**Zeitschrift:** Commentarii Mathematici Helvetici

**Herausgeber:** Schweizerische Mathematische Gesellschaft

**Band:** 66 (1991)

**Artikel:** On a simplicial complex associated with tilting modules.

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**DOI:** https://doi.org/10.5169/seals-50391

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Comment. Math. Helvetici 66 (1991) 70–78

0010-2571/91/010070-09$1.50 + 0.20/0

# On a simplicial complex associated with tilting modules

CHRISTINE RIEDTMANN AND AIDAN SCHOFIELD

## Introduction

Let $\Lambda$ be a finite-dimensional associative algebra over an algebraically closed field, and denote by mod $\Lambda$ the category of finite-dimensional $\Lambda$-modules. We fix the number of pairwise non-isomorphic simple $\Lambda$-modules to be $n + 1$.

Denote by $\mathscr{E}$ a set of fixed representatives for the isomorphism classes of indecomposable $\Lambda$-modules $T$ satisfying the following conditions:

(i) The projective dimension of $T$ is at most 1.

(ii) $T$ does not extend itself, *i.e.* $\operatorname{Ext}^1_\Lambda(T, T) = 0$.

Following Ringel, we define a simplicial complex $\mathscr{C}_\Lambda$ on the set $\mathscr{E}$ of vertices: $(T_0, \ldots, T_r)$ is an $r$-simplex if $\operatorname{Ext}^1_\Lambda(T_0 \oplus \cdots \oplus T_r, T_0 \oplus \cdots \oplus T_r) = 0$. Ringel told us that $\mathscr{C}_\Lambda$ is a triangulated ball for certain hereditary algebras. Our goal is to prove the following result:

THEOREM. *If $\mathscr{E}$ is finite, the geometric realization of $\mathscr{C}_\Lambda$ is an $n$-dimensional ball.*

We wish to thank C. Ringel for drawing our attention to $\mathscr{C}_\Lambda$ and N. A'Campo for discussing with us the topological aspects of the question.

## 1. The Bongartz completion

**1.1.** Recall from [3], [5] that a $\Lambda$-module $T$ is a *tilting module* if it satisfies:

(i) $\operatorname{projdim}_\Lambda T \leq 1$,

(ii) $\operatorname{Ext}^1_\Lambda(T, T) = 0$,

(iii) There is an exact sequence

$$0 \to \Lambda \to T' \to T'' \to 0,$$

with modules $T'$, $T''$ that belong to the full subcategory add $T$ of mod $\Lambda$ whose objects are direct summands of $T^N$ for some $N$.

The simplest example of a tilting module is $\Lambda$ itself, and for some algebras, e.g. the selfinjective ones, there are no others (aside from those obtained by changing the multiplicities of the indecomposable direct summands). Bongartz proved in [2] that a module $T$ satisfying (i) and (ii) is a tilting module if and only if the number of its pairwise non-isomorphic indecomposable direct summands equals the number $n+1$ of isomorphism classes of simple modules. He also showed that any module $T$ satisfying (i) and (ii) is a direct summand of a tilting module. We recall his construction: write $T = \bigoplus_{i=0}^{r} T_i^{\lambda_i}$ as a direct sum of pairwise non-isomorphic indecomposables $T_0, \ldots, T_r$ with multiplicities $\lambda_0, \ldots, \lambda_r$. Choose an exact sequence

$$0 \to \Lambda \to X \to \bigoplus_{i=0}^{r} T_i^{\mu_i} \to 0$$

with the property that, for any $k = 0, \ldots, r$, the induced map

$$\operatorname{Hom}_\Lambda\left(T_k, \bigoplus_{i=0}^{r} T_i^{\mu_i}\right) \to \operatorname{Ext}_\Lambda^1(T_k, \Lambda), \tag{$*$}$$

is surjective. Then $T \oplus X$ is the desired tilting module.

Of course the condition $(*)$ does not determine $X$ uniquely. But it is easy to see that possible choices for $X$ only differ by direct summands in add $T$, up to isomorphism. Hence $T$ determines a multiplicity-free tilting module $\tilde{T} = \bigoplus_{i=0}^{n} T_i$, which is unique up to isomorphism. We call $T_B = T_{r+1} \oplus \cdots \oplus T_n$ the *Bongartz completion* of $T$.

**1.2.** Let $T_0, \ldots, T_n$ be pairwise non-isomorphic indecomposables, and suppose that $\bigoplus_{i=0}^{n} T_i$ is a tilting module.

PROPOSITION. *The following statements are equivalent*:

(a) $\bigoplus_{i=r+1}^{n} T_i$ *is the Bongartz completion of* $\bigoplus_{i=0}^{r} T_i$.

(b) *For* $j = r+1, \ldots, n$, *there is no surjection from any module in* add $(T_0 \oplus \cdots \oplus T_{j-1} \oplus T_{j+1} \oplus \cdots \oplus T_n)$ *to* $T_j$.

*Proof.* Let $\bigoplus_{i=r+1}^{n} T_i$ be the Bongartz completion of $\bigoplus_{i=0}^{r} T_i$, and suppose there is a' surjection $f: \bigoplus_{i \neq j} T_i^{\nu_i} \to T_j$ for some $j > r$. Consider the following commutative diagram:

$$\begin{tikzcd}
0 \arrow[r] & \Lambda \arrow[r] & T_j^{\rho_j} \arrow[r, phantom, "\oplus"] & \bigoplus_{i\neq j} T_i^{\rho_i} \arrow[r] & \bigoplus_{i=0}^{r} T_i^{\mu_i} \arrow[r] & 0 \\
0 \arrow[r] & \Lambda \arrow[u, equal] \arrow[r, "g"'] & \left(\bigoplus_{i\neq j} T_i^{\nu_i}\right)^{\rho_j} \arrow[u, "\begin{bmatrix} f\cdot & 0 \\ 0 & \cdot f \end{bmatrix}"] \arrow[r, phantom, "\oplus"] & \bigoplus_{i\neq j} T_i^{\rho_i} \arrow[u, equal] \arrow[r] & X \arrow[u] \arrow[r] & 0.
\end{tikzcd}$$

The first row is an exact sequence used to construct the Bongartz completion, and the existence of $g$ follows from the projectivity of $\Lambda$. The square on the right yields another exact sequence:

$$0 \to \bigoplus_{i\neq j} T_i^{\nu_i\rho_j+\rho_i} \to \bigoplus_{i=0}^{n} T_i^{\rho_i} \oplus X \to \bigoplus_{i=0}^{r} T_i^{\mu_i} \to 0,$$

which must split. But then $T_j$ is isomorphic to some $T_i$ for $i\neq j$, and this is impossible.

As to the converse, we choose an exact sequence

$$0 \to \Lambda \to \bigoplus_{i=0}^{n} T_i^{\alpha_i} \xrightarrow{h} \bigoplus_{i=0}^{n} T_i^{\beta_i} \to 0.$$

For any $j>r$ with $\beta_j>0$, the composition of $h$ with the canonical projection from $\bigoplus_{i=0}^{n} T_i^{\beta_i}$ to $T_j^{\beta_j}$ must be retraction by (b). So we can choose another such sequence with $\beta_j=0$ for $j>r$. As our sequence then satisfies (*), $\bigoplus_{i=r+1}^{n} T_i$ must be the Bongartz completion of $\bigoplus_{i=0}^{r} T_i$.

*Remark.* The same arguments show that $T=\bigoplus_{i=0}^{n} T_i$ is a projective tilting module if and only if there is no surjection from any modules in $\operatorname{add}(T_0\oplus\cdots\oplus T_{j-1}\oplus T_{j+1}\oplus\cdots\oplus T_n)$ to $T_j$, for $j=0,\ldots,n$.

**1.3.** Let $T_0,\ldots,T_{n-1}$ be pairwise non-isomorphic indecomposables of projective dimension 1 at most, and assume that $\operatorname{Ext}^1_\Lambda(T,T)=0$ for $T=\bigoplus_{i=0}^{n-1} T_i$. Denote by $T_n$ the Bongartz completion of $T$.

The following result has been obtained independently by Happel in [4]. In case $\Lambda$ is hereditary, it was proved in [7] and later in [6].

PROPOSITION. *There is at most one indecomposable $T'_n$ not isomorphic to $T_n$ such that $T\oplus T'_n$ is a tilting module. If such a $T'_n$ exists, there is an exact sequence*

$$0 \to T_n \to \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \to T'_n \to 0.$$

We first have to recall the definitions of a source map and a sink map used in [7]. Closely related concepts have been introduced in [1]. Let $X_1, \ldots, X_r$ be pairwise non-isomorphic indecomposables and let $Y$ be a module not having any direct summands in add $X$, where $X = \bigoplus_{i=1}^r X_i$.

A map $f: Y \to \bigoplus_{i=1}^r X_i^{\lambda_i}$ is a *source map from* $Y$ to add $X$ if

(i) for any $X'$ in add $X$, any map from $Y$ to $X'$ factors through $f$, and

(ii) $f$ is minimal with respect to property (i); *i.e.* if $\alpha \circ f$ still has property (i) for an endomorphism $\alpha$ of $\bigoplus_{i=1}^r X_i^{\lambda_i}$, then $\alpha$ is an automorphism.

Source maps exist and are unique up to isomorphism. If a map $g: Y \to \bigoplus_{i=1}^r X_i^{\mu_i}$ has property (i), it is isomorphic to $\begin{bmatrix} f \\ 0 \end{bmatrix}: Y \to \bigoplus_{i=1}^r X_i^{\lambda_i} \oplus X'$ for any source map $f$, where $X'$ lies in add $X$.

*Sink maps from* add $X$ *to* $Y$ are defined by dualizing the definition of source maps.

*Proof of the proposition.* Let $T'_n$ be an indecomposable not isomorphic to $T_n$ such that $T \oplus T'_n$ is a tilting module. By the preceding proposition, there is a surjection from some module in add $T$ to $T'_n$. In particular, any sink map

$$g: \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \to T'_n,$$

from add $T$ to $T'_n$ is surjective. Consider the exact sequence

$$0 \to Z \xrightarrow{f} \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \xrightarrow{g} T'_n \to 0,$$

where $Z = \ker g$.

Since $g$ is a sink map, $f$ lies in the radical of mod $\Lambda$; *i.e.*, its restriction to any indecomposable direct summand of $Z$ is never a section. Moreover, any map from $Z$ to $T_j$ factors through $f$, since we have $\mathrm{Ext}^1(T'_n, T_j) = 0$, for $j = 0, \ldots, n-1$. Therefore $Z$ has no direct summand that belongs to add $T$. As $g$ lies in the radical of mod $\Lambda$, $f$ is a source map from $Z$ to add $T$.

Obviously the projective dimension of $Z$ is 1 at most, and by construction we have $\mathrm{Ext}^1_\Lambda(T_j, Z) = 0$, for $j = 0, \ldots, n-1$. Considering maps from our sequence to $Z$ and $T_j$, respectively, and using that $\mathrm{projdim}_\Lambda T'_n \leq 1$, we find that $\mathrm{Ext}^1_\Lambda(Z, Z) = 0$ and $\mathrm{Ext}^1_\Lambda(T_j, Z) = 0$, for $j = 0, \ldots, n-1$. As $Z$ does not belong to add $T$, $T \oplus Z$ is a tilting module.

If there were a surjection from some $T'$ in add $T$ to $Z$, it would induce a surjection from $\mathrm{Ext}^1_\Lambda(T'_n, T')$ to $\mathrm{Ext}^1(T'_n, Z)$, since $\mathrm{projdim}_\Lambda T'_n \leq 1$. But this is impossible, as the first group is zero and our sequence does not split. By the preceding proposition, we know that $Z$ is isomorphic to $T_n^\lambda$ for some $\lambda \geq 1$, and we may suppose $Z = T_n^\lambda$.

We now want to show that $\lambda = 1$. Let $h : T_n \to T'$ be a source map from $T_n$ to add $T$. The map

$$\begin{bmatrix} h & & 0 \\ & \ddots & \\ 0 & & h \end{bmatrix} : T_n^{\lambda} \to T'^{\lambda},$$

still has the first property of a source map, and it is therefore isomorphic to

$$\begin{bmatrix} f \\ 0 \end{bmatrix} : T_n^{\lambda} \to \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \oplus T'',$$

for some $T''$ in add $T$. Comparing cokernels, we find that $(\operatorname{coker} h)^{\lambda}$ is isomorphic to $T'' \oplus T'_n$, which implies $\lambda = 1$, by Krull–Schmidt.

Finally, since $f : T_n \to \bigoplus_{i=0}^{n-1} T_i^{\lambda_i}$ is a source map, its cokernel $T'_n$ is determined uniquely, up to isomorphism, by $T_n$. Our proposition is proved.

*Remark.* There exist modules $T$ as in the proposition whose only completion is the Bongartz completion $T_n$. Indeed, if $\bigoplus_{i=0}^{n} P_i$ is a projective tilting module, at least one of the modules $\bigoplus_{i \neq j} P_i$ has this property, since chains of injections in the radical of mod $\Lambda$ between projectives have bounded length.

## 2. Proof of the theorem

**2.1.** We associate a quiver $K$ with the complex $\mathscr{C}_\Lambda$ defined in the introduction in the following way: the vertices of $K$ are the $n$-simplices of $\mathscr{C}_\Lambda$. For each $(n-1)$-simplex $(T_0, \ldots, T_{n-1})$ which is face of two $n$-simplices, $K$ contains an arrow $\sigma = (T_0, \ldots, T_n) \to \sigma' = (T_0, \ldots, T_{n-1}, T'_n)$, where $T_n$ is the Bongartz completion of $\bigoplus_{i=0}^{n-1} T_i$. For any simplex $\tau$ of $\mathscr{C}_\Lambda$, we let $K_\tau$ denote the full subquiver of $K$ whose vertices are then $n$-simplices of $\mathscr{C}_\Lambda$ containing $\tau$.

LEMMA. *Let $\tau$ be a simplex of $\mathscr{C}_\Lambda$. If there is a path $\sigma_1 \to \sigma_2 \to \cdots \to \sigma_s$ in $K$ with $\sigma_1, \sigma_s$ in $K_\tau$, then the whole path lies in $K_\tau$.*

*Proof.* Recall that, for a tilting module $T$, the category $\mathscr{T}(T)$ of torsion modules with respect to $T$ is the full subcategory of mod $\Lambda$ whose objects are quotients of $T^N$ for some $N$. Set $\mathscr{T}(\sigma) = \mathscr{T}(\bigoplus_{i=0}^{n} T_i)$ for $\sigma = (T_0, \ldots, T_n)$.

If $K$ contains an arrow $\sigma = (T_0, \ldots, T_n) \to \sigma' = (T_0, \ldots, T_{n-1}, T'_n)$, there is an exact sequence

$$0 \to T_n \to \bigoplus_{i=0}^{n-1} T_i^{\lambda_i} \to T'_n \to 0,$$

by 1.3, and therefore any module in $\mathcal{T}(\sigma')$ belongs to $\mathcal{T}(\sigma)$. However by 1.2, $T_n$ does not lie in $\mathcal{T}(\sigma')$. Moreover, for any path $\sigma \to \sigma' \to \cdots \to \sigma''$ in $K$, $\mathcal{T}(\sigma'')$ lies in $\mathcal{T}(\sigma')$ and thus does not contain $T_n$.

The lemma follows by applying these considerations to $\sigma = \sigma_k \to \sigma' = \sigma_{k+1} \to \cdots \to \sigma'' = \sigma_s$ in case $\sigma_1 \to \cdots \to \sigma_s$ does not lie in $K_\tau$, where $k$ is the maximal index for which $\sigma_1 \to \cdots \to \sigma_k$ is in $K_\tau$. Then $\tau$ contains $T_n$, by the choice of $k$, but $\sigma_s$ cannot.

**2.2.** Applying the lemma to an $n$-simplex we find:

PROPOSITION. *K does not contain oriented cycles.*

This allows us to define an *order relation* for the $n$-simplices of $\mathcal{C}_\Lambda : \sigma \leq \sigma'$ if there is an oriented path $\sigma = \sigma_1 \to \sigma_2 \to \cdots \to \sigma_s = \sigma'$ in $K$.

*Remarks.* (a) The *Hasse diagram* of this order relation is the quiver whose vertices are the $n$-simplices of $\mathcal{C}_\Lambda$ and which contains an arrow $\sigma \to \sigma'$ if $\sigma \leq \sigma'$, $\sigma \neq \sigma'$ and $\sigma \leq \sigma'' \leq \sigma'$ implies either $\sigma'' = \sigma$ or $\sigma'' = \sigma'$. Applying the lemma to an $(n-1)$-simplex which is face of two $n$-simplices, it is easy to see that the Hasse diagram coincides with $K$.

(b) Our order relation is in general distinct from the one defined by: $\sigma \leq \sigma'$ if $\mathcal{T}(\sigma) \supseteq \mathcal{T}(\sigma')$. The projective and the injective tilting module of a hereditary algebra of infinite representation type furnish an example. We don't know, however, whether the Hasse diagrams coincide.

**2.3.** Suppose now that $\mathcal{E}$ is finite. Number the $n$-simplices $\sigma_1, \sigma_2, \ldots, \sigma_M$ of $\mathcal{C}_\Lambda$ in such a way that $\sigma_i \leq \sigma_j$ implies $i \leq j$. For $N \leq M$, let $\mathcal{B}_N$ be the union of $\sigma_1, \sigma_2, \ldots, \sigma_N$.

The following proposition implies our theorem.

PROPOSITION. *The geometric realization of $\mathcal{B}_N$ is an n-ball, for all N.*

*Proof.* The result is true for $n = 0$, as a local algebra admits no modules of projective dimension 1.

For $n > 0$, we proceed by induction on $N$, the case $N = 1$ being obvious. Suppose that the geometric realization of $\mathcal{B}_{N-1}$ is an $n$-ball for some $N \geq 2$. Our goal is to show that the intersection $\sigma_N \cap \mathcal{B}_{N-1}$, which lies in the boundary of $\mathcal{B}_{N-1}$, is a union of $(n-1)$-faces of $\sigma_N$. Then the geometric realization of $\mathcal{B}_N$ is either an $n$-sphere or an $n$-ball, according as $\sigma_N \cap \mathcal{B}_{N-1}$ is the whole boundary of $\sigma_N$ or not. The case of a sphere can be ruled out, as we know that $\mathcal{B}_N$ has a non-empty boundary by the remark in 1.3.

The intersection $\sigma_N \cap \mathscr{B}_{N-1}$ contains at least one $(n-1)$-face of $\sigma_N$, and hence $\mathscr{B}_N$ is connected. Indeed, $\sigma_N$ is distinct from the unique minimal $n$-simplex of $\mathscr{C}_\Lambda$, whose vertices are the indecomposable projectives (remark 1.2). Any predecessor of $\sigma_N$ in $K$, and in particular the tail of any arrow in $K$ whose head in $\sigma_N$, belongs to $\mathscr{B}_{N-1}$.

Now let $\tau = (T_0, \ldots, T_r)$ be a simplex in $\sigma_N \cap \mathscr{B}_{N-1}$, and let $\bigoplus_{i=r+1}^{n} T_i$ be the Bongartz completion of $\bigoplus_{i=0}^{r} T_i$. By proposition 1.2, the $n$-simplex $\sigma = (T_0, \ldots, T_n)$ is the unique minimal vertex of $K_\tau$. Note that $\sigma_N$ is a vertex of $K_\tau$. As any path in $K$ from $\sigma$ to $\sigma_N$ lies in $K_\tau$ by lemma 2.1, and since any predecessor of $\sigma_N$ belongs to $\mathscr{B}_{N-1}$, there is an $(n-1)$-simplex in $\sigma_N \cap \mathscr{B}_{N-1}$ containing $\tau$.

*Remark.* If $\mathscr{C}_\Lambda$ is infinite, the same argument shows that the geometric realization of a union $\sigma_1 \cup \cdots \cup \sigma_M$ is an $n$-ball, provided that the full subquiver of $K$ whose vertices are $\sigma_1, \ldots, \sigma_M$ is closed under predecessors in $K$.

## 3. Examples

**3.1.** Let $Q$ be the quiver $\cdot \rightrightarrows \cdot$ and $\Lambda$ its quiver algebra. Denote by $P_m$ and $I_m$ the preprojective and preinjective indecomposables, respectively, given by

$$\begin{tikzcd} P_m = k^m \arrow[r, shift left, "\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \\ 0 & \cdots & 0 \end{bmatrix}"] \arrow[r, shift right, "\begin{bmatrix} 0 & & 0 \\ 1 & & \\ & \ddots & \\ & & 1 \end{bmatrix}"'] & k^{m+1} \end{tikzcd} \qquad \begin{tikzcd} I_m = k^{m+1} \arrow[r, shift left, "\begin{bmatrix} 1 & 0 & 0 \\ & \ddots & \vdots \\ 0 & 1 & 0 \end{bmatrix}"] \arrow[r, shift right, "\begin{bmatrix} 0 & 1 & 0 \\ \vdots & & \ddots \\ 0 & 0 & 1 \end{bmatrix}"'] & k_m \end{tikzcd}$$

for $m \geq 0$. These are the only indecomposables that do not extend themselves. As $\mathscr{E}$ is infinite, our theorem does not apply. In fact, the complex $\mathscr{C}_\Lambda$ has two connected components:

$$P_0 - P_1 - P_2 - \cdots$$

$$\cdots I_2 - I_1 - I_0.$$

The arrows of $K$ are:

$$(P_m, P_{m+1}) \to (P_{m+1}, P_{m+2})$$

and

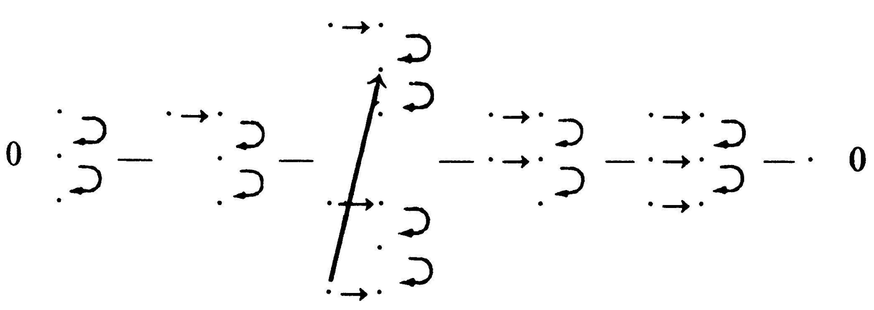
$$(I_{m+2}, I_{m+1}) \to (I_{m+1}, I_m),$$

for $m \geq 0$. They all correspond to almost split sequences.

**3.2.** Let $\Lambda$ be the quiver algebra of $Q = 1 \to 2 \to 3 \leftarrow 4$, and denote by $\overline{ij}$ a representative of the indecomposable whose support are the vertices $i, i+1, \ldots, j$, for $1 \leq i \leq j \leq 4$. We only draw $K$ as it contains all information necessary to build $\mathscr{C}_\Lambda$.

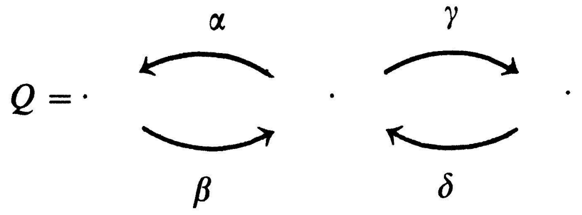
$$\begin{tikzcd}
 & & (\overline{12}, \overline{13}, \overline{14}, \overline{22}) \arrow[dr] \arrow[rddd, bend left=60] & & \\
(\overline{13}, \overline{22}, \overline{23}, \overline{24}) \arrow[r] & (\overline{13}, \overline{14}, \overline{22}, \overline{24}) \arrow[r] \arrow[ur] & (\overline{14}, \overline{22}, \overline{24}, \overline{44}) \arrow[r] & (\overline{12}, \overline{14}, \overline{22}, \overline{44}) \arrow[dr] & \\
(\overline{13},\overline{23},\overline{24},\overline{34}) \arrow[r] \arrow[u] & (\overline{13},\overline{14},\overline{24},\overline{34}) \arrow[r] \arrow[u] \arrow[dr] & (\overline{14},\overline{24},\overline{34},\overline{44}) \arrow[r] \arrow[u] & (\overline{11},\overline{14},\overline{34},\overline{44}) \arrow[r] & (\overline{11},\overline{12},\overline{14},\overline{44}) \\
(\overline{13}, \overline{23}, \overline{33}, \overline{34}) \arrow[r] \arrow[u] & (\overline{11}, \overline{13}, \overline{33}, \overline{34}) \arrow[r] & (\overline{11}, \overline{13}, \overline{14}, \overline{34}) \arrow[r] \arrow[ur] & (\overline{11}, \overline{12}, \overline{13}, \overline{14}) \arrow[ur] &
\end{tikzcd}$$

**3.3.** Consider the quiver $Q = \cdot \xrightarrow{\alpha} \cdot \circlearrowleft \beta$, let $I$ be the two-sided ideal in the quiver-algebra $kQ$ generated by $\beta^3$, and set $\Lambda = kQ/I$. Then $C_\Lambda$ is an interval:

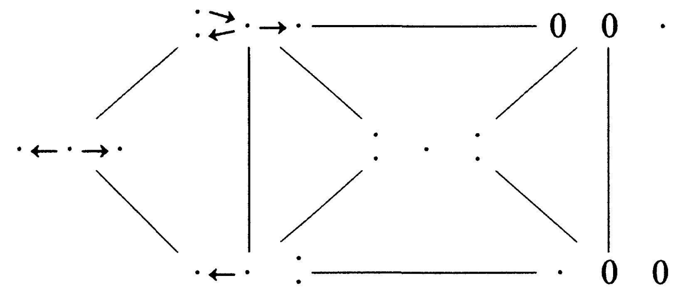


To picture representations, we represent each basis vector by a dot. The linear map $V(\gamma) : V(i) \to V(j)$ corresponding to an arrow $\gamma : i \to j$ sends a dot in $V(i)$ to the sum of the heads of all arrows of type $\gamma$ starting at the dot, and to zero if there is no such arrow.

**3.4.** Finally, we give an example of an algebra $\Lambda$ of infinite representation type and for which the complex $\mathscr{C}_\Lambda$ is finite. Let $Q$ be the quiver



and $I$ the two-sided ideal in $kQ$ generated by $\alpha\beta$ and $\gamma\delta$. The complex $\mathscr{C}_\Lambda$ for the algebra $\Lambda = kQ/I$ is the following:



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Received October 18, 1989