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On a simplicial complex associated with tilting modules

CHRISTINE RIEDTMANN AND AIDAN SCHOFIELD

Introduction

Let Λ be a finite-dimensional associative algebra over an algebraically closed field, and denote by mod Λ the category of finite-dimensional Λ -modules. We fix the number of pairwise non-isomorphic simple Λ -modules to be n+1.

Denote by \mathscr{E} a set of fixed representatives for the isomorphism classes of indecomposable Λ -modules T satisfying the following conditions:

- (i) The projective dimension of T is at most 1.
- (ii) T does not extend itself, i.e. $\operatorname{Ext}_{A}^{1}(T, T) = 0$.

Following Ringel, we define a simplicial complex \mathscr{C}_A on the set \mathscr{E} of vertices: (T_0, \ldots, T_r) is an r-simplex if $\operatorname{Ext}^1_A(T_0 \oplus \cdots \oplus T_r, T_0 \oplus \cdots \oplus T_r) = 0$. Ringel told us that \mathscr{C}_A is a triangulated ball for certain hereditary algebras. Our goal is to prove the following result:

THEOREM. If $\mathscr E$ is finite, the geometric realization of $\mathscr C_\Lambda$ is an n-dimensional ball.

We wish to thank C. Ringel for drawing our attention to \mathscr{C}_{Λ} and N. A'Campo for discussing with us the topological aspects of the question.

1. The Bongartz completion

- **1.1.** Recall from [3], [5] that a Λ -module T is a *tilting module* if it satisfies:
 - (i) projdim_A $T \leq 1$,
 - (ii) $\operatorname{Ext}_{A}^{1}(T, T) = 0$,
 - (iii) There is an exact sequence

$$0 \rightarrow \Lambda \rightarrow T' \rightarrow T'' \rightarrow 0$$
,

with modules T', T'' that belong to the full subcategory add T of mod Λ whose objects are direct summands of T^N for some N.

The simplest example of a tilting module is Λ itself, and for some algebras, e.g. the selfinjective ones, there are no others (aside from those obtained by changing the multiplicities of the indecomposable direct summands). Bongartz proved in [2] that a module T satisfying (i) and (ii) is a tilting module if and only if the number of its pairwise non-isomorphic indecomposable direct summands equals the number n+1 of isomorphism classes of simple modules. He also showed that any module T satisfying (i) and (ii) is a direct summand of a tilting module. We recall his construction: write $T = \bigoplus_{i=0}^{r} T_{i}^{\lambda_i}$ as a direct sum of pairwise non-isomorphic indecomposables T_0, \ldots, T_r with multiplicities $\lambda_0, \ldots, \lambda_r$. Choose an exact sequence

$$0 \to \Lambda \to X \to \bigoplus_{i=0}^r T_i^{\mu_i} \to 0$$

with the property that, for any k = 0, ..., r, the induced map

$$\operatorname{Hom}_{\Lambda}\left(T_{k},\bigoplus_{i=0}^{r}T_{i}^{\mu_{i}}\right) \to \operatorname{Ext}_{\Lambda}^{1}\left(T_{k},\Lambda\right),\tag{*}$$

is surjective. Then $T \oplus X$ is the desired tilting module.

Of course the condition (*) does not determine X uniquely. But it is easy to see that possible choices for X only differ by direct summands in add T, up to isomorphism. Hence T determines a multiplicity-free tilting module $\tilde{T} = \bigoplus_{i=0}^{n} T_i$, which is unique up to isomorphism. We call $T_B = T_{r+1} \oplus \cdots \oplus T_n$ the Bongartz completion of T.

1.2. Let T_0, \ldots, T_n be pairwise non-isomorphic indecomposables, and suppose that $\bigoplus_{i=0}^{n} T_i$ is a tilting module.

PROPOSITION. The following statements are equivalent:

- (a) $\bigoplus_{i=r+1}^{n} T_i$ is the Bongartz completion of $\bigoplus_{i=0}^{r} T_i$.
- (b) For j = r + 1, ..., n, there is no surjection from any module in add $(T_0 \oplus \cdots \oplus T_{i-1} \oplus T_{i+1} \oplus \cdots \oplus T_n)$ to T_j .

Proof. Let $\bigoplus_{i=r+1}^n T_i$ be the Bongartz completion of $\bigoplus_{i=0}^r T_i$, and suppose there is a surjection $f: \bigoplus_{i\neq j} T_i^{\nu_i} \to T_j$ for some j > r. Consider the following commutative diagram:

$$0 \longrightarrow \Lambda \longrightarrow T_{j}^{\rho_{j}} \bigoplus \bigoplus_{i \neq j} T_{i}^{\rho_{i}} \longrightarrow \bigoplus_{i=0}^{r} T_{i}^{\mu_{i}} \to 0$$

$$\left\| \begin{bmatrix} f & 0 \\ 0 & f \end{bmatrix} \uparrow & \left\| \right\| \uparrow$$

$$0 \longrightarrow \Lambda \longrightarrow \left(\bigoplus_{i \neq j} T_{i}^{\nu_{i}} \right)^{\rho_{j}} \bigoplus \bigoplus_{i \neq j} T_{i}^{\rho_{i}} \longrightarrow X \longrightarrow 0.$$

The first row is an exact sequence used to construct the Bongartz completion, and the existence of g follows from the projectivity of Λ . The square on the right yields another exact sequence:

$$0 \to \bigoplus_{i \neq i} T_i^{\nu_i \rho_j + \rho_i} \to \bigoplus_{i=0}^n T_i^{\rho_i} \oplus X \to \bigoplus_{i=0}^r T_i^{\mu_i} \to 0,$$

which must split. But then T_j is isomorphic to some T_i for $i \neq j$, and this is impossible.

As to the converse, we choose an exact sequence

$$0 \to \Lambda \to \bigoplus_{i=0}^n T_i^{\alpha_i} \xrightarrow{h} \bigoplus_{i=0}^n T_i^{\beta_i} \to 0.$$

For any j > r with $\beta_j > 0$, the composition of h with the canonical projection from $\bigoplus_{i=0}^n T_i^{\beta_i}$ to $T_j^{\beta_j}$ must be retraction by (b). So we can choose another such sequence with $\beta_j = 0$ for j > r. As our sequence then satisfies (*), $\bigoplus_{i=r+1}^n T_i$ must be the Bongartz completion of $\bigoplus_{i=0}^r T_i$.

Remark. The same arguments show that $T = \bigoplus_{i=0}^{n} T_i$ is a projective tilting module if and only if there is no surjection from any modules in add $(T_0 \oplus \cdots \oplus T_{i-1} \oplus T_{i+1} \oplus \cdots \oplus T_n)$ to T_i , for $i = 0, \ldots, n$.

1.3. Let T_0, \ldots, T_{n-1} be pairwise non-isomorphic indecomposables of projective dimension 1 at most, and assume that $\operatorname{Ext}_{A}^{1}(T, T) = 0$ for $T = \bigoplus_{i=0}^{n-1} T_i$. Denote by T_n the Bongartz completion of T.

The following result has been obtained independently by Happel in [4]. In case Λ is hereditary, it was proved in [7] and later in [6].

PROPOSITION. There is at most one indecomposable T'_n not isomorphic to T_n such that $T \oplus T'_n$ is a tilting module. If such a T'_n exists, there is an exact sequence

$$0 \to T_n \to \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \to T_n' \to 0.$$

We first have to recall the definitions of a source map and a sink map used in [7]. Closely related concepts have been introduced in [1]. Let X_1, \ldots, X_r be pairwise non-isomorphic indecomposables and let Y be a module not having any direct summands in add X, where $X = \bigoplus_{i=1}^r X_i$.

A map $f: Y \to \bigoplus_{i=1}^r X_i^{\lambda_i}$ is a source map from Y to add X if

- (i) for any X' in add X, any map from Y to X' factors through f, and
- (ii) f is minimal with respect to property (i); *i.e.* if $\alpha \circ f$ still has property (i) for an endomorphism α of $\bigoplus_{i=1}^r X_i^{\lambda_i}$, then α is an automorphism.

Source maps exist and are unique up to isomorphism. If a map $g: Y \to \bigoplus_{i=1}^r X_i^{\mu_i}$ has property (i), it is isomorphic to $\begin{bmatrix} f \\ 0 \end{bmatrix}: Y \to \bigoplus_{i=1}^r X_i^{\lambda_i} \oplus X'$ for any source map f, where X' lies in add X.

Sink maps from add X to Y are defined by dualizing the definition of source maps.

Proof of the proposition. Let T'_n be an indecomposable not isomorphic to T_n such that $T \oplus T'_n$ is a tilting module. By the preceding proposition, there is a surjection from some module in add T to T'_n . In particular, any sink map

$$g: \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \to T_n',$$

from add T to T'_n is surjective. Consider the exact sequence

$$0 \to Z \xrightarrow{f} \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \xrightarrow{g} T_n' \to 0,$$

where $Z = \ker g$.

Since g is a sink map, f lies in the radical of mod Λ ; *i.e.*, its restriction to any indecomposable direct summand of Z is never a section. Moreover, any map from Z to T_j factors through f, since we have $\operatorname{Ext}^1(T'_n, T_j) = 0$, for $j = 0, \ldots, n-1$. Therefore Z has no direct summand that belongs to add T. As g lies in the radical of mod Λ , f is a source map from Z to add T.

Obviously the projective dimension of Z is 1 at most, and by construction we have $\operatorname{Ext}_{\Lambda}^{1}(T_{j}, Z) = 0$, for $j = 0, \ldots, n-1$. Considering maps from our sequence to Z and T_{j} , respectively, and using that $\operatorname{projdim}_{\Lambda} T'_{n} \leq 1$, we find that $\operatorname{Ext}_{\Lambda}^{1}(Z, Z) = 0$ and $\operatorname{Ext}_{\Lambda}^{1}(T_{j}, Z) = 0$, for $j = 0, \ldots, n-1$. As Z does not belong to add T, $T \oplus Z$ is a tilting module.

If there were a surjection from some T' in add T to Z, it would induce a surjection from $\operatorname{Ext}^1_A(T'_n, T')$ to $\operatorname{Ext}^1(T'_n, Z)$, since $\operatorname{projdim}_A T'_n \leq 1$. But this is impossible, as the first group is zero and our sequence does not split. By the preceding proposition, we know that Z is isomorphic to T_n^{λ} for some $\lambda \geq 1$, and we may suppose $Z = T_n^{\lambda}$.

We now want to show that $\lambda = 1$. Let $h: T_n \to T'$ be a source map from T_n to add T. The map

$$\begin{bmatrix} h & 0 \\ 0 & h \end{bmatrix} : T_n^{\lambda} \to T^{\prime \lambda},$$

still has the first property of a source map, and it is therefore isomorphic to

$$\begin{bmatrix} f \\ 0 \end{bmatrix} : T_n^{\lambda} \to \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \oplus T'',$$

for some T'' in add T. Comparing cokernels, we find that $(\operatorname{coker} h)^{\lambda}$ is isomorphic to $T'' \oplus T'_n$, which implies $\lambda = 1$, by Krull-Schmidt.

Finally, since $f: T_n \to \bigoplus_{i=0}^{n-1} T_{i}^{\lambda_i}$ is a source map, its cokernel T_n' is determined uniquely, up to isomorphism, by T_n . Our proposition is proved.

Remark. There exist modules T as in the proposition whose only completion is the Bongartz completion T_n . Indeed, if $\bigoplus_{i=0}^n P_i$ is a projective tilting module, at least one of the modules $\bigoplus_{i\neq j} P_i$ has this property, since chains of injections in the radical of mod Λ between projectives have bounded length.

2. Proof of the theorem

2.1. We associate a quiver K with the complex \mathscr{C}_{Λ} defined in the introduction in the following way: the vertices of K are the n-simplices of \mathscr{C}_{Λ} . For each (n-1)-simplex (T_0, \ldots, T_{n-1}) which is face of two n-simplices, K contains an arrow $\sigma = (T_0, \ldots, T_n) \to \sigma' = (T_0, \ldots, T_{n-1}, T'_n)$, where T_n is the Bongartz completion of $\bigoplus_{i=0}^{n-1} T_i$. For any simplex τ of \mathscr{C}_{Λ} , we let K_{τ} denote the full subquiver of K whose vertices are then n-simplices of \mathscr{C}_{Λ} containing τ .

LEMMA. Let τ be a simplex of \mathscr{C}_{Λ} . If there is a path $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_s$ in K with σ_1, σ_s in K_{τ} , then the whole path lies in K_{τ} .

Proof. Recall that, for a tilting module T, the category $\mathcal{F}(T)$ of torsion modules with respect to T is the full subcategory of mod Λ whose objects are quotients of T^N for some N. Set $\mathcal{F}(\sigma) = \mathcal{F}(\bigoplus_{i=0}^n T_i)$ for $\sigma = (T_0, \ldots, T_n)$.

If K contains an arrow $\sigma = (T_0, \ldots, T_n) \rightarrow \sigma' = (T_0, \ldots, T_{n-1}, T'_n)$, there is an exact sequence

$$0 \to T_n \to \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \to T_n' \to 0,$$

by 1.3, and therefore any module in $\mathcal{F}(\sigma')$ belongs to $\mathcal{F}(\sigma)$. However by 1.2, T_n does not lie in $\mathcal{F}(\sigma')$. Moreover, for any path $\sigma \to \sigma' \to \cdots \to \sigma''$ in K, $\mathcal{F}(\sigma'')$ lies in $\mathcal{F}(\sigma')$ and thus does not contain T_n .

The lemma follows by applying these considerations to $\sigma = \sigma_k \to \sigma' = \sigma_{k+1} \to \cdots \to \sigma'' = \sigma_s$ in case $\sigma_1 \to \cdots \to \sigma_s$ does not lie in K_τ , where k is the maximal index for which $\sigma_1 \to \cdots \to \sigma_k$ is in K_τ . Then τ contains T_n , by the choice of k, but σ_s cannot.

2.2. Applying the lemma to an n-simplex we find:

PROPOSITION. K does not contain oriented cycles.

This allows us to define an *order relation* for the *n*-simplices of \mathscr{C}_{Λ} : $\sigma \leq \sigma'$ if there is an oriented path $\sigma = \sigma_1 \rightarrow \sigma_2 \rightarrow \cdots \rightarrow \sigma_s = \sigma'$ in K.

Remarks. (a) The Hasse diagram of this order relation is the quiver whose vertices are the *n*-simplices of \mathscr{C}_A and which contains an arrow $\sigma \to \sigma'$ if $\sigma \le \sigma'$, $\sigma \ne \sigma'$ and $\sigma \le \sigma'' \le \sigma'$ implies either $\sigma'' = \sigma$ or $\sigma'' = \sigma'$. Applying the lemma to an (n-1)-simplex which is face of two *n*-simplices, it is easy to see that the Hasse diagram coincides with K.

- (b) Our order relation is in general distinct from the one defined by: $\sigma \leq \sigma'$ if $\mathcal{F}(\sigma) \supseteq \mathcal{F}(\sigma')$. The projective and the injective tilting module of a hereditary algebra of infinite representation type furnish an example. We don't know, however, whether the Hasse diagrams coincide.
- **2.3.** Suppose now that \mathscr{E} is finite. Number the *n*-simplices $\sigma_1, \sigma_2, \ldots, \sigma_M$ of \mathscr{C}_{Λ} in such a way that $\sigma_i \leq \sigma_j$ implies $i \leq j$. For $N \leq M$, let \mathscr{B}_N be the union of $\sigma_1, \sigma_2, \ldots, \sigma_N$.

The following proposition implies our theorem.

PROPOSITION. The geometric realization of \mathcal{B}_N is an n-ball, for all N.

Proof. The result is true for n = 0, as a local algebra admits no modules of projective dimension 1.

For n > 0, we proceed by induction on N, the case N = 1 being obvious. Suppose that the geometric realization of \mathcal{B}_{N-1} is an n-ball for some $N \ge 2$. Our goal is to show that the intersection $\sigma_N \cap \mathcal{B}_{N-1}$, which lies in the boundary of \mathcal{B}_{N-1} , is a union of (n-1)-faces of σ_N . Then the geometric realization of \mathcal{B}_N is either an n-sphere or an n-ball, according as $\sigma_N \cap \mathcal{B}_{N-1}$ is the whole boundary of σ_N or not. The case of a sphere can be ruled out, as we know that \mathcal{B}_N has a non-empty boundary by the remark in 1.3.

The intersection $\sigma_N \cap \mathcal{B}_{N-1}$ contains at least one (n-1)-face of σ_N , and hence \mathcal{B}_N is connected. Indeed, σ_N is distinct from the unique minimal *n*-simplex of \mathcal{C}_A , whose vertices are the indecomposable projectives (remark 1.2). Any predecessor of σ_N in K, and in particular the tail of any arrow in K whose head in σ_N , belongs to \mathcal{B}_{N-1} .

Now let $\tau = (T_0, \ldots, T_r)$ be a simplex in $\sigma_N \cap \mathcal{B}_{N-1}$, and let $\bigoplus_{i=r+1}^n T_i$ be the Bongartz completion of $\bigoplus_{i=0}^r T_i$. By proposition 1.2, the *n*-simplex $\sigma = (T_0, \ldots, T_n)$ is the unique minimal vertex of K_τ . Note that σ_N is a vertex of K_τ . As any path in K from σ to σ_N lies in K_τ by lemma 2.1, and since any predecessor of σ_N belongs to \mathcal{B}_{N-1} , there is an (n-1)-simplex in $\sigma_N \cap \mathcal{B}_{N-1}$ containing τ .

Remark. If \mathscr{C}_{Λ} is infinite, the same argument shows that the geometric realization of a union $\sigma_1 \cup \cdots \cup \sigma_M$ is an *n*-ball, provided that the full subquiver of K whose vertices are $\sigma_1, \ldots, \sigma_M$ is closed under predecessors in K.

3. Examples

3.1. Let Q be the quiver $\cdot \rightrightarrows \cdot$ and Λ its quiver algebra. Denote by P_m and I_m the preprojective and preinjective indecomposables, respectively, given by

$$\begin{bmatrix} 1 & 0 \\ \cdot \cdot \\ 0 & 1 \\ 0 & \cdot \cdot 0 \end{bmatrix}$$

$$P_{m} = k^{m} \xrightarrow{k^{m+1}} k^{m+1}$$

$$I_{m} = k^{m+1} \xrightarrow{k_{m}} k_{m}$$

$$\begin{bmatrix} 0 & 0 \\ 1 & \cdot \cdot \\ 0 & 1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 1 & 0 \\ \vdots & \ddots & \vdots \\ 0 & 1 & 0 \end{bmatrix}$$

for $m \ge 0$. These are the only indecomposables that do not extend themselves. As \mathscr{E} is infinite, our theorem does not apply. In fact, the complex \mathscr{C}_{Λ} has two connected components:

$$P_0 - P_1 - P_2 - \cdots$$

$$\cdots I_2 - I_1 - I_0.$$

The arrows of K are:

$$(P_m, P_{m+1}) \rightarrow (P_{m+1}, P_{m+2})$$

and

$$(I_{m+2}, I_{m+1}) \rightarrow (I_{m+1}, I_m),$$

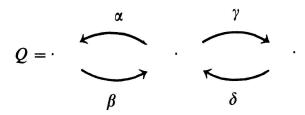
for $m \ge 0$. They all correspond to almost split sequences.

3.2. Let Λ be the quiver algebra of $Q = 1 \rightarrow 2 \rightarrow 3 \leftarrow 4$, and denote by ij a representative of the indecomposable whose support are the vertices $i, i+1, \ldots, j$, for $1 \le i \le j \le 4$. We only draw K as it contains all information necessary to build \mathscr{C}_{Λ} .

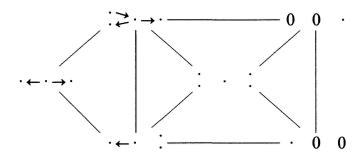
3.3. Consider the quiver $Q = \stackrel{\alpha}{\longrightarrow} \stackrel{\alpha}{\longrightarrow} \beta$, let I be the two-sided ideal in the quiveralgebra kQ generated by β^3 , and set A = kQ/I. Then C_A is an interval:

To picture representations, we represent each basis vector by a dot. The linear map $V(\gamma): V(i) \to V(j)$ corresponding to an arrow $\gamma: i \to j$ sends a dot in V(i) to the sum of the heads of all arrows of type γ starting at the dot, and to zero if there is no such arrow.

3.4. Finally, we give an example of an algebra Λ of infinite representation type and for which the complex \mathscr{C}_{Λ} is finite. Let Q be the quiver



and I the two-sided ideal in kQ generated by $\alpha\beta$ and $\gamma\delta$. The complex \mathscr{C}_{Λ} for the algebra $\Lambda = kQ/I$ is the following:



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