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On the reflection representation in Springer's theory

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Summary. The theory of arrangements of hyperplanes allows to attach to every parabolic subgroup of a finite Coxeter group some numbers which look very much like exponents of the Coxeter group. In the case of Weyl groups similar numbers arise from the character theory of finite groups of Lie type, and more generally from the theory of Springer's representations. For exceptional Weyl groups these numbers were known to coincide, at least if the characteristic is not a small prime. Lehrer and Shoji [8] have shown that in characteristic 0 the same is true for classical Weyl groups, by computing the multiplicity of the reflection representation in the Springer representations associated to various nilpotent orbits. According to a note added in proof, they can handle all nilpotent orbits which are relevant to the connection with arrangements of hyperplanes, but some nilpotent orbits still evade their investigation.

In this paper we show that there is an additional structure on the cohomology spaces they consider. This allows to recover their results in a more direct way, to complete the determination of the multiplicity of the reflection representation in the Springer representations for classical groups, and to extend these results to arbitrary characteristic, including characteristic 2, for which we consider both unipotent elements in the group and nilpotent elements in the Lie algebra. We refer to [loc. cit.] for a description of the problem as far as arrangements of hyperplanes are concerned and deal here only with Springer representations.

1. Introduction

1.1. Let $\mathfrak{g}$ be the Lie algebra of a connected reductive algebraic group G defined over an algebraically closed field k. Let $\mathscr{B}$ be the variety of all Borel subgroups of G. For $A \in \mathfrak{g}$ let $\mathscr{B}_A = \{B \in \mathscr{B} \mid A \in \mathrm{Lie}(B)\}$. The Springer representation for A is a natural action of the Weyl group W of G on the cohomology $H^*(\mathscr{B}_A)$. The cohomology theory used here is l-adic cohomology, where l is a prime distinct from the characteristic of k. A construction of the representation is sketched in 2.1.

Let ρ be the reflection representation of W. What we actually determine in this paper is the multiplicity of ρ in the cohomology groups $H^i(\mathscr{B}_A)$ for A nilpotent, when G is a classical group.

Let q be an indeterminate and let $Q_A = Q_A^G = \Sigma_{i \geq 0}(-1)^i H^i(\mathscr{B}_A) q^{i/2}$, a polynomial in $q^{1/2}$ with virtual representations of W as coefficients. We want to compute $\langle Q_A, \rho\rangle = \Sigma_{i \geq 0}(-1)^i \langle H^i(\mathscr{B}_A), \rho\rangle q^{i/2}$, a polynomial in $q^{1/2}$ with coefficients in $\mathbb{Z}$.

Let B be a Borel subgroup of G, $T \subset B$ a maximal torus. Then W can be identified with $N_G(T)/T$. The action of $N_G(T)/T$ on G/T by right multiplication induces a left action of W on $H^*(G/T) \cong H^*(G/B)$. This action coincides with the

Springer representation on the cohomology of $\mathscr{B}_0 = \mathscr{B}$ [3], [10]. This can be used to show that $\langle Q_0, \rho\rangle = \Sigma_{i=1}^r q^{m_i}$, where r is the semisimple rank of G and $m_1, \ldots, m_r$ are the exponents of G.

Remark. It is known under mild restrictions on the characteristic that $\mathscr{B}_A$ has no odd cohomology. When this is the case, in the alternating sum defining Q_A the terms with i odd vanish and Q_A is therefore a polynomial in q with representations of W as coefficients. We shall not need this result here.

For $x \in G$ we can consider in a similar way the variety $\mathscr{B}_x = \{B \in \mathscr{B} \mid x \in B\}$. There is an action of the Weyl group W on $H^*(\mathscr{B}_x)$. If G is defined over a finite field K, the polynomials Q_x with x unipotent are tightly related to the Green functions of the finite group $G(K)$ [7], [9].

As long as the characteristic is good there is no essential difference between the unipotent and the nilpotent case, as follows from [2, Prop. 9.3.3], and even for bad characteristic many arguments apply to both cases with obvious changes. We shall usually treat only one of them.

1.2. There is a natural bijection between W and the set of all G-orbits in $\mathscr{B} \times \mathscr{B}$. Let $\mathscr{O}_w \subset \mathscr{B} \times \mathscr{B}$ be the orbit corresponding to $w \in W$.

Let $\mathscr{P}$ be a conjugacy class of parabolic subgroups of G. Then $\{(B_1, B_2) \in \mathscr{B} \times \mathscr{B} \mid \text{there exists } P \in \mathscr{P} \text{ such that } B_1, B_2 \subset P\}$ is a G-stable subset of $\mathscr{B} \times \mathscr{B}$, hence of the form $\bigcup_{w \in W(\mathscr{P})} \mathscr{O}_w$ for a well-defined subset $W(\mathscr{P})$ of W, and $W(\mathscr{P})$ is a subgroup of W which can be thought of as the Weyl group of $P \in \mathscr{P}$. Let also $N(\mathscr{P}) = N_W(W(\mathscr{P}))$, $W^{\mathscr{P}} = N(\mathscr{P})/W(\mathscr{P})$. For $A \in \mathfrak{g}$ let $\mathscr{P}_A = \{P \in \mathscr{P} \mid A \in \operatorname{Lie}(P)\}$. Then $W^{\mathscr{P}}$ acts on the cohomology of $\mathscr{P}_A$ (see 2.1). Let $\pi_A : \mathscr{B}_A \to \mathscr{P}_A$ be the map which associates to $B \in \mathscr{B}_A$ the unique element of $\mathscr{P}$ which contains B. The following result, a proof of which is outlined in 2.1, could (or should) have been stated by Borho and MacPherson [4].

THEOREM. *The homomorphism $\pi_A^* : H^*(\mathscr{P}_A) \to H^*(\mathscr{B}_A)$ induces an isomorphism of $W^{\mathscr{P}}$-modules*

$$H^*(\mathscr{P}_A) \cong H^*(\mathscr{B}_A)^{W(\mathscr{P})}. \tag{1.2.1}$$

In view of this result we define also $P_A = \Sigma_{i \geq 0} (-1)^i H^i(\mathscr{P}_A) q^{i/2}$, a polynomial in $q^{1/2}$ with virtual representations of $W^{\mathscr{P}}$ as coefficients. When $W^{\mathscr{P}} = 1$ we consider P_A as a polynomial with coefficients in $\mathbb{Z}$.

1.3. The approach used by Lehrer and Shoji to determine $\langle Q_A, \rho\rangle$ is to use 1.2.1, considered as an isomorphism of graded vector spaces, with a conjugacy class of parabolic subgroups $\mathscr{P}$ such that the cohomology groups $H^*(\mathscr{P}_A)$ can be determined explicitly and for which the representation $\operatorname{ind}_{W(\mathscr{P})}^W 1$ is small. For $\mathfrak{sl}_n$ (or $\mathfrak{gl}_n$)

we can arrange to have $\mathrm{ind}_{W(\mathscr{P})}^{W} 1 = 1 + \rho$. As $\langle Q_A, 1 \rangle = 1$ (this can be deduced from 1.2.1 with $\mathscr{P} = \{G\}$), we get $\langle Q_A, \rho \rangle = P_A - 1$. For the remaining classical groups the best one can achieve is $\mathrm{ind}_{W(\mathscr{P})}^{W} 1 = 1 + \rho + \xi$ for some irreducible representation ξ of W. Then in $P_A - 1$ we have contributions from both ρ and ξ. Shoji and Lehrer can separate them to a large extent by using the commutative diagram

$$\begin{array}{ccc} H^*(\mathscr{P}) & \longrightarrow & H^*(\mathscr{P}_A) \\ \downarrow & & \downarrow \\ H^*(\mathscr{B})^{W(\mathscr{P})} & \longrightarrow & H^*(\mathscr{B}_A)^{W(\mathscr{P})} \end{array} \tag{1.3.1}$$

in which the vertical arrows are isomorphisms. They however overlook the fact that 1.3.1 is a commutative diagram of $W^{\mathscr{P}}$-modules, as follows from the following lemma which is proved in 2.3.

LEMMA 1.4. *For every $A \in \mathfrak{g}$, the map $H^*(\mathscr{P}) \to H^*(\mathscr{P}_A)$ induced by the inclusion $\mathscr{P}_A \subset \mathscr{P}$ is $W^{\mathscr{P}}$-equivariant.*

1.5. Let now G be one of the groups GL_N, Sp_N or SO_N. For Sp_N and SO_N we set $n = [N/2]$. In all cases we have a natural representation of G in $V = k^N$, Sp_{2n} is defined by a non-degenerate alternating bilinear form on V, and SO_N is defined by a non-degenerate quadratic form Q on V. Let $\mathscr{P}$ be the conjugacy class of parabolic subgroups of G which consists of the stabilizers of isotropic lines in V (in the case of GL_N all lines are considered to be isotropic). Then $\mathscr{P}$ is isomorphic to the subvariety of $\mathbb{P}(V)$ formed by the isotropic lines. For GL_N and Sp_N we get in this way the full projective space $\mathbb{P}(V)$. In the orthogonal case we get a quadratic hypersurface $\mathscr{Q}$ in $\mathbb{P}(V)$. In the case of GL_N we have $\mathrm{ind}_{W(\mathscr{P})}^{W} 1 = 1 + \rho$ and $W^{\mathscr{P}} = 1$. For Sp_N and SO_N, $\mathrm{ind}_{W(\mathscr{P})}^{W} 1 = 1 + \rho + \xi$, where ξ is a permutation representation of degree $n - 1$, and $W^{\mathscr{P}}$ has order 2. If σ is the generator of $W^{\mathscr{P}}$, the contribution coming from ρ in $H^*(\mathscr{P}_A)$ is the (-1)-eigenspace of σ, and the fixed points of σ correspond to ξ and the trivial representation of W.

The following terminology is used in the orthogonal case. Let q be a quadratic form on a vector space U. By the radical of q we mean the set of all vectors $x \in U$ such that $q(y + x) = q(y)$ for all $y \in U$. Let β be the bilinear form on U defined by $\beta(u, v) = q(u + v) - q(u) - q(v)$. The radical of q is also the set of all q-isotropic vectors in $U^\perp$, where $U^\perp$ is defined using β. If $\mathrm{char}\,(k) \neq 2$, then the radical of q is simply $U^\perp$.

The following results are due to a large extent to Lehrer and Shoji. They assume that $\mathrm{char}\,(k) = 0$, but for 1.6, 1.7 and 1.8 their proofs carry over to arbitrary characteristic. In the even orthogonal case there are some orbits which they cannot handle, and the characteristic 2 case requires some additional arguments.

PROPOSITION 1.6. (cf. [8, §6]) (a) *Let* $x \in GL_N$ *be unipotent and let* $d = \dim \operatorname{Ker}(x-1)$. *Then*

$$\langle Q_x, \rho \rangle = \sum_{i=1}^{d-1} q^i.$$

(b) *The same result holds for* $A \in \mathfrak{gl}_N$ *nilpotent, with* $d = \dim \operatorname{Ker}(A)$.

Proof. We take $\mathscr{P}$ as in 1.5. Then $W^{\mathscr{P}} = 1$ and we consider P_A as a polynomial with coefficients in $\mathbb{Z}$. Moreover $\operatorname{ind}_{W(\mathscr{P})}^{W} 1 = 1 + \rho$ and therefore $\langle Q_A, \rho \rangle = P_A - 1$. Let $U = \operatorname{Ker}(A)$. Then $\mathscr{P}_A$ can be identified with $\mathbb{P}(U)$. It follows immediately that $P_A = \Sigma_{i=0}^{d-1} q^i$. Thus $\langle Q_A, \rho \rangle = \Sigma_{i=1}^{d-1} q^i$.

PROPOSITION 1.7. (cf. [8, 6.5, 6.6]) (a) *Let* $x \in Sp_{2n}$ *be unipotent and let* $d = \dim \operatorname{Ker}(x-1)$. *Then*

$$\langle Q_x, \rho \rangle = \sum_{\substack{i=1 \\ i \text{ odd}}}^{d-1} q^i, \qquad \langle Q_x, \xi \rangle = \sum_{\substack{i=2 \\ i \text{ even}}}^{d-1} q^i.$$

(b) *The same results hold for* $A \in \mathfrak{sp}_{2n}$ *nilpotent, with* $d = \dim \operatorname{Ker}(A)$.

Proof. In this case σ acts as multiplication by $(-1)^i$ on $H^{2i}(\mathscr{P})$. Let $U = \operatorname{Ker}(A)$. Then $\mathscr{P}_A$ can be identified with $\mathbb{P}(U) \subset \mathbb{P}(V)$, and the natural map $H^*(\mathbb{P}(V)) \to H^*(\mathbb{P}(U))$ is surjective. The result follows.

PROPOSITION 1.8. (cf. [8, 7.15]) (a) *Let* $x \in SO_{2n+1}$ *be unipotent and let* $d = \dim \operatorname{Ker}(x-1)$, r *the dimension of the radical of the restriction of* Q *to* $\operatorname{Ker}(x-1)$. *Then the following hold.*

(i) *If* r *is even or* char $(k) = 2$, *then*

$$\langle Q_x, \rho \rangle = \sum_{\substack{i=1 \\ i \text{ odd}}}^{d-2} q^i, \qquad \langle Q_x, \xi \rangle = \sum_{\substack{i=2 \\ i \text{ even}}}^{d-2} q^i.$$

(ii) *If* r *is odd and* char $(k) \neq 2$, *then*

$$\langle Q_x, \rho \rangle = \sum_{\substack{i=1 \\ i \text{ odd}}}^{d-2} q^i, \qquad \langle Q_x, \xi \rangle = \sum_{\substack{i=2 \\ i \text{ even}}}^{d-2} q^i + q^{(d+r-2)/2}.$$

(b) *The same results hold for* $A \in \mathfrak{so}_{2n+1}$ *nilpotent, with* $\operatorname{Ker}(x-1)$ *replaced by* $\operatorname{Ker}(A)$.

Proof. In this case σ acts as multiplication by $(-1)^i$ on $H^{2i}(\mathscr{P})$. Let $U = \operatorname{Ker}(A)$. Then $\mathscr{P}_A$ can be identified with $\mathscr{Q}_U = \mathbb{P}(U) \cap \mathscr{Q}$. If char$(k) = 2$, then U contains $V^{\perp}$. As we shall see in 3.3 and 3.5 we have then

$$\dim H^i(\mathscr{Q}_U) = \begin{cases} 1 & \text{if } i \text{ is even and } 0 \le i \le 2(d-2), \\ 0 & \text{otherwise,} \end{cases}$$

and the natural map $H^*(\mathscr{Q}) \to H^*(\mathscr{Q}_U)$ is surjective. The result follows. The proof for char$(k) \neq 2$ is given in 4.3.

PROPOSITION 1.9. (cf. [8, 7.28]) (a) *Let $x \in SO_{2n}$ be unipotent and let $d = \dim \operatorname{Ker}(x-1)$, r the dimension of the radical of the restriction of Q to* $\operatorname{Ker}(x-1)$. *Then the following hold.*

(i) *If r is odd, then*

$$\langle Q_x, \rho \rangle = \sum_{\substack{i=1 \\ i \text{ odd}}}^{d-2} q^i, \qquad \langle Q_x, \xi \rangle = \sum_{\substack{i=2 \\ i \text{ even}}}^{d-2} q^i.$$

(ii) *If r is even, then*

$$\langle Q_x, \rho \rangle = \sum_{\substack{i=1 \\ i \text{ odd}}}^{d-2} q^i + q^{(d+r-2)/2}, \qquad \langle Q_x, \xi \rangle = \sum_{\substack{i=2 \\ i \text{ even}}}^{d-2} q^i.$$

(b) *The same results hold for $A \in \mathfrak{so}_{2n}$ nilpotent, with* $\operatorname{Ker}(x-1)$ *replaced by* $\operatorname{Ker}(A)$.

The proof is given in 4.4 and 4.5.

1.10. One of the ingredients used in [8] is the following result of Lusztig, which is stated in [1].

PROPOSITION. *Let L be a Levi factor of some parabolic subgroup of G. Let $x \in L$ be unipotent. Then $Q_x(1) = \operatorname{ind}_{W(L)}^{W} Q_x^L(1)$, where $W(L)$ is the Weyl group of L, considered as a subgroup of W.*

This result is characteristic free. A crucial ingredient in Lusztig's unpublished proof is the fact that Springer's Green functions, which are defined in terms of the W-action on $H^*(\mathscr{B}_x)$, are integer valued, and this is now known to hold in every characteristic [9]. The corresponding result for nilpotent elements is however not proven.

COROLLARY. *In this situation, let x be a regular unipotent element of L. Then* $\langle Q_x(1), \rho\rangle = \langle 1, \rho\rangle_{W(L)} = \mathrm{rank}_{ss}\, G - \mathrm{rank}_{ss}\, L$.

Proof. In this case $Q_x^L = 1$. The first equality follows then from Frobenius reciprocity and the second from geometric properties of ρ.

The corresponding results for nilpotent elements in $\mathfrak{g}$ hold at least when the characteristic is good. For nilpotent elements in bad characteristic, we will use instead the following result.

PROPOSITION 1.11. *Let M be a connected reductive subgroup of G containing a maximal torus of G, $\mathfrak{m}$ its Lie algebra and W' its Weyl group. Let $A \in \mathfrak{m}$ have Jordan decomposition $A_s + A_n$. Suppose that $\mathfrak{c}_{\mathfrak{g}}(A_s) \subset \mathfrak{m}$. Then $Q_A^G = \mathrm{ind}_{W'}^W Q_A^M$. If moreover $\mathfrak{c}_{\mathfrak{g}}(A_s) = \mathfrak{m}$, then $Q_A^G = \mathrm{ind}_{W'}^W Q_{A_n}^M$.*

The proof is given in 2.7.

2. Partial resolutions

2.1. We sketch a proof of Theorem 1.2, following essentially [4]. The key tool is intersection cohomology. Recall that intersection cohomology is a functor which assigns to a pair $(X, \mathscr{L})$ consisting of an irreducible algebraic variety X and a local system of $\mathbb{Q}_l$-modules $\mathscr{L}$ defined over a smooth dense open subset of X a complex of $\mathbb{Q}_l$-sheaves $\mathbf{IC}(X; \mathscr{L})$, viewed as an object in a suitable triangulated category.

Let $X = \{(x, B) \in \mathfrak{g} \times \mathscr{B} \mid x \in \mathrm{Lie}\,(B)\}$, $Y = \{(x, P) \in \mathfrak{g} \times \mathscr{P} \mid x \in \mathrm{Lie}\,(P)\}$, $p : X \to \mathfrak{g}$ and $q : Y \to \mathfrak{g}$ the first projections and $\pi : X \to Y$ the map which associates to $(x, B) \in X$ the pair $(x, P) \in Y$ such that $P \supset B$. Let also $\mathfrak{g}_{rs}$ be the set of all regular semisimple elements in $\mathfrak{g}$, $X_{rs} = p^{-1}(\mathfrak{g}_{rs})$, $Y_{rs} = q^{-1}(\mathfrak{g}_{rs})$.

At this point let us note that $\mathfrak{g}_{rs}$ may be empty, as is the case for symplectic Lie algebras in characteristic 2. This difficulty, which does not arise in the group case, can be overcome by enlarging the center of G in a suitable way. For example, in characteristic 2 we may replace Sp_N by the subgroup of GL_N generated by Sp_N and the center of GL_N.

Let us assume henceforth that $\mathfrak{g}_{rs} \neq \varnothing$. The restriction p_{rs} of p to $X_{rs} \to \mathfrak{g}_{rs}$ is a Galois covering with Galois group W and the restriction q_{rs} of q to $Y_{rs} \to \mathfrak{g}_{rs}$ is a covering on which $W^{\mathscr{P}}$ acts by deck transformations. Let also $\pi_{rs} : X_{rs} \to Y_{rs}$ be the restriction of π.

If Z is any variety, let $\mathbf{1}_Z$, or simply $\mathbf{1}$, denote the constant local system on Z with $\mathbb{Q}_l$ as stalk. The results above imply that $p_{rs*}\mathbf{1}$ and $q_{rs*}\mathbf{1}$ are local systems on

$\mathfrak{g}_{rs}$. Moreover $p_{rs*}\mathbf{1}$ is equipped with a W-action which affords the regular representation on the stalks and $q_{rs*}\mathbf{1}$ is equipped with a $W^{\mathscr{P}}$-action, hence also with an $N(\mathscr{P})$-action. Let $\varepsilon : \mathbf{1} \to R\pi_*\mathbf{1} = R\pi_*\pi^*\mathbf{1}$ be the adjunction morphism and let ε_{rs} be its restriction to Y_{rs}. Then $q_{rs*}(\varepsilon_{rs}) : q_{rs*}\mathbf{1} \to q_{rs*}\pi_{rs*}\mathbf{1}$ is $N(\mathscr{P})$-equivariant and induces a $W^{\mathscr{P}}$-equivariant isomorphism

$$q_{rs*}\mathbf{1} \cong (p_{rs*}\mathbf{1})^{W(\mathscr{P})}. \tag{2.1.1}$$

The group actions can be taken out of the local systems in the following way. Let $W(\mathscr{P})^\wedge$ be a complete set of representatives for the isomorphism classes of irreducible $\mathbb{Q}_l W(\mathscr{P})$-modules, and for $\theta \in W(\mathscr{P})^\wedge$ let $\mathscr{M}_\theta = \mathscr{H}om_{W(\mathscr{P})}(\boldsymbol{\theta}, p_{rs*}\mathbf{1})$, where $\boldsymbol{\theta}$ is the constant local system on $\mathfrak{g}_{rs}$ with stalk θ. Then $\mathscr{M}_\theta$ is a local system of $\mathbb{Q}_l$-vector spaces on $\mathfrak{g}_{rs}$, and there is a natural isomorphism

$$p_{rs*}\mathbf{1} \cong \bigoplus_{\theta \in W(\mathscr{P})^\wedge} \mathscr{M}_\theta \otimes_{\mathbb{Q}_l} \boldsymbol{\theta}. \tag{2.1.2}$$

Notice that $\mathscr{M}_1 \cong (p_{rs*}\mathbf{1})^{W(\mathscr{P})}$, and hence $\mathscr{M}_1 \cong q_{rs*}\mathbf{1}$ by 2.1.1.

The maps p and q are small in the sense of [5, 6.2]. They are also proper and both X and Y are smooth. Following [loc. cit.] we have then natural isomorphisms

$$Rp_*\mathbf{1} \cong \mathbf{IC}(\mathfrak{g}; p_{rs*}\mathbf{1}) \quad \text{and} \quad Rq_*\mathbf{1} \cong \mathbf{IC}(\mathfrak{g}; q_{rs*}\mathbf{1}). \tag{2.1.3}$$

The morphism of local systems $q_{rs*}\varepsilon_{rs} : q_{rs*}\mathbf{1} \to p_{rs*}\mathbf{1}$ induces a morphism

$$\mathbf{IC}(\mathfrak{g}; q_{rs*}\varepsilon_{rs}) : \mathbf{IC}(\mathfrak{g}; q_{rs*}\mathbf{1}) \to \mathbf{IC}(\mathfrak{g}; p_{rs*}\mathbf{1}).$$

Moreover both $\mathbf{IC}(\mathfrak{g}; q_{rs*}\varepsilon_{rs})$ and $Rq_*\varepsilon : Rq_*\mathbf{1} \to Rq_*R\pi_*\mathbf{1} = Rp_*\mathbf{1}$ restrict to $q_{rs*}\varepsilon_{rs}$ over $\mathfrak{g}_{rs}$. Hence $\mathbf{IC}(\mathfrak{g}; q_{rs*}\varepsilon_{rs})$ agrees with $Rq_*\varepsilon$ under the isomorphisms 2.1.3.

By functoriality of intersection cohomology, W acts on $\mathbf{IC}(\mathfrak{g}; p_{rs*}\mathbf{1})$ and $N(\mathscr{P})$ acts on $\mathbf{IC}(\mathfrak{g}; q_{rs*}\mathbf{1})$. Therefore W acts on $Rp_*\mathbf{1}$ and $N(\mathscr{P})$ acts on $Rq_*\mathbf{1}$. Since $q_{rs*}\varepsilon_{rs}$ is $N(\mathscr{P})$-equivariant, so are $\mathbf{IC}(\mathfrak{g}; q_{rs*}\varepsilon_{rs})$ and $Rq_*\varepsilon$. For $\theta \in W(\mathscr{P})^\wedge$ we have $\mathbf{IC}(\mathfrak{g}; \mathscr{M}_\theta \otimes_{\mathbb{Q}_l} \boldsymbol{\theta}) \cong \mathbf{IC}(\mathfrak{g}; \mathscr{M}_\theta) \otimes_{\mathbb{Q}_l} \boldsymbol{\theta}$, and in view of 2.1.1 we have therefore

$$\mathbf{IC}(\mathfrak{g}; p_{rs*}\mathbf{1})^{W(\mathscr{P})} \cong \mathbf{IC}(\mathfrak{g}; (p_{rs*}\mathbf{1})^{W(\mathscr{P})}).$$

It follows then from 2.1.1 that $\mathbf{IC}(\mathfrak{g}; q_{rs*}\varepsilon_{rs})$ induces a $W^{\mathscr{P}}$-equivariant isomorphism

$$\mathbf{IC}(\mathfrak{g}; q_{rs*}\mathbf{1}) \cong \mathbf{IC}(\mathfrak{g}; p_{rs*}\mathbf{1})^{W(\mathscr{P})}$$

and hence that $Rq_*\varepsilon$ induces a $W^{\mathscr{P}}$-equivariant isomorphism

$$Rq_*\mathbf{1} \cong Rp_*\mathbf{1}^{W(\mathscr{P})}. \tag{2.1.4}$$

This is the form of the Borho–MacPherson theorem which is needed here.

Let $A \in \mathfrak{g}$. Since p is proper, $h^*((Rp_*\mathbf{1})_A) \cong H^*(\mathscr{B}_A)$, and the action of W on $Rp_*\mathbf{1}$ induces therefore an action of W on $H^*(\mathscr{B}_A)$. This is the Springer representation. Similarly, q is proper and therefore $h^*((Rq_*\mathbf{1})_A) \cong H^*(\mathscr{P}_A)$, and we get in this way an action of $W^{\mathscr{P}}$ on $H^*(\mathscr{P}_A)$. The homomorphism $h^*((Rq_*\mathbf{1})_A) \to h^*((Rp_*\mathbf{1})_A)$ induced by $Rq_*\varepsilon$ corresponds under the above isomorphisms to the homomorphism $\pi_A^* : H^*(\mathscr{P}_A) \to H^*(\mathscr{B}_A)$ induced by $\pi_A : \mathscr{B}_A \to \mathscr{P}_A$. Thus π_A^* is $N(\mathscr{P})$-equivariant and induces a $W^{\mathscr{P}}$-equivariant isomorphism $H^*(\mathscr{P}_A) \cong H^*(\mathscr{B}_A)^{W(\mathscr{P})}$. This proves 1.2.

For later use, let us note that if $Z \in \mathfrak{g}$ is central, then the map $A \mapsto A + Z$ from $\mathfrak{g}$ to $\mathfrak{g}$ leaves $\mathfrak{g}_{rs}$ invariant and lifts in a canonical way to X and Y. It follows that for every $A \in \mathfrak{g}$ the Springer representations for A and $A + Z$ are isomorphic.

2.2. Consider an algebraic variety Z and a morphism $f : Z \to \mathfrak{g}$. Let $X' = Z \times_{\mathfrak{g}} X$, $Y' = Z \times_{\mathfrak{g}} Y$. By proper base change we have canonical isomorphisms $H^*(X') \cong H^*(Z; f^*Rp_*\mathbf{1})$ and $H^*(Y') \cong H^*(Z; f^*Rq_*\mathbf{1})$. It follows that W acts on $H^*(X')$ and $W^{\mathscr{P}}$ acts on $H^*(Y')$. Moreover $f^*Rq_*\varepsilon$ is $N(\mathscr{P})$-equivariant and induces a $W^{\mathscr{P}}$-equivariant isomorphism $H^*(Y') \cong H^*(X')^{W(\mathscr{P})}$. Let Z_1 be a second variety and let $g : Z_1 \to Z$ be a morphism, $f_1 = f \circ g$, $X_1' = Z_1 \times_{\mathfrak{g}} X$, $Y_1' = Z_1 \times_{\mathfrak{g}} Y$. Then the homomorphisms $(g \times_{\mathfrak{g}} X)^* : H^*(X') \to H^*(X_1')$ and $(g \times_{\mathfrak{g}} Y)^* : H^*(Y') \to H^*(Y_1')$ are respectively W-equivariant and $W^{\mathscr{P}}$-equivariant.

Suppose for example that we have $Z_1 \subset Z \subset \mathfrak{g}$ and that f and g are the inclusion morphisms. Then $W^{\mathscr{P}}$ acts on $H^*(q^{-1}(Z))$ and $H^*(q^{-1}(Z_1))$ and the restriction homomorphism $H^*(q^{-1}(Z)) \to H^*(q^{-1}(Z_1))$ is $W^{\mathscr{P}}$-equivariant. This can be used in some cases to compare the $W^{\mathscr{P}}$-module structures on $H^*(\mathscr{P}_A)$ and $H^*(\mathscr{P}_{A'})$, for elements $A, A' \in \mathfrak{g}$, using a suitable subvariety Z of $\mathfrak{g}$ containing both A and A', and taking for Z_1 successively $\{A\}$ and $\{A'\}$. We give here two simple applications.

2.3. *Proof of Lemma* 1.4

Taking $A' = 0$ and $Z = \mathfrak{g}$, we have equivariant homomorphisms

$$H^*(\mathscr{P}_0) \leftarrow H^*(Y) \to H^*(\mathscr{P}_A).$$

But the projection $\mathrm{pr}_2 : Y \to \mathscr{P}$ is a vector bundle map and induces an isomorphism $(\mathrm{pr}_2)^* : H^*(\mathscr{P}) \cong H^*(Y)$ which is inverse to $H^*(Y) \to H^*(\mathscr{P}_0)$. This implies 1.4.

LEMMA 2.4. *Let Z be a connected subvariety of* $\mathfrak{g}$ *such that* $\mathscr{P}_A = \mathscr{P}_{A'}$ *for all* $A, A' \in Z$. Then $H^*(\mathscr{P}_A)$ and $H^*(\mathscr{P}_{A'})$ *coincide as* $W^{\mathscr{P}}$*-modules for all* $A, A' \in Z$.

Proof. Let F be the common value of $\mathscr{P}_A$ for $A \in Z$. Then $Y' = q^{-1}(Z) = Z \times F$. For $A \in Z$, let $j_A : F \to Z \times F$ be defined by $P \mapsto (A, P)$. Since Z is connected, $j_A^* : H^*(Y') \to H^*(F)$ is independent of $A \in Z$ and surjective. We know also that j_A^* is W-equivariant when $H^*(F)$ is given the same structure of W-module as $H^*(\mathscr{P}_A)$. The result follows.

2.5. Let $P \in \mathscr{P}$, $B \subset P$ a Borel subgroup, $T \subset B$ a maximal torus, and $L \supset T$ a Levi factor of P. We have an obvious commutative diagram

$$\begin{array}{ccccc} G/T & \longrightarrow & G/B & \cong & \mathscr{B} \\ \downarrow & & \downarrow & & \downarrow \\ G/L & \longrightarrow & G/P & \cong & \mathscr{P} \end{array}$$

which induces a commutative diagram

$$\begin{array}{ccccc} H^*(G/T) & \longleftarrow & H^*(G/B) & \cong & H^*(\mathscr{B}) \\ \uparrow & & \uparrow & & \uparrow \\ H^*(G/L) & \longleftarrow & H^*(G/P) & \cong & H^*(\mathscr{P}) \end{array}$$

in which the horizontal arrows are isomorphisms. The groups $W \cong N_G(T)/T$ and $W^{\mathscr{P}} \cong N_G(L)/L$ act by right multiplication on G/T and G/L respectively, and hence also on $H^*(\mathscr{B})$ and $H^*(\mathscr{P})$ respectively. As mentioned already in 1.1, this action of W on $H^*(\mathscr{B}_0)$ is the Springer representation of W on $H^*(\mathscr{B}_0)$, and it follows therefore from 1.2 that this action of $W^{\mathscr{P}}$ on $H^*(\mathscr{P})$ coincides with the representation of $W^{\mathscr{P}}$ on $H^*(\mathscr{P}_0)$ defined in 2.1.

2.6. In the case of orthogonal groups and quadrics (cf. 1.5), we can replace the variety G/L by the subset X of $\mathscr{Q} \times \mathscr{Q}$ consisting of the pairs (x, y) such that the subspace $x + y$ of V is not isotropic, and the action of σ on G/L corresponds to the involution $s : (x, y) \mapsto (y, x)$. The first projection from X to $\mathscr{Q}$ is an affine bundle, hence induces an isomorphism $H^*(\mathscr{Q}) \cong H^*(X)$, and the action of σ on $H^*(\mathscr{Q})$ corresponds to s^*. Another way to look at the action of σ is to use also the second projection from X to $\mathscr{Q}$. The action of σ is then obtained by composing the isomorphisms

$$H^*(\mathscr{Q}) \overset{\mathrm{pr}_1^*}{\cong} H^*(X) \overset{\mathrm{pr}_2^*}{\cong} H^*(\mathscr{Q}). \tag{2.6.1}$$

2.7. Let M be a connected reductive subgroup of maximal rank in G, $\mathfrak{m}$ its Lie algebra and W' its Weyl group. Then each Borel subgroup of M is contained in exactly $[W : W']$ Borel subgroups of G. Let $\mathfrak{m}'$ be the set of all elements $A \in \mathfrak{m}$ whose semisimple part A_s satisfies $\mathfrak{c}_{\mathfrak{g}}(A_s) \subset \mathfrak{m}$. Then for every $A \in \mathfrak{m}'$ and $B \in \mathscr{B}_A$, $B \cap M$ is a Borel subgroup of M whose Lie algebra contains A.

Let $\mathfrak{m}'_{\mathrm{rs}} = \mathfrak{m}' \cap \mathfrak{g}_{\mathrm{rs}}$. Then $\mathfrak{m}'_{\mathrm{rs}}$ is also the set of all regular semisimple elements of $\mathfrak{m}$ which are contained in $\mathfrak{m}'$. Let $\bar{\mathfrak{m}}'$ be the variety of all pairs (A, B') with $A \in \mathfrak{m}'$ and B' a Borel subgroup of M whose Lie algebra contains A, and let $\bar{p}' : \bar{\mathfrak{m}}' \to \mathfrak{m}', (A, B') \mapsto A$ be the first projection. Let also $\bar{\mathfrak{m}}'_{\mathrm{rs}} = \bar{p}'^{-1}(\mathfrak{m}'_{\mathrm{rs}})$ and let $\bar{p}'_{\mathrm{rs}} : \bar{\mathfrak{m}}'_{\mathrm{rs}} \to \mathfrak{m}'_{\mathrm{rs}}$ be the restriction of $\bar{p}'$. Suppose that $\mathfrak{g}_{\mathrm{rs}} \neq \varnothing$. Then

$$R\bar{p}'_* \mathbf{1} \cong \mathbf{IC}(\mathfrak{m}'; \bar{p}'_{\mathrm{rs}*}\mathbf{1}) \tag{2.7.1}$$

is the intersection complex which defines the Springer representations relative to M and W' for the elements of $\mathfrak{m}'$.

Let X and $p : X \to \mathfrak{g}$ be as in 2.1. Let $X' = p^{-1}(\mathfrak{m}')$, $X'_{\mathrm{rs}} = p^{-1}(\mathfrak{m}'_{\mathrm{rs}})$, and let $p' : X' \to \mathfrak{m}'$, $p'_{\mathrm{rs}} : X'_{\mathrm{rs}} \to \mathfrak{m}'_{\mathrm{rs}}$ be the restrictions of p. Consider $(A_0, B_0) \in X'$. Then $(A_0, B_0 \cap M) \in \bar{\mathfrak{m}}'$. Moreover there is a unique irreducible component X'_0 of X' which contains (A_0, B_0), X'_0 is obtained by applying M-conjugation to $(\mathrm{Lie}\,(B_0) \cap \mathfrak{m}') \times \{B_0\}$, and $(A, B) \mapsto (A, B \cap M)$ defines an isomorphism from X'_0 to $\bar{\mathfrak{m}}'$. Let $p'_0 : X'_0 \to \mathfrak{m}'$ and $p'_{0\mathrm{rs}} : X'_0 \cap X_{\mathrm{rs}} \to \mathfrak{m}'_{\mathrm{rs}}$ be the restrictions of p'. Comparing with 2.7.1, we get an isomorphism

$$Rp'_{0*}\mathbf{1} \cong \mathbf{IC}(\mathfrak{m}'; p'_{0\mathrm{rs}*}\mathbf{1}). \tag{2.7.2}$$

The description of the components of X' shows also that they are disjoint. Summing 2.7.2 over the components of X', we get an isomorphism

$$Rp'_*\mathbf{1} \cong \mathbf{IC}(\mathfrak{m}'; p'_{\mathrm{rs}*}\mathbf{1}). \tag{2.7.3}$$

We get a W-action on the right hand side of 2.7.3 by using the functoriality of intersection cohomology together with the W-action on the local system $p'_{\mathrm{rs}*}\mathbf{1}$. However the definition of Springer representations for $\mathfrak{g}$ requires that we use as W-action the restriction to $Rp'_*\mathbf{1}$ of the W-action on $Rp_*\mathbf{1}$ defined in 2.1. Fortunately these two actions of W have the same restriction to $p'_{\mathrm{rs}*}\mathbf{1}$, and hence coincide.

Choose a maximal torus $T \subset M$ and a Borel subgroup $B \in \mathscr{B}$ containing T. Using B to identify W with $N_G(T)/T$ and $B \cap M$ to identify W' with $N_M(T)/T$, W' becomes a subgroup of W. We get then an isomorphism of local systems

$$p'_{\mathrm{rs}*}\mathbf{1} \cong \mathbb{Q}_l W \otimes_{\mathbb{Q}_l W'} \bar{p}'_{\mathrm{rs}*}\mathbf{1}. \tag{2.7.4}$$

Separating group actions from local systems as in 2.1, we find that 2.7.4 induces W-equivariant isomorphisms

$$\mathbf{IC}(\mathfrak{m}'; p'_{\mathrm{rs}*}\mathbf{1}) \cong \mathbf{IC}(\mathfrak{m}; \mathbb{Q}_l W \otimes_{\mathbb{Q}_l W'} \bar{p}'_{\mathrm{rs}*}\mathbf{1}) \cong \mathbb{Q}_l W \otimes_{\mathbb{Q}_l W'} \mathbf{IC}(\mathfrak{m}'; \bar{p}'_{\mathrm{rs}*}\mathbf{1}). \quad (2.7.5)$$

Combining with 2.7.1 and 2.7.3, we get a W-equivariant isomorphism

$$Rp'_*\mathbf{1} \cong \mathbb{Q}_l W \otimes_{\mathbb{Q}_l W'} R\bar{p}'_*\mathbf{1}. \quad (2.7.6)$$

Looking at the stalks at $A \in \mathfrak{m}'$ and taking cohomology in 2.7.6, we get an isomorphism of graded W-modules

$$H^*(\mathcal{B}_A) \cong \mathbb{Q}_l W \otimes_{\mathbb{Q}_l W'} H^*(\mathcal{B}(M)_A),$$

where $\mathcal{B}(M)$ is the variety of all Borel subgroups of M. Thus $Q_A^G = \mathrm{ind}_{W'}^W Q_A^M$. When the semisimple part A_s of A satisfies $\mathfrak{c}_{\mathfrak{g}}(A_s) = \mathfrak{m}$, then A_s is central in $\mathfrak{m}$, and therefore $Q_A^M = Q_{A-A_s}^M$. This proves 1.11.

3. Cohomology of quadrics

In this section we review some classical results on the cohomology of quadrics.

3.1. Let V be a finite dimensional vector space over k, of dimension $N \in \{2n, 2n+1\}$, equipped with a non-degenerate quadratic form Q, and let $\mathcal{Q} \subset \mathbb{P}(V)$ be the corresponding quadric, $\tilde{G}$ the full orthogonal group defined by Q, and $G = \tilde{G}^0$ the special orthogonal group. Let also $\beta(x, y) = Q(x+y) - Q(x) - Q(y)$ be the bilinear form associated to Q. The radical of the restriction of Q to a subspace U of V is denoted U_0, and we say that U is of type (d, r) if $\dim U = d$ and $\dim U_0 = r$ (recall that by the radical we mean the set of all isotropic vectors in $U \cap U^\perp$). Given integers $d \geq r \geq 0$, there are subspaces of type (d, r) in V if and only if $d + r \leq N$. Moreover the subspaces of type (d, r) are all $\tilde{G}$-conjugate, except when char $(k) = 2$, both N and $d + r$ are odd and $d + r < N$, in which case we must know whether they contain $V^\perp$. For a subspace U of V, let $\mathcal{Q}_U = \mathcal{Q} \cap \mathbb{P}(U)$. Our object in this section is to describe $H^*(\mathcal{Q}_U)$ and to understand the restriction homomorphism $H^*(\mathcal{Q}) \to H^*(\mathcal{Q}_U)$.

3.2. It is convenient to consider sequences of subspaces

$$U^{(0)} \subset U^{(1)} \subset \cdots \subset U^{(m)} \quad (3.2.1)$$

in V, with $U^{(j)}$ of type (d_j, r_j), such that $d_j + r_j = M$ is constant $(0 \leq j \leq m)$. Given such a sequence, we can find a linearly independent family $(e_1, \ldots, e_M)$ such that $Q(\Sigma_{j \le M} x_j e_j) = \Sigma_{j \le (M+1)/2} x_j x_{M-j+1}$ and such that each $U^{(i)}$ is generated by $(e_j)_{j \le d_i}$. In particular, the sequence (3.2.1) can be extended to a similar sequence in which $d_0 - r_0 \le 1$, $m = r_0$ and $d_i = d_0 + i$, and all such sequences are $\tilde{G}$-conjugate, unless char $(k) = 2$, both N and M are odd and $M < N$, in which case there are two orbits.

In particular, let U be a subspace of V of type (d, r). If $d - r \ge 2$, then U contains a subspace U' of type $(d-1, r+1)$, and U' is unique up to conjugation under the stabilizer of U in $\tilde{G}$. Similarly, if $r \ge 1$, then U is contained in a subspace U'' of type $(d+1, r-1)$, and all such subspaces are conjugate under the stabilizer of U in $\tilde{G}$.

LEMMA 3.3. *Let $U \subset V$ be a subspace of type (d, r). Then the following hold.*

(a) *If $U' \subset U$ is a subspace of type $(d-1, r+1)$, then $\mathcal{Q}_U \backslash \mathcal{Q}_{U'} \cong \mathbb{A}^{d-2}$.*

(b) *If $d - r$ is odd, then*

$$\dim H^i(\mathcal{Q}_U) = \begin{cases} 1 & \text{if } i \le 2(d-2) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

(c) *If $d - r$ is even and $d > r$, then*

$$\dim H^i(\mathcal{Q}_U) = \begin{cases} 1 & \text{if } i \le 2(d-2) \text{ is even and } i \ne d + r - 2, \\ 2 & \text{if } i = d + r - 2, \\ 0 & \text{otherwise.} \end{cases}$$

(d) *If $d - r = 0$, then*

$$\dim H^i(\mathcal{Q}_U) = \begin{cases} 1 & \text{if } i \le 2(d-1) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This follows easily from the discussion in 3.2. Notice that $\mathcal{Q}_U = \mathbb{P}(U)$ if $d - r = 0$ and $\mathcal{Q}_U = \mathbb{P}(U_0)$ if $d - r = 1$.

3.4. If V_1 and V_2 are two finite dimensional vector spaces, we say that a linear map $f: V_1 \to V_2$ has *maximal rank* if rank $(f) = \min\{\dim V_1, \dim V_2\}$.

LEMMA. *Let U be a subspace of V. Then for every $m \in \mathbb{N}$ the restriction map $H^m(\mathbb{P}(V)) \to H^m(\mathcal{Q}_U)$ has maximal rank.*

Proof. Let (d, r) be the type of U. Choose a subspace $V' \supset U$ of dimension $d+r$ and to which Q has a non-degenerate restriction. Using 3.3 and the factorization

$$H^m(\mathbb{P}(V)) \to H^m(\mathbb{P}(V')) \to H^m(\mathcal{Q}_{V'}) \to H^m(\mathcal{Q}_V),$$

the problem reduces to the well-known fact that the restriction map $H^m(\mathbb{P}(V')) \to H^m(\mathcal{Q}_{V'})$ has maximal rank.

LEMMA 3.5. *For every $m \in \mathbb{N}$, the restriction map $H^m(\mathcal{Q}) \to H^m(\mathcal{Q}_U)$ has maximal rank.*

Proof. Let $e = \min\{\dim H^m(\mathcal{Q}), \dim H^m(\mathcal{Q}_U)\}$. The result is obvious if $e = 0$, and it is an immediate consequence of 3.3 if $e = 2$. We may thus assume that $e = 1$. Then $\dim H^m(\mathbb{P}(V)) = 1$. Using the obvious commutative diagram

$$\begin{tikzcd} H^*(\mathcal{Q}_U) & & H^*(\mathcal{Q}) \arrow[ll] \\ & H^*(\mathbb{P}(V)) \arrow[ul] \arrow[ur] & \end{tikzcd} \tag{3.5.1}$$

the result follows from 3.4.

4. The orthogonal case

As in Section 3, V is a vector space of dimension $N \in \{2n, 2n+1\}$ equipped with a non-degenerate quadratic form Q and $\tilde{G} = O(Q)$, $G = SO(Q)$ are the corresponding orthogonal and special orthogonal groups. We choose $\mathcal{P}$ as in 1.5. Then $\mathcal{P}$ is isomorphic to the quadratic hypersurface $\mathcal{Q} \subset \mathbb{P}(V)$ defined by Q, $W^{\mathcal{P}} = \{1, \sigma\}$, $\operatorname{ind}_{W(\mathcal{P})}^{W}(1) = 1 + \rho + \xi$, where ξ is a permutation representation of degree $n-1$, and σ acts trivially on $\xi^{W(\mathcal{P})}$ and as -1 on $\rho^{W(\mathcal{P})}$. Let $\mathcal{N} = \{A \in \mathfrak{g} \mid A \text{ is nilpotent}\}$ be the nilpotent variety of $\mathfrak{g}$. If $A \in \mathcal{N}$ and $U = \operatorname{Ker}(A)$, we can identify $\mathcal{P}_A$ with $\mathcal{Q}_U$ and we let i_A denote both inclusion maps $\mathcal{P}_A \to \mathcal{P}$ and $\mathcal{Q}_U \to \mathcal{Q}$.

4.1. Suppose that $N = 2n$. Then $\dim H^{2(n-1)}(\mathcal{Q}) = 2$. We can identify the action of σ on $H^{2(n-1)}(\mathcal{Q}) = H^{2(n-1)}(\mathcal{Q}_0)$ by using the subvariety X of $\mathcal{Q} \times \mathcal{Q}$ defined in 2.6. Let V' and V'' be maximal isotropic subspaces of V such that $V = V' \oplus V''$. Let $U = (\mathbb{P}(V') \times \mathbb{P}(V'')) \cap X$. Then both projections $Y \to \mathbb{P}(V')$ and $Y \to \mathbb{P}(V'')$ are affine bundles. We have therefore a commutative diagram

$$\begin{array}{ccccc} H^{2(n-1)}(\mathbb{P}(V')) & \overset{\text{pr}_1^*}{\cong} & H^{2(n-1)}(Y) & \overset{\text{pr}_2^*}{\cong} & H^{2(n-1)}(\mathbb{P}(V'')) \\ \uparrow & & \uparrow & & \uparrow \\ H^{2(n-1)}(\mathcal{Q}) & \overset{\text{pr}_1^*}{\cong} & H^{2(n-1)}(X) & \overset{\text{pr}_2^*}{\cong} & H^{2(n-1)}(\mathcal{Q}) \end{array} \tag{4.1.1}$$

in which the vertical arrows are restriction morphisms. Notice that $H^*(\mathbb{P}(V'))$ and $H^*(\mathbb{P}(V''))$ are naturally isomorphic. The subvariety X of $\mathbb{P}(V') \times \mathbb{P}(V'')$ is defined by the perfect pairing $V' \times V'' \to k$ induced by Q. The same pairing is also induced by a non-degenerate alternating form. From the symplectic case, the composition of the isomorphisms in the top row of 4.1.1 is multiplication by $(-1)^{n-1}$.

If n is even, then V' and V'' are G-conjugate. It follows that σ acts as multiplication by -1 on the image of $H^{2(n-1)}(\mathbb{P}(V'))$ in $H^{2(n-1)}(\mathcal{Q})$. Since the same holds for every maximal isotropic subspace of V, σ acts as multiplication by -1 on the whole of $H^{2(n-1)}(\mathcal{Q})$.

If n is odd, then V' and V'' are not G-conjugate and the restriction map

$$H^{2(n-1)}(\mathcal{Q}) \to H^{2(n-1)}(\mathbb{P}(V')) \oplus H^{2(n-1)}(\mathbb{P}(V''))$$

is an isomorphism. Since $(-1)^{n-1} = 1$, the action of σ on the left hand side corresponds to the permutation of the two factors in the right hand side. Thus σ acts as the identity on the image of $H^{2(n-1)}(\mathbb{P}(V))$ in $H^{2(n-1)}(\mathcal{Q})$ and as multiplication by -1 on the cokernel.

More generally we have for every N, even or odd:

LEMMA. *σ acts as multiplication by $(-1)^i$ on the image of $H^{2i}(\mathbb{P}(V))$ in $H^{2i}(\mathcal{Q})$ and as multiplication by -1 on the cokernel.*

Proof. Using 1.2 with $A = 0$, we get $\langle P_0, -1\rangle = \langle Q_0, \rho\rangle = \Sigma_{j=1}^{n} q^{m_j}$, where $m_1, \ldots, m_n$ are the exponents of G. We know therefore the dimension of the (-1)-eigenspace of σ on $H^{2i}(\mathcal{Q})$. Comparing with 3.3, and using 3.4, we find that the results stated in the lemma hold, except maybe when $i = n-1$, $N = 2n$ and n is odd, a case we have just discussed above.

4.2. We need a few facts about nilpotent infinitesimal orthogonal transformations. Let $A \in \mathcal{N}$.

Suppose that char $(k) \neq 2$. Then V can be written as an orthogonal direct sum of A-stable subspaces such that the restriction of A to any of these subspaces has either only one Jordan block (necessarily of odd size) or two Jordan blocks of even size (the sizes are necessarily equal). In particular, the partition λ of N whose parts are the dimensions of the Jordan blocks of A is an orthogonal partition, that is, for each even integer $m > 0$ the number of parts of λ equal to m is even. Each block of dimension 1 is a one-dimensional non-degenerate subspace of Ker (A) and the other blocks contribute to the radical of Ker (A).

Suppose that char $(k) = 2$. Following Hesselink [6], we can decompose V as an orthogonal direct sum of A-stable subspaces on which the restriction of A has one of the following forms. In Hesselink's notation, the various possibilities are $W(m)$ $(m \geq 1)$, $W_s(m)$ $(m \geq 1, [(m+1)/2] < s \leq m)$ and $D(m)$ $(m \geq 1)$. There is exactly one factor of type $D(m)$ (for some m) if N is odd, and none if N is even. A factor $D(1)$ is a one-dimensional non-degenerate subspace. The other factors consist all of two Jordan blocks, both of dimension m for $W(m)$ and $W_s(m)$, one of dimension m and one of dimension $m-1$ for $D(m)$ $(m \geq 2)$. The kernel of the restriction of A to any of these factors has thus dimension 2. It is non-degenerate for $W(1)$, completely isotropic for $W(m)$ $(m \geq 2)$ and $W_s(m)$ $(s < m)$, and has a radical of dimension 1 for $W_m(m)$ and $D(m)$ $(m \geq 2)$. The factors of type $W(m)$ are characterized by the fact that they can be written as the direct sum of two A-stable completely isotropic subspaces.

LEMMA. *For every* $A \in \mathcal{N}$, *the* $W^{\mathcal{P}}$*-structure on* $H^*(\mathcal{P}_A)$ *depends only on* Ker (A).

Proof. Let U be a subspace of V which is the kernel of some nilpotent element of $\mathfrak{g}$ and let (d, r) be its type. If $U^\perp \cap U \neq U_0$, then $d - r$ is odd, as follows from the discussion above, and using 4.1 and the results in Section 3 we find that σ acts as multiplication by $(-1)^i$ on $H^{2i}(\mathcal{P}_A)$ for every $A \in \mathcal{N}$ such that Ker $(A) = U$.

We may therefore assume that $U^\perp \cap U = U_0$. Let $E = \{A \in \mathcal{N} \mid \mathrm{Ker}(A) = U\}$. In view of 2.4 it is enough to show that E is irreducible.

Let V' be a complement to U_0 in U and let $V'' = V'^\perp$. Let $\mathfrak{g}'' \subset \mathfrak{gl}(V'')$ be the orthogonal Lie algebra defined by the restriction of Q to V'' and let $E'' = \{A'' \in \mathfrak{g}'' \mid A'' \text{ is nilpotent and } \mathrm{Ker}(A'') = U_0\}$. Then E is the set of all elements $A \in \mathfrak{g}$ which leave both V' and V'' stable, restrict to 0 on V' and to an element of E'' on V''. Thus $E \cong E''$. We may therefore assume that U is completely isotropic.

Assuming now that $U = U_0$, the stabilizer H of U in G is a parabolic subgroup of G. Let B be a Borel subgroup of H and let $E_B = E \cap \mathrm{Lie}(B)$. Then E_B is an open subset of a subspace of Lie (B), and hence is irreducible or empty. Moreover E is the image of the morphism $H \times E_B \to \mathfrak{g}, (h, A) \mapsto \mathrm{Ad}\, h(A)$. Since $E \neq \varnothing$, E is therefore irreducible.

COROLLARY 1. *Suppose that* char $(k) \neq 2$. *Then for every* $A \in \mathcal{N}$, *there exists an element* $A_0 \in \mathcal{N}$ *such that* $H^*(\mathcal{P}_A) = H^*(\mathcal{P}_{A_0})$ *as* $W^{\mathcal{P}}$*-modules, with* A_0 *corre-*

sponding to a partition which has either no part ≥ 3, *or exactly one part* ≥ 3 *(necessarily odd), or exactly two parts* ≥ 3 *and these parts are equal. In particular* A_0 *is a regular nilpotent element in a Levi factor* $\mathfrak{l}$ *of some parabolic subalgebra of* $\mathfrak{g}$.

COROLLARY 2. *Suppose that* char $(k) = 2$ *and* N *is even. Let* $A \in \mathcal{N}$ *be such that* Ker (A) *is of type* (d, r) *with* r *even and* $d + r < N$. *Then there exists an element* $A_0 \in \mathcal{N}$ *such that* $H^*(\mathscr{P}_A) = H^*(\mathscr{P}_{A_0})$ *as* $W^{\mathscr{P}}$*-modules and such that* A_0 *has a decomposition into factors of the form* $W(1)$, $W(2)$ *and exactly one factor of the form* $W(m)$ *with* $m \geq 3$.

4.3. Suppose that char $(k) \neq 2$ and let N be odd. Then σ acts as multiplication by $(-1)^i$ on $H^{2i}(\mathscr{P})$. Let $A \in \mathfrak{g}$ be nilpotent, (d, r) the type of $U = \operatorname{Ker}(A)$. Notice that d is odd.

Suppose first that r is even. Then $i_A^* : H^*(\mathscr{P}) \to H^*(\mathscr{P}_A)$ is surjective. It follows that σ acts as multiplication by $(-1)^i$ on $H^{2i}(\mathscr{P}_A)$, and we get the result stated in 1.8(i).

Suppose next that r is odd and $r \neq d$. In view of 4.2 we may assume that A is a regular nilpotent element in a Levi factor $\mathfrak{l}$ of some parabolic subalgebra of $\mathfrak{g}$, with $\operatorname{rank}_{ss} \mathfrak{g} - \operatorname{rank}_{ss} \mathfrak{l} = (d-1)/2$. By 1.10 we have $\langle P_A, -1 \rangle(1) = \langle Q_A, \rho \rangle(1) = (d-1)/2$. As for r even we can identify the action of σ on Im i_A^*. Considering Im i_A^* as a part of P_A, we have

$$\langle \operatorname{Im} i_A^*, -1 \rangle = \sum_{\substack{0 \leq i \leq d-2 \\ i \text{ odd}}} q^i, \qquad \langle \operatorname{Im} i_A^*, 1 \rangle = \sum_{\substack{0 \leq i \leq d-2 \\ i \text{ even}}} q^i.$$

In particular $\langle \operatorname{Im} i_A^*, \rho \rangle(1) = (d-1)/2$. This implies that σ acts trivially on the cokernel of i_A^* which has dimension 1, and we get 1.8(ii) for $r \neq d$.

Finally suppose that $r = d$. Then U is totally isotropic, $\mathscr{Q}_U = \mathbb{P}(U)$ and $\dim \mathscr{P}_A = d - 1$. In this case i_A^* is surjective and therefore σ acts as multiplication by $(-1)^i$ on $H^{2i}(\mathscr{P}_A)$. We find

$$\langle Q_A, \rho \rangle = \sum_{\substack{1 \leq i \leq d-1 \\ i \text{ odd}}} q^i, \qquad \langle Q_A, \xi \rangle = \sum_{\substack{2 \leq i \leq d-1 \\ i \text{ even}}} q^i.$$

Since d is odd and $r = d$ these formulas coincide with those in 1.8(ii). This completes the proof of 1.8.

4.4. Let N be even, and assume that char $(k) \neq 2$. Let $A \in \mathfrak{g}$ be nilpotent, (d, r) the type of $U = \operatorname{Ker}(A)$. Then d is even. The involution σ acts as multiplication by $(-1)^i$ on $H^{2i}(\mathscr{P})$, except for $i = n - 1$ when n is odd. In this case $H^{2(n-1)}(\mathscr{P})$ has dimension 2, and as observed in 4.1 σ acts trivially on the image of $H^{2(n-1)}(\mathbb{P}(V))$ in $H^{2(n-1)}(\mathcal{Q})$ and as multiplication by -1 on the cokernel.

Suppose first that $\dim H^{2(n-1)}(\mathcal{Q}_U) \leq 1$. Then σ acts as multiplication by $(-1)^i$ on $i_A^*(H^{2i}(\mathcal{Q})) \subset H^{2i}(\mathcal{Q}_U)$. If r is odd, then i_A^* is surjective and we get 1.9(i). Suppose that r is even. Using 4.2 we find that we may assume that A is regular nilpotent in a Levi factor of some parabolic subalgebra of $\mathfrak{g}$. By 1.10 we find then that $\langle P_A, -1\rangle = d/2$. As there are only $(d/2)-1$ odd numbers between 1 and $d-2$, σ must act as -1 on the cokernel of i_A^*, and 1.9(ii) follows.

We are left with the case where $\dim H^{2(n-1)}(\mathcal{Q}_U) = 2$. Then $d+r=2n$ and i_A^* is surjective. It follows easily that 1.9(ii) holds also in this case.

4.5. Consider the same situation as in 4.4, but assume now that char $(k)=2$. As in 4.4, the problem reduces to proving that σ acts as -1 on the cokernel of i_A^*. We may thus assume that r is even and $d+r<N$, and we must find a substitute foı 1.10.

In view of Corollary 2 in 4.2, we may assume that there exists an A-stable orthogonal decomposition $V = V' \oplus V''$ such that the restriction of A to V' is of type $W(m)$ and the restriction of A to V'' has only factors of type $W(1)$ or $W(2)$. We define a morphism $t \mapsto A_t$ from $\mathbb{A}^1$ to $\mathfrak{g}$ such that $A_0 = A$ as follows. We require first that each A_t stabilizes both V' and V''. For every t, the restriction of A_t to V'' coincides with the restriction of A. Choose a basis $(e_1, \ldots, e_{2m})$ of V' such that the restriction of Q to V' corresponds to $\Sigma_{1 \leq i \leq m} x_i x_{2m-i+1}$ and such that the matrix of the restriction A' of A to V' has coefficients

$$a_{ij} = \begin{cases} 1 & \text{if } j = i+1 \text{ and } i \neq m, \\ 0 & \text{otherwise.} \end{cases}$$

Choose also distinct non-zero elements $\lambda_1, \ldots, \lambda_{m-2} \in k$. Setting $\lambda_{m-1} = \lambda_m = 0$, let D be the endomorphism of V' whose matrix is diagonal with diagonal coefficients $(\lambda_1, \ldots, \lambda_m, \lambda_m, \ldots, \lambda_1)$. Then for every $t \in \mathbb{A}^1$ the restriction of A_t to V' is defined to be $A' + tD$.

Let $U_t = \operatorname{Ker}(A_t)$, $U_t' = U_t \cap V'$, $U'' = U \cap V'' = U_t \cap V''$. Then $U_t = U_t' \oplus U_t''$, where the two factors are mutually orthogonal, and U_t' is generated by the vectors e_{m+1} and

$$u_t = \sum_{j=1}^{m-1} \left(\prod_{h=1}^{j-1} t\lambda_h \right) e_j.$$

Notice that U_t' is completely isotropic. For $t \neq 0$ and $1 \leq i \leq m-2$, the $(t\lambda_i)$-eigenspace of A_t has dimension 2 and contains exactly two isotropic lines which are generated respectively by the vectors

$$v'_{t,i} = \sum_{j=1}^{i} \left(\prod_{h=1}^{j-1} t(\lambda_i + \lambda_h) \right) e_j \quad \text{and} \quad v''_{t,i} = \sum_{j=1}^{m-i+1} \left(\prod_{h=1}^{j-1} t(\lambda_i + \lambda_{m+1-h}) \right) e_{m+j}.$$

It follows that for $t \neq 0$, $\mathscr{P}_{A_t}$ is the union of $\mathscr{Q}_{U_t}$ with $2(m-2)$ isolated points given by the isotropic lines $kv'_{t,i}$ and $kv''_{t,i}$ $(1 \leq i \leq m-2)$.

Let Y be as in 2.1. Making t vary, we find that $Y' = \mathbb{A}^1 \times_{\mathfrak{g}} Y$ has $2m-3$ irreducible components. Using $\mathscr{Q}$ instead of $\mathscr{P}$, the irreducible components of Y' have the following description.

(1) The subvariety $Z = \bigcup_{t \in \mathbb{A}^1} \{t\} \times \mathscr{Q}_{U_t}$, which is isomorphic to $\mathbb{A}^1 \times \mathscr{Q}_U$.
(2) For $1 \leq i \leq m-2$, the subvariety $Z'_i = \{(t, kv'_{t,i}) \mid t \in \mathbb{A}^1\} \cong \mathbb{A}^1$.
(3) For $1 \leq i \leq m-2$, the subvariety $Z''_i = \{(t, kv''_{t,i}) \mid t \in \mathbb{A}^1\} \cong \mathbb{A}^1$.

The components Z and Z'_i $(1 \leq i \leq m-2)$ all contain the point $(0, ke_1)$. The components Z and Z''_i $(1 \leq i \leq m-2)$ all contain the point $(0, ke_{m+1})$. The components of Y' do not meet otherwise. This implies that for $j \geq 1$ the restriction morphism $H^j(Y') \to H^j(\mathscr{P}_{A_t})$ is an isomorphism. In particular $H^j(\mathscr{P}_A)$ and $H^j(\mathscr{P}_{A_1})$ are isomorphic as $W^{\mathscr{P}}$-modules, for $j \geq 1$. Let $B = A_1$ and let $B = B_s + B_n$ be the Jordan decomposition of B, $M = C_G^0(B_s)$, $\mathfrak{m} = \mathfrak{c}_{\mathfrak{g}}(B_s) = \mathrm{Lie}\,(M)$. Then M is of type D_{n-m+2}. Moreover $B_n \in \mathfrak{m}$ corresponds to an element of $\mathfrak{so}_{N-2m+4}$ which has $r/2$ factors $W(2)$ and $(d-r)/2$ factors $W(1)$. Let W' be the Weyl group of M, ρ' the reflection representation of W'. Then

$$\langle Q_{B_n}^M, \rho' \rangle = \sum_{\substack{1 \leq i \leq d-2 \\ i \text{ odd}}} q^i + q^{(d+r-2)/2}$$

and $\mathrm{res}_{W'}^W \rho = \rho' + (m-2)1_{W'}$. By 1.11, $Q_B^G = \mathrm{ind}_{W'}^W Q_{B_n}^M$. Therefore

$$\begin{aligned}
\langle Q_B^G, \rho \rangle = \langle \mathrm{ind}_{W'}^W Q_{B_n}^M, \rho \rangle &= \langle Q_{B_n}^M, \mathrm{res}_{W'}^W \rho \rangle \\
&= \langle Q_{B_n}^M, \rho' \rangle + (m-2)\langle Q_{B_n}^M, 1 \rangle \\
&= \sum_{\substack{1 \leq i \leq d-2 \\ i \text{ odd}}} q^i + q^{(d+r-2)/2} + (m-2).
\end{aligned}$$

In view of the isomorphism between $H^j(\mathscr{P}_A)$ and $H^j(\mathscr{P}_B)$ for $j \geq 1$, we find that 1.9(ii) holds for A, and in particular σ acts as multiplication by -1 on the cokernel of i_A^*. This completes the proof of 1.9.

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