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Perturbatively unstable eigenvalues of a periodic Schrödinger operator

JOEL FELDMAN, HORST KNÖRRER AND EUGENE TRUBOWITZ

1. Introduction

Let Γ be a lattice of maximal rank in \mathbb{R}^d , $d \leq 3$, and

$$\Gamma^{\#} = \{b \in \mathbb{R}^d \mid \langle b, \gamma \rangle \in 2\pi \mathbb{Z} \text{ for all } \gamma \in \Gamma \}$$

the lattice dual to Γ . For $q \in L^2(\mathbb{R}^d/\Gamma)$ and $k \in \mathbb{R}^d$ the spectrum of $-\Delta + q$ acting on the space

$$\mathscr{F}_k = \{ \psi \in H^2_{loc}(\mathbf{R}^d) \mid \psi(x+\gamma) = e^{i\langle k, \gamma \rangle} \psi(x) \text{ for all } \gamma \in \Gamma \},$$

or equivalently, the spectrum of $-\Delta_k + q$, where

$$\Delta_k = \Delta + 2ik \cdot \nabla - k^2$$

acting on

$$L^2(\mathbf{R}^d/\Gamma)$$

is called the Floquet spectrum of q with crystal momentum k. For example, the Floquet spectrum with crystal momentum k when q = 0, is the set

$$\{(k+b)^2 \mid b \in \Gamma^\# \}.$$

The corresponding eigenfunctions are

$$e^{i\langle k+b,x\rangle}, b\in\Gamma^{\#}.$$

It is shown in [FKT] that for almost every $k \in \mathbb{R}^d$, and any sufficiently regular q, there is a density zero subset S(k) of $k + \Gamma^*$ such that for all $l \in (k + \Gamma^*) - S(k)$

there is exactly one point in the spectrum of $-\Delta_k + q$ lying in the interval

$$\left[l^{2} + \int_{\mathbf{R}^{d/\Gamma}} q \, dx - \frac{1}{|l|^{2-\epsilon}}, \, l^{2} + \int_{\mathbf{R}^{d/\Gamma}} q \, dx + \frac{1}{|l|^{2-\epsilon}}\right].$$

Moreover, the corresponding eigenfunctions are close to $e^{i\langle l,x\rangle}$. We called the eigenvalues l^2 , $l \in (k + \Gamma^*) - S(k)$ of $-\Delta_k + q$ stable under the perturbation q. The purpose of this paper is to discuss some of the Floquet eigenvalues l^2 , $l \in S(k)$ that are *unstable* under the perturbation q.

We now recall the construction of [ERT] Section 3.b. It yields a class of unstable eigenvalues. Let $\gamma \in \Gamma - \{0\}$, and set

$$q_{\gamma}(x) = \int_{0}^{1} q(x + s\gamma) ds$$
$$= \sum_{\substack{b \in \Gamma \text{ *}\\ \langle b, \gamma \rangle = 0}} \hat{q}(b) e^{i\langle b, x \rangle},$$

where

$$\hat{q}(b) = \frac{1}{|\mathbf{R}^d/\Gamma|} \int_{\mathbf{R}^d/\Gamma} q(x) e^{-i\langle b, x \rangle} dx$$

is the "b'th" Fourier coefficient of q. The averaged potential $q_{\gamma}(x)$ is constant on all translates of the line $\mathbf{R} \cdot \gamma$.

Fix $k' \in \mathbb{R}^d$. Let ϕ be an eigenfunction of $-\Delta + q_{\gamma}(x)$ with crystal momentum k' and eigenvalue μ that is constant on all translates of the line $\mathbb{R} \cdot \gamma$. Then,

$$\psi(x) = e^{it\langle \gamma, x \rangle} \phi(x)$$

is in the space $\mathcal{F}_{(t\gamma+k')}$ and satisfies

$$\frac{1}{\|\psi\|} \|(-\Delta + q)\psi - (t^2\gamma^2 + \mu)\psi\| = O(t^{-2}).$$

The last estimate, combined with the spectral theorem, guarantees that there is a genuine Floquet eigenvalue λ of q with crystal momentum $t\gamma + k'$ close to $t^2\gamma^2 + \mu$. Consequently, the unperturbed eigenvalues l^2 , l near the line $\mathbf{R} \cdot \gamma$, may be moved

far out of the interval

$$\left[l^{2} + \int_{\mathbb{R}^{d/\Gamma}} q \, dx - \frac{1}{|l|^{2-\epsilon}}, \, l^{2} + \int_{\mathbb{R}^{d/\Gamma}} q \, dx + \frac{1}{|l|^{2-\epsilon}}\right]$$

by μ and are therefore unstable in the sense of [FKT]. This phenomenon is consistent with the observation made in [FKT], Section 4, that points of $k + \Gamma^*$ close to a line $\mathbf{R} \cdot \gamma$ for some $\gamma \in \Gamma$ lie in S(k).

The main object of this paper is to show that for each primitive $\gamma \in \Gamma$ and almost every k' satisfying $\langle k', \gamma \rangle = 0$ and almost every sufficiently large t the "WKB" Floquet eigenvalue λ produced in the last paragraph is bounded away from all other points of the Floquet spectrum of q with crystal momentum $t\gamma + k'$, and that the corresponding eigenfunction is close to the quasimode

$$\psi(x) = e^{it\langle \gamma, x \rangle} \phi(x).$$

We first, using the techniques of [FKT], make the WKB construction above more quantitative, giving estimates for the allowed values of t and the accuracy with which Floquet eigenvalues of q are determined. See, (i) of the Theorem below for a precise statement.

Next, for d=2, counting carefully, it is shown ((ii) of the Theorem) that there is a constant Q, depending only on a norm of q, such that for all k lying in a density one subset of the line $k' + \mathbf{R} \cdot \gamma$ the eigenvalues of q with crystal momentum k in the interval

$$[k^2 - Q, k^2 + Q]$$

are all accounted for by stable eigenvalues of $-\Delta$ and eigenvalues constructed as above from $-\Delta + q_{\nu}$.

Finally (part (iii)) for most k, the eigenvalues in the interval $[k^2 - Q, k^2 + Q]$ accounted for by $-\Delta$ are effectively separated from those accounted for by $-\Delta + q_{\gamma}$. This allows us to estimate how well the true eigenfunctions are approximated by the quasi-modes $\psi(x) = e^{it\langle \gamma, x \rangle} \phi(x)$.

2. Construction of eigenvalues and eigenfunctions

As in [FKT] we introduce a monotonically increasing function $f \ge 1$ on \mathbb{R}_+ such that $f(s) f(t) \ge f(s+t)$ and use the f-weighted l_1 -norm $||q||_f = \sum_{b \in \Gamma} \# f(|b|) |\hat{q}(b)|$. Furthermore choose constants $p < \frac{1}{2}$, Q > 0. We restrict ourselves to potentials q with mean zero and $||q||_f \le Q$.

THEOREM. Let γ be a primitive vector of Γ and $k' \in \mathbb{R}^d$ with $\langle k', \gamma \rangle = 0$. Let q be a function on \mathbb{R}^d/Γ with mean zero and $\|q\|_f \leq Q$.

(i) Let t_0 obey $t_0 \ge 2^{1/2p}$, $|t_0\gamma|^p \ge 1/(2\sqrt{Q})|k'|$ and $|t_0\gamma| \ge ((72Q|\gamma|/\pi) + 12\sqrt{Q}) \cdot ((1+k'^2)^p + |t_0\gamma|^{2p})$. Let μ be any Floquet eigenvalue of $-\Delta + q_{\gamma}$ (acting on functions on the hyperplane $\{x \in \mathbb{R}^d \mid \langle x, \gamma \rangle = 0\}$) with multiplier k' of finite multiplicity m fulfilling $|\mu - k'^2| \le Q - \tau$ where

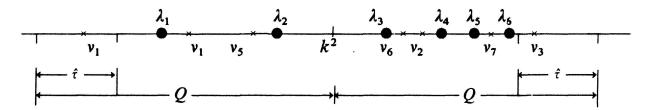
$$\tau := 4Q \left(\frac{1}{|t_0 \gamma|^{2p}} + \frac{1}{f(6\sqrt{Q}|t_0 \gamma|^p)} \right).$$

Then there are at least m Floquet eigenvalues λ (counted with multiplicity) of $-\Delta + q$ with multiplier $k' + t_0 \gamma$ satisfying $|\mu + t_0^2 \gamma^2 - \lambda| \le \tau$.

(ii) Suppose in addition that d=2, p<1/2. Let $h(t)=1+\min(t^{1/2(1/2-p)},t^{2p})$. Then there is a subset $K=K(k',\gamma,Q,p,h)$ of density one in $k'+\mathbf{R}\gamma$ such that for any $k=k'+t_0\gamma\in K$ the following holds. Let $\lambda_1,\ldots,\lambda_r$ be Floquet eigenvalues of $-\Delta+q$ with multiplier k in the interval $[k^2-Q+\hat{\tau},k^2+Q-\hat{\tau}]$ where

$$\hat{\tau}(k) = 4Q \left(\frac{1}{|t_0 \gamma|^{2p}} + \frac{1}{f(6\sqrt{Q}|t_0 \gamma|^p)} + \frac{1}{f(h(|k|))} \right).$$

Let μ_1, \ldots, μ_m be the Floquet eigenvalues of $-\Delta + q_{\gamma}$ with multiplier k' in the interval $[k'^2 - Q, k'^2 + Q]$, $v_i := \mu_i + k^2 - k'^2$, and v_{m+1}, \ldots, v_n the numbers $(k+b)^2$, $b \in \Gamma^*$ with $\langle b, \gamma \rangle \neq 0$ and $(k+b)^2 \in [k^2 - Q, k^2 + Q]$. All these numbers



are counted with multiplicity. Then there is an injection $\sigma: \{1, \ldots, r\} \rightarrow \{1, \ldots, n\}$ such that for $i = 1, \ldots, r$

$$|\lambda_i - \nu_{\sigma(i)}| \leq \hat{\tau}.$$

Furthermore v_j is in the image of σ whenever $|v_j - k^2| \le Q - \hat{\tau}$.

(iii) Suppose that for large t

$$f(6\sqrt{Q}|t\gamma|^p) \ge |t\gamma|^{2p}$$
 and $f(h(\sqrt{t})) \ge |t\gamma|^{2p}$.

Then for any $0 < \delta < 2p$ there is $K' \subset K$ of density one such that for every $k \in K'$ the sets $\bigcup_{j=m+1}^n [v_j - \hat{\tau}, v_j + \hat{\tau}]$ and $[\mu_{i_1} + k^2 - k'^2 - \hat{\tau}, \mu_{i_1} + k^2 - k'^2 + \hat{\tau}], [\mu_{i_2} + k^2 - k'^2 - \hat{\tau}, \mu_{i_2} + k^2 - k'^2 + \hat{\tau}], \ldots, [\mu_{i_s} + k^2 - k'^2 - \hat{\tau}, \mu_{i_s} + k^2 - k'^2 + \hat{\tau}],$ where $\mu_{i_1}, \ldots, \mu_{i_s}$ runs over the different Floquet eigenvalues of $-\Delta + q_\gamma$ to the multiplier k', are mutually disjoint and have distance at least $1/|k|^{2p-\delta}$ from each other. If for some $i=1,\ldots,m$ one takes a Floquet eigenvalue λ of $-\Delta + q$ in $[\mu_i + k^2 - k'^2 - \hat{\tau}, \mu_i + k^2 - k'^2 + \hat{\tau}]$ with multiplier k and if ψ is a normalized eigenfunction for that eigenvalue then there is a Floquet eigenfunction ϕ of $-\Delta + q_\gamma$ for the eigenvalue μ_{i_j} and multiplier k' that is constant in γ -direction such that $\|\psi - e^{i\langle k - k', x \rangle}\phi\| \leq \text{const. } Q/|t_0\gamma|^{\delta}$.

Remarks

- (1) In [ERT] and also [KT] it was shown that the Floquet spectrum of $-\Delta + q$ determines that of $-\Delta + q_{\gamma}$. The proofs given there were non-constructive. For d=2 the theorem above gives a constructive way of determining the Floquet spectrum of $-\Delta + q$ from that of $-\Delta + q_{\gamma}$. Suppose you want to determine the Floquet eigenvalues of $-\Delta + q_{\gamma}$ with multiplier k' ($\langle k', \gamma \rangle = 0$) up to accuracy ϵ . By minimax they are contained in $\bigcup_{b \in \Gamma \neq \langle b, \gamma \rangle = 0} [(k' + b)^2 Q, (k' + b)^2 + Q]$. We show how one determines the desired spectrum up to accuracy ϵ in one of these intervals. Without loss of generality we may assume that this is the interval $[k'^2 Q, k'^2 + Q]$. Choose R so big that
 - (a) the set $\{k' + t\gamma \mid |t||\gamma| \le R\} \cap K(k', \gamma, Q, p, h)$ has measure at least 3R/2 in $k' + \mathbf{R}\gamma$.
 - (b) For each $\mu \in [-Q, Q]$ the set

$$\{k' + t\gamma \mid |t||\gamma| \le R$$
, there is $b \in \Gamma^{\#}$ with $\langle b, \gamma \rangle \ne 0$ such that
$$|(k' + t\gamma + b)^2 - (k' + t\gamma)^2 - \mu| \le 2\hat{\tau}|k' + t\gamma|\}$$

has measure at most R/2 in $k' + \mathbf{R}\gamma$.

(c)
$$\hat{\tau}|k'+R\gamma|<\epsilon/2$$
.

It is possible to find such an R by part (ii) of the Theorem above and Proposition 2 of Section 3. We will see that bounded pieces of the set K can be determined by finitely many operations. Similarly the constants involved in Proposition 2 of Section 3 can be estimated in terms of k', γ and the lattice. So the choice of R is constructive.

Now choose $k_0 \in (k' + \mathbf{R}\gamma) \cap K$ with $|k_0 - k'| \ge R$. By part (ii) of the Theorem the Floquet spectrum of $-\Delta + q_{\gamma}$ in $[k'^2 - Q, k'^2 + Q]$ is contained in the union of

the intervals of length ϵ around the points $\lambda + k'^2 - k_0^2$, where λ runs over all points of the Floquet spectrum of $-\Delta + q_{\gamma}$ with multiplier k_0 in $[k_0^2 - Q, k_0^2 + Q]$. To test whether the interval around such a point $\lambda + k'^2 - k_0^2$ actually contains a point of the Floquet spectrum of $-\Delta + q_{\gamma}$ we proceed as follows. Put $\mu = \lambda - k_0^2$. By (a) and (b) there is $k_1 \in (k' + \mathbf{R}\gamma) \cap K$ with $|k_1 - k'| \le R$ such that for all $b \in \Gamma^{\#}$ with $\langle b, \gamma \rangle \ne 0$ one has $|(k_1 + b)^2 - k_1^2 - \mu| > 2\hat{\tau}(k_1)$. Again k_1 can be found by finitely many operations. By part (ii) of the Theorem the interval around $\lambda + k'^2 - k_0^2$ of length ϵ contains a point of the Floquet spectrum of $-\Delta + q_{\gamma}$ if and only if the interval of length $2\hat{\tau}(k_1)$ around the point $(\lambda + k' - k_0^2) + k_1^2 - k'^2 = k_1^2 + \mu$ contains a point of the spectrum of $-\Delta + q$ with multiplier k_1 .

- (2) If q is sufficiently regular then the higher terms in the asymptotic expansion for the eigenvalues generated by the WKB-Ansatz (cf. [ERT2]) can also be determined by this method.
- (3) With some extra work it should be possible to put all the sets $K(k', \gamma, Q, p)$ together in a subset of full density in a set of the form $\{k' + t\gamma \mid \langle k', \gamma \rangle = 0, |t| \geq C_{\gamma} \cdot |k'|^{N}\}$ for some C_{γ} , N > 0.

In the proof of the Theorem we use the techniques and results of [FKT]. For $k_0 \in \mathbb{R}^d$ we put $\Delta_{k_0} := \Delta + 2ik_0 \cdot \nabla - k_0^2$. Then $\psi(x)$ is a periodic eigenfunction of $-\Delta_{k_0} + q$ for the eigenvalue λ if and only if the function $e^{i\langle k_0, x\rangle}\psi(x)$ is a Floquet eigenfunction for the eigenvalue λ with multiplier k_0 . We showed in [FKT] that the eigenvalues of $-\Delta_{k_0} + q$ in a neighborhood of k_0^2 are the zeroes of the second regularized determinant of a certain infinite matrix. Precisely for r > 0 put

$$G = G_r := \{ (k_0 + b) \mid b \in \Gamma^{\#}, \left| (k_0 + b)^2 - k_0^2 \right| \le r \}$$

$$R_r := (k^2 \delta_{kl} + \hat{q}(k - l))_{k,l \in G_r}.$$

If r is sufficiently big then the eigenvalues of $-\Delta_{k_0} + q$ in the interval $[k_0^2 - Q, k_0^2 + Q]$ are the zeroes of \det_2 of

$$\frac{G_r}{G_r} = \frac{1 \in k_0 + \Gamma^*}{G_r} \left[\begin{array}{ccc} R_r - \lambda & \hat{q}(k-l) \\ \frac{\hat{q}(k-l)}{k^2 - \lambda} & \delta_{kl} + \frac{\hat{q}(k-l)}{k^2 - \lambda} \end{array} \right]. \tag{1}$$

Furthermore if $(v_k)_{k \in k_0 + \Gamma}$ lies in the kernel of this matrix then $\sum_{k \in k_0 + \Gamma} v_k e^{i\langle k - k_0, x \rangle}$ is in the kernel of $-\Delta_k + q - \lambda$. As $r \to \infty$ the eigenvalues and eigenfunctions of R_r approximate those of the whole infinite matrix above.

PROPOSITION. Assume that $||q||_f \le Q \le \frac{1}{6}r$.

- (i) Let $\lambda_1, \ldots, \lambda_n$ be eigenvalues (allowing multiplicities) of $-\Delta_{k_0} + q$ that obey $|\lambda_j k_0^2| \le Q 3Q^2/(r Q)$. Then R_r has at least n eigenvalues (counting multiplicity) in $\binom{n}{j-1} [\lambda_j 3Q^2/(r Q), \lambda_j + 3Q^2/(r Q)]$.
- (ii) Let $\lambda_1', \ldots, \lambda_n'$ be eigenvalues (allowing multiplicities) of R_r that obey $|\lambda_j' k_0^2| \le Q 3Q^2/(r Q)$. Then $\Delta_{k_0} + q$ has at least n eigenvalues (counting multiplicity) in $\bigcup_{j=1}^n [\lambda_j' 3Q^2/(r Q), \lambda_j' + 3Q^2/(r Q)]$.
- (iii) Let $I \subset [k_0^2 Q + 3Q^2/(r Q), k_0^2 + Q 3Q^2/(r Q)]$ be an interval of length ϵ , such that all eigenvalues of $-\Delta_{k_0} + q$ and of R_r either lie in I or have distance at least ρ from I. Let π resp. π' be the orthogonal projections to $\vartheta := \bigoplus_{\lambda \in I} \ker (-\Delta_{k_0} + q \lambda) \operatorname{resp}. \ \vartheta' := \bigoplus_{\lambda' \in I} \{ \sum_{k \in G_r} v_k e^{i\langle k k_0, x \rangle} | v \in \ker (R_r \lambda') \}.$ Then for any $\Psi \in \vartheta$, $\Psi' \in \vartheta'$

$$\begin{split} &\frac{\left\|\Psi'-\pi(\Psi')\right\|}{\left\|\Psi'\right\|} \leq \frac{1}{\rho}\left(\epsilon + \frac{2Q^2}{r-Q}\right) + \frac{2Q}{r-Q},\\ &\frac{\left\|\Psi-\pi'(\Psi)\right\|}{\left\|\Psi\right\|} \leq \frac{1}{\rho}\left(\epsilon + \frac{2Q^2}{r-Q}\right) + \frac{2Q}{r-Q}. \end{split}$$

Proof. We put $W(\lambda) := (\hat{q}(k-l)/(k^2-\lambda))_{k,l \in (k_0+\Gamma^*)\backslash G_r}$. Since $|k^2-\lambda| \ge r-Q$ for all $\lambda \in \Lambda := [k_0^2-Q, k_0^2+Q]$ and $k \in k_0+\Gamma^*\backslash G_r$ one has

$$\|W(\lambda)\|_{f} \le \frac{Q}{r-Q} \le \frac{1}{5}, \qquad \left\|\frac{d}{d\lambda} W(\lambda)\right\|_{f} \le \frac{Q}{(r-Q)^{2}} \qquad \text{for } \lambda \in \Lambda.$$
 (2)

(The operator norm $\|\cdot\|_f$ and its properties are introduced in [FKT eq. (3.4)].) In particular $1 + W(\lambda)$ is invertible for $\lambda \in \Lambda$. So the eigenvalues of $-\Delta_{k_0} + q$ in Λ are the zeroes of

$$\det (R_r - \lambda \mathbf{1} - VU),$$

where

$$\begin{split} U &:= \left(\sum_{k' \in (k_0 + \Gamma^{\#}) \setminus G_r} (1 + W)_{k,h'}^{-1} \cdot \frac{\hat{q}(k' - l)}{k'^2 - \lambda} \right)_{k \notin G_r, l \in G_r}, \\ V &:= (\hat{q}(k - l))_{k \in G_r, l \notin G_r}. \end{split}$$

Furthermore, for a vector y in the kernel of $R - \lambda \mathbf{1} - VU$ the vector $\begin{bmatrix} y \\ -Uy \end{bmatrix}$ lies in the kernel of the matrix (1).

Similar to [FKT], Lemma 3.2, one gets the bounds

$$||U|| \le \frac{2Q}{r - Q}, \qquad ||VU|| \le \frac{2Q^2}{r - Q},$$

$$||\frac{d}{d\lambda}(VU)|| \le \frac{Q^3}{(r - Q)^3} + \frac{2Q^2}{(r - Q)^2} \le \frac{1}{4}.$$
(3)

As in the proof of [FKT], Theorem 3.3, we define the matrix $\tilde{R}(\lambda, \nu) := R - \lambda \mathbf{1} + \nu V U$ and call the eigenvalues of this matrix

$$\rho_1(\lambda, \nu) \leq \rho_2(\lambda, \nu) \leq \cdots \leq \rho_k(\lambda, \nu).$$

Then

$$\begin{split} \left| \, \rho_i(\lambda, \, v) - \rho_i(\lambda, \, v') \right| & \leq \frac{2Q^2}{r - Q} \, \left| v - v' \right| \qquad \text{for } \lambda \in \Lambda; \quad v, \, v' \in [0, \, 1], \\ \\ \rho_i(\lambda, \, v) - \rho_i(\lambda', \, v) & \leq -\frac{3}{4}(\lambda - \lambda') \qquad \text{for } \lambda \geq \lambda'; \quad \lambda, \, \lambda' \in \Lambda, \quad v \in [0, \, 1]. \end{split}$$

The zeroes of $\rho_i(\lambda, 0)$ are the eigenvalues of R while the zeroes of $\rho_i(\lambda, 1)$ in Λ are the eigenvalues of $-\Delta_{k_0} + q$ in this interval. The estimates above show that for all $v \in [0, 1]$ the function $\rho_i(-, v)$ has at most one zero in Λ , and that this zero moves with speed at most $\frac{8}{3}(Q^2/(r-Q))$ with v. This proves part (i) and (ii) of the Proposition.

To prove (iii) let $\tilde{\pi}$ resp. $\tilde{\pi}'$ be the orthogonal projections onto $\mathfrak{I} := \bigoplus_{\lambda \in I} \ker (R_r - \lambda \mathbf{1} - V(\lambda)U(\lambda))$ resp. $\mathfrak{I}' := \bigoplus_{\lambda \in I} \ker (R_r - \lambda \mathbf{1})$. First we show that for all $v \in \mathfrak{I}$, $v' \in \mathfrak{I}'$

$$\frac{\left\|v'-\tilde{\pi}(v')\right\|}{\left\|v'\right\|} \le \frac{1}{\rho} \left(\epsilon + \frac{2Q^2}{r-Q}\right), \qquad \frac{\left\|v-\tilde{\pi}'(v)\right\|}{\left\|v\right\|} \le \frac{1}{\rho} \left(\epsilon + \frac{2Q^2}{r-Q}\right). \tag{4}$$

Let for example $v \in \ker (R_r - \lambda \mathbf{1} - V(\lambda)U(\lambda))$ with $\lambda \in I$. Then

$$\|(R_r-\lambda \mathbf{1})(\tilde{\pi}'(v)-v+v)\| \leq \epsilon \|\tilde{\pi}'(v)\|,$$

hence

$$\begin{split} \left\| (R_r - \lambda \mathbf{1}) (\tilde{\pi}'(v) - v) \right\| &\leq \epsilon \left\| \tilde{\pi}'(v) \right\| + \left\| (R_r - \lambda \mathbf{1}) \cdot v \right\| \\ &\leq \epsilon \left\| v \right\| + \left\| V(\lambda) U(\lambda) \cdot v \right\| \leq \left(\epsilon + \frac{2Q^2}{r - Q} \right) \left\| v \right\| \end{split}$$

by (3). Since $v - \tilde{\pi}'(v)$ is orthogonal to \mathfrak{I}' and the norm of $(R_r - \lambda \mathbf{1})^{-1}$ on \mathfrak{I}'^{\perp} is at most ρ^{-1} this gives the estimate (4).

Since for all $v \in \ker(R_r - \lambda \mathbf{1} - VU)$ the vector $\begin{bmatrix} v \\ -Uv \end{bmatrix}$ lies in the kernel of the matrix (1) and $||Uv|| \le (2Q/(r-Q))||v||$ by (3) we get the estimates stated in part (iii) of the Proposition.

We now proceed to the proof of the theorem. Let

$$k = k' + t_0 \gamma$$
 in \mathbf{R}^d .

We will apply the Proposition with $k_0 = k$, $r = 4Q(1 + k^2)^p$. Split G_r into the union of

$$\mathcal{L}_1 := \{ l \in G_r \mid \langle k - l, \gamma \rangle = 0 \},$$

$$\mathcal{L}_2 := \{ l \in G_r \mid \langle k - l, \gamma \rangle \neq 0 \}.$$

Let $B_i := (l^2 \delta_{lm} + \hat{q}(l-m))_{l,m \in \mathcal{L}_i}$ be the subblock of R, corresponding to \mathcal{L}_i . The key observation is that $B_1 - (k^2 - k'^2)\mathbf{1}$ is equal to a subblock of the matrix describing $-\Delta_{k'} + q_{\gamma}$. Precisely put

$$G'_{r'} := \{ (k'+b) \mid b \in \Gamma^{\#}, \langle b, \gamma \rangle = 0, |(k'+b)^{2} - k'^{2}| \le r' \},$$

$$R'_{r'} := (l'^{2}\delta_{l'm'} + \hat{q}(l'-m'))_{l',m' \in G'_{r'}}.$$

Then

$$B_1 - (k^2 - k'^2)\mathbf{1} = R'_r$$

and the proposition above also applies to the operator $-\Delta_{k'} + q_{\gamma}$ and r'. Thus eigenvalues and eigenvectors of B_1 are related to those of $-\Delta_{k'} + q_{\gamma}$. In order to also relate them to eigenvalues and eigenvectors of R_r (and then of $-\Delta_k + q$) we use that the entries $\hat{q}(l-l')$ of R_r with $l \in \mathcal{L}_1$, $l' \in \mathcal{L}_2$ are small. This will be a consequence of

LEMMA. Assume that
$$|k'| \le 2\sqrt{Q}(1+k^2)^{p/2}$$
, and

$$|t_0\gamma| \ge 12\sqrt{Q}(1+k^2)^{p/2} + 72|\gamma|Q(1+k^2)^p/\pi.$$

Then for all $b \in \Gamma^{\#}$ with $|(k+b)^2 - k^2| \le 4Q(1+k^2)^p$ one has either

$$\langle b, \gamma \rangle = 0$$
 and $|b| \le 5\sqrt{Q}(1+k^2)^{p/2}$

or

$$\langle b, \gamma \rangle \neq 0$$
 and $|b| \geq 6\sqrt{Q}(1+k^2)^{p/2}$.

In particular for any $l \in \mathcal{L}_1$, $l' \in \mathcal{L}_2$ one has $|l - l'| \ge \sqrt{Q}(1 + k^2)^{p/2}$.

Proof. Let $b \in \Gamma^{\#}$ such that $|(k+b)^2 - k^2| \le 4Q(1+k^2)^p$. First assume that $\langle b, \gamma \rangle = 0$. Then $(k+b)^2 - k^2 = (k'+b)^2 - k'^2$ so that $(k'+b)^2 \le 4Q(1+k^2)^p + k'^2 \le 9Q(1+k^2)^p$ so $|b| \le 3\sqrt{Q}(1+k^2)^{p/2} + |k'| \le 5\sqrt{Q}(1+k^2)^{p/2}$.

Now assume that $\langle b, \gamma \rangle \neq 0$. Write $b = b' + s\gamma$ with $\langle b', \gamma \rangle = 0$. Since γ is primitive $|s\gamma| \geq 2\pi/|\gamma|$. If $|s\gamma| \geq 6\sqrt{Q}(1+k^2)^{p/2}$ then there is nothing to prove, so assume that $|s\gamma| \leq 6\sqrt{Q}(1+k^2)^{p/2}$. Then

$$(k + b^2) - k^2 = (k' + b')^2 - k'^2 + (t_0 + s)^2 \gamma^2 - t_0^2 \gamma^2$$

SO

$$(k' + b')^{2} \ge |(t_{0} + s)^{2} - t_{0}^{2}|\gamma^{2} - k'^{2} - 4Q(1 + k^{2})^{p}$$

$$\ge |2t_{0} + s||s|\gamma^{2} - 8Q(1 + k^{2})^{p}$$

$$\ge 2\pi |2t_{0} + s| - 8Q(1 + k^{2})^{p}$$

$$\ge \pi |t_{0}| - \frac{12\pi \sqrt{Q}}{|\gamma|} (1 + k^{2})^{p/2} - 8Q(1 + k^{2})^{p} \ge 64Q(1 + k^{2})^{p}.$$

Therefore

$$|b'| \ge 8\sqrt{Q}(1+k^2)^{p/2} - |k'| \ge 6\sqrt{Q}(1+k^2)^{p/2}.$$

From now on we assume that t_0 fulfills the hypotheses of part (i) of the theorem. Then the lemma above applies.

Put

$$g(t) := 6\sqrt{Q(1+t)^{p/2}}.$$

The lemma above implies that

$$\left\| R_r - \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \right\| \le \frac{Q}{f(g(k^2))}.$$
 (5)

Now let μ be a Floquet eigenvalue of $-\Delta_{k'} + q_{\gamma}$ of multiplicity m fulfilling $|\mu - k'^2| \le \tau$. By the proposition applied to $-\Delta + q_{\gamma}$ there are at least m eigenvalues of R'_r in the interval $[\mu - 3Q^2/(r-Q), \mu + 3Q^2/(r-Q)]$. So there are at least m eigenvalues of B_1 in the interval around $\mu + k^2 - k'^2$ of length $3Q^2/(r-Q)$. By (5) there are then at least m eigenvalues of R_r in the interval around $\mu + k^2 - k'^2$ of length $3Q^2/(r-Q) + Q/f(g(k^2))$. Applying the proposition to $-\Delta + q_{\gamma}$ we see that there are at least m eigenvalues of $-\Delta_{k_0} + q$ satisfying

$$\begin{aligned} |\mu + k^{2} - k'^{2} - \lambda| &\leq \frac{6Q^{2}}{r - Q} + \frac{Q}{f(g(k^{2}))} \\ &\leq 4Q \left(\frac{1}{|t_{0}\gamma|^{2p}} + \frac{1}{f(6\sqrt{Q}|t_{0}\gamma|^{p})} \right) = \tau. \end{aligned}$$

This proves part (i) of the theorem.

For part (ii) we put

$$M := \{k \in \mathbb{R}^d \mid \text{there are } c \neq c' \text{ in } \Gamma^\# \text{ with } \langle c, \gamma \rangle \neq 0, \ \langle c', \gamma \rangle \neq 0 \text{ such that}$$
$$|(k+c)^2 - k^2| \leq 2Q, \ |(k+c')^2 - k^2| \leq 4Q(1+k^2)^p \text{ and } |c-c'| \leq h(|k|)\}.$$

Then we define K as the intersection of $\{k = k' + t\gamma \mid |t| \ge 2^{1/2p}, |t\gamma|^p \ge 1/(2\sqrt{Q})|k'|, |t\gamma| \ge ((72/\pi)|\gamma| + 12\sqrt{Q})((1 + k'^2)^p + |t\gamma|^{2p})\}$ with $\mathbb{R}^d \setminus M$. In Section 3, Proposition 1, we show that

$$|\{k \in (k' + \mathbf{R}\gamma) \cap M \mid |k - k'| \le s\}| = O(s^{1-\epsilon})$$

for some $\epsilon > 0$, so K is of density one in $k' + \mathbf{R}\gamma$.

Now assume that $k = k' + t_0 \gamma$ lies in K. We keep the notation used in the proof of part (i) of the theorem. Put

$$\mathcal{L}_{2}' := \{l \in \mathcal{L}_{2} \ \big| \ \big| l^{2} - k^{2} \big| \leq 2Q \}, \qquad \mathcal{L}_{2}'' := \mathcal{L}_{2} \backslash \mathcal{L}_{2}',$$

and let B_2' resp. B_2'' be the subblocks of B_2 corresponding to \mathcal{L}_2' resp. \mathcal{L}_2'' . Furthermore let D be the diagonal part of B_2' . Since for all $l \in \mathcal{L}_2$, $l' \in \mathcal{L}_2'$ one has $|l-l'| \ge h(|k|)$, by the definition of M

$$\left\|\begin{pmatrix} \mathcal{L}_{2}' & \mathcal{L}_{2}'' \\ D & 0 \\ 0 & B_{2}'' \end{pmatrix} - B_{2} \right\| \leq \frac{Q}{f(h(|k|))},$$

hence

$$\left\| R_r - \begin{pmatrix} B_1 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & B_2'' \end{pmatrix} \right\| \le \frac{Q}{f(g(k^2))} + \frac{Q}{f(h(|k|))}.$$
 (6)

By minimax B_2'' has no eigenvalues in $[k^2 - Q, k^2 + Q]$. Therefore the eigenvalues of $\begin{pmatrix} B_1 & 0 & 0 \\ 0 & D & 0 \end{pmatrix}$ in the interval $[k^2 - Q, k^2 + Q]$ are the eigenvalues of B_1 in this

$$\begin{pmatrix} B_1 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & B_2'' \end{pmatrix}$$
 in the interval $[k^2 - Q, k^2 + Q]$ are the eigenvalues of B_1 in this

interval, and the numbers $(k+b)^2$, $b \in \Gamma^\#$, $\langle b, \gamma \rangle \neq 0$ that lie in this interval. We already know that the eigenvalues of B_1 in the interval under consideration are obtained from the eigenvalues of $-\Delta + q_{\gamma}$ by adding $k^2 - k'^2$ and shifting by at most $3Q^2/(r-Q)$. Similarly the eigenvalues of R_r are obtained from those of $-\Delta + q$ by shifting by at most $3Q^2/(r-Q)$. This yields part (ii) of the theorem.

To prove part (iii) put

$$K'_{i} := \left\{ k \in k' + \mathbf{R}\gamma \mid \text{ for all } b \in \Gamma^{\#} \text{ with } \langle b, \gamma \rangle \neq 0 \text{ one has} \right.$$

$$\left| (k+b)^{2} - k^{2} + k'^{2} - \mu_{i} \right| \geq \frac{1}{|k|^{2p-\delta}} + 2\hat{\tau}(k) \right\},$$

$$K' := \bigcap_{i=1}^{m} K'_{i} \cap K.$$

In Section 3 we will show that each K_i' and hence also K' has density one in $k' + \mathbf{R}\gamma$ (Proposition 2 of Section 3). Now suppose that $k \in K'$ is big enough that $1/|k|^{2p-\delta} + 2\hat{\tau}(k) \le |\mu_i - \mu_j|$ for all i, j such that $\mu_i \ne \mu_j$. Then the first statement of part (iii) of the Theorem is trivially true.

Now let $\lambda \in [\mu_i + k^2 - k'^2 - \hat{\tau}, \mu_i + k^2 - k'^2 + \hat{\tau}]$ and $\tilde{\Psi}(x) = \sum_{l \in k + \Gamma} v_l e^{i\langle l - k, x \rangle}$ be a unit vector in $\ker(-\Delta_k + q - \lambda)$. Put $I := [\mu_i - \hat{\tau}, \mu_i + \hat{\tau}]$ and $\tilde{\mathfrak{I}}' := \bigoplus_{\mu' \in I} \ker(R'_r - \mu')$. Then $v := (v_l)_{l \in G_r}$ is an eigenvector of $R_r - VU$ to the eigenvalue λ and $\|(v_l)_{l \in (k + \Gamma \#) \setminus G_r}\| = \|Uv\| \le \hat{\tau}$. Put $I := [\mu_i - \hat{\tau}, \mu_i + \hat{\tau}]$ and $\tilde{\mathfrak{I}} := \bigoplus_{\mu \in I} \ker(R'_r - \mu)$. Let w be the projection of v onto $\tilde{\mathfrak{I}}$. Then

$$\left\| \left(\begin{pmatrix} B_1 & D & \\ & B_2'' \end{pmatrix} - v_i \right) w \right\| \le \hat{\tau} \| w \|.$$

Since all eigenvalues of $\binom{B_1}{D_{B_2''}}$ that lie in $[k^2 - Q, k^2 + Q]$ are actually contained in $\binom{n}{i-1}[v_i - \hat{\tau}, v_i + \hat{\tau}]$

$$\left\| \left(\binom{B_1}{B_2''} - \nu_i \right) (v - w) \right\| \ge \frac{1}{|k|^{2p - \delta}} \| (v - w) \|.$$

As in the proof of part (iii) of the proposition we get (using (3) and (6)) that

$$||v-w|| \leq 4\hat{\tau}|k|^{2p-\delta}.$$

After part (iii) of the proposition there is $\tilde{\phi} \in \ker (-\Delta_{k'} + q_{\gamma} - \mu_i)$ such that $\|\tilde{\phi} - \Sigma_{l \in G_r} w_l e^{i\langle l - k', x \rangle}\| \le 4\hat{\tau} |k|^{2p - \delta} + \hat{\tau}$. Hence

$$\|\tilde{\phi} - \tilde{\Psi}\| \le 8\hat{\tau} |k|^{2p-\delta} + 2\hat{\tau}.$$

Under the hypotheses in the theorem $\hat{\tau} \leq 12Q(1/|t_0\gamma|^{2p})$. So if t_0 was chosen big enough we get the claimed estimate.

3. Lattice properties

In Section 2 we used two purely lattice theoretic results, which we are going to prove now. As before fix $0 \le p \le \frac{1}{2}$ and Q > 0, and choose a monotonically increasing function $h(t) \ge 1$. With this notation put

$$M(P, Q, h) := \{k \in \mathbb{R}^2 \mid \exists b, c \in \Gamma^{\#} \text{ with } \langle c, \gamma \rangle \neq 0, \langle b + c, \gamma \rangle \neq 0 \text{ such that}$$
$$|(k + c)^2 - k^2| \leq 2Q, \ b \neq 0, \ |(k + b + c)^2 - k^2| \leq 4Q(1 + k^2)^p$$
and $|b| \leq h(|k|) \}.$

PROPOSITION 1. Assume that $p < \frac{1}{2}$, $h(t) = O(\min(t^{1/2(1/2-p)}, t^{2p}))$. Then for each $k' \in \mathbb{R}^2$

$$\left|\left\{k \in k' + \mathbb{R}\gamma \mid \left|k\right| \le r\right\} \cap M(p, Q, h)\right| = O(r^{1-\epsilon})$$

for some $\epsilon > 0$.

The other result we needed can be phrased as follows. For any $0 \le \alpha < 1$ and $\mu \in \mathbb{R}$ put

$$M'(\alpha, \mu) := \left\{ k \in \mathbb{R}^2 \mid \text{there is } b \in \Gamma^\# \text{ with } \langle b, \gamma \rangle \neq 0 \text{ such that} \right.$$

$$\left| (k+b)^2 - k^2 - \mu \right| \leq \frac{1}{|k|^{\alpha}} \right\}.$$

PROPOSITION 2. Let $k' \in \mathbb{R}^2$ and m > 0. Then there is a constant C > 0 such that for all $\mu \in \mathbb{R}$ with $|\mu| \le m$

$$|\{k \in k' + \mathbf{R}\gamma \mid |k| \le r\} \cap M'(\alpha, \mu)| \le C \cdot (1 + r^{1-\alpha}).$$

Remark. The proofs given below are constructive, i.e. each bounded piece of the sets M(p, Q, h) resp. $M'(\alpha, \mu)$ can be determined by finitely many operations.

For the proof of Proposition 1 and Proposition 2 we may, after rotating and scaling the lattice, assume that $\gamma = (0, 2\pi)$. We prove the propositions in the case k' = 0, the general case is similar. To simplify notation write $B_r := \{x \in \mathbb{R}^2 \mid |x| \le r\}$.

Proof of Proposition 1. Split M(p, Q, h) into the union of

$$M_1(p, Q, h) := \{ k \in \mathbb{R}^2 \mid \exists b, c \in \Gamma^\# \text{ with } b_2 = 0, b \neq 0, c_2 \neq 0 \text{ and } |b| \leq h(|k|),$$
$$|(k+c)^2 - k^2| \leq 2Q, |(k+b+c)^2 - k^2| \leq 4Q(1+|k|^2)^p \},$$

and

$$M_2(p, Q, h) := \{ k \in \mathbb{R}^2 \mid \exists b, c \in \Gamma^* \text{ with } b_2 \neq 0, c_2 \neq 0, b_2 + c_2 \neq 0 \text{ and } |b| \leq h(|k|), \\ |(k+c)^2 - k^2| \leq 2Q, |(k+b+c)^2 - k^2| \leq 4Q(1+|k|^2)^p \}.$$

LEMMA 1. Suppose that $h(t) \le t^{2p}$. Then for any $\epsilon > 0$

$$|\mathbf{R}\gamma \cap M_1(p, Q, h) \cap B_r| = O(r^{2p+\epsilon}).$$

Proof. Take $\epsilon > 0$ and put

$$N := \{ k \in \mathbb{R} \gamma \mid \exists c \in \Gamma^{\#} \setminus \{0\} \text{ such that } |(k+c)^2 - k^2| \le 2Q \text{ and}$$
$$|(k_2 + c_2)^2 - k_2^2| \le |k|^{4p + 2c} \}.$$

Below we show that $|\{k \in \mathbb{N} \mid |k| \le r\}| = O(r^{2p+c})$. We claim that there is an R > 0 such that $M_1 \cap \{k \in \mathbb{R}\gamma \mid |k| \ge R\} \subset N$. So suppose that $k \in \mathbb{R}\gamma \cap M_1$ but $k \notin N$. By definition there are $b, c \in \Gamma^{\#}$ with $b_2 = 0$, $c_2 \ne 0$ and $|b| \le h(|k|)$ such that with l := k + c

$$|l^2 - k^2| \le 2Q$$
 and $|(l+b)^2 - l^2| \le 6Q(1+k^2)^p$.

Since $k \notin N$ this implies

$$\left| l_2^2 - k_2^2 \right| \ge |k|^{4p + 2\epsilon}$$

and therefore

$$|l_1^2 \ge |k|^{4p+2\epsilon} - 2Q. \tag{1}$$

On the other hand, the inequality $|(l+b)^2 - l^2| \le 6Q(1+k^2)^p$ gives

$$|2l_1+b_1| \leq \frac{6Q}{|b_1|} (1+k^2)^p.$$

Since $|b_1| \le h(|k|) \le |k|^{2p}$ we get

$$|2l_1| \le \frac{6Q}{|b_1|} (1+k^2)^p + |k|^{2p},$$

which is a contradiction to (1) whenever k is big enough.

It remains to prove the estimate for N. For each $c \in \Gamma^{\#} \setminus \{0\}$ the intersection of $\{k \in \mathbb{R}^2 \mid |(k+c)^2 - k^2| \le 2Q\}$ with the line $\mathbb{R}\gamma$ is contained in the interval J_c of length $2Q/|c_2|$ around the point $(0, -\frac{1}{2}(|c|^2/c_2))$. The inequalities $|(k_2+c_2)^2 - k_2^2| \le |k|^{4p+2\epsilon}$ and $|(k+c)^2 - k^2| \le 2Q$ imply $c_1^2 \le |k|^{4p+2\epsilon} + 2Q$. Therefore there is a compact subset C of N such that for all r > 0

$$\left\{k \in N \setminus C \mid |k| \le r\right\} \subset \bigcup_{\substack{c \in \Gamma \# \\ c_1^2 \le 2r^{4p+2\epsilon} \\ |c_2| \le r+1}} J_c.$$

The measure of the latter set is bounded by

$$4\sum_{c_2=1}^{r} \left(\sqrt{2} \frac{r^{2p+\epsilon}}{L} + 2\right) \frac{4Q}{c_2}$$
,

where L is the length of the shortest non-zero vector in Γ . This proves Lemma 1.

We now discuss the set M_2 . Again for $c \in \Gamma^*$ the intersection of $\{k \in \mathbb{R}^2 \mid |(k+c)^2 - k^2| \le 2Q\}$ with the line $\mathbb{R}\gamma$ is contained in the interval J_c of length $2Q/|c_2|$ around $(0, -\frac{1}{2}(|c|^2/c_2))$. If $c^2/|c_2|$ is big enough then for any $b \in \Gamma^*$

with $b_2 + c_2 \neq 0$ this interval meets

$${k \in \mathbb{R}^2 \mid |(k+b+c)^2 - k^2| \le 4Q(1+k^2)^p}$$

only if

$$\left|\frac{c^2}{c_2} - \frac{(c+b)^2}{c_2 + b_2}\right| \le 6Q \, \frac{|c+b|^{4p}}{|c_2 + b_2|^{1+2p}} \, .$$

So up to a finite interval $\mathbb{R}_{\gamma} \cap M_2$ is contained in the union of the intervals J_c over all c in the set

$$P := \left\{ c \in \Gamma^{\#} \mid c_2 \neq 0 \text{ and there is } b \in \Gamma^{\#} \text{ with } b_2 \neq 0, \ b_2 + c_2 \neq 0 \text{ and} \right.$$

$$\left| \frac{c^2}{c_2} - \frac{(c+b)^2}{c_2 + b_2} \right| \le 6Q \frac{|c+b|^{4p}}{|c_2 + b_2|^{1+p}}, \ |b| \le h \left(\frac{c^2}{|c_2|} \right) + 1 \right\}.$$

Therefore we put for each $b \in \Gamma^{\#}$

$$P_b := \left\{ x \in \mathbb{R}^2 \left| \left| \frac{x^2}{x_2} - \frac{(x+b)^2}{x_2 + b_2} \right| \le 6Q \frac{|x+b|^{4p}}{|x_2 + b_2|^{1+2p}}, |x_2| \ge 1, |x_2 + b_2| \ge 1, \\ x^2 \ge |x_2|(h^{-1}(|b|) - 1) \right\}.$$

Then

$$P = \bigcup_{\substack{b \in \Gamma \# \\ b_2 \neq 0}} (P_b \cap \Gamma^\#).$$

By elementary computation

$$\begin{split} P_b &= \bigg\{ x \in \mathbb{R}^2 \, \bigg| \, \bigg| \bigg(x + \frac{b}{2} \bigg) \bigg(\frac{b_2}{-b_1} - \frac{b_1}{-b_2} \bigg) \bigg(x + \frac{b}{2} \bigg) + \frac{1}{4} \, b_2 |b|^2 \bigg| \\ &\leq 6 Q \, \frac{|x+b|^{4p}}{|x_2+b_2|^{2p}} \, |x_2|, \, \, x^2 \geq |x_2| (h^{-1}(|b|)-1), \, \, |x_2| \geq 1, \, \, |x_2+b_2| \geq 1 \bigg\}. \end{split}$$

LEMMA 2. Suppose that $p < \frac{1}{2}$, $\lim_{t \to \infty} (h^{-1}(t)/t^2) = \infty$.

(i) There is a constant A such that for all but finitely many $b \in \Gamma^*$ with $b_2 \neq 0$,

$$P_b \cap \left\{ x \in \mathbb{R}^2 \mid |x_2| \le A \frac{h^{-1}(|b|)}{|b|^2} \right\} = \emptyset.$$

(ii) There is a constant μ such that for all $b \in \Gamma^{\#}$ with $b_2 \neq 0$ and all $\eta \in \mathbb{R}$ with $|\eta| \geq \mu |b_2|$ the intersection of P_b with the line $\{x \in \mathbb{R}^2 \mid x_2 = \eta\}$ is contained in the union of at most two intervals, each of length at most const. $(|b|^{4p-1}/b_2^{4p})|\eta|^{2p}$. Here const. is a constant independent of b and η .

Let us first explain how Lemma 1 and Lemma 2 imply Proposition 1. By Lemma 2 and the assumption on h there is a finite set $S \subset \Gamma^{\#}$ such that for all $b \in \Gamma^{\#} \setminus S$ with $b_2 \neq 0$,

$$P_b \cap \left\{ x \in \mathbf{R}^2 \mid |x_2| \le \frac{h^{-1}(|b|)}{|b|^2} \right\} = \emptyset$$

and

$$\mu|b_2| \ge A \frac{h^{-1}(|b|)}{|b|^2}.$$

Put $\rho := \max \{ \mu | b_2 | \mid b \in S \}$, and for $b \in \Gamma^{\#}$ with $b_2 \neq 0$,

$$\tilde{P}_b := \left\{ x \in P_b \mid |x_2| \ge \max\left(\mu |b_2|, A \frac{h^{-1}(|b|)}{|b|^2}\right) \right\}.$$

Then

$$P \subset \{x \in \mathbf{R}^2 \mid |x_2| \le \rho\} \cup \bigcup_{\substack{b \in \Gamma \\ b_2 \ne 0}} \widetilde{P}_b.$$

Now by Lemma 2 for each $|\eta| \ge \rho$ and each $b \in \Gamma^{\#}$ with $b_2 \ne 0$, $\tilde{P}_b \cap \{x \in \mathbb{R}^2 \mid x_2 = \eta\}$ contains at most const. $(1 + |b|^{4p-1}/b_2^{4p})|\eta|^{2p}$ points of $\Gamma^{\#}$. Let l(t) be the inverse function of $Ah^{-1}(t)/t^2$. The assumptions on h imply that $l(t) = O(t^{1/2-p-\epsilon})$ for some $\epsilon > 0$.

Then for sufficiently large r

$$\sum_{\substack{c \in P \\ J_c \cap B_r \neq \varnothing}} \left| J_c \right| \leq \text{const.} \sum_{\substack{c \in \Gamma \# \\ 1 \leq |c_2| \leq \rho \\ |c|^2 \leq 2r|c_2|}} \frac{1}{|c_2|} + \text{const.} \sum_{\substack{b \in \Gamma \# \\ b_2 \neq 0}} \sum_{\substack{c \in \Gamma \# \cap \beta_b \\ |c|^2 \leq 2r|c_2|}} \frac{1}{|c_2|}.$$

The first sum clearly is $O(r^{1/2})$. By what we said above the second sum is bounded by

const.
$$\sum_{c_{2}=1}^{2r} \sum_{\substack{b \in \Gamma * \\ |b| \le l(c_{2})}} \left(1 + \frac{|b|^{4p-1}}{|b_{2}|^{4p}}\right) c_{2}^{2p-1} \le$$

$$\le \text{const.} \sum_{c_{2}=1}^{2r} c_{c}^{2p-1} l(c_{2})^{2} = 0 (r^{1-\varepsilon})$$

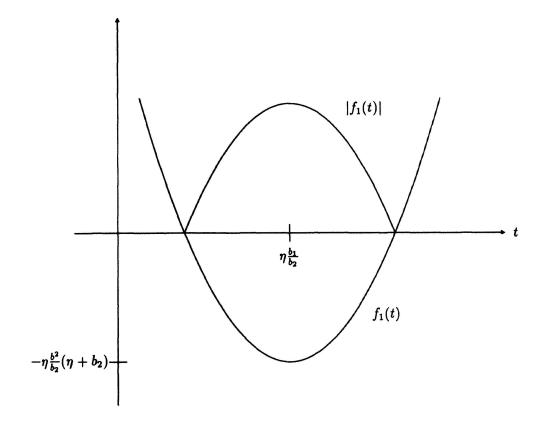
So $|M_2 \cap B_r| = O(r^{1-\varepsilon})$. This, together with Lemma 1, implies Proposition 1. \square

We now prove Lemma 2. Fix any $\eta \in \mathbb{R}$, $b \in \Gamma^*$ with $b_2 \neq 0$. Without loss of generality we may assume that $b_2 > 0$. Parametrise the line $\{x \in R^2 | x_2 = \eta\}$ by Φ : $t \to (t, \eta)$, and denote by $f_1(t)$, resp. $f_2(t)$, the restriction of the functions $\left(x + \frac{b}{2}\right) \left(-\frac{b_2}{-b_2} - \frac{b_1}{-b_2}\right) \left(x + \frac{b}{2}\right) + \frac{1}{4}b_2b^2$, resp. $6Q|x + b|^{4p} \frac{|\eta|}{|\eta + b_2|^{2p}}$, to this line. Then

$$\{t \in \mathbb{R} | \Phi(t) \in P_b \} = \{t \in \mathbb{R} | |f_1(t)| \le f_2(t) \text{ and } t^2 \ge \eta(h^{-1}(|b|) - 1) - \eta^2 \}$$

The matrix $\begin{pmatrix} b_2 & -b_1 \\ -b_2 & -b_2 \end{pmatrix}$ has $\pm |b|$ as eigenvalues. Its isotropic subspaces are spanned by the vectors $(b_1 \pm |b|, b_2)$. The zeros of the restriction of $(x + \frac{b}{2}) \begin{pmatrix} b_2 & -b_1 \\ -b_1 & -b_2 \end{pmatrix} (x + \frac{b}{2})$ to $\{x \in \mathbb{R}^2 | x_2 = \eta\}$ are at $t = \eta \frac{b_1}{b_2} \pm |b| \left(\frac{\eta}{b_2} + \frac{1}{2}\right)$. The restriction of $(x + \frac{b}{2}) \begin{pmatrix} b_2 & -b_1 \\ -b_1 & -b_2 \end{pmatrix} (x + \frac{b}{2})$ to the line $\{x \in \mathbb{R}^2 | x_2 = \eta\}$ is a quadratic polynomial in t with leading coefficient b_2 and the zeroes described above, so it equals $b_2(t - \eta(b_1/b_2))^2 - b_2b^2(\eta/b_2 + \frac{1}{2})^2$. Therefore

$$f_1(t) = b_2 \left(t - \eta \frac{b_1}{b_2}\right)^2 - \frac{\eta b^2}{b_2} (\eta + b_2).$$



The function

$$f_2(t) = 6Q[(t+b_1)^2 + (\eta+b_2)^2]^{2p} \frac{|\eta|}{(\eta+b_2)^{2p}}$$

is symmetric about $t = -b_1$ and increasing monotonically but slower than quadratically in $|t + b_1|$.

We now show that any intersection point T with $f_1(T) = f_2(T)$ obeys

$$\left|T - \frac{b_1}{b_2}\eta\right| \le \text{const.} \begin{cases} |b|, & |\eta| \le 2b_2, \\ \frac{|b|}{b_2}|\eta|, & |\eta| \ge 2b_2. \end{cases}$$
 (2)

To prove (2) we introduce $\tau = T - (b_1/b_2)\eta$ and observe that the equation

$$f_1\left(\tau + \eta \frac{b_1}{b_2}\right) = f_2\left(\tau + \eta \frac{b_1}{b_2}\right),$$

i.e.

$$b_2\tau^2 = \eta \frac{b^2}{b_2}(\eta + b_2) + 6Q \left[\tau^2 + 2\frac{b_1}{b_2}\tau(\eta + b_2) + \frac{|b|^2}{b_2^2}(\eta + b_2)^2\right]^{2p} \frac{|\eta|}{(\eta + b_2)^{2p}}$$

implies

$$\tau^{2} \leq \frac{\text{const.}}{b_{2}} \max \left\{ \frac{|\eta|}{(\eta + b_{2})^{2p}} \tau^{4p}, \ |\eta| \left| \frac{b_{1}}{b_{2}} \tau \right|^{2p}, \ \eta \frac{b^{2}}{b_{2}} (\eta + b_{2}), \ \frac{|b|^{4p}}{b_{2}^{4p}} \left| \eta \right| (\eta + b_{2})^{2p} \right\}.$$

When $|\eta| \ge 2b_2$ we get

$$\tau^{2} \leq \text{const. max} \left\{ \frac{|\eta|^{1-2p}}{b_{2}} \tau^{4p}, \frac{|b_{1}|^{2p}}{|b_{2}|^{1+2p}} |\eta| |\tau|^{2p}, \frac{b^{2}}{b_{2}^{2}} \eta^{2}, \frac{|b|^{4p}}{b_{2}^{1+4p}} |\eta|^{1+2p} \right\},$$

which yields

$$\begin{split} & \left| \tau \right| \leq \text{const. max } \left\{ \left| \eta \right|^{1/2}, \left| \frac{b_1}{b_2} \right|^{p/(1-p)} \left| \eta \right|^{1/(2-2p)}, \frac{\left| b \right|}{b_2} \left| \eta \right|, \left(\frac{\left| b \right|}{b_2} \right)^{2p} \left| \eta \right|^{1/2+p} \right\} \\ & \leq \text{const. } \frac{\left| b \right|}{b_2} \left| \eta \right|. \end{split}$$

The case $|\eta| \le 2b_2$ is treated similarly.

The inequality (2) implies that for any $t \in P_b \{x_2 = \eta\}$

$$|\tau| \le |\tau| + \left|\eta \frac{b_1}{b_2}\right| \le C \begin{cases} |b| & \text{if } |\eta| \le 2b_2, \\ \frac{|b|}{b_2}|\eta| & \text{if } |\eta| \ge 2b, \end{cases}$$

where the constant C is independent of b and η . So if $P_b \cap \{x \in \mathbb{R}^2 \mid x_2 = \eta\}$ is not empty then

$$|\eta|(h^{-1}(|b|)-1)-\eta^2 \le C^2 \begin{cases} |b|^2 & \text{if } |\eta| \le 2b_2, \\ \frac{|b|^2}{b_2^2}\eta^2 & \text{if } |\eta| \ge 2b_2. \end{cases}$$

When $1 \le |\eta| \le 2b_2$

$$h^{-1}(|b|) \le (C^2+4)|b|^2+1$$
,

which is satisfied only by finitely many b's since $h^{-1}(|b|)/|b|^2$ tends to infinity with |b|. When $2b_2 \le |\eta| \le A(h^{-1}(|b|)/|b|^2)$ with $A = 1/(2C^2)$ this would imply

$$h^{-1}(|b|) - 1 - |\eta| \le \frac{\eta^{-1}(|b|)}{2b_2^2},$$

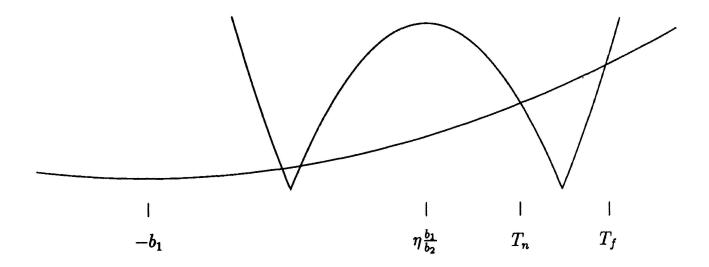
which is impossible. We have thus shown part (i) of Lemma 2.

We now prove part (ii). Assume that $|\eta| \ge \mu b_2$. Observe that

$$f_1\left(\eta \frac{b_1}{b_2}\right) = -\eta \frac{b^2}{b_2}(\eta + b_2) < 0,$$

$$f_2\left(\eta \frac{b_1}{b_2}\right) \le \left|f_1\left(\eta \frac{b_1}{b_2}\right)\right|$$

provided μ is chosen sufficiently large. (Consequently $P_b \cap \{\eta \in \mathbb{R}^2 \mid x_2 = \eta\}$ is contained in the union of two intervals, one to the left and one to the right of $\eta(b_1/b_2)$. The longer of these two intervals is that on the side of $\eta(b_1/b_2)$ opposite to $-b_1$. See the figure. Define the end points T_n , resp. T_f , of this interval to be the solution of $|f_1(t)| = f_2(t)$ nearest to, resp. farthest, from $\eta(b_1/b_2)$ on the side of



 $\eta(b_1/b_2)$ opposite $-b_1$. To bound $|T_f - T_n|$ observe that

$$f_{1}(T_{f}) = f_{2}(T_{f}),$$

$$f_{1}(T_{n}) = -f_{2}(T_{n}),$$

$$\Rightarrow f_{1}(T_{f}) - f_{1}(T_{n}) = f_{2}(T_{f}) + f_{2}(T_{n}),$$

$$\Rightarrow b_{2}(T_{f} - T_{n}) \left(T_{f} + T_{n} - 2\eta \frac{b_{1}}{b_{2}} \right) = f_{2}(T_{f}) + f_{2}(T_{n}),$$

$$\Rightarrow |T_{f} - T_{n}| \leq \frac{2}{b_{2}} \frac{f_{2}(T_{f})}{\left| T_{f} - \eta \frac{b_{1}}{b_{2}} \right|}.$$

Setting $\tau = T_f - \eta \frac{b_1}{b_2}$ we have that

const.
$$\frac{|b|}{b_2}|\eta| \le |\tau| \le \text{const.} \frac{|b|}{b_2}|\eta|,$$

with the upper bound coming from (2) and the lower bound coming from the fact that T_f is farther from $\eta \frac{b_1}{b_2}$ than the zeroes $\eta \frac{b_1}{b_2} \pm \left[\eta \frac{b^2}{b_2^2} (\eta + b_2) \right]^{1/2}$ of $f_1(t)$. Consequently

$$|T_{f} - T_{n}| \leq \operatorname{const} \frac{1}{b_{2}} \frac{\left[\left(\tau + \eta \frac{b_{1}}{b_{2}} + b_{1} \right)^{2} + (\eta + b_{2})^{2} \right]^{2p}}{|\tau|} \frac{|\eta|}{(n + b_{2})^{2p}}$$

$$\leq \operatorname{const} \frac{1}{b_{2}} \tau^{4p - 1} |\eta|^{1 - 2p}$$

$$\leq \operatorname{const} \frac{|b|^{4p - 1}}{b_{2}^{4p}} |\eta|^{2p}$$

Proof of Proposition 2

Choose a finite set $S \subset \Gamma^*$ such that for all $b \in \Gamma^* \setminus S$ with $b_2 \neq 0$

(i)
$$b^2 \ge 2m$$
, $4 \frac{|b_2|^{\alpha - 1}}{(b^2 - \mu)^{\alpha}} \le m$

(ii) for all $\mu \in [-m, m]$ the intersection of $\{k \in \mathbb{R}^2 | (k+b)^2 - k^2 - \mu | \le \frac{1}{|k|^{\alpha}} \text{ with } \mathbb{R}^{\gamma} \text{ is contained in the interval on this axis around this point}$

$$\left(0, \frac{-b^2 - \mu}{2b_2}\right)$$
 of radius $4\frac{|b_2|^{\alpha - 1}}{(b^2 - \mu)^{\alpha}}$.

Then it suffices to show that there is a constant C and that for all $\mu \in [-m, m]$

$$\sum_{\substack{b \in \Gamma \neq \backslash S, b_2 \neq 0 \\ I_b \cap B_r \neq \emptyset}} \left| I_b \right| \leq C r^{1-\alpha}.$$

The sum under consideration is bounded by

$$8 \sum_{\substack{b \in \Gamma \# \\ \frac{b^2 - \mu}{4|b_2|} \le r + m,}} \frac{|b_2|^{\alpha - 1}}{(b^2 - \mu)^{\alpha}} \le 16 \sum_{\substack{b \in \Gamma \# \\ b^2 \le 4(r + 2m)|b_2|}} \frac{|b_2|^{\alpha - 1}}{|b|^{\alpha}} \le O((r + 2m)^{1 - \alpha}), \qquad \Box$$

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