \geq 2$ *and* $\mathcal{P} := \{\{p, 3\}, \{3, 6\}_{(s,0)}\}$. *Then* $\mathcal{H}(\mathcal{P}) = \{p, s\}$, *and thus is finite if and only if* $1/p + 1/s > 1/2$.

Theorem 4 follows immediately from the above lemma and our previous remarks on $\mathcal{H}(\mathcal{P})$. It is interesting to note here that a polytope $\mathcal{P}$ can be infinite even though its tessellation $\mathscr{H}(\mathscr{P})$ is finite. In Theorem 4, if $p \geq 3$, this occurs precisely for the polytopes $\{\{3,3\},\{3,6\}_{(5,0)}\}$, $\{\{4,3\},\{3,6\}_{(3,0)}\}$ and $\{\{5,3\},\{3,6\}_{(3,0)}\}$ which have (locally) "inscribed" finite tessellations $\{3,5\}$, $\{4,3\}$ and $\{5,3\}$, respectively. The proof of Lemma 1 explains why this happens. In fact, in the geometric representation of the corresponding group W, the subgroup which defines $\mathscr{H}(\mathscr{P})$ is a finite unitary (indeed even euclidean) reflexion group, while W itself is not a finite unitary group. In other words, the hermitian form restricts to a positive definite form on the corresponding 3-space but is not positive definite on the whole space.

We continue our discussion with the polytopes $\mathscr{P} = \{\{6,3\}_{(q,r)}, \{3,6\}_{(s,0)}\}$. Here $\mathscr{H}(\mathscr{P})$ is of type $\{6, s\}$. First we consider the case $r = 0$. Recall from [22], Section 5.3, that $\mathscr{P} = \{\{6,3\}_{(q,0)}, \{3,6\}_{(s,0)}\}$ can be constructed from the abstract group $W = \langle \alpha_1, \ldots, \alpha_4 \rangle$ with diagram



(35)

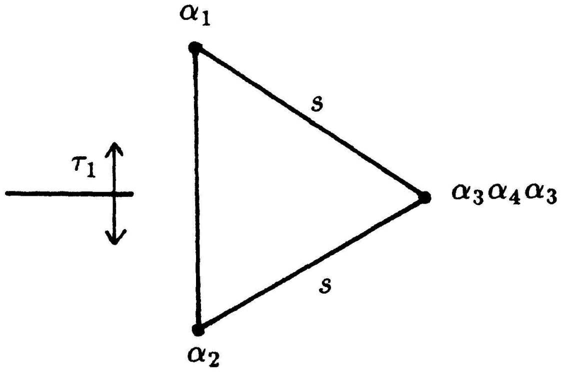
by the twisting operation

$$\kappa : (\alpha_1, \ldots, \alpha_4; \tau_1, \tau_2) \mapsto (\tau_1, \alpha_2, \alpha_3, \tau_2) =: (\rho_0, \ldots, \rho_3). \tag{36}$$

Then $A(\mathscr{P}) = W \rtimes (C_2 \times C_2)$. For the generators of $A(\mathscr{H}(\mathscr{P}))$ this implies

$$(\psi_0, \psi_1, \psi_2) = (\tau_1, \alpha_2, \alpha_3\alpha_4\alpha_3). \tag{37}$$

Hence, $A(\mathscr{H}(\mathscr{P})) = Z \rtimes C_2$ with $Z := \langle \alpha_1, \alpha_2, \alpha_3\alpha_4\alpha_3 \rangle$ and $C_2 = \langle \tau_1 \rangle$. Note that the subgroup Z of $A(\mathscr{H}(\mathscr{P}))$ can be associated with the diagram



(38)

The map $\mathscr{H}(\mathscr{P})$ itself is constructed from Z by the twisting operation

$$\kappa : (\alpha_1, \alpha_2, \alpha_3\alpha_4\alpha_3; \tau_1) \mapsto (\psi_0, \psi_1, \psi_2). \tag{39}$$

As in Lemma 1, to obtain more information on $\mathscr{H}(\mathscr{P})$ we consider the geometric representation $f: W \mapsto U = \langle R_1, \ldots, R_4 \rangle$ given by a sesquilinear form h as in (32), but now with coefficients

$$c_{12} = e^{2\pi i/q} =: c_q, \qquad c_{13} = c_{14} = c_{23} = c_{24} = 1, \qquad c_{34} = e^{2\pi i/s} =: c_s \tag{40}$$

([22], Section 5.3). This representation is known to be faithful if W is finite, that is, if h is positive definite. We conjecture that this is true for all q and s.

As in the proof of Lemma 1, $S_1 := R_1$, $S_2 := R_2$ and $S_3 := R_3R_4R_3$ are the reflexions orthogonal to $b_1 := a_1$, $b_2 := a_2$ and $b_3 = a_4 + \bar{c}_s a_3$, respectively. Now the Gram matrix of b_1, b_2, b_3 is given by

$$(h(b_i, b_j))_{i,j} = \begin{bmatrix} 1 & -\frac{1}{2}c_q & -\frac{1}{2}-\frac{1}{2}c_s \\ -\frac{1}{2}\bar{c}_q & 1 & -\frac{1}{2}-\frac{1}{2}c_s \\ -\frac{1}{2}-\frac{1}{2}\bar{c}_s & -\frac{1}{2}-\frac{1}{2}\bar{c}_s & 1 \end{bmatrix}. \tag{41}$$

This time we cannot simply rescale the base vectors to transform the matrix into a real matrix. In fact, the determinant Δ of the Gram matrix is given by

$$8\Delta = 6 - 8\left(1 + 2\cos^2\frac{\pi}{q}\right)\cos^2\frac{\pi}{s},$$

so it genuinely depends on both s and q. It follows that

$$\begin{cases} \Delta > 0 & \text{if } s = 2 \text{ (and } q \text{ arbitrary); } s = 3 \text{ and } q < \infty\text{; or, } (s, q) = (4, 2) \\ & \text{or } (5, 2); \\ \Delta = 0 & \text{if } (s, q) = (3, \infty), (4, 3) \text{ or } (6, 2); \\ \Delta < 0 & \text{otherwise.} \end{cases}$$

Accordingly, the restriction h_1 (say) of h to the subspace $\langle b_1, b_2, b_3 \rangle$ is positive definite, positive semi-definite, or indefinite, respectively.

If $\Delta > 0$ then h_1 defines a unitary metric and $\langle S_1, S_2, S_3 \rangle$ becomes a unitary reflexion group. If $s = 2$ then it is clear that $Z \simeq S_3 \times C_2$, the Coxeter group with diagram (38). This remains true for $\langle S_1, S_2, S_3 \rangle = f(Z)$.

Let $s = 3$. By computing the eigenvalues of $S_1 S_2 S_3 S_2$ we find that $S_1 S_2 S_3 S_2$ has order q; for similar computations see [22], equation (12). It follows that $\langle S_1, S_2, S_3 \rangle \simeq [1\ 1\ 1]^q$, or order $6q^2$. Note that h_1 is positive semi-definite if $q = \infty$, in agreement with the fact that the unmarked triangle is the diagram of a euclidean Coxeter group; here, the Gram matrix can be made real. We do not know if indeed $Z \simeq \langle S_1, S_2, S_3 \rangle$.

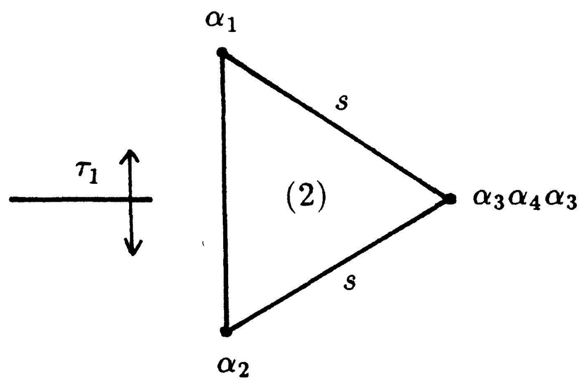
Let $q = 2$ (and s be arbitrary). Then in Z we have

$$(\alpha_2 \alpha_1 \alpha_2 \alpha_3) \alpha_4 \alpha_3 = \alpha_3 \alpha_2 \alpha_1 \alpha_2 \alpha_4 \alpha_3 \sim \alpha_2 \alpha_1 \alpha_2 \alpha_4,$$

so that

$$(\alpha_2 \alpha_1 \alpha_2 \cdot \alpha_3 \alpha_4 \alpha_3)^2 = 1. \tag{42}$$

It follows that Z can be associated with the diagram



(43)

(Here the parentheses around the mark 2 indicate that the corresponding relation is not necessarily equivalent to any of the five other relations obtained by permuting the generators.) Now, choosing the new generators $\beta_0 := \alpha_3\alpha_4\alpha_3$, $\beta_1 := \alpha_1$ and $\beta_2 := \alpha_1\alpha_2\alpha_1$ for Z, and using (42), shows that Z is a quotient of the group with diagram $\overset{0}{\bullet}\underset{s}{\text{—}}\overset{1}{\bullet}\text{—}\overset{2}{\bullet}$. We shall prove that $Z = [s, 3]$.

First, note that $T_0 := f(\beta_0) = R_3R_4R_3$, $T_1 := f(\beta_1) = R_1$ and $T_2 := f(\beta_2) = R_1R_2R_1$ generate $\langle S_1, S_2, S_3\rangle$. They are the reflexions orthogonal to $d_1 := a_4 + \bar{c}_s a_3 = b_3$, $d_2 := a_1 = b_1$ and $d_3 := a_2 + \bar{c}_q a_1 = a_2 - a_1$, respectively. The Gram matrix of d_1, d_2, d_3 is given by

$$\begin{bmatrix} 1 & -\frac{1}{2}(1+\bar{c}_s) & 0 \\ -\frac{1}{2}(1+c_s) & 1 & -\frac{1}{2} \\ 0 & -\frac{1}{2} & 1 \end{bmatrix}.$$

As in the proof of Lemma 1 we can change the basis d_1, d_2, d_3 to obtain the Gram matrix for $[s, 3]$. Hence $[s, 3] = \langle S_1, S_2, S_3\rangle = Z$.

Interesting special cases arise for $(s, q) = (3, 2)$, $(4, 2)$, $(5, 2)$ and $(6, 2)$. Then $Z = [3, 3]$, $[4, 3]$, $[5, 3]$ or $[6, 3]$, respectively. Note that only in the first two cases the form h itself is positive definite. For $(s, q) = (3, 2)$ we have $[3, 3] = S_4 = [1\ 1\ 1]^2$, in agreement with our above results.

Another way to find the structure of $\mathscr{H}(\mathscr{P})$ for $q = 2$ is obtained as follows. Recall from [22], Lemma 1, that $\mathscr{P} = \{\{6, 3\}_{(2,0)}, \{3, 6\}_{(s,0)}\}$ is related to $\mathscr{P}_{(s,0)} = \{\{3, 3\}, \{3, 6\}_{(s,0)}\}$ by $A(\mathscr{P}) = A(\mathscr{P}_{(s,0)}) \times C_2$; if $A(\mathscr{P}_{(s,0)}) = \langle \kappa_0, \ldots, \kappa_3\rangle$ and $C_2 = \langle \alpha \rangle$ (say), then

$$(\rho_0, \ldots, \rho_3) = (\kappa_0\alpha, \kappa_1, \kappa_2, \kappa_3)$$

are the generators for $A(\mathscr{P})$ and $\alpha = (\rho_0\rho_1)^3$. It follows that $\mathscr{H}(\mathscr{P})$ and $\mathscr{H}(\mathscr{P}_{(s,0)})$ are similarly related. But then Theorem 4 implies $A(\mathscr{H}(\mathscr{P})) = A(\mathscr{H}(\mathscr{P}_{(s,0)})) \times C_2 = [3, s] \times C_2$. Note that the semi-direct product $Z \rtimes C_2$ becomes direct; in fact, the automorphism τ_1 in (38) can be realized by conjugation with $\alpha_1\alpha_2\alpha_1 = \beta_2$.

In the remaining cases for (s, q) the group $\langle S_1, S_2, S_3\rangle$ becomes infinite, since h_1 is positive semi-definite or indefinite. For $(s, q) = (4, 3)$ the form h_1 is positive semi-definite and $\langle S_1, S_2, S_3\rangle$ acts as a (possibly non-discrete) unitary reflexion group in a unitary plane. In the other cases the non-finiteness follows from the irreducibility of $\langle S_1, S_2, S_3\rangle$ ([22], Lemma 3). Now we can prove the following result. See [3] for the notation in part (a).

THEOREM 5. *Let $q, s \geq 2$ and $\mathscr{P} := \{\{6, 3\}_{(q,0)}, \{3, 6\}_{(s,0)}\}$. Then $\mathscr{H}(\mathscr{P})$ is of type $\{6, s\}$.*

(a) *If $q = 2$ then $\mathscr{H}(\mathscr{P}) = \{s, 6|, 2\}^*$, the dual of $\{s, 6|, 2\}$. Also $A(\mathscr{H}(\mathscr{P})) = [s, 3] \times C_2$, and thus $\mathscr{H}(\mathscr{P})$ is finite if and only if $s \leq 5$.*

(b) *If* $s = 2$ *then* $\mathcal{H}(\mathcal{P}) = \{6, 2\}$. *If* $s = 3$ *and* $q < \infty$, *then* $\mathcal{H}(\mathcal{P})$ *has a projection onto* $\{6, 3\}_{(q,0)}$. (*Most likely*, $\mathcal{H}(\mathcal{P}) = \{6, 3\}_{(q,0)}$.) *If* $s = 3$ *and* $q = \infty$, *then* $\mathcal{H}(\mathcal{P}) = \{6, 3\}$.

(c) *In all other cases* $\mathcal{H}(\mathcal{P})$ *is infinite.*

Proof. Recall that $\mathcal{H}(\mathcal{P})$ is constructed from Z by (39). For (a) we can use (42) to obtain

$$\psi_2\psi_1(\psi_0\psi_1)^2 = \alpha_3\alpha_4\alpha_3\alpha_2\alpha_1\alpha_2 \sim \alpha_1\alpha_2(\alpha_3\alpha_4\alpha_3)\alpha_2$$

and thus $(\psi_2\psi_1(\psi_0\psi_1)^2)^2 = 1$. Since $Z = [s, 3]$, this is the only extra relation used to define $\mathcal{H}(\mathcal{P})$. This proves (a). The remaining parts follow immediately from what was said above.

Note that in Theorem 5(b) the faithfulness of the representation of W would imply $\mathcal{H}(\mathcal{P}) = \{6, 3\}_{(q,0)}$. (In fact, for this it suffices to generalize (42) to $(\alpha_2\alpha_1\alpha_2\alpha_3\alpha_4\alpha_3)^q = 1$.) Note further that there are again instances where $\mathcal{H}(\mathcal{P})$ is finite but $\mathcal{P}$ is infinite. An interesting example arises for $(s, q) = (5, 2)$.

We shall not proceed to discuss the case $\mathcal{P} = \{\{6, 3\}_{(q,q)}, \{3, 6\}_{(s,0)}\}$ in full generality. Limited information is available from the fact that the homomorphism $A(\{6, 3\}_{(q,q)}) \mapsto A(\{6, 3\}_{(q,0)})$ induces a homomorphism between the groups of the corresponding maps. In particular it follows that parts (b) and (c) of Theorem 5 carry over (but $\mathcal{H}(\mathcal{P})$ might possibly be $\{6, 3\}_{(q,q)}$).

5.3. *Polytopes with vertex-figures* $\{3, 6\}_{(s,s)}$

We continue our discussion with the polytopes $\mathcal{P} = \{\{p, 3\}, \{3, 6\}_{(s,s)}\}$, but restrict ourselves mainly to the case $p = 3$. Now $\mathcal{H}(\mathcal{P})$ is of type $\{p, 3s\}$. Here the length $3s$ of the 3-chains does not determine the vertex-figure (among maps of type $\{3, 6\}$), so that it is not obvious what the structure of $\mathcal{H}(\mathcal{P})$ is.

Now let $p = 3$. Recall that $\mathcal{P} = \mathcal{P}_{(s,s)} = \{\{3, 3\}, \{3, 6\}_{(s,s)}\}$ was constructed from the group $W = \langle\alpha_1, \ldots, \alpha_4\rangle$ with diagram (17) by the twisting operation (18). Rename the generators $\sigma_0, \ldots, \sigma_3$ of (18) by $\rho_0, \ldots, \rho_3$, so that $A(\mathcal{P}) = \langle\rho_0, \ldots, \rho_3\rangle$, as above. Then

$$(\rho_0, \ldots, \rho_3) = (\tau_3, \tau_2, \tau_1, \alpha_1),$$

so that the generators of $A(\mathcal{H}(\mathcal{P}))$ are given by

$$(\psi_0, \psi_1, \psi_2) = (\tau_3, \tau_2, \tau_1\alpha_1\alpha_2).$$

As we remarked above, $\rho_3 = \alpha_1$ commutes with ψ_0, ψ_1 and ψ_2 and thus with each element of $Z := A(\mathscr{H}(\mathscr{P})) \cap W$.

To obtain more information on $\mathscr{H}(\mathscr{P})$ we use the geometric representation for W described in [22], Section 4.2. Again we consider a hermitian form h as in (32), but now with coefficients

$$c_{12} = c_{34} = c_{31} = e^{2\pi i/s} =: c_s, \qquad c_{23} = c_{24} = c_{41} = \bar{c}_s. \tag{44}$$

Then h is positive definite, positive semi-definite or indefinite if $s = 2$, $s = 3$ or $s \geq 4$, respectively. As in the proof of Lemma 1 we have a homomorphism $f\colon W \mapsto U := \langle R_1, \ldots, R_4 \rangle$, with R_k as in (34).

But now $R_1 = f(\alpha_1)$ commutes with each element of $f(Z)$, so that $f(Z)$ stabilizes the reflexion hyperplane of R_1, that is, the orthogonal complement $a_1^{\perp}$ of a_1. It follows that $f(Z)$ acts (probably faithfully) on $a_1^{\perp}$ as a group U_1 (say) of isometries with respect to the hermitian form h_1 (say) which is the restriction of h to $a_1^{\perp}$. But the geometry on $a_1^{\perp}$ is completely determined by h; that is, h_1 is positive definite, positive semi-definite or indefinite if $s = 2$, $s = 3$ or $s \geq 4$, respectively.

If $s = 2$, then $W = S_5$ and $A(\mathscr{P}) = S_5 \times S_4$, with $\alpha_i = (i\ 5)$ for $i = 1, \ldots, 4$, $\tau_1 = (1\ 2)(6\ 7)$, $\tau_2 = (2\ 3)(7\ 8)$ and $\tau_3 = (3\ 4)(8\ 9)$. Here it is easy to check directly that $\mathscr{H}(\mathscr{P}) = \{3, 6\}_{(2,2)}$. In this case $Z = \langle (1\ 5)(2\ 3), (1\ 5)(3\ 4) \rangle \simeq S_3$.

If $s \geq 4$ then U_1 acts irreducibly on $a_1^{\perp}$. Note for this that elements like $(R_3 R_1 R_2)^2 = f((\psi_1 \psi_2)^3)$ or $(R_4 R_1 R_3)^2 = f((\psi_0 \psi_1 \psi_2 \psi_1)^3)$ belong to $f(Z)$ and have order s. But an irreducible isometry group for a non-degenerate indefinite hermitian form must necessarily be infinite; see Lemma 3 of [22]. It follows that U_1, Z and $A(\mathscr{H}(\mathscr{P}))$ are infinite groups. In particular, $\mathscr{H}(\mathscr{P})$ is an infinite map.

In case $s = 3$ we have a positive semi-definite form h. Here W has an infinite discrete unitary representation g (say) on C^3. The same kind of arguments as for f show that $g(Z)$ acts on the reflexion plane of $g(\alpha_1)$ as a 2-dimensional discrete unitary group. Since no point is invariant under $g(Z)$, we must have an infinite group. It follows again that $\mathscr{H}(\mathscr{P})$ is infinite. We summarize our results in the following theorem.

THEOREM 6. *For the polytopes $\mathscr{P}_{(s,s)} = \{\{3, 3\}, \{3, 6\}_{(s,s)}\}$ the map $\mathscr{H}(\mathscr{P}_{(s,s)})$ is of type $\{3, 3s\}$. In particular, $\mathscr{H}(\mathscr{P}_{(2,2)}) = \{3, 6\}_{(2,2)}$ and $\mathscr{H}(\mathscr{P}_{(s,s)})$ is infinite if $s \geq 3$.*

Note that in Theorem 6 the polytope $\mathscr{P}_{(s,s)}$ is finite if and only if $\mathscr{H}(\mathscr{P}_{(s,s)})$ is finite. For $s \geq 6$ another proof of the non-finiteness of $\mathscr{H}(\mathscr{P}_{(s,s)})$ can be obtained as follows. First observe that for $s \geq 2$ the polytope $\mathscr{P}_{(s,0)} := \{\{3, 3\}, \{3, 6\}_{(s,0)}\}$ is a quotient of $\mathscr{P}_{(s,s)}$. The corresponding homomorphism $A(\mathscr{P}_{(s,s)}) \mapsto A(\mathscr{P}_{(s,0)})$ induces a homomorphism $A(\mathscr{H}(\mathscr{P}_{(s,s)})) \mapsto A(\mathscr{H}(\mathscr{P}_{(s,0)}))$. Hence, $\mathscr{H}(\mathscr{P}_{(s,s)})$ (of type $\{3, 3s\}$)

projects onto $\mathscr{H}(\mathscr{P}_{(s,0)}) = \{3, s\}$. Note that similarly $\mathscr{H}(\mathscr{P}_{(s,s)})$ is derived from $\mathscr{H}(\mathscr{P}_{(3s,0)}) = \{3, 3s\}$ by identifications induced by a homomorphism $A(\mathscr{P}_{(3s,0)}) \mapsto A(\mathscr{P}_{(s,s)})$.

It is likely that Theorem 6 and its proof carry over to the polytopes $\mathscr{P} = \{\{p, 3\}, \{3, 6\}_{(s,s)}\}$ with arbitrary p. For $p \geq 4$ and $s \geq 2$ these polytopes are infinite. For the corresponding geometric representation see [22], Section 5.2. Note that by Theorem 4 the map $\mathscr{H}(\mathscr{P})$ (of type $\{p, 3s\}$) has a projection onto $\{p, s\}$.

A similar remark applies to the (infinite) polytopes $\{\{6, 3\}_{(q,r)}, \{3, 6\}_{(s,s)}\}$. Here $\mathscr{H}(\mathscr{P})$ is of type $\{6, 3s\}$ and is likely to be infinite except for $(q, r, s) = (2, 0, 2)$. For the representation see [22], Section 5. Here projections are obtained from Theorems 5 and 6.

5.4. *Other tessellations for polytopes of types* $\{3, 3, 6\}$ *and* $\{3, 6, 3\}$

There are other ways of associating a regular map with a regular polytope $\mathscr{P}$ in $\langle\{3, 3\}, \{3, 6\}_{(s,t)}\rangle$. Consider the operation

$$(\rho_0, \dots, \rho_3) \mapsto (\rho_0, \rho_1\rho_2\rho_3\rho_2\rho_1, \rho_3\rho_2\rho_3\rho_2\rho_3) =: (\kappa_0, \kappa_1, \kappa_2) \tag{45}$$

on $A(\mathscr{P}) = \langle \rho_0, \dots, \rho_3 \rangle$. Then

$$\begin{aligned}\kappa_0\kappa_1 &\sim \rho_1\rho_0\rho_1 \cdot \rho_2\rho_3\rho_2 = \rho_0\rho_1\rho_0\rho_2\rho_3\rho_2 = \rho_0 \cdot \rho_1\rho_2\rho_3\rho_2 \cdot \rho_0 \\ &\sim \rho_1\rho_2\rho_3\rho_2 \sim \rho_2\rho_1\rho_2\rho_3 = \rho_1\rho_2\rho_1\rho_3 = \rho_1\rho_2\rho_3\rho_1 \sim \rho_2\rho_3\end{aligned}$$

and

$$\kappa_1\kappa_2 = \rho_1\rho_2\rho_3\rho_2\rho_1 \cdot \rho_3\rho_2\rho_3\rho_2\rho_3 = (\rho_1(\rho_2\rho_3)^2)^2.$$

It follows that $A := \langle \kappa_0, \kappa_1, \kappa_2 \rangle$ is the group of a regular map $\mathscr{M}(\mathscr{P})$ of type $\{6, s\}$. Note that the period of $\kappa_1\kappa_2$ is s in both cases $t = 0$ and $s = t$; see (3).

An application of Wythoff's construction with initial vertex the base vertex F_0 of $\mathscr{P}$ shows that $\mathscr{M}(\mathscr{P})$ is a map with vertices and edges among those of $\mathscr{P}$. Note that ρ_2 commutes with κ_0, κ_1 and κ_2, so that in a sense $\mathscr{M}(\mathscr{P})$ lies on the reflexion wall of ρ_2.

Let $\mathscr{P} = \mathscr{P}_{(s,s)} = \{\{3, 3\}, \{3, 6\}_{(s,s)}\}$. Then by (3) the condition $(\kappa_1\kappa_2)^s = 1$ specifies the group of the original polytope. Using the construction of $\mathscr{P}_{(s,s)}$ in (17) and (18) we find that

$$(\kappa_0, \kappa_1, \kappa_2) = (\tau_3, \alpha_3, \alpha_1\alpha_2\alpha_1),$$

so that $A = \langle \alpha_1\alpha_2\alpha_1, \alpha_3, \alpha_4 \rangle \ltimes \langle \tau_3 \rangle$. This situation is analogous to that of (35) and (37) with $q = s$. In particular,

$$\mathcal{M}(\mathcal{P}_{(s,s)}) = \mathcal{H}(\{\{6, 3\}_{(s,0)}, \{3, 6\}_{(s,0)}\}),$$

so that Theorem 5 applies.

For $\mathcal{P} = \{\{3, 3\}, \{3, 6\}_{(s,0)}\}$ the condition $(\kappa_1\kappa_2)^s = 1$ does not specify the group of the original polytope, so that the structure of $\mathcal{M}(\mathcal{P})$ is less obvious.

Concluding, let us remark on polytopes $\mathcal{P}$ of type $\{3, 6, 3\}$. Here the "sections" of $\mathcal{P}$ by its "reflexion walls" do not appear to yield useful information. However, in certain cases, limited information is available by other means.

Consider the operation

$$(\rho_0, \ldots, \rho_3) \mapsto (\rho_0, (\rho_1\rho_2)^3, \rho_3) =: (\chi_0, \chi_1, \chi_2) \tag{46}$$

on $A(\mathcal{P}) = \langle \rho_0, \ldots, \rho_3 \rangle$. This gives the group of a regular map $\mathcal{L}(\mathcal{P})$. Note that $\mathcal{L}(\mathcal{P}^*) = \mathcal{L}(\mathcal{P})^*$. If the facets of $\mathcal{P}$ are of type $\{3, 6\}_{(q,r)}$, then the period of $\chi_0\chi_1$ is

$$\begin{cases} q, & \text{if } r = 0, \quad q \text{ even;} \\ 2q, & \text{if } r = 0, \quad q \text{ odd;} \\ 3q, & \text{if } r = s, \quad q \text{ even;} \\ 6q, & \text{if } r = s, \quad q \text{ odd.} \end{cases}$$

Moreover, if q is odd, then $\rho_2 \in \langle \chi_0, \chi_1 \rangle$. All this is most easily seen geometrically. A similar remark applies to the vertex-figures $\{6, 3\}_{(s,t)}$. In particular, if q and s are odd, then $A(\mathcal{L}(\mathcal{P})) = A(\mathcal{P})$. It is thus only in the case $\mathcal{P} = \{\{3, 6\}_{(2m,0)}, \{6, 3\}_{(2n,0)}\}$ that the periods of $\chi_0\chi_1$ and $\chi_1\chi_2$ specify the type of the polytope; then $\mathcal{L}(\mathcal{P})$ is of type $\{2m, 2n\}$. However, no explicit construction of $\{\{3, 6\}_{(2m,0)}, \{6, 3\}_{(2n,0)}\}$ is known yet, so that we do not know the structure of the map. (It seems that Theorem 1 does not really help here.)

It is worth mentioning that the analogous operation

$$(\rho_0, \ldots, \rho_3) \mapsto (\rho_0, (\rho_1\rho_2)^2, \rho_3)$$

applied to a polytope of type $\{\{4, 4\}_{(2m,0)}, \{4, 4\}_{(2n,0)}\}$ indicates that it is finite if and only if $1/m + 1/n > 1$; we know this to be true [23]. Applied to $\{\{3, 4\}, \{4, 4\}_{(2n,0)}\}$ it similarly yields the known criterion $n \leq 1$ for finiteness; here $\rho_0(\rho_1\rho_2)^2$ has period 4 [23].

6. Flat polytopes

Recall that a regular n-polytope $\mathscr{P}$ is called *(combinatorially) flat* if each of its vertices is a vertex of each of its facets. Note that the dual of a flat polytope is also flat. Flat regular polytopes have been constructed in [30] using the so-called *degenerate amalgamation property*, or briefly, the *DAP*.

Recall that a polytope $\mathscr{P}$ with group $A(\mathscr{P}) = \langle \rho_0, \ldots, \rho_{n-1} \rangle$ is said to have the *DAP with respect to its vertex-figure* $\mathscr{P}_2$ (say) if and only if $A(\mathscr{P})$ is a semi-direct product of the normal closure $N(\rho_0)$ of ρ_0 in $A(\mathscr{P})$ by the vertex stabilizer $\langle \rho_1, \ldots, \rho_{n-1} \rangle$ (with $\langle \rho_1, \ldots, \rho_{n-1} \rangle$ acting on $N(\rho_0)$ in the obvious way); that is, if and only if $A(\mathscr{P}) = N(\rho_0) \rtimes A(\mathscr{P}_2)$. Similarly, $\mathscr{P}$ has the *DAP with respect to its facet* $\mathscr{P}_1$ (say) if and only if $A(\mathscr{P}) = N(\rho_{n-1}) \rtimes A(\mathscr{P}_1)$, with $N(\rho_{n-1})$ the normal closure of ρ_{n-1} in $A(\mathscr{P})$. Clearly, $\mathscr{P}$ has the DAP with respect to its facets if and only if $\mathscr{P}^*$ has the DAP with respect to its vertex-figures.

Note that the DAP can also be defined in terms of collapses. For example, $\mathscr{P}$ has the DAP with respect to its facet $\mathscr{P}_1$ if and only if the mapping $\rho_{n-1} \mapsto 1$ (and $\rho_i \mapsto \rho_i$ for $i \leq n-2$) induces a "collapse" of $\mathscr{P}$ onto $\mathscr{P}_1$.

The following are examples of regular maps which have the DAP with respect to their vertex-figures: $\{4, 3\}$, $\{6, 3\}_{(s,s)}$ with $s \geq 1$, and $\{6, 3\}_{(3r,0)}$ with $r \geq 1$. No other spherical or toroidal maps (excluding $\{2, p\}$) have the DAP with respect to their vertex-figures.

Below we make use of the following result ([30], Theorem 2). Let $\mathscr{L}$ and $\mathscr{R}$ be two regular $(n-1)$-polytopes such that the vertex-figures of $\mathscr{L}$ are isomorphic to the facets of $\mathscr{R}$. Assume that $\mathscr{L}$ has the DAP with respect to its vertex-figures, and $\mathscr{R}$ has the DAP with respect to its facets. Then $\langle \mathscr{L}, \mathscr{R} \rangle$ contains a flat regular n-polytope $\mathscr{P}$ which has the DAP with respect to both the facets and the vertex-figures. For an example see the end of this section.

In this section we mainly discuss locally toroidal 4-polytopes $\mathscr{P}$ with flat facets and vertex-figures. We begin with the following simple observation.

PROPOSITION 1. *Let $\mathscr{L}$ be a flat facet of a regular n-polytope $\mathscr{P}$.*

(a) *Then $\mathscr{P}$ is also flat. (In fact, $\mathscr{P}$ is flat in the stronger sense that each vertex of $\mathscr{P}$ is a vertex of each $(n-2)$-face of $\mathscr{P}$.)*

(b) *If $\mathscr{P}$ has vertex-figures $\mathscr{R}$, then $\mathscr{P} = \{\mathscr{L}, \mathscr{R}\}$ and $\mathscr{P}$ is the only polytope in $\langle \mathscr{L}, \mathscr{R} \rangle$.*

Proof. The proof of (a) is obvious, because $\mathscr{P}$ is connected and any two adjacent facets of $\mathscr{P}$ meet in an $(n-2)$-face which contains all the vertices of each facet. For (b) we know by part (a) that $\{\mathscr{L}, \mathscr{R}\}$ is flat. But any identifications in a flat universal polytope must lead to a collapse of the facets or vertex-figures. This proves (b).

THEOREM 7. *Let $\mathscr{L}$ and $\mathscr{R}$ be regular $(n-1)$-polytopes such that the vertex-figures of $\mathscr{L}$ are isomorphic to the facets of $\mathscr{R}$. Assume that $\mathscr{L}$ is flat and $A(\mathscr{L})$ acts faithfully on the vertices of $\mathscr{L}$.*

(a) *Then $\langle \mathscr{L}, \mathscr{R} \rangle \neq \varnothing$ if and only if $\mathscr{R}$ has the DAP with respect to its facets.*

(b) *If $\langle \mathscr{L}, \mathscr{R} \rangle \neq \varnothing$, then $\{\mathscr{L}, \mathscr{R}\}$ has the DAP with respect to its facets.*

Proof. Let $\mathscr{P} \in \langle \mathscr{L}, \mathscr{R} \rangle$. By Proposition 1 each vertex of $\mathscr{P}$ is a vertex of the base $(n-2)$-face F_{n-2} of $\mathscr{P}$. But the generator ρ_{n-1} acts trivially on F_{n-2}/F_{-1} and thus fixes each vertex of $\mathscr{P}$. It follows that each element of $N(\rho_{n-1})$ fixes each vertex of $\mathscr{P}$ (or equivalently, each vertex of the base facet F_{n-1} of $\mathscr{P}$). Hence, by our assumptions on $\mathscr{L}$ we must have $N(\rho_{n-1}) \cap \langle \rho_0, \ldots, \rho_{n-2} \rangle = \{1\}$. But by definition of $N(\rho_{n-1})$ we have $A(\mathscr{P}) = N(\rho_{n-1}) \cdot \langle \rho_0, \ldots, \rho_{n-2} \rangle$, so that this product is semi-direct. Therefore $\mathscr{P}$ must have the DAP with respect to its facets. But the DAP is hereditary; that is, each co-face of $\mathscr{P}$ must also have the DAP with respect to its facets. In particular this is true for the vertex-figures $\mathscr{R}$ of $\mathscr{P}$.

Now to prove (b) and one direction of (a) apply these considerations with $\mathscr{P} = \{\mathscr{L}, \mathscr{R}\}$. The other direction of (a) follows from [30]. This completes the proof.

The following result was already proved in [22] by other means.

COROLLARY. *The universal $\{\{6, 3\}_{(1,1)}, \mathscr{R}\}$*

(a) *exists for $\mathscr{R} = \{3, 4\}$, $\{3, 6\}_{(s,s)}$ with $s \geq 1$, and $\{3, 6\}_{(3r,0)}$ with $r \geq 1$;*

(b) *does not exist for $\mathscr{R} = \{3, 3\}$, $\{3, 5\}$, and $\{3, 6\}_{(s,0)}$ with $3 \nmid s$.*

Of the flat torus maps, $\mathscr{L} = \{4, 4\}_{(2,0)}$ and $\{3, 6\}_{(1,1)}$ do not satisfy the condition of Theorem 7 but $\mathscr{L} = \{6, 3\}_{(1,1)}$ does. See [23] for the discussion of the case $\{4, 4\}_{(2,0)}$.

As an example we illustrate the case $\mathscr{R} = \{3, 6\}_{(s,t)}$ geometrically. However, rather than taking $\{6, 3\}_{(1,1)}$ as a facet we shall find it more convenient to take its dual $\{3, 6\}_{(1,1)}$ as a potential vertex-figure.

First, consider the universal $\mathscr{P} = \{\{6, 3\}, \{3, 6\}_{(1,1)}\}$ (which exists by Theorem 7). We know that $\mathscr{P}$ is flat and thus has only 6 facets. Each edge belongs to each of the 6 facets. But the facets $1, \ldots, 6$ (say) cycle around the edges in three different orders, namely

$$\begin{cases} A\colon & 1\,2\,3\,4\,5\,6 \\ B\colon & 1\,6\,3\,2\,5\,4 \\ C\colon & 1\,4\,3\,6\,5\,2. \end{cases} \tag{47}$$

(Note that these are the three different ways of interleaving 1 3 5 and 2 4 6.) Figure 6 shows facet 1 which meets only facets 2, 4, 6, in the fashion of the figure; the edges

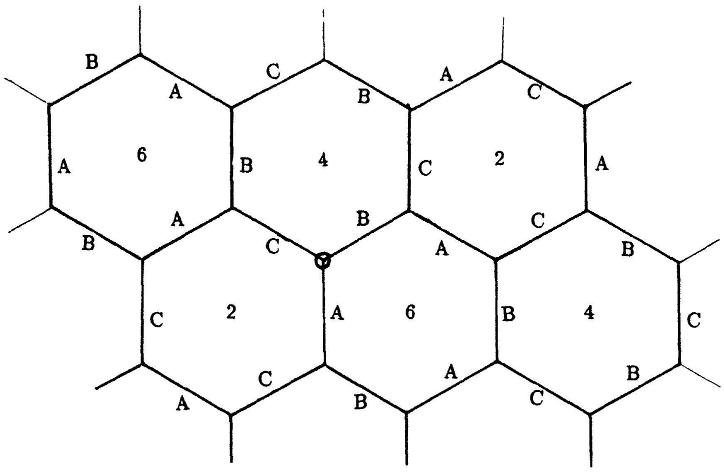


Figure 6

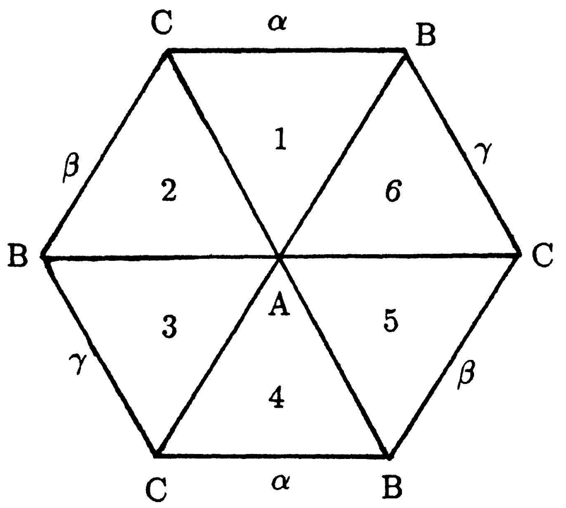


Figure 7

are labelled according to (47). Figure 7 shows the vertex-figure corresponding to the circled vertex in Figure 6.

Now, the identification $\{6,3\} \mapsto \{6,3\}_{(s,t)}$ is only compatible with this labelling in the cases $s = t \geq 1$, and $t = 0, s = 3r \geq 3$. These are precisely the cases where $\{6,3\}_{(s,t)}$ has the DAP with respect to its vertex-figures. If $t = 0$ and $3 \nmid s$, the facets $2, 4, 6$ (and $1, 3, 5$) are forced to coincide, and the polytope collapses to $\{\{6,3\}_{(s,t)}, \{3,2\}\}$. Otherwise the 6 copies of $\{6,3\}_{(s,t)}$ can be glued together according to the appropriate incidences (corresponding vertices of the copies coincide, as do their edges).

We continue our discussion with polytopes $\mathscr{P}$ of type $\{3,6,3\}$ which have vertex-figures $\{6,3\}_{(1,1)}$. Since the vertex-figures are flat, $\mathscr{P}$ can only have 3 facets.

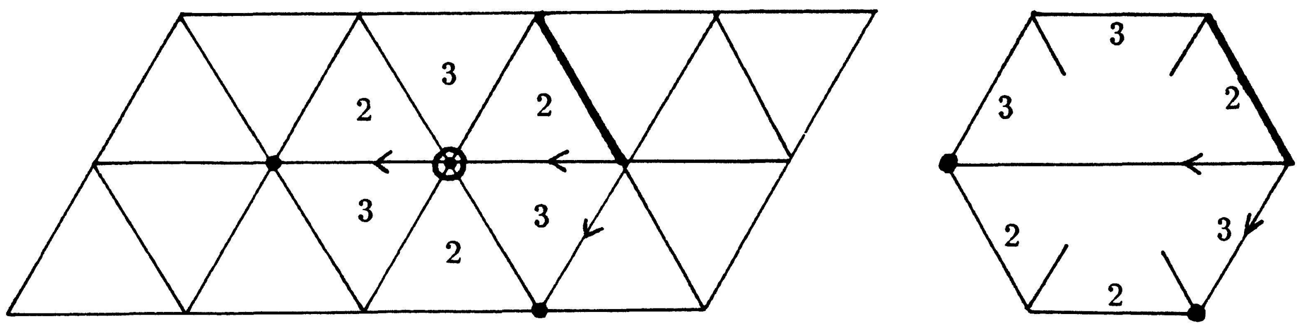


Figure 8

On a given facet 1, the incidence pattern with the two other facets 2, 3 has to be as indicated in Figure 8. Now, while facets 1, 2, 3 must have the same vertices, the reflexion ρ_3 in the group $A(\mathscr{P}) = \langle \rho_0, \ldots, \rho_3 \rangle$ does not fix all these vertices. In fact, while fixing the given (circled) vertex and also the emphasized edge, it changes edges joining "adjacent" vertices of the vertex-figure to edges joining "opposite" vertices. Thus, in the adjacent facets 2 and 3, vertices two steps away along a "straight" path are joined, which forces the facet to be a collapse of $\{3, 6\}_{(3,0)}$. It is easily verified, on the other hand, that $\{\{3, 6\}_{(3,0)}, \{6, 3\}_{(1,1)}\}$ and $\{\{3, 6\}_{(1,1)}, \{6, 3\}_{(1,1)}\}$ exist as polytopes. For the former polytope see also our Theorem 1, or [22].

More generally, let $\mathscr{P}$ be a regular 4-polytope of type $\{p, 6, 3\}$ whose vertex-figure is $\{6, 3\}_{(1,1)}$. By (3) we have $((\rho_1\rho_2)^2\rho_3)^2 = 1$, or equivalently, $(\rho_1\rho_2)^2\rho_3 = \rho_3(\rho_2\rho_1)^2$. Now, conjugation of ρ_0 and ρ_1 by $\rho_2\rho_3\rho_2 = \rho_3\rho_2\rho_3$ gives

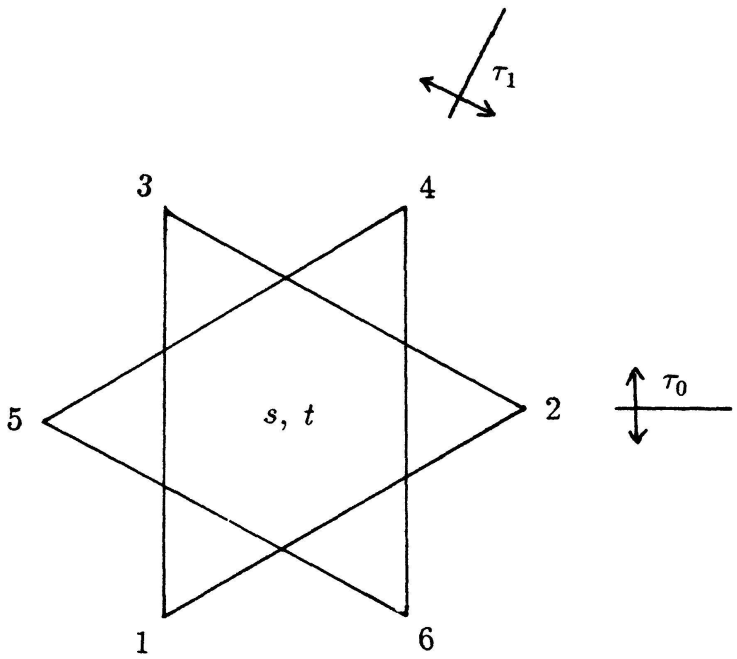
$$\rho_3\rho_2\rho_3 \cdot \rho_0 \cdot \rho_3\rho_2\rho_3 = \rho_0,$$

$$\begin{aligned}\rho_3\rho_2\rho_3 \cdot \rho_1 \cdot \rho_3\rho_2\rho_3 = \rho_3\rho_2\rho_1\rho_2\rho_3 = \rho_3(\rho_2\rho_1)^2\rho_1\rho_3 &= (\rho_1\rho_2)^2\rho_3\rho_1\rho_3 \\ &= \rho_1\rho_2\rho_1\rho_2\rho_1.\end{aligned}$$

It follows that $\rho_0\rho_1(\rho_2\rho_1)^2$ is conjugate to $\rho_0\rho_1$, and thus has order p. Hence the facets of $\mathscr{P}$ have 3-chains of length p; see Section 2. In other words, the facets can be obtained from the map $\{p, 6|, p\}$ by identifications (preserving the length of the 3-chains); see [3] for notation. Note that this map is $\{3, 6\}_{(3,0)}$ if $p = 3$. It would be interesting to know if indeed each map of type $\{p, 6\}$ with 3-chains of length p occurs as the facet of a (necessarily flat) regular 4-polytope with vertex-figures $\{6, 3\}_{(1,1)}$.

Concluding this section we describe a construction of flat regular 4-polytopes $\mathscr{P}$ in $\langle \{6, 3\}_{(s,s)}, \{3, 6\}_{(t,t)} \rangle$ with $s, t \geq 1$. Take the unitary groups $[1\ 1\ 1]^s = \langle \sigma_1, \sigma_2, \sigma_3 \rangle$ and $[1\ 1\ 1]^t = \langle \sigma_4, \sigma_5, \sigma_6 \rangle$, and consider their direct product

$W = \langle \sigma_1, \ldots, \sigma_6 \rangle$ with diagram



(48)

Here the mark s refers to the triangle 123, the mark t to the triangle 456. Consider the twisting operation

$$(\sigma_1, \ldots, \sigma_6; \tau_1, \tau_2) \mapsto (\sigma_1, \tau_0, \tau_1, \sigma_2) =: (\rho_0, \ldots, \rho_3). \tag{49}$$

This gives us the group of a regular 4-polytope $\mathscr{P}$ in $\langle \{6, 3\}_{(s,s)}, \{3, 6\}_{(t,t)} \rangle$. More precisely, $A(\mathscr{P}) = ([1\ 1\ 1]^s \times [1\ 1\ 1]^t) \rtimes S_3$, of order $216s^2t^2$. The polytope $\mathscr{P}$ has $6s^2$ vertices and $6t^2$ facets, and so must be flat.

Note that the construction generalizes to flat polytopes $\mathscr{P}$ in $\langle \mathscr{L}, \mathscr{R} \rangle$, where $\mathscr{L}$ has the DAP with respect to its vertex-figure $\mathscr{K}$ (say), and $\mathscr{R}$ has the DAP with respect to its facet $\mathscr{K}$. In fact, if $A(\mathscr{L}) = \langle \alpha_0, \ldots, \alpha_{n-2} \rangle$ and $A(\mathscr{R}) = \langle \beta_0, \ldots, \beta_{n-2} \rangle$, then we have $A(\mathscr{L}) = N(\alpha_0) \ltimes A(\mathscr{K})$ and $A(\mathscr{R}) = N(\beta_{n-2}) \rtimes A(\mathscr{K})$, so that we can choose the group $(N(\alpha_0) \times N(\beta_{n-2})) \rtimes A(\mathscr{K})$ to construct $\mathscr{P}$. Note that this construction is equivalent to that of [30].

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