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On a simplicial complex associated with tilting modules

CHRISTINE RIEDTMANN AND AIDAN SCHOFIELD

Introduction

Let Λ be a finite-dimensional associative algebra over an algebraically closed field, and denote by $\text{mod } \Lambda$ the category of finite-dimensional Λ -modules. We fix the number of pairwise non-isomorphic simple Λ -modules to be $n + 1$.

Denote by \mathcal{E} a set of fixed representatives for the isomorphism classes of indecomposable Λ -modules T satisfying the following conditions:

- (i) The projective dimension of T is at most 1.
- (ii) T does not extend itself, *i.e.* $\text{Ext}_\Lambda^1(T, T) = 0$.

Following Ringel, we define a simplicial complex \mathcal{C}_Λ on the set \mathcal{E} of vertices: (T_0, \dots, T_r) is an r -simplex if $\text{Ext}_\Lambda^1(T_0 \oplus \dots \oplus T_r, T_0 \oplus \dots \oplus T_r) = 0$. Ringel told us that \mathcal{C}_Λ is a triangulated ball for certain hereditary algebras. Our goal is to prove the following result:

THEOREM. *If \mathcal{E} is finite, the geometric realization of \mathcal{C}_Λ is an n -dimensional ball.*

We wish to thank C. Ringel for drawing our attention to \mathcal{C}_Λ and N. A'Campo for discussing with us the topological aspects of the question.

1. The Bongartz completion

1.1. Recall from [3], [5] that a Λ -module T is a *tilting module* if it satisfies:

- (i) $\text{projdim}_\Lambda T \leq 1$,
- (ii) $\text{Ext}_\Lambda^1(T, T) = 0$,
- (iii) There is an exact sequence

$$0 \rightarrow \Lambda \rightarrow T' \rightarrow T'' \rightarrow 0,$$

with modules T', T'' that belong to the full subcategory $\text{add } T$ of $\text{mod } A$ whose objects are direct summands of T^N for some N .

The simplest example of a tilting module is A itself, and for some algebras, e.g. the selfinjective ones, there are no others (aside from those obtained by changing the multiplicities of the indecomposable direct summands). Bongartz proved in [2] that a module T satisfying (i) and (ii) is a tilting module if and only if the number of its pairwise non-isomorphic indecomposable direct summands equals the number $n + 1$ of isomorphism classes of simple modules. He also showed that any module T satisfying (i) and (ii) is a direct summand of a tilting module. We recall his construction: write $T = \bigoplus_{i=0}^r T_i^{\lambda_i}$ as a direct sum of pairwise non-isomorphic indecomposables T_0, \dots, T_r with multiplicities $\lambda_0, \dots, \lambda_r$. Choose an exact sequence

$$0 \rightarrow A \rightarrow X \rightarrow \bigoplus_{i=0}^r T_i^{\mu_i} \rightarrow 0$$

with the property that, for any $k = 0, \dots, r$, the induced map

$$\text{Hom}_A \left(T_k, \bigoplus_{i=0}^r T_i^{\mu_i} \right) \rightarrow \text{Ext}_A^1(T_k, A), \quad (*)$$

is surjective. Then $T \oplus X$ is the desired tilting module.

Of course the condition $(*)$ does not determine X uniquely. But it is easy to see that possible choices for X only differ by direct summands in $\text{add } T$, up to isomorphism. Hence T determines a multiplicity-free tilting module $\tilde{T} = \bigoplus_{i=0}^n T_i$, which is unique up to isomorphism. We call $T_B = T_{r+1} \oplus \dots \oplus T_n$ the *Bongartz completion* of T .

1.2. Let T_0, \dots, T_n be pairwise non-isomorphic indecomposables, and suppose that $\bigoplus_{i=0}^n T_i$ is a tilting module.

PROPOSITION. *The following statements are equivalent:*

- (a) $\bigoplus_{i=r+1}^n T_i$ is the Bongartz completion of $\bigoplus_{i=0}^r T_i$.
- (b) For $j = r+1, \dots, n$, there is no surjection from any module in $\text{add}(T_0 \oplus \dots \oplus T_{j-1} \oplus T_{j+1} \oplus \dots \oplus T_n)$ to T_j .

Proof. Let $\bigoplus_{i=r+1}^n T_i$ be the Bongartz completion of $\bigoplus_{i=0}^r T_i$, and suppose there is a surjection $f: \bigoplus_{i \neq j} T_i \rightarrow T_j$ for some $j > r$. Consider the following commutative diagram:

$$\begin{array}{ccccccc}
0 \rightarrow \Lambda & \rightarrow & T_j^{\rho_j} \oplus \bigoplus_{i \neq j} T_i^{\rho_i} & \rightarrow & \bigoplus_{i=0}^r T_i^{\mu_i} & \rightarrow & 0 \\
\parallel & & \left[\begin{smallmatrix} f & 0 \\ 0 & f \end{smallmatrix} \right] \uparrow & & \parallel & & \uparrow \\
0 \rightarrow \Lambda & \xrightarrow{g} & \left(\bigoplus_{i \neq j} T_i^{\nu_i} \right)^{\rho_j} \oplus \bigoplus_{i \neq j} T_i^{\rho_i} & \rightarrow & X & \rightarrow & 0.
\end{array}$$

The first row is an exact sequence used to construct the Bongartz completion, and the existence of g follows from the projectivity of Λ . The square on the right yields another exact sequence:

$$0 \rightarrow \bigoplus_{i \neq j} T_i^{\nu_i, \rho_j + \rho_i} \rightarrow \bigoplus_{i=0}^n T_i^{\rho_i} \oplus X \rightarrow \bigoplus_{i=0}^r T_i^{\mu_i} \rightarrow 0,$$

which must split. But then T_j is isomorphic to some T_i for $i \neq j$, and this is impossible.

As to the converse, we choose an exact sequence

$$0 \rightarrow \Lambda \rightarrow \bigoplus_{i=0}^n T_i^{\alpha_i} \xrightarrow{h} \bigoplus_{i=0}^n T_i^{\beta_i} \rightarrow 0.$$

For any $j > r$ with $\beta_j > 0$, the composition of h with the canonical projection from $\bigoplus_{i=0}^n T_i^{\beta_i}$ to $T_j^{\beta_j}$ must be retraction by (b). So we can choose another such sequence with $\beta_j = 0$ for $j > r$. As our sequence then satisfies (*), $\bigoplus_{i=r+1}^n T_i$ must be the Bongartz completion of $\bigoplus_{i=0}^r T_i$.

Remark. The same arguments show that $T = \bigoplus_{i=0}^n T_i$ is a projective tilting module if and only if there is no surjection from any modules in $\text{add}(T_0 \oplus \cdots \oplus T_{j-1} \oplus T_{j+1} \oplus \cdots \oplus T_n)$ to T_j , for $j = 0, \dots, n$.

1.3. Let T_0, \dots, T_{n-1} be pairwise non-isomorphic indecomposables of projective dimension 1 at most, and assume that $\text{Ext}_\Lambda^1(T, T) = 0$ for $T = \bigoplus_{i=0}^{n-1} T_i$. Denote by T_n the Bongartz completion of T .

The following result has been obtained independently by Happel in [4]. In case Λ is hereditary, it was proved in [7] and later in [6].

PROPOSITION. *There is at most one indecomposable T'_n not isomorphic to T_n such that $T \oplus T'_n$ is a tilting module. If such a T'_n exists, there is an exact sequence*

$$0 \rightarrow T_n \rightarrow \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \rightarrow T'_n \rightarrow 0.$$

We first have to recall the definitions of a source map and a sink map used in [7]. Closely related concepts have been introduced in [1]. Let X_1, \dots, X_r be pairwise non-isomorphic indecomposables and let Y be a module not having any direct summands in $\text{add } X$, where $X = \bigoplus_{i=1}^r X_i$.

A map $f: Y \rightarrow \bigoplus_{i=1}^r X_i^{\lambda_i}$ is a *source map* from Y to $\text{add } X$ if

- (i) for any X' in $\text{add } X$, any map from Y to X' factors through f , and
- (ii) f is minimal with respect to property (i); *i.e.* if $\alpha \circ f$ still has property (i) for an endomorphism α of $\bigoplus_{i=1}^r X_i^{\lambda_i}$, then α is an automorphism.

Source maps exist and are unique up to isomorphism. If a map $g: Y \rightarrow \bigoplus_{i=1}^r X_i^{\mu_i}$ has property (i), it is isomorphic to $\begin{bmatrix} f \\ 0 \end{bmatrix}: Y \rightarrow \bigoplus_{i=1}^r X_i^{\lambda_i} \oplus X'$ for any source map f , where X' lies in $\text{add } X$.

Sink maps from $\text{add } X$ to Y are defined by dualizing the definition of source maps.

Proof of the proposition. Let T'_n be an indecomposable not isomorphic to T_n such that $T \oplus T'_n$ is a tilting module. By the preceding proposition, there is a surjection from some module in $\text{add } T$ to T'_n . In particular, any sink map

$$g: \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \rightarrow T'_n,$$

from $\text{add } T$ to T'_n is surjective. Consider the exact sequence

$$0 \rightarrow Z \xrightarrow{f} \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \xrightarrow{g} T'_n \rightarrow 0,$$

where $Z = \ker g$.

Since g is a sink map, f lies in the radical of $\text{mod } \Lambda$; *i.e.*, its restriction to any indecomposable direct summand of Z is never a section. Moreover, any map from Z to T_j factors through f , since we have $\text{Ext}^1(T'_n, T_j) = 0$, for $j = 0, \dots, n-1$. Therefore Z has no direct summand that belongs to $\text{add } T$. As g lies in the radical of $\text{mod } \Lambda$, f is a source map from Z to $\text{add } T$.

Obviously the projective dimension of Z is 1 at most, and by construction we have $\text{Ext}_\Lambda^1(T_j, Z) = 0$, for $j = 0, \dots, n-1$. Considering maps from our sequence to Z and T_j , respectively, and using that $\text{projdim}_\Lambda T'_n \leq 1$, we find that $\text{Ext}_\Lambda^1(Z, Z) = 0$ and $\text{Ext}_\Lambda^1(T_j, Z) = 0$, for $j = 0, \dots, n-1$. As Z does not belong to $\text{add } T$, $T \oplus Z$ is a tilting module.

If there were a surjection from some T' in $\text{add } T$ to Z , it would induce a surjection from $\text{Ext}_\Lambda^1(T'_n, T')$ to $\text{Ext}^1(T'_n, Z)$, since $\text{projdim}_\Lambda T'_n \leq 1$. But this is impossible, as the first group is zero and our sequence does not split. By the preceding proposition, we know that Z is isomorphic to T_n^λ for some $\lambda \geq 1$, and we may suppose $Z = T_n^\lambda$.

We now want to show that $\lambda = 1$. Let $h : T_n \rightarrow T'$ be a source map from T_n to $\text{add } T$. The map

$$\begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} : T_n^\lambda \rightarrow T'^\lambda,$$

still has the first property of a source map, and it is therefore isomorphic to

$$\begin{bmatrix} f \\ 0 \end{bmatrix} : T_n^\lambda \rightarrow \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \oplus T'',$$

for some T'' in $\text{add } T$. Comparing cokernels, we find that $(\text{coker } h)^\lambda$ is isomorphic to $T'' \oplus T'_n$, which implies $\lambda = 1$, by Krull–Schmidt.

Finally, since $f : T_n \rightarrow \bigoplus_{i=0}^{n-1} T_i^{\lambda_i}$ is a source map, its cokernel T'_n is determined uniquely, up to isomorphism, by T_n . Our proposition is proved.

Remark. There exist modules T as in the proposition whose only completion is the Bongartz completion T_n . Indeed, if $\bigoplus_{i=0}^n P_i$ is a projective tilting module, at least one of the modules $\bigoplus_{i \neq j} P_i$ has this property, since chains of injections in the radical of $\text{mod } \Lambda$ between projectives have bounded length.

2. Proof of the theorem

2.1. We associate a quiver K with the complex \mathcal{C}_Λ defined in the introduction in the following way: the vertices of K are the n -simplices of \mathcal{C}_Λ . For each $(n-1)$ -simplex (T_0, \dots, T_{n-1}) which is face of two n -simplices, K contains an arrow $\sigma = (T_0, \dots, T_n) \rightarrow \sigma' = (T_0, \dots, T_{n-1}, T'_n)$, where T_n is the Bongartz completion of $\bigoplus_{i=0}^{n-1} T_i$. For any simplex τ of \mathcal{C}_Λ , we let K_τ denote the full subquiver of K whose vertices are then n -simplices of \mathcal{C}_Λ containing τ .

LEMMA. *Let τ be a simplex of \mathcal{C}_Λ . If there is a path $\sigma_1 \rightarrow \sigma_2 \rightarrow \dots \rightarrow \sigma_s$ in K with σ_1, σ_s in K_τ , then the whole path lies in K_τ .*

Proof. Recall that, for a tilting module T , the category $\mathcal{T}(T)$ of torsion modules with respect to T is the full subcategory of $\text{mod } \Lambda$ whose objects are quotients of T^N for some N . Set $\mathcal{T}(\sigma) = \mathcal{T}(\bigoplus_{i=0}^n T_i)$ for $\sigma = (T_0, \dots, T_n)$.

If K contains an arrow $\sigma = (T_0, \dots, T_n) \rightarrow \sigma' = (T_0, \dots, T_{n-1}, T'_n)$, there is an exact sequence

$$0 \rightarrow T_n \rightarrow \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \rightarrow T'_n \rightarrow 0,$$

by 1.3, and therefore any module in $\mathcal{T}(\sigma')$ belongs to $\mathcal{T}(\sigma)$. However by 1.2, T_n does not lie in $\mathcal{T}(\sigma')$. Moreover, for any path $\sigma \rightarrow \sigma' \rightarrow \dots \rightarrow \sigma''$ in K , $\mathcal{T}(\sigma'')$ lies in $\mathcal{T}(\sigma')$ and thus does not contain T_n .

The lemma follows by applying these considerations to $\sigma = \sigma_k \rightarrow \sigma' = \sigma_{k+1} \rightarrow \dots \rightarrow \sigma'' = \sigma_s$ in case $\sigma_1 \rightarrow \dots \rightarrow \sigma_s$ does not lie in K_τ , where k is the maximal index for which $\sigma_1 \rightarrow \dots \rightarrow \sigma_k$ is in K_τ . Then τ contains T_n , by the choice of k , but σ_s cannot.

2.2. Applying the lemma to an n -simplex we find:

PROPOSITION. *K does not contain oriented cycles.*

This allows us to define an *order relation* for the n -simplices of \mathcal{C}_A : $\sigma \leq \sigma'$ if there is an oriented path $\sigma = \sigma_1 \rightarrow \sigma_2 \rightarrow \dots \rightarrow \sigma_s = \sigma'$ in K .

Remarks. (a) The *Hasse diagram* of this order relation is the quiver whose vertices are the n -simplices of \mathcal{C}_A and which contains an arrow $\sigma \rightarrow \sigma'$ if $\sigma \leq \sigma'$, $\sigma \neq \sigma'$ and $\sigma \leq \sigma'' \leq \sigma'$ implies either $\sigma'' = \sigma$ or $\sigma'' = \sigma'$. Applying the lemma to an $(n-1)$ -simplex which is face of two n -simplices, it is easy to see that the Hasse diagram coincides with K .

(b) Our order relation is in general distinct from the one defined by: $\sigma \leq \sigma'$ if $\mathcal{T}(\sigma) \supseteq \mathcal{T}(\sigma')$. The projective and the injective tilting module of a hereditary algebra of infinite representation type furnish an example. We don't know, however, whether the Hasse diagrams coincide.

2.3. Suppose now that \mathcal{C} is finite. Number the n -simplices $\sigma_1, \sigma_2, \dots, \sigma_M$ of \mathcal{C}_A in such a way that $\sigma_i \leq \sigma_j$ implies $i \leq j$. For $N \leq M$, let \mathcal{B}_N be the union of $\sigma_1, \sigma_2, \dots, \sigma_N$.

The following proposition implies our theorem.

PROPOSITION. *The geometric realization of \mathcal{B}_N is an n -ball, for all N .*

Proof. The result is true for $n = 0$, as a local algebra admits no modules of projective dimension 1.

For $n > 0$, we proceed by induction on N , the case $N = 1$ being obvious. Suppose that the geometric realization of \mathcal{B}_{N-1} is an n -ball for some $N \geq 2$. Our goal is to show that the intersection $\sigma_N \cap \mathcal{B}_{N-1}$, which lies in the boundary of \mathcal{B}_{N-1} , is a union of $(n-1)$ -faces of σ_N . Then the geometric realization of \mathcal{B}_N is either an n -sphere or an n -ball, according as $\sigma_N \cap \mathcal{B}_{N-1}$ is the whole boundary of σ_N or not. The case of a sphere can be ruled out, as we know that \mathcal{B}_N has a non-empty boundary by the remark in 1.3.

The intersection $\sigma_N \cap \mathcal{B}_{N-1}$ contains at least one $(n-1)$ -face of σ_N , and hence \mathcal{B}_N is connected. Indeed, σ_N is distinct from the unique minimal n -simplex of \mathcal{C}_A , whose vertices are the indecomposable projectives (remark 1.2). Any predecessor of σ_N in K , and in particular the tail of any arrow in K whose head in σ_N , belongs to \mathcal{B}_{N-1} .

Now let $\tau = (T_0, \dots, T_r)$ be a simplex in $\sigma_N \cap \mathcal{B}_{N-1}$, and let $\bigoplus_{i=r+1}^n T_i$ be the Bongartz completion of $\bigoplus_{i=0}^r T_i$. By proposition 1.2, the n -simplex $\sigma = (T_0, \dots, T_n)$ is the unique minimal vertex of K_τ . Note that σ_N is a vertex of K_τ . As any path in K from σ to σ_N lies in K_τ by lemma 2.1, and since any predecessor of σ_N belongs to \mathcal{B}_{N-1} , there is an $(n-1)$ -simplex in $\sigma_N \cap \mathcal{B}_{N-1}$ containing τ .

Remark. If \mathcal{C}_A is infinite, the same argument shows that the geometric realization of a union $\sigma_1 \cup \dots \cup \sigma_M$ is an n -ball, provided that the full subquiver of K whose vertices are $\sigma_1, \dots, \sigma_M$ is closed under predecessors in K .

3. Examples

3.1. Let Q be the quiver $\cdot \rightrightarrows \cdot$ and A its quiver algebra. Denote by P_m and I_m the preprojective and preinjective indecomposables, respectively, given by

$$\begin{array}{ccc}
 \begin{matrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \cdots & 0 \end{matrix} & & \begin{matrix} 1 & 0 & 0 \\ & \ddots & \vdots \\ 0 & 1 & 0 \end{matrix} \\
 P_m = k^m \xrightarrow{\quad} k^{m+1} & & I_m = k^{m+1} \xrightarrow{\quad} k_m \\
 \begin{matrix} 0 & 0 \\ 1 \\ \vdots \\ \cdot \\ 1 \end{matrix} & & \begin{matrix} 0 & 1 & 0 \\ \vdots & \ddots & \cdot \\ 0 & 0 & 1 \end{matrix}
 \end{array}$$

for $m \geq 0$. These are the only indecomposables that do not extend themselves. As \mathcal{E} is infinite, our theorem does not apply. In fact, the complex \mathcal{C}_A has two connected components:

$$P_0 - P_1 - P_2 - \dots$$

$$\dots - I_2 - I_1 - I_0.$$

The arrows of K are:

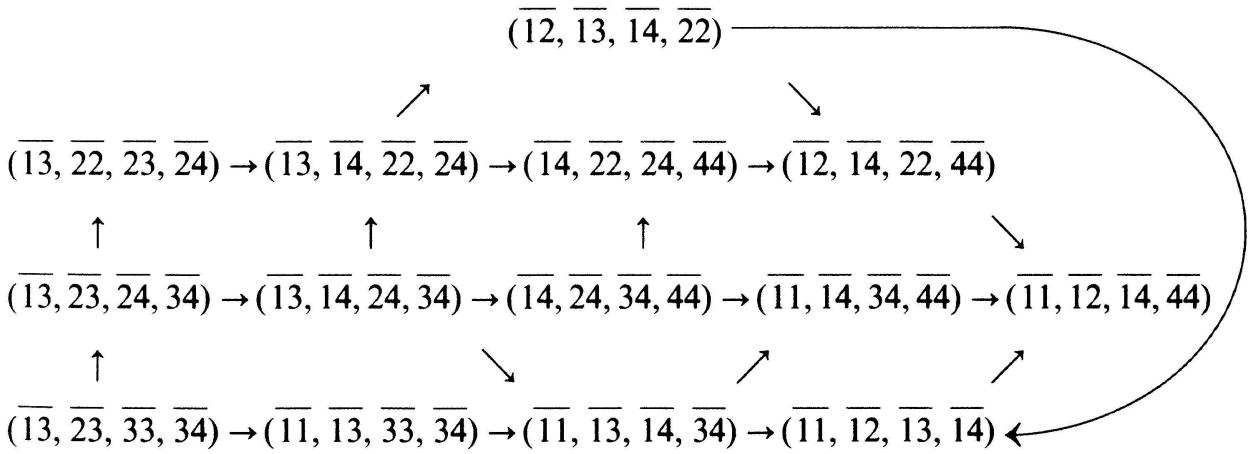
$$(P_m, P_{m+1}) \rightarrow (P_{m+1}, P_{m+2})$$

and

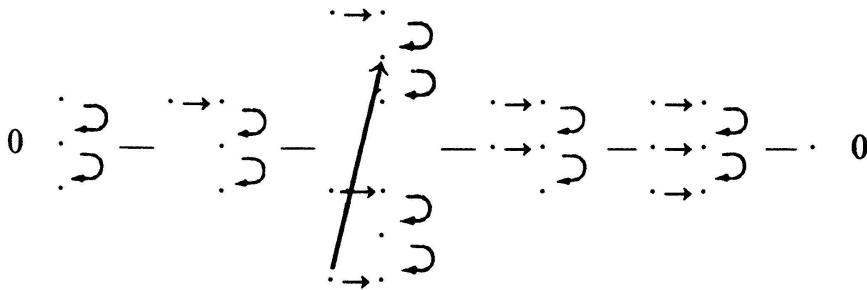
$$(I_{m+2}, I_{m+1}) \rightarrow (I_{m+1}, I_m),$$

for $m \geq 0$. They all correspond to almost split sequences.

3.2. Let Λ be the quiver algebra of $Q = 1 \rightarrow 2 \rightarrow 3 \leftarrow 4$, and denote by \bar{ij} a representative of the indecomposable whose support are the vertices $i, i+1, \dots, j$, for $1 \leq i \leq j \leq 4$. We only draw K as it contains all information necessary to build \mathcal{C}_Λ .

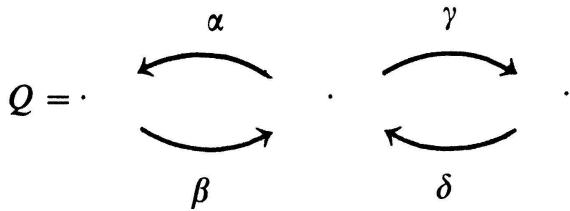


3.3. Consider the quiver $Q = \cdot \xrightarrow{\alpha} \cdot \circlearrowleft \beta$, let I be the two-sided ideal in the quiver-algebra kQ generated by β^3 , and set $\Lambda = kQ/I$. Then \mathcal{C}_Λ is an interval:

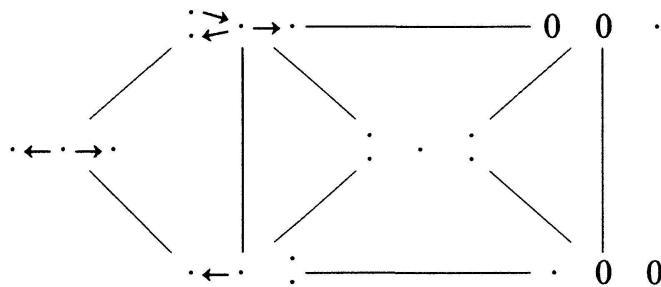


To picture representations, we represent each basis vector by a dot. The linear map $V(\gamma) : V(i) \rightarrow V(j)$ corresponding to an arrow $\gamma : i \rightarrow j$ sends a dot in $V(i)$ to the sum of the heads of all arrows of type γ starting at the dot, and to zero if there is no such arrow.

3.4. Finally, we give an example of an algebra A of infinite representation type and for which the complex \mathcal{C}_A is finite. Let Q be the quiver



and I the two-sided ideal in kQ generated by $\alpha\beta$ and $\gamma\delta$. The complex \mathcal{C}_A for the algebra $A = kQ/I$ is the following:



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