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Smith theory and the functor T

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§1. Introduction

J. Lannes has introduced and studied a remarkable functor T [L1] which takes an unstable module (or algebra) over the Steenrod algebra to another object of the same type. This functor has played an important role in several proofs of the generalized Sullivan Conjecture [L1] [L2] [DMN] and has led to homotopical rigidity theorems for classifying spaces [DMW1] [DMW2]. In this paper we will use techniques of Smith theory [DW] to calculate the functor T explicitly in certain key special situations (see 1.1 and 1.3). On the one hand, our calculation gives general structural information (1.4) about T itself. On the other hand, up to a convergence question which we will not discuss here our calculation produces a direct analogue of Smith theory (1.2) for actions of elementary abelian p-groups on certain infinite-dimensional complexes; this analogue differs from Smith theory only in that "homotopy fixed point set" is substituted for "fixed point set".

We will now state the main results, which are completely algebraic in nature although they have a geometric motivation. Fix a prime p; the field $\mathbf{F}_p$ with p elements will be the coefficient ring for all cohomology. Let $\mathbf{A}_p$ denote the mod p Steenrod algebra, and $\mathcal{U}$ (resp. $\mathcal{K}$) the category of unstable modules (resp. unstable algebras) over $\mathbf{A}_p$ (see [L1]). If R is an object of $\mathcal{K}$, an *unstable* $\mathbf{A}_p \odot R$ *module* M is by definition an object of $\mathcal{U}$ which is also an R module in such a way that the multiplication map $R \otimes M \to M$ obeys the Cartan formula; we will denote the category of $\mathbf{A}_p \odot R$ modules by $\mathcal{U}(R)$. An object of $\mathcal{U}(R)$ typically arises from a map $q : E \to B$ of spaces; in this case the induced cohomology map q^* makes H^*E an object of $\mathcal{U}(R)$ for $R = H^*B$.

Let V be an elementary abelian p-group, i.e., a finite-dimensional vector space over $\mathbf{F}_p$, and H^V the classifying space cohomology H^*BV. Lannes [L1] has constructed a functor $T^V : \mathcal{U} \to \mathcal{U}$ which is left adjoint to the functor given by tensor product (over $\mathbf{F}_p$) with H^V and has shown that T^V lifts to a functor $\mathcal{K} \to \mathcal{K}$ which is also left adjoint to tensoring with H^V. The adjointness property of T^V

produces for any space X a natural map

$$\lambda_X : T^V(H^*X) \to H^* \operatorname{Hom}(BV, X)$$

which is often an isomorphism [L1] [L2] [DS]. Given an object R of $\mathscr{K}$ there is a simple way (see §2) of using a particular $\mathscr{K}$-map $f : R \to H^V$ to single out a quotient $T_f^V(R)$ of $T^V(R)$ or for $M \in \mathscr{U}(R)$ a quotient $T_f^V(M)$ of $T^V(M)$. These quotients correspond via λ to subspaces of function spaces; more precisely, if $q : E \to B$ is a map of spaces and $f : H^*B \to H^V$ is a map $\mathscr{K}$, then λ_E induces a quotient map $\lambda_{E,f}$ from $T_f^V(H^*E)$ to the cohomology of the subspace of $\operatorname{Hom}(BV, E)$ consisting of maps $r : BV \to E$ with $r^*q^* = f$.

For any $\mathscr{K}$-map $R \to H^V$, we will let $S_f \subset R$ denote the multiplicative subset of R generated by the Bockstein images in R^2 of the elements of R^1 which map non-trivially under f. Recall from [W1] [Si] that if M is an object of $\mathscr{U}(R)$ any localization of the form $S_f^{-1}M$ inherits an action of $\mathbf{A}_p$, although this action is not necessarily unstable. Denote the largest unstable $\mathbf{A}_p$-submodule [DW, 2.2] of such a localization by $\operatorname{Un}(S_f^{-1}\mathrm{M})$.

THEOREM 1.1. *Let W be an elementary abelian p-group, V a subgroup of W, and $f : H^W \to H^V$ the map induced by subgroup inclusion. Suppose that M is an object of $\mathscr{U}(H^W)$ which is finitely-generated as a module over H^W. Then there is a natural isomorphism*

$$T_f^V(M) \cong \operatorname{Un} S_f^{-1}(M).$$

This theorem has a geometric background. Let V and W be as in 1.1. Suppose that X is a finite CW-complex with a cellular action of W and let M be the cohomology of the Borel construction $E_W X = EW \times_W X$ on this action. According to [L2] there is an isomorphism

$$T_f^V(M) \cong H^*E_W(X^V)$$

where X^V is the fixed point set of the action of V on X. Similarly, by Smith theory [DW, 2.3] there is an isomorphism

$$H^*E_W(X^V) \cong \operatorname{Un} S_f^{-1}(M).$$

The composite of these two maps is the isomorphism of 1.1; the present paper sprang in part from a desire to produce this isomorphism in a purely algebraic way without assuming that $M \cong H^*E_W X$ for a finite complex X.

With Theorem 1.1 in hand, though, it is possible to work backwards through the above geometric example. Suppose that X is a space with an action of W but instead of assuming that X is finite assume only that H^*X is finite. Again let $M = H^*(E_W X)$; a spectral sequence argument shows that M is finitely generated as a module over H^W. Let X^{hV} denote the *homotopy fixed point set* [M, p. 581] of the action of V on X. It is not hard to see that the Borel construction $E_W(X^{hV})$ is homotopy equivalent to the space of maps $BV \to E_W X$ which up to cohomology cover the map $BV \to BW$ induced by $V \subset W$, so that $\lambda_{E_W x, f}$ produces a map $T_f^V(H^*E_W X) \to H^*E_W(X^{hV})$. Theorem 1.1 now computes the domain of $\lambda_{E_W x, f}$; this gives the following corollary.

COROLLARY 1.2. *Let W be an elementary abelian p-group, V a subgroup of W and $f: H^W \to H^V$ the map induced by subgroup inclusion. Let X a space on which W acts and assume that H^*X is finite and that $\lambda_{E_W x, f}$ is an isomorphism. Then there is a natural isomorphism.*

$$H^*E_W(X^{hV}) \cong \operatorname{Un} S_f^{-1} H^*(E_W X).$$

REMARK. The conclusion of Corollary 1.2 implies that the localized restriction map

$$S_f^{-1} H^*(E_W X) \to S_f^{-1} H^* E_W(X^{hV})$$

is an isomorphism. This is the promised extension of Smith theory [H, Chap. III] to infinite dimensional complexes (the only finiteness condition on X is a cohomological one). The question of whether $\lambda_{E_w X, f}$ is an isomorphism is essentially a spectral sequence convergence problem [DS] [B] and we intend to consider it in a future note.

REMARK. As in [DW, 2.5], for $V = W$ the conclusion of Corollary 1.2 gives an isomorphism $H^*(X^{hW}) \cong \mathbf{F}_p \otimes_{H^W} \operatorname{Un} S_f^{-1} H^*(E_W X)$.

An *unstable* $\mathbf{A}_p \odot H^V$ *algebra* is an object R of $\mathscr{K}$ together with a $\mathscr{K}$-map $H^V \to R$. Denote the category of $\mathbf{A}_p \odot H^V$ algebra by $\mathscr{K}(H^V)$. (We will also occasionally consider the category $\mathscr{K}(R)$ for other objects R of $\mathscr{K}$.) If M is an R module and $I \subset R$ is an ideal, let $M_I^\wedge$ stand for the completion of M with respect to powers of I, i.e. for the inverse limit $\varprojlim_s M/I^s M$.

THEOREM 1.3. *Let V be an elementary abelian p-group and $f: R \to H^V$ a map in $\mathscr{K}(H^V)$ with kernel I. Assume that R is finitely-generated as a ring and that M is* *an object of $\mathcal{U}(R)$ which is finitely-generated as a module over R. Then there is a natural isomorphism*

$$T_f^V(M) \cong \mathrm{Un}((S_f^{-1}M)_I^{\wedge}).$$

REMARK. Example 4.6 shows that the appearance of something like a completion is necessary in Theorem 1.3. It is easy to set (4.4) that in the statement of Theorem 1.3 the localization $S_f^{-1}M$ can be replaced by the localization $S^{-1}M$ of M with respect to the multiplicative set S generated by the Bockstein images in H^2BV of the non-zero elements of H^1BV.

The restriction in Theorem 1.3 that R be an algebra over H^V is not too serious, since it is possible to make any object of $\mathcal{K}$ into an object of $\mathcal{K}(H^V)$ by tensoring with H^V. If $g: R \to H^V$ is a map of $\mathcal{K}$ and $f: H^V \otimes_{F_p} R \to H^V$ is the map of $\mathcal{K}(H^V)$ which extends g, then (see 2.2 and the proof of 1.4) there is a natural isomorphism $T_f^V(H^V \otimes_{F_p} R) \cong H^V \otimes_{F_p} T_g^V(R)$. In conjunction with Theorem 1.3 this calculation leads to the following result.

THEOREM 1.4. *Let V be an elementary abelian p-group, R an object of $\mathcal{K}$ which is finitely generated as a ring and M an object of $\mathcal{U}(R)$ which is finitely generated as a module over R. Then $T^V(R)$ is finitely generated as a ring and $T^V(M)$ is finitely generated as a module over $T^V(R)$.*

Organization of the paper. In sections 2 through 4 we will prove the above theorems in the special case in which V is the rank one elementary abelian p-group $\mathbf{Z}/p$; for this particular V, T^V is simply denoted by T and H^V by H. Section 5 describes the argument that extends the results to general V. Section 6 contains some auxiliary algebraic material on $\mathcal{K}$.

§2. Some properties of the functor T

The purpose of this section is to set up some machinery involving the functor T.

Let $f: R \to H$ be a $\mathcal{K}$-map. The adjoint of f is a $\mathcal{K}$-map $T(R) \to \mathbf{F}_p$, which amounts to a ring homomorphism $\hat{f}: T(R)^0 \to \mathbf{F}_p$. For $M \in \mathcal{U}(R)$, let $T_f(M)$ denote the tensor product $T(M) \otimes_{T(R)^0} \mathbf{F}_p$, where the action of $T(R)^0$ on $\mathbf{F}_p$ is given by $\hat{f}$. Note that $T_f(R) \in \mathcal{K}$.

PROPOSITION 2.1. *For any $\mathcal{K}$-map $f: R \to H$ the construction $T_f(-)$ induces functors $\mathcal{U}(R) \to \mathcal{U}(T_f(R))$ and $\mathcal{K}(R) \to \mathcal{K}(T_f(R))$. Moreover, T_f is exact, and preserves tensor products in the sense that if M and N are objects of $\mathcal{U}(R)$ there is a* *natural isomorphism*

$$T_f(M \otimes_R N) \cong T_f(M) \otimes_{T_f(R)} T_f(N).$$

Proof. Most of what is asserted follows from the fact that T is exact and preserves tensor products [L1]. To see that T_f is exact, note in addition that $\hat{f}$ makes $\mathbf{F}_p$ into a flat module over $T(R)^0$; this flatness is an algebraic consequence of the fact that $T(R)^0$ is a *p-boolean ring* [L1, 3.5] (that is, each element x in $T(R)^0$ satisfies the equation $x^p = x$). In fact, if $T(R)^0$ is finite, then $T_f(R)$ is a summand of $T(R)$.

PROPOSITION 2.2. *Let $f_i : R_i \to H$, $i = 1, 2$ be $\mathscr{K}$-maps with $M_i \in \mathscr{U}(R_i)$ and let f be the product map $f_1 \cdot f_2 : R_1 \otimes_{F_p} R_2 \to H$. Then there is a natural isomorphism*

$$T_f(M_1 \otimes_{F_p} M_2) \to T_{f_1}(M_1) \otimes_{F_p} T_{f_2}(M_2).$$

Proof. This is again a consequence of the fact that T preserves tensor products.

PROPOSITION 2.3. *Suppose that $R \in \mathscr{K}$, $M \in \mathscr{U}(R)$, and that $x \in R$ is an element such that $x \cdot M = 0$. Let $f : R \to H$ be a $\mathscr{K}$-map such that $f(x) \neq 0$. Then $T_f(M) = 0$.*

Proof. By the Cartan formula the annihilator ideal I of M in R is invariant under $\mathbf{A}_p$ and hence the quotient map $R \to R/I$ is a morphism of $\mathscr{K}$. By Proposition 2.1 $T_f(M) \cong T_f(R/I) \otimes_{T_f(R)} T_f(M)$ so we will be done if we can show that $T_f(R/I)$ vanishes in dimension 0. Now $T_f(R)^0 \cong \mathbf{F}_p$ and the map $T_f(R)^0 \to T_f(R/I)^0$ is surjective, so non-vanishing of $T_f(R/I)^0$ implies that $T_f(R)^0 \to T_f(R/I)^0$ is an isomorphism and therefore that $\hat{f}: T(R)^0 \to T_f(R)^0 \cong \mathbf{F}_p$ extends to a map $T(R/I)^0 \to \mathbf{F}_p$. This is impossible, because by assumption the adjoint map $f : R \to H$ does not extend to a map $R/I \to H$.

For any object M of $\mathscr{U}$ the adjunction map $M \to H \otimes_{\mathbf{F}_p} T(M)$ can be combined with the unique algebra map $H \to \mathbf{F}_p$ to give a map $M \to T(M)$; call this map ϵ. (If $M = H^*X$ for some space X, then ϵ corresponds via λ_X to the cohomology homomorphism induced by the basepoint evaluation map $\mathrm{Hom}(B\mathbf{Z}/p, X) \to X$.) If $R \in \mathscr{K}$, $M \in \mathscr{U}(R)$ and $f : R \to H$ is a $\mathscr{K}$-map, we will denote the composite $M \xrightarrow{\epsilon} T(M) \to T_f(M)$ by ϵ_f.

If $f : H \to H$ is the identity map then $\epsilon_f : H \to T_f(H)$ is an isomorphism [L1, 4.2], so that by (2.1) T_f lifts to a functor $\mathscr{U}(H) \to \mathscr{U}(H)$ (or $\mathscr{K}(H) \to \mathscr{K}(H)$).

PROPOSITION 2.4 (SPLITTING PROPERTY). *Let $\varphi : H \to H$ be the identity map and $I \subset H$ the kernel of the unique $\mathscr{K}$-map $H \to \mathbf{F}_p$. Then for any object R of $\mathscr{K}(H)$ there is a natural $\mathscr{K}(H)$-isomorphism*

$$T_\varphi(R) \cong H \otimes_{\mathbf{F}_p} (T_\varphi(R)/I \cdot T_\varphi(R))$$

where the action of H on the tensor product is by multiplication on the left-hand factor.

REMARK. The proof of Proposition 2.4 also shows that if $R \in \mathscr{K}(H)$ and $M \in \mathscr{U}(R)$ there is an isomorphism $T_\varphi(M) \cong H \otimes_{\mathbf{F}_p} (T_\varphi(M)/I.\ T_\varphi(M))$. The action of $T_\varphi(R)$ on $T_\varphi(M)$ then respects the tensor product splittings of both objects.

Proof of 2.4. The adjunction map $R \to H \otimes_{\mathbf{F}_p} T(R)$ can be combined with the Hopf algebra coproduct map $H \to H \otimes_{F_p} H$ to give a map $R \to H \otimes_{\mathbf{F}_p} H \otimes_{\mathbf{F}_p} T(R)$; the adjoint to this is a map $T(R) \to H \otimes_{\mathbf{F}_p} T(R)$ which has as quotient a $\mathscr{K}(H)$-map $T_\varphi(R) \to H \otimes_{\mathbf{F}_p} T_\varphi(R)$. The desired isomorphism σ is the composite of this map with the projection

$$H \otimes_{\mathbf{F}_p} T_\varphi(R) \to H \otimes_{\mathbf{F}_p} (T_\varphi(R)/I \cdot T_\varphi(R)).$$

The fact that φ is the identity map insures that the action of H on the target tensor product is the desired one. Note that σ induces an isomorphism on $\mathrm{Tor}_0^H(H/I, -)$ and an epimorphism on $\mathrm{Tor}_1^H(H/I, -)$ (the latter because $\mathrm{Tor}_1^H(H/I, -)$ vanishes on the free H-module which is the target of σ). Let C be the cokernel of σ and k the kernel. The fact that $\mathrm{Tor}_0^H(H/I, C) = C/I \cdot C = 0$ implies that $C = 0$, since I is a connected ideal. The long exact sequence for $\mathrm{Tor}_*^H(H/I, -)$ then shows that $\mathrm{Tor}_0^H(H/I, K)$ vanishes, which similarly implies $K = 0$.

§3. Spherical elements

The purpose of this section is to prove Proposition 3.1, which is the algebraic basis of all of the results in this paper.

Let $S \subset H$ be the multiplicative subset generated by the non-zero elements of degree 2. The *closure* of an object M of $\mathscr{U}(H)$ is defined to be $\mathrm{Un}(S^{-1}M)$; M is *closed* if the natural map from M to the closure of m is an isomorphism. An element x of an unstable $\mathbf{A}_p$ module is *spherical* if $\alpha x = 0$ for each strictly positive-dimensional element $\alpha \in \mathbf{A}_p$.

PROPOSITION 3.1. *If $M \in \mathcal{U}(H)$ is non-zero, closed and finitely generated as a module over H then M contains a non-zero spherical element.*

Let H_+ be the subalgebra of H generated by the elements of S; H_+ is an unstable $\mathbf{A}_p$ algebra which is isomorphic to the cohomology ring of CP^∞. The multiplicative set S is contained in H_+ and so it is possible to speak of the *closure* of an object M of $\mathcal{U}(H_+)$; as above, M is *closed* if the natural map from M to the closure of M is an isomorphism. An object of $\mathcal{U}$ is *even-dimensional* if it vanishes in odd degrees. Proposition 3.1 is a consequence of the following result.

PROPOSITION 3.2. *If $M \in \mathcal{U}(H_+)$ is non-trivial, closed, even-dimensional, and finitely generated as a module over H_+, then M contains a nonzero spherical element.*

Proof of 3.1 (*given* 3.2). We will assume that p is an odd prime since the case $p = 2$ is a little simpler. Consider the map $\mathscr{V} : M \to M$ given by

$$\mathscr{V}(x) = \begin{cases} \mathscr{P}^k(x) & |x| = 2k \\ \beta\mathscr{P}^k(x) & |x| = 2k+1 \end{cases}.$$

The map $\mathscr{V}$ is not an $\mathbf{A}_p$-map, but the image of $\mathscr{V}$ is an $\mathbf{A}_p$-submodule of M [Li]. Let N be the closure of the H_+-submodule of M generated by the image of $\mathscr{V}$. It is clear that N is an $\mathbf{A}_p \odot H_+$ submodule of M which is closed, even-dimensional and finitely generated over H_+. If N is non-trivial we are done, since by Proposition 3.2 N contains a non-zero spherical element. If N is trivial, then $\mathscr{V}$ is the zero map and M is the *suspension* [Li] of a closed $\mathbf{A}_p \odot H$ module M'. We can argue by induction on the largest degree in which $M \otimes_{H_+} \mathbf{F}_p$ fails to vanish that M' contains a non-zero spherical element; the suspension of this element is then a non-zero spherical element of M. The induction begins with the fact that any zero-dimensional class is spherical.

In order to prove Proposition 3.2 we will need some additional notation. Let R be an even-dimensional object of $\mathscr{K}$ and M an even-dimensional object of $\mathcal{U}(R)$. Given a degree 2 element a of R and a degree $2k$ element x of M, define $\Phi_a(x)$ by the formula

$$\Phi_a(x) = \sum_{i=0}^{k} (-1)^i a^{i(p-1)} \mathscr{P}^{k-i}(x)$$

(cf. [DW, 3.2]). The Cartan formula shows that $\Phi_a(x)$ is multiplicative in x whenever this makes sense; in particular, $\Phi_a(a) = 0$ implies that $\Phi_a(x) = 0$ if x is

a-decomposable in M. We will be particularly interested in the case $R = H_+$ and will let c denote a chosen fixed degree-two generator of H_+.

The following lemma is essentially a reformulation of the Adem relations for the reduced p-th powers.

LEMMA 3.3. *Let R be an even-dimensional object of $\mathcal{K}$, M an even-dimensional object of $\mathcal{U}(R)$, and a, b degree 2 elements of R. Then, for any $x \in M$, $\Phi_a \Phi_b(x) = \Phi_b \Phi_a(x)$.*

Proof. Suppose that $|x| = 2k$. Let F_{2k} be the free object in $\mathcal{U}$ on a single generator ι_{2k} of dimension $2k$; F_{2k} is isomorphic to the submodule of $H^*K(\mathbf{Z}/p, 2k)$ generated by the fundamental class. By a universality argument we can assume that x is the element $1 \otimes \iota_{2k}$ of $R \otimes_{\mathbf{F}_p} F_{2k}$. By [AW, 2.7] we can assume that x is the element $1 \otimes (c^{\otimes k})$ of $R \otimes_{\mathbf{F}_p} (H_+)^{\otimes k}$. In fact, by multiplicativity of Φ we can even prove the lemma by checking the desired relation on the element $x = 1 \otimes c$ of $R \otimes H_+$. In this case explicit calculation gives

$$\Phi_a(x) = x^p - a^{p-1}x = \prod_{i=0}^{p-1} (x + ia)$$

and hence

$$\Phi_a \Phi_b(x) = \prod_{i=0}^{p-1} \prod_{j=0}^{p-1} (x + ia + jb).$$

The lemma follows from the fact that this expression is symmetric in a and b.

LEMMA 3.4. *Let $M \in \mathcal{U}(H_+)$ be closed and even-dimensional, and let x be an element of M. Then there exists an element y of M such that $x = c \cdot y$ if and only if $\Phi_c(x) = 0$.*

Proof. If such a y exists, then $\Phi_c(x) = \Phi_c(c)\Phi_c(y) = 0$. On the other hand, suppose that $\Phi_c(x) = 0$ and let $|x| = 2k$. The Cartan formula shows that $\mathcal{P}^{k+i}(x/c) = (-1)^i c^{i-1} \Phi_c(x)$, so that the vanishing of $\Phi_c(x)$ implies that $\mathcal{P}^j(x/c)$ vanishes for $j > k - 1$. Since M is even-dimensional, this easily ([Li] [AW, §2]) leads to the conclusion that $y = x/c \in \mathrm{Un}(S^{-1}M) = M$. Then $x = c \cdot (x/c) = c \cdot y$.

LEMMA 3.5. *Let M be an even-dimensional object of $\mathcal{U}(H_+)$ and $x \in M$ an element of degree $2k$ with the property that $\mathcal{P}^i x$ is c-decomposable for each $i > 0$.*

Then the action of $\mathbf{A}_p$ *on* $\Phi_c(x)$ *is given by the formula*

$$\mathscr{P}^i\Phi_c(x) = \binom{k(p-1)}{i} c^{i(p-1)}\Phi_c(x).$$

Proof. Identify M with the submodule $1 \otimes M$ of $N = H_+ \otimes_{\mathbf{F}_p} M$; it is clear that N is an object of $\mathscr{U}(H_+ \otimes_{\mathbf{F}_p} H_+)$. Let a denote the element $c \otimes 1$ of $H_+ \otimes H_+$ and b the element $1 \otimes c$. Since $\mathscr{P}^i(x)$ is b-decomposable for $i > 0$, it follows from the multiplicativity of Φ that $\Phi_b\Phi_a(x)$ depends only on the leading term of $\Phi_a(x)$, in particular,

$$\Phi_b\Phi_a(x) = \Phi_b((-1)^k a^{k(p-1)}x) = (-1)^k(a^p - b^{p-1}a)^{k(p-1)}\Phi_b(x).$$

By definition, however,

$$\Phi_a\Phi_b(x) = \sum_{i=1}^{pk} (-1)^i a^{i(p-1)}\mathscr{P}^{pk-i}\Phi_b(x).$$

The proof is finished by equating these two expressions (Lemma 3.3) and matching up the coefficients of corresponding powers of a.

Proof of 3.2. Let $2k$ be the largest dimension in which $M/(c \cdot M)$ is not zero, and choose an element $z \in M$ of dimension $2k$ which is not divisible by c. It is clear that $\mathscr{P}^i(x)$ is c-decomposable for each $i > 0$, so the action of $\mathbf{A}_p$ on $\Phi_c(x)$ is given by the formula of Lemma 3.5. A check with the Cartan formula shows that $y = c^{-k(p-1)}\Phi_c(x)$ is a spherical element of M. The element y is non-zero by Lemma 3.4.

§4. The rank one case

This section contains the proofs of Theorems 1.1, 1.3 and 1.4 in the special case in which the elementary abelian p-group involved is $\mathbf{Z}/p$. Recall from §3 that S denotes the multiplicative subset of H generated by the non-trivial elements of degree 2. Throughout this section $\varphi : H \to H$ will denote the identity map.

PROPOSITION 4.1. *If* $M \in \mathscr{U}(H)$ *is finitely generated as a module over* H *there is a natural isomorphism*

$$T_\varphi(M) \cong \mathrm{Un}(S^{-1}M).$$

REMARK. An unstable $\mathbf{A}_p$ module F is *finite* if F^j is finite-dimensional for all j and zero for almost all j. Suppose that $M \in \mathcal{U}(H)$ is finitely generated as a module over H. Propositions 2.4 and 4.1 combine to give the surprising fact that $\mathrm{Un}(S^{-1}M)$ splits as a tensor product $H \otimes_{\mathbf{F}_p} F$ for some finite $\mathbf{A}_p$ module F.

Recall from §2 that $T_\varphi(H) \cong H$ so that $T_\varphi(M) \in \mathcal{U}(H)$ if $M \in \mathcal{U}(H)$. It is easy to see that the map $\epsilon_\varphi : M \to T_\varphi(M)$ is a map of H-modules.

LEMMA 4.2. *Let $M \in \mathcal{U}(H)$ be the tensor product $H \otimes_{\mathbf{F}_p} F$, where $F \in \mathcal{U}$ is finite. Then $\epsilon_\varphi : M \to T_\varphi(M)$ is an isomorphism.*

Proof. Combine Proposition 2.2 and the fact [L1, 4.1] that $\epsilon : F \to T(F)$ is an isomorphism.

LEMMA 4.3. *If $M \in \mathcal{U}(H)$ is finitely generated as a module over H, then the map $S^{-1}\epsilon_\varphi : S^{-1}M \to S^{-1}T_\varphi(M)$ is an isomorphism.*

Proof. Work by induction on the rank $\mathrm{rk}(S^{-1}M)$ of $S^{-1}M$ as a module over $S^{-1}(H_+)$. Let $M' = \mathrm{Un}(S^{-1}M)$, so that $\mathrm{rk}(S^{-1}M') = \mathrm{rk}(S^{-1}M)$ and M' is closed. If $M' = 0$ then $T_\varphi(M)$ vanishes by Proposition 2.3; this case begins the induction.

Suppose then that M' is not trivial. By Proposition 3.1 M' contains a non-zero spherical element x. The annihilator ideal I of x in H is closed under the action of the Steenrod algebra, but, since M' embeds in $S^{-1}M'$, I contains no element of S. It follows from Proposition 6.4 that I is trivial and thus that the cyclic $\mathbf{A}_p \odot H$ submodule $\langle x \rangle$ of M' generated by x is a free H module of rank one. This shows both that $\mathrm{rk}(S^{-1}M') > 0$ and, by Lemma 4.2, that the map $\langle x \rangle \to T_\varphi\langle x \rangle$ is an isomorphism. Let $M'' = M/\langle x \rangle$. By induction the map $S^{-1}M'' \to S^{-1}T_\varphi(M'')$ is an isomorphism. Exactness of T_φ and exactness of localization now together imply that $S^{-1}M' \to S^{-1}T_\varphi(M')$ is an isomorphism. The inductive step is completed by observing that the map $M \to M'$ induces isomorphisms $S^{-1}M \cong S^{-1}M'$ and $T_\varphi(M) \cong T_\varphi(M')$ (the last by Proposition 2.3).

Proof of 4.1. By lemma 4.3 the map $S^{-1}\epsilon_\varphi : S^{-1}M \to S^{-1}T_\varphi(M)$ is an isomorphism. By Proposition 2.4 $T_\varphi(M)$ is a tensor product $H \otimes_{\mathbf{F}_p} F$ for some $F \in \mathcal{U}$, so [DW, 3.6] guarantees that the map $T_\varphi(M) \to \mathrm{Un}(S^{-1}T_\varphi(M))$ is an isomorphism. The proposition follows.

Let $f : R \to H$ be a map in $\mathcal{K}(H)$. Any object M of $\mathcal{U}(R)$ is an H module as well as an R module, so it is possible to form $T_\varphi(M)$ as well as $T_f(M)$. There is a natural surjection $T_\varphi(M) \to T_f(M)$.

LEMMA 4.4. *Let $f: R \to H$ be a map in $\mathscr{K}(H)$ with kernel I, and let M be an object of $\mathscr{U}(R)$. Then for each $s \geqslant 0$*

(1) *the map $T_f(M) \to T_f(M/I^sM)$ is an isomorphism up through dimension $s-1$ and*

(2) *the map $T_\varphi(M/I^sM) \to T_f(M/I^sM)$ is an isomorphism.*

Suppose moreover that R is finitely generated as a ring and M is finitely generated as an R module. Then for each $s \geqslant 0$

(3) *there is a natural isomorphism $T_f(M/I^sM) \cong \mathrm{Un}\, S_f^{-1}(M/I^sM)$.*

Proof. To prove (1), observe that the map

$$\mathbf{F}_p \cong T_f(R)^0 \to T_f(R/I)^0 \cong \mathbf{F}_p$$

is an isomorphism, so by exactness $T_f(I)$ vanishes in dimension 0. By Proposition 2.1, $T_f(I \otimes_R I \otimes_R \cdots \otimes_R I)$ (s factors) vanishes through dimension $s-1$. The rest follows from exactness of T_f. To prove (2), argue from the fact that I/I^s is nilpotent to conclude [L1, 4.3.2] that the projection map $R/I^s \to R/I \cong H$ induces an isomorphism $\hom_{\mathscr{K}}(H, H) \to \hom_{\mathscr{K}}(R/I^3, H)$ and thus by adjointness an isomorphism $T(R/I^s)^0 \cong T(H)^0$. It follows that there are isomorphisms

$$\begin{aligned} T_f(M/I^sM) &= T(M/I^sM) \otimes_{T(R)^0} \mathbf{F}_p \\ &\cong T(M/I^sM) \otimes_{T(R/I^s)^0} \mathbf{F}_p \\ &\cong T(M/I^sM) \otimes_{T(H)^0} \mathbf{F}_p = T_\varphi(M/I^sM). \end{aligned}$$

Statement (3) now follows easily from (2) and Proposition 4.1. The hypotheses imply that M/I^sM is finitely generated as a module over H. It is clear that given an element x of S_f there is an element y of S such that the image of x in R/I^s differs from the image of y by a nilpotent element; this implies that the natural map $S^{-1}(M/I^sM) \to S_f^{-1}(M/I^sM)$ is an isomorphism.

Proof of 1.3 (*for $V = \mathbf{Z}/p$*). Lemma 4.4(1) implies that there is an isomorphism $T_f(M) \cong \varprojlim_s T_f(M/I^sM)$ and it is clear by inspection that there is an isomorphism $\mathrm{Un}((S_f^{-1}M)_f^\wedge) \cong \varprojlim_s \mathrm{Un}\, S_f^{-1}(M/I^sM)$. The theorem follows from Lemma 4.4(3).

Proof of 1.4. (*for $V = \mathbf{Z}/p$*): We will prove only the statement about $T(R)$. Suppose first that $R \in \mathscr{K}(H)$ and that $f: R \to H$ is a $\mathscr{K}(H)$ map. By Lemma 4.4(1) and exactness there are isomorphisms

$$T_f(R) \cong \varprojlim_s T_f(R/I^s) \cong \varprojlim_s T_f(R)/T_f(I^s)$$

and the proof of 4.4(1) shows that the associated graded ring gr $T_f(R) = \Sigma_s T_f(I^s)/T_f(I^{s+1})$ is generated as an H algebra by $T_f(I)/T_f(I^2) = T_f(I/I^2)$. Lemma 4.3(3) and exactness of T_f imply that $T_f(I/I^2)$ is finitely generated as an H module. It follows that $T_f(R)$ is finitely generated as a ring.

Now choose $R \in \mathscr{K}$, pick a $\mathscr{K}$-map $g : R \to H$, and let $f : H \otimes_{\mathbf{F}_p} R \to H$ be the product of g with the identity map φ of H. The above considerations show that $T_f(H \otimes_{\mathbf{F}_p} R)$ is finitely generated as a ring, and Proposition 2.2 shows that $(T_f H \otimes_{F_p} R)$ is isomorphic to $T_\varphi(H) \otimes_{\mathbf{F}_p} T_g(R) \cong H \otimes_{\mathbf{F}_p} T_g(R)$. This proves that $T_g(R)$ is finitely generated. There are only a finite number of choices for the map g (since R is finitely generated as a ring) and $T(R)$ is isomorphic to a product $\Pi_g T_g(R)$ indexed by these choices (cf. [L1, 3.5]). This implies that $T_g(R)$ is finitely generated.

The next lemma requires an additional bit of notation. For any $\mathscr{K}$-map $f : R \to H^V$ let $\Sigma_f \subset R$ denote the multiplicative subset of R consisting of elements x such that $f(x)$ is not a zero-divisor in H^V.

LEMMA 4.5. *Let $f : R \to H$ be a map in $\mathscr{K}(H)$ and M an object of $\mathscr{U}(R)$. Suppose that R is finitely generated as a ring, that M is finitely generated as an R module, and that $\epsilon_f : R \to T_f(R)$ is an isomorphism. Then there is a natural isomorphism $T_f(M) \cong \mathrm{Un}(\Sigma_f^{-1}M)$.*

Proof. Calculating with Lemma 4.4 shows that the map $\epsilon_f : M \to T_f(M)$ induces an isomorphism $T_f(M) \to T_f T_f(M)$. It follows that $T_f(N) = 0$ if N is either the kernel or cokernel of $M \to T_f(M)$. Theorem 1.4 guarantees that N is finitely-generated as an R module, so by Lemma 4.4(3) the localization $\Sigma_f^{-1}(N/IN)$ vanishes, and hence by Nakayama's lemma [AM, p. 21] the localization $\Sigma_f^{-1}N$ itself vanishes. In other words, the map ϵ_f induces an isomorphism $\Sigma_f^{-1}M \to \Sigma_f^{-1}T_f(M)$. Now Proposition 2.3 (together with a finite generation argument) shows that there is an isomorphism $T_f(M) \to T_f \,\mathrm{Un}\, \Sigma_f^{-1}(M)$ and hence that there is a map ϵ_f from $\mathrm{Un}\, \Sigma_f^{-1}M$ to $T_f(\mathrm{Un}\, \Sigma_f^{-1}M) \cong T_f(M)$. The above considerations produce a map in the other direction from $T_f M$ to $\mathrm{Un}\, \Sigma_f^{-1}T_f(M) \cong \mathrm{Un}\, \Sigma_f^{-1}M$. The lemma follows easily.

Proof of 1.1 (*for* $V = \mathbf{Z}/p$). Any subgroup of W is a summand, so the map $f : H^W \to H$ is split and can be treated as a map in $\mathscr{K}(H)$. Since $\epsilon_f : H^W \to T_f(H^W)$ is an isomorphism [L1, 4.2], Lemma 4.5 applies and reduces the proof of the theorem to showing that the natural map $\mathrm{Un}\, S_f^{-1}(M) \to \mathrm{Un}\, \Sigma_f^{-1}(M)$ is an isomorphism. Let N denote either the kernel or the cokernel of the map $\epsilon_f : M \to T_f(M) \cong \mathrm{Un}\, \Sigma_f^{-1}(M)$. It is clear that $N \in \mathscr{U}(H^W)$ is a finitely generated module over H^W with the property that $\Sigma_f^{-1}(n) = 0$; to finish the proof it is enough to show that any such module N has $S_f^{-1}(N) = 0$, or in other words, it is enough to find an element w of S_f which annihilates N. By the finite generation of N there exists an element x of Σ_f which annihilates N. Write $x = u + v$ where u belongs to the polynomial subalgebra H_+^W of H^W generated by the image of the Bockstein $\beta : H^1BW \to H^2BW$ and v is nilpotent. By replacing x with x^{p^L} for large L we can in fact assume that $x = u \in H_+^W$; note that $x \neq 0$ because the image of x in H is non-nilpotent. Let $I \subset H_+^W$ be the annihilator ideal of N, and write the radical of I in H_+^W as an intersection $\bigcap_i \rho_i$ of prime ideals ρ_i which are closed under the action of the Steenrod algebra (Proposition 6.1). By Proposition 6.3 each ρ_i is generated as an ideal by two-dimensional classes. If any ρ_i has all of its two-dimensional generators contained in the kernel of f then $I \subset \ker(f)$, which is impossible because $x \in I$. Consequently, it is possible to choose from each ρ_i a two-dimensional generator w_i such that the image of w_i in H is non-zero. It is clear that $w_i \in S_f$. If w is set equal to a sufficient high power of the product $\Pi_i w_i$, then w is the desired element in $S_f \cap I$.

EXAMPLE 4.6. The following example from Smith theory illustrates that it is necessary to include a completion of some type in the statement of Theorem 1.3. Let $p = 2$, let $R = H^*BO(2)$ and let $f : R \to H$ be the map induced by an homomorphism $Z/2 \to O(2)$ sending the non-trivial element of $Z/2$ to a matrix of determinant -1. Note that the determinant map $O(2) \to Z/2$ lifts R to an H algebra and f to a morphism in $\mathscr{K}(H)$. It follows from [L2], say, that $T_f(R)$ is isomorphic to $H^*(RP^\infty \times RP^\infty)$, but there is only one unstable one-dimensional generator in $S_f^{-1}R$. Let w_1 and w_2 be the Stiefel-Whitney classes which generate $H^*BO(2)$ as a polynomial algebra. The additional necessary one-dimensional generator appears in the completion of the localization of R as the finite sum

$$\sum_{i=0}^{\infty} w_1(w_2/w_1^2)^{2^i}$$

(a formula which can be derived by the *a posteriori* knowledge that this generator α satisfies the equation $\alpha^2 + w_1\alpha + w_2 = 0$).

§5. The general case

In this section we will describe the argument which is used to prove Theorems 1.1, 1.3 and 1.4 for a general elementary abelian p-group V. We will use the fact that for $V = \mathbf{Z}/p$ these theorems have already been proven in §4.

Let $S^V \subset H^V$ be the multiplicative subset generated by the non-zero elements in the image of the Bockstein map $\beta: H^1BV \to H^2BV$ and H_+^V the subalgebra of H^V generated by the elements of S^V. It is possible to check that the proofs in §2 and §4 hold almost verbatim with H replaced by H^V, H_+ by H_+^V, T by T^V and S by S^V; the only exception is the proof of Lemma 4.3 to the extent to which it relies on Lemma 3.1. To complete the proof of the desired results for V, then, it is enough to give an appropriate generalization of Lemma 3.1.

The *closure* of an object $M \in \mathcal{U}(H^V)$ is defined to be $\mathrm{Un}(S^V)^{-1}M$; M is *closed* if the natural map from M to the closure of M is an isomorphism.

LEMMA 5.1. *If $M \in \mathcal{U}(H^V)$ is non-zero, closed and finitely generated as a module over H^V then M contains a non-zero spherical element.*

Proof. The argument is by induction on the dimension of V as an $\mathbf{F}_p$ vector space. By Lemma 3.1, we can assume that this dimension is greater than one. Write V as a direct sum $\mathbf{Z}/p \oplus W$ for some W; this gives a map $g: H \to H^V$ which lifts H^V to $\mathcal{K}(H)$, as well as a map $f: H^V \to H$ such that $f \cdot g$ is the identity map of H. By the special case of Theorem 1.3 proven in §4 there is an isomorphism $T_f(M) \cong \mathrm{Un}\, S_f^{-1}(M)$. It follows from Proposition 2.4 that there is a tensor product splitting $T_f(M) \cong H \otimes_{\mathbf{F}_p} N$ which is compatible with the evident [L1, 4.2] splitting $T_f(H^V) \cong H^V \cong H^V \cong H \otimes_{\mathbf{F}_p} H^W$. Since M is closed, $\mathrm{Un}\, S_f^{-1}(M)$ is isomorphic to M. The conclusion is that M splits as a tensor product $H \otimes_{\mathbf{F}_p} N$ where N is an object of $\mathcal{U}(H^W)$ which is necessarily non-zero, closed and finitely generated as a module over H^W. By induction n contains a non-zero spherical element x. The element $1 \otimes x \in H \otimes_{\mathbf{F}_p} N \cong M$ is the desired spherical element of M.

§6. Some algebraic facts

The purpose of this section is to gather together some standard algebraic results. Proposition 6.3 is used in the proof of Theorem 1.1 (see §4) and Proposition 6.4 in the proof of Lemma 4.3 (particularly in the inductive setting described in §4).

PROPOSITION 6.1. *Suppose that $R \in \mathcal{K}$ is evenly graded and finitely generated as a ring, and that $I \subset R$ is a homogeneous ideal which is closed under the action of $\mathbf{A}_p$. Then the radical of I in R can be written as an intersection $\bigcap_i \rho_i$ of homogeneous prime ideals ρ_i closed under the action of $\mathbf{A}_p$.*

Proof. Let J be the radical of I (that is, the ideal of all elements $x \in R$ such that some power of x lies in i). It is easy to prove by induction on j and the Cartan formula that $\mathscr{P}^j x \in J$ if $x \in J$, in other words, that j is closed under the action of $\mathbf{A}_p$. Write J as an intersection $\bigcap_i \sigma_i$ of (homogeneous) prime ideals (see [AM, Chap. 7]), For each i let $\sigma_i^{(1)}$ be the set of elements $x \in \sigma_i$ such that $\mathscr{P}^j x \in \sigma_i$ for all $j \geqslant 0$. The argument of [AW, p. 138] shows that $\sigma_i^{(1)}$ is a prime ideal contained in σ_i, although it is not evident that $\sigma_i^{(1)}$ is closed under the action of $\mathbf{A}_p$. Iterate the procedure of passing from σ_i to σ_i^1 to obtain a descending chain

$$\sigma_i \supseteq \sigma_i^{(1)} \supseteq \sigma_i^{(2)} \supseteq \cdots$$

of prime ideals in R. Such a chain must eventually become constant [ZS, p. 241]; let ρ_i denote the limiting constant value. It is clear that ρ_i is a prime ideal of R which is closed under the action of $\mathbf{A}_p$ and that $J = \bigcap_i \rho_i$.

The following propositions use some of the notation of §5.

PROPOSITION 6.2. *Let V be an elementary abelian p-group, and suppose that $I \subset H^V$ is a non-trivial homogeneous ideal closed under the action of $\mathbf{A}_p$. Then I contains a non-zero element of H_+^V.*

Proof. Let $e_1, \ldots, e_k$ be a collection of generators for H^1BV and $t_1, \ldots, t_k$ their Bockstein images in H^2BV, so that H_+^V is the polynomial algebra $\mathbf{F}_p[t_1, \ldots, t_k]$ and H^V is isomorphic as an algebra to the tensor product of H_+^V and an exterior algebra on $e_1, \ldots, e_k$. Pick a non-zero element x in I. By multiplying x, if necessary, by a suitable product of e_i's we can assume that x has the form $te_1e_2 \cdots e_k$ where $t \in H_+^V$ is non-zero. Define a sequence $d_1, \ldots d_k$ of integers inductively by setting $d_1 = |t|/2 + 1$ and $d_{i+1} = pd_i + 1$ $(i \geq 1)$. Define elements $\alpha_0, \ldots, \alpha_{k-1}$ of $\mathbf{A}_p$ by setting $\alpha_0 = \beta$ and $\alpha_{i+1} = \beta\mathscr{P}^{d_i+1}\alpha_i (i \geq 0)$. A short calculation then shows that

$$\alpha_{k-1}x = t^{p^{k-1}} \sum_{\sigma} \operatorname{sgn}(\sigma) t_{\sigma(1)}^{p^{k-1}} t_{\sigma(2)}^{p^{k-2}} \cdots t_{\sigma(k)}$$

where σ runs through the permuation group on k letters. It is clear that $\alpha_{k-1}x$ is a non-zero element of $I \cap H_+^V$.

PROPOSITION 6.3. *Suppose that V is an elementary abelian p-group and that $I \subset H_+^V$ is a homogeneous prime ideal which is closed under the action of $\mathbf{A}_p$. Then I is generated as an ideal by its elements of dimension 2.*

Proof. The result is due to Serre ([Se], cf. [AW, 1.11]).

PROPOSITION 6.4. *Suppose that V is an elementary abelian p-group and that $I \subset H^V$ is a non-trivial homogeneous ideal closed under the action of $\mathbf{A}_p$. Then I contains an element of S^V.*

Proof. Let J be the intersection of the radical of I with H_+^V. By Proposition 6.1 and 6.3, J can be written as an intersection $\bigcap_i \rho_i$ of prime ideals in H_+^V each of which is generated by elements of dimension 2. Proposition 6.3, guarantees that J is not trivial, so each of the ρ_i is also a non-trivial ideal. Pick non-zero two-dimensional elements $x_i \in \rho_i$ and let x be the product of the x_i's. It is clear that some power of x lies in $I \cap S^V$.

REMARK. The top Dickson invariant c_0 [W2] in H^V is by definition the product of the two-dimensional elements of S^V. It follows from Proposition 6.4 that any ideal I of the indicated type actually contains some power of c_0.

REFERENCES

[AM] M. F. ATIYAH and I. G. MACDONALD, "Introduction to Commutative Algebra," Addison-Wesley, New York, 1969.

[AW] J. F. ADAMS and C. W. WILKERSON, *Finite H-spaces and algebras over the Steenrod algebra*, Annals of Math. *111* (1980), 95–143.

[B] A. K. BOUSFIELD, *On the homology spectral sequence of a cosimplicial space*, Amer J. Math. *109* (1987), 361–394.

[DMN] W. G. DWYER, H. R. MILLER and J. NEISENDORFER, *Fibrewise completion and unstable Adams spectral sequences*, Israel J. Math. *66* (1989), 160–178.

[DMW1] W. G. DWYER, H. R. MILLER and C. W. WILKERSON, *The homotopic uniqueness of BS^3*, in "Algebraic Topology, Barcelona 1986," Lecture Notes in Math. 1298, Springer, Berlin, 1987, pp. 90–105.

[DMW2] W. G. DWYER, H. R. MILLER and C. W. WILKERSON, *Homotopical uniqueness of classifying space*, preprint (Notre Dame) 1990.

[DS] E. DROR-FARJOUN and J. SMITH, *A geometric interpretation of Lannes' functor T*, Israel J. Math. *66* (1989).

[DW] W. G. DWYER and C. W. WILKERSON, *Smith theory revisited*, Annals of Math. *127* (1988), 191–198.

[H] W.-Y. HSIANG, "Cohomological Theory of Transformation Groups," Springer, Berlin, 1975.

[L1] J. LANNES, *Sur la cohomologie modulo p des p-groupes. Abéliens élémentaires*, in "Homotopy Theory, Proc. Durham Symp. 1985," edited by E. Rees and J. D. S. Jones, Cambridge Univ. Press. Cambridge, 1987.

[L2] J. LANNES, *Cohomology of groups and function spaces*, preprint (École Polytechnique) 1987.

[Li] H. LI, *Iterated loop functors and the homology of the Steenrod algebra*, Ph.D. thesis (Fordham University) (1980).

[M] H. R. MILLER, *The Sullivan conjecture and homotopical representation theory*, in "Proceedings of the International Congress of Mathematicians, Berkeley, California," Amer. Math. Soc., Providence, 1987, pp. 580–589.

[Se] J.-P. Serre, *Sur la dimension cohomologique des groupes profinis*, Topology *3* (1965), 413–420.

[Si] W. M. SINGER, *On the localization of modules over the Steenrod algebra*, J. Pure and Appl. Alg. *16* (1980), 75–84.

[W1] C. W. WILKERSON, *Classifying spaces, Steenrod operations and algebraic closure*, Topology *16* (1977), 227–237.

[W2] C. W. WILKERSON, *A primer on the Dickson invariants*, in "Contemporary Math. Vol. 19," A.M.S., Providence, 1983, pp. 233–246.

[ZS] O. ZARISKI and P. SAMUEL, "Commutative Algebra," Volume I, Van Nostrand, Princeton, 1958.

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