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Perturbatively unstable eigenvalues of a periodic Schrödinger operator

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1. Introduction

Let Γ be a lattice of maximal rank in $\mathbf{R}^d$, $d \le 3$, and

$$\Gamma^{\#} = \{b \in \mathbf{R}^d \mid \langle b, \gamma \rangle \in 2\pi\mathbf{Z} \text{ for all } \gamma \in \Gamma\}$$

the lattice dual to Γ. For $q \in L^2(\mathbf{R}^d/\Gamma)$ and $k \in \mathbf{R}^d$ the spectrum of $-\Delta + q$ acting on the space

$$\mathscr{F}_k = \{\psi \in H^2_{\text{loc}}(\mathbf{R}^d) \mid \psi(x+\gamma) = e^{i\langle k,\gamma \rangle}\psi(x) \text{ for all } \gamma \in \Gamma\},$$

or equivalently, the spectrum of $-\Delta_k + q$, where

$$\Delta_k = \Delta + 2ik \cdot \nabla - k^2$$

acting on

$$L^2(\mathbf{R}^d/\Gamma)$$

is called the Floquet spectrum of q with crystal momentum k. For example, the Floquet spectrum with crystal momentum k when $q = 0$, is the set

$$\{(k+b)^2 \mid b \in \Gamma^{\#}\}.$$

The corresponding eigenfunctions are

$$e^{i\langle k+b,x \rangle}, \qquad b \in \Gamma^{\#}.$$

It is shown in [FKT] that for almost every $k \in \mathbf{R}^d$, and any sufficiently regular q, there is a density zero subset $S(k)$ of $k + \Gamma^{\#}$ such that for all $l \in (k + \Gamma^{\#}) - S(k)$

there is exactly one point in the spectrum of $-\Delta_k + q$ lying in the interval

$$\left[l^2 + \int_{\mathbf{R}^d/\Gamma} q\,dx - \frac{1}{|l|^{2-\epsilon}},\ l^2 + \int_{\mathbf{R}^d/\Gamma} q\,dx + \frac{1}{|l|^{2-\epsilon}}\right].$$

Moreover, the corresponding eigenfunctions are close to $e^{i\langle l,x\rangle}$. We called the eigenvalues l^2, $l \in (k + \Gamma^\#) - S(k)$ of $-\Delta_k + q$ stable under the perturbation q. The purpose of this paper is to discuss some of the Floquet eigenvalues l^2, $l \in S(k)$ that are *unstable* under the perturbation q.

We now recall the construction of [ERT] Section 3.b. It yields a class of unstable eigenvalues. Let $\gamma \in \Gamma - \{0\}$, and set

$$\begin{aligned} q_\gamma(x) &= \int_0^1 q(x + s\gamma)\,ds \\ &= \sum_{\substack{b \in \Gamma^\# \\ \langle b,\gamma\rangle = 0}} \hat{q}(b)\, e^{i\langle b,x\rangle}, \end{aligned}$$

where

$$\hat{q}(b) = \frac{1}{|\mathbf{R}^d/\Gamma|} \int_{\mathbf{R}^d/\Gamma} q(x)\, e^{-i\langle b,x\rangle}\,dx$$

is the "b'th" Fourier coefficieint of q. The averaged potential $q_\gamma(x)$ is constant on all translates of the line $\mathbf{R}\cdot\gamma$.

Fix $k' \in \mathbf{R}^d$. Let ϕ be an eigenfunction of $-\Delta + q_\gamma(x)$ with crystal momentum k' and eigenvalue μ that is constant on all translates of the line $\mathbf{R}\cdot\gamma$. Then,

$$\psi(x) = e^{it\langle\gamma,x\rangle}\phi(x)$$

is in the space $\mathscr{F}_{(t\gamma + k')}$ and satisfies

$$\frac{1}{\|\psi\|}\,\|(-\Delta + q)\psi - (t^2\gamma^2 + \mu)\psi\| = O(t^{-2}).$$

The last estimate, combined with the spectral theorem, guarantees that there is a genuine Floquet eigenvalue λ of q with crystal momentum $t\gamma + k'$ close to $t^2\gamma^2 + \mu$. Consequently, the unperturbed eigenvalues l^2, l near the line $\mathbf{R}\cdot\gamma$, may be moved far out of the interval

$$\left[l^2 + \int_{\mathbf{R}^d/\Gamma} q\,dx - \frac{1}{|l|^{2-\epsilon}},\, l^2 + \int_{\mathbf{R}^d/\Gamma} q\,dx + \frac{1}{|l|^{2-\epsilon}}\right]$$

by μ and are therefore unstable in the sense of [FKT]. This phenomenon is consistent with the observation made in [FKT], Section 4, that points of $k + \Gamma^{\#}$ close to a line $\mathbf{R} \cdot \gamma$ for some $\gamma \in \Gamma$ lie in $S(k)$.

The main object of this paper is to show that for each primitive $\gamma \in \Gamma$ and almost every k' satisfying $\langle k', \gamma \rangle = 0$ and almost every sufficiently large t the "WKB" Floquet eigenvalue λ produced in the last paragraph is bounded away from all other points of the Floquet spectrum of q with crystal momentum $t\gamma + k'$, and that the corresponding eigenfunction is close to the quasimode

$$\psi(x) = e^{it\langle \gamma, x \rangle}\phi(x).$$

We first, using the techniques of [FKT], make the WKB construction above more quantitative, giving estimates for the allowed values of t and the accuracy with which Floquet eigenvalues of q are determined. See, (i) of the Theorem below for a precise statement.

Next, for $d = 2$, counting carefully, it is shown ((ii) of the Theorem) that there is a constant Q, depending only on a norm of q, such that for all k lying in a density one subset of the line $k' + \mathbf{R} \cdot \gamma$ the eigenvalues of q with crystal momentum k in the interval

$$[k^2 - Q, k^2 + Q]$$

are all accounted for by stable eigenvalues of $-\Delta$ and eigenvalues constructed as above from $-\Delta + q_\gamma$.

Finally (part (iii)) for most k, the eigenvalues in the interval $[k^2 - Q, k^2 + Q]$ accounted for by $-\Delta$ are effectively separated from those accounted for by $-\Delta + q_\gamma$. This allows us to estimate how well the true eigenfunctions are approximated by the quasi-modes $\psi(x) = e^{it\langle \gamma, x \rangle}\phi(x)$.

2. Construction of eigenvalues and eigenfunctions

As in [FKT] we introduce a monotonically increasing function $f \geq 1$ on $\mathbf{R}_+$ such that $f(s)f(t) \geq f(s+t)$ and use the f-weighted l_1-norm $\|q\|_f = \Sigma_{b \in \Gamma^{\#}} f(|b|)|\hat{q}(b)|$. Furthermore choose constants $p < \frac{1}{2}$, $Q > 0$. We restrict ourselves to potentials q with mean zero and $\|q\|_f \leq Q$.

THEOREM. *Let γ be a primitive vector of Γ and $k' \in \mathbf{R}^d$ with $\langle k', \gamma\rangle = 0$. Let q be a function on $\mathbf{R}^d/\Gamma$ with mean zero and $\|q\|_f \le Q$.*

(i) *Let t_0 obey $t_0 \ge 2^{1/2p}$, $|t_0\gamma|^p \ge 1/(2\sqrt{Q})|k'|$ and $|t_0\gamma| \ge ((72Q|\gamma|/\pi) + 12\sqrt{Q}) \cdot ((1+k'^2)^p + |t_0\gamma|^{2p})$. Let μ be any Floquet eigenvalue of $-\Delta + q_\gamma$ (acting on functions on the hyperplane $\{x \in \mathbf{R}^d \mid \langle x, \gamma\rangle = 0\}$) with multiplier k' of finite multiplicity m fulfilling $|\mu - k'^2| \le Q - \tau$ where*

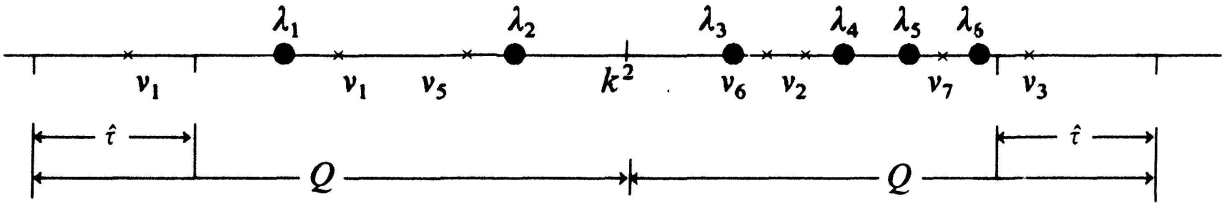
$$\tau := 4Q\left(\frac{1}{|t_0\gamma|^{2p}} + \frac{1}{f(6\sqrt{Q}|t_0\gamma|^p)}\right).$$

Then there are at least m Floquet eigenvalues λ (counted with multiplicity) of $-\Delta + q$ with multiplier $k' + t_0\gamma$ satisfying $|\mu + t_0^2\gamma^2 - \lambda| \le \tau$.

(ii) *Suppose in addition that $d = 2$, $p < 1/2$. Let $h(t) = 1 + \min(t^{1/2(1/2-p)}, t^{2p})$. Then there is a subset $K = K(k', \gamma, Q, p, h)$ of density one in $k' + \mathbf{R}\gamma$ such that for any $k = k' + t_0\gamma \in K$ the following holds. Let $\lambda_1, \ldots, \lambda_r$ be Floquet eigenvalues of $-\Delta + q$ with multiplier k in the interval $[k^2 - Q + \hat{\tau}, k^2 + Q - \hat{\tau}]$ where*

$$\hat{\tau}(k) = 4Q\left(\frac{1}{|t_0\gamma|^{2p}} + \frac{1}{f(6\sqrt{Q}|t_0\gamma|^p)} + \frac{1}{f(h(|k|))}\right).$$

Let $\mu_1, \ldots, \mu_m$ be the Floquet eigenvalues of $-\Delta + q_\gamma$ with multiplier k' in the interval $[k'^2 - Q, k'^2 + Q]$, $\nu_i := \mu_i + k^2 - k'^2$, and $\nu_{m+1}, \ldots, \nu_n$ the numbers $(k+b)^2$, $b \in \Gamma^*$ with $\langle b, \gamma\rangle \ne 0$ and $(k+b)^2 \in [k^2 - Q, k^2 + Q]$. All these numbers



are counted with multiplicity. Then there is an injection $\sigma : \{1, \ldots, r\} \to \{1, \ldots, n\}$ such that for $i = 1, \ldots, r$

$$|\lambda_i - \nu_{\sigma(i)}| \le \hat{\tau}.$$

Furthermore ν_j is in the image of σ whenever $|\nu_j - k^2| \le Q - \hat{\tau}$.

(iii) *Suppose that for large t*

$$f(6\sqrt{Q}\,|t\gamma|^p) \ge |t\gamma|^{2p} \quad \text{and} \quad f(h(\sqrt{t})) \ge |t\gamma|^{2p}.$$

Then for any $0<\delta<2p$ *there is* $K'\subset K$ *of density one such that for every* $k\in K'$ *the sets* $\bigcup_{j=m+1}^{n}[\nu_j-\hat{\tau},\nu_j+\hat{\tau}]$ *and* $[\mu_{i_1}+k^2-k'^2-\hat{\tau},\mu_{i_1}+k^2-k'^2+\hat{\tau}]$, $[\mu_{i_2}+k^2-k'^2-\hat{\tau},\mu_{i_2}+k^2-k'^2+\hat{\tau}],\ldots,[\mu_{i_s}+k^2-k'^2-\hat{\tau},\mu_{i_s}+k^2-k'^2+\hat{\tau}]$, *where* $\mu_{i_1},\ldots,\mu_{i_s}$ *runs over the different Floquet eigenvalues of* $-\Delta+q_\gamma$ *to the multiplier* k', *are mutually disjoint and have distance at least* $1/|k|^{2p-\delta}$ *from each other. If for some* $i=1,\ldots,m$ *one takes a Floquet eigenvalue* λ *of* $-\Delta+q$ *in* $[\mu_i+k^2-k'^2-\hat{\tau},\mu_i+k^2-k'^2+\hat{\tau}]$ *with multiplier* k *and if* ψ *is a normalized eigenfunction for that eigenvalue then there is a Floquet eigenfunction* ϕ *of* $-\Delta+q_\gamma$ *for the eigenvalue* μ_{i_j} *and multiplier* k' *that is constant in* γ*-direction such that* $\|\psi-e^{i\langle k-k',x\rangle}\phi\|\le\text{const. }Q/|t_0\gamma|^\delta$.

Remarks

(1) In [ERT] and also [KT] it was shown that the Floquet spectrum of $-\Delta+q$ determines that of $-\Delta+q_\gamma$. The proofs given there were non-constructive. For $d=2$ the theorem above gives a constructive way of determining the Floquet spectrum of $-\Delta+q$ from that of $-\Delta+q_\gamma$. Suppose you want to determine the Floquet eigenvalues of $-\Delta+q_\gamma$ with multiplier k' ($\langle k',\gamma\rangle=0$) up to accuracy ϵ. By minimax they are contained in $\bigcup_{b\in\Gamma^*,\langle b,\gamma\rangle=0}[(k'+b)^2-Q,(k'+b)^2+Q]$. We show how one determines the desired spectrum up to accuracy ϵ in one of these intervals. Without loss of generality we may assume that this is the interval $[k'^2-Q,k'^2+Q]$. Choose R so big that

(a) the set $\{k'+t\gamma\mid |t||\gamma|\le R\}\cap K(k',\gamma,Q,p,h)$ has measure at least $3R/2$ in $k'+\mathbf{R}\gamma$.

(b) For each $\mu\in[-Q,Q]$ the set

$$\{k'+t\gamma\mid |t||\gamma|\le R,\text{ there is } b\in\Gamma^*\text{ with }\langle b,\gamma\rangle\ne 0\text{ such that}$$

$$|(k'+t\gamma+b)^2-(k'+t\gamma)^2-\mu|\le 2\hat{\tau}|k'+t\gamma|\}$$

has measure at most $R/2$ in $k'+\mathbf{R}\gamma$.

(c) $\hat{\tau}|k'+R\gamma|<\epsilon/2$.

It is possible to find such an R by part (ii) of the Theorem above and Proposition 2 of Section 3. We will see that bounded pieces of the set K can be determined by finitely many operations. Similarly the constants involved in Proposition 2 of Section 3 can be estimated in terms of k', γ and the lattice. So the choice of R is constructive.

Now choose $k_0\in(k'+\mathbf{R}\gamma)\cap K$ with $|k_0-k'|\ge R$. By part (ii) of the Theorem the Floquet spectrum of $-\Delta+q_\gamma$ in $[k'^2-Q,k'^2+Q]$ is contained in the union of the intervals of length ϵ around the points $\lambda + k'^2 - k_0^2$, where λ runs over all points of the Floquet spectrum of $-\Delta + q_\gamma$ with multiplier k_0 in $[k_0^2 - Q, k_0^2 + Q]$. To test whether the interval around such a point $\lambda + k'^2 - k_0^2$ actually contains a point of the Floquet spectrum of $-\Delta + q_\gamma$ we proceed as follows. Put $\mu = \lambda - k_0^2$. By (a) and (b) there is $k_1 \in (k' + \mathbf{R}\gamma) \cap K$ with $|k_1 - k'| \leq R$ such that for all $b \in \Gamma^\#$ with $\langle b, \gamma \rangle \neq 0$ one has $|(k_1 + b)^2 - k_1^2 - \mu| > 2\hat{\tau}(k_1)$. Again k_1 can be found by finitely many operations. By part (ii) of the Theorem the interval around $\lambda + k'^2 - k_0^2$ of length ϵ contains a point of the Floquet spectrum of $-\Delta + q_\gamma$ if and only if the interval of length $2\hat{\tau}(k_1)$ around the point $(\lambda + k' - k_0^2) + k_1^2 - k'^2 = k_1^2 + \mu$ contains a point of the spectrum of $-\Delta + q$ with multiplier k_1.

(2) If q is sufficiently regular then the higher terms in the asymptotic expansion for the eigenvalues generated by the WKB-Ansatz (cf. [ERT2]) can also be determined by this method.

(3) With some extra work it should be possible to put all the sets $K(k', \gamma, Q, p)$ together in a subset of full density in a set of the form $\{k' + t\gamma \mid \langle k', \gamma \rangle = 0, |t| \geq C_\gamma \cdot |k'|^N\}$ for some $C_\gamma, N > 0$.

In the proof of the Theorem we use the techniques and results of [FKT]. For $k_0 \in \mathbf{R}^d$ we put $\Delta_{k_0} := \Delta + 2ik_0 \cdot \nabla - k_0^2$. Then $\psi(x)$ is a periodic eigenfunction of $-\Delta_{k_0} + q$ for the eigenvalue λ if and only if the function $e^{i\langle k_0, x\rangle}\psi(x)$ is a Floquet eigenfunction for the eigenvalue λ with multiplier k_0. We showed in [FKT] that the eigenvalues of $-\Delta_{k_0} + q$ in a neighborhood of k_0^2 are the zeroes of the second regularized determinant of a certain infinite matrix. Precisely for $r > 0$ put

$$G = G_r := \{(k_0 + b) \mid b \in \Gamma^\#, |(k_0 + b)^2 - k_0^2| \leq r\}$$
$$R_r := (k^2 \delta_{kl} + \hat{q}(k - l))_{k,l \in G_r}.$$

If r is sufficieintly big then the eigenvalues of $-\Delta_{k_0} + q$ in the interval $[k_0^2 - Q, k_0^2 + Q]$ are the zeroes of $\det_2$ of

$$\begin{array}{ll} & \xrightarrow{\qquad\qquad} l \in k_0 + \Gamma^\# \\ & \qquad G_r \\ \big\downarrow \quad G_r & \begin{bmatrix} R_r - \lambda & \hat{q}(k - l) \\ \dfrac{\hat{q}(k - l)}{k^2 - \lambda} & \delta_{kl} + \dfrac{\hat{q}(k - l)}{k^2 - \lambda} \end{bmatrix} \\ k \in k_0 + \Gamma^\# & \end{array} \qquad (1)$$

Furthermore if $(v_k)_{k \in k_0 + \Gamma^\#}$ lies in the kernel of this matrix then $\Sigma_{k \in k_0 + \Gamma^\#} v_k e^{i\langle k - k_0, x\rangle}$ is in the kernel of $-\Delta_k + q - \lambda$. As $r \to \infty$ the eigenvalues and eigenfunctions of R_r approximate those of the whole infinite matrix above.

PROPOSITION. *Assume that* $\|q\|_f \le Q \le \frac{1}{6}r$.

(i) *Let* $\lambda_1, \ldots, \lambda_n$ *be eigenvalues (allowing multiplicities) of* $-\Delta_{k_0} + q$ *that obey* $|\lambda_j - k_0^2| \le Q - 3Q^2/(r-Q)$. *Then* R_r *has at least n eigenvalues (counting multiplicity) in* $\bigcup_{j=1}^n [\lambda_j - 3Q^2/(r-Q), \lambda_j + 3Q^2/(r-Q)]$.

(ii) *Let* $\lambda'_1, \ldots, \lambda'_n$ *be eigenvalues (allowing multiplicities) of* R_r *that obey* $|\lambda'_j - k_0^2| \le Q - 3Q^2/(r-Q)$. *Then* $\Delta_{k_0} + q$ *has at least n eigenvalues (counting multiplicity) in* $\bigcup_{j=1}^n [\lambda'_j - 3Q^2/(r-Q), \lambda'_j + 3Q^2/(r-Q)]$.

(iii) *Let* $I \subset [k_0^2 - Q + 3Q^2/(r-Q), k_0^2 + Q - 3Q^2/(r-Q)]$ *be an interval of length* ϵ, *such that all eigenvalues of* $-\Delta_{k_0} + q$ *and of* R_r *either lie in I or have distance at least* ρ *from I. Let* π *resp.* π' *be the orthogonal projections to* $\vartheta := \bigoplus_{\lambda \in I} \ker(-\Delta_{k_0} + q - \lambda)$ *resp.* $\vartheta' := \bigoplus_{\lambda' \in I} \{\Sigma_{k \in G_r} v_k e^{i\langle k - k_0, x\rangle} \mid v \in \ker(R_r - \lambda')\}$. *Then for any* $\Psi \in \vartheta$, $\Psi' \in \vartheta'$

$$\frac{\|\Psi' - \pi(\Psi')\|}{\|\Psi'\|} \le \frac{1}{\rho}\left(\epsilon + \frac{2Q^2}{r-Q}\right) + \frac{2Q}{r-Q},$$

$$\frac{\|\Psi - \pi'(\Psi)\|}{\|\Psi\|} \le \frac{1}{\rho}\left(\epsilon + \frac{2Q^2}{r-Q}\right) + \frac{2Q}{r-Q}.$$

Proof. We put $W(\lambda) := (\hat{q}(k-l)/(k^2-\lambda))_{k,l \in (k_0 + \Gamma^\#)\setminus G_r}$. Since $|k^2 - \lambda| \ge r - Q$ for all $\lambda \in \Lambda := [k_0^2 - Q, k_0^2 + Q]$ and $k \in k_0 + \Gamma^\# \setminus G_r$ one has

$$\|W(\lambda)\|_f \le \frac{Q}{r-Q} \le \frac{1}{5}, \qquad \left\|\frac{d}{d\lambda} W(\lambda)\right\|_f \le \frac{Q}{(r-Q)^2} \qquad \text{for } \lambda \in \Lambda. \tag{2}$$

(The operator norm $\|\cdot\|_f$ and its properties are introduced in [FKT eq. (3.4)].) In particular $\mathbf{1} + W(\lambda)$ is invertible for $\lambda \in \Lambda$. So the eigenvalues of $-\Delta_{k_0} + q$ in Λ are the zeroes of

$$\det(R_r - \lambda\mathbf{1} - VU),$$

where

$$U := \left(\sum_{k' \in (k_0 + \Gamma^\#)\setminus G_r} (\mathbf{1} + W)^{-1}_{k,k'} \cdot \frac{\hat{q}(k'-l)}{k'^2 - \lambda}\right)_{k \notin G_r, l \in G_r},$$

$$V := (\hat{q}(k-l))_{k \in G_r, l \notin G_r}.$$

Furthermore, for a vector y in the kernel of $R - \lambda\mathbf{1} - VU$ the vector $\begin{bmatrix} y \\ -Uy \end{bmatrix}$ lies in the kernel of the matrix (1).

Similar to [FKT], Lemma 3.2, one gets the bounds

$$\|U\| \leq \frac{2Q}{r-Q}, \qquad \|VU\| \leq \frac{2Q^2}{r-Q},$$
$$\left\|\frac{d}{d\lambda}(VU)\right\| \leq \frac{Q^3}{(r-Q)^3} + \frac{2Q^2}{(r-Q)^2} \leq \frac{1}{4}. \tag{3}$$

As in the proof of [FKT], Theorem 3.3, we define the matrix $\tilde{R}(\lambda, \nu) := R - \lambda\mathbf{1} + \nu VU$ and call the eigenvalues of this matrix

$$\rho_1(\lambda, \nu) \leq \rho_2(\lambda, \nu) \leq \cdots \leq \rho_k(\lambda, \nu).$$

Then

$$|\rho_i(\lambda, \nu) - \rho_i(\lambda, \nu')| \leq \frac{2Q^2}{r-Q}|\nu - \nu'| \qquad \text{for } \lambda \in \Lambda; \quad \nu, \nu' \in [0, 1],$$

$$\rho_i(\lambda, \nu) - \rho_i(\lambda', \nu) \leq -\frac{3}{4}(\lambda - \lambda') \qquad \text{for } \lambda \geq \lambda'; \quad \lambda, \lambda' \in \Lambda, \quad \nu \in [0, 1].$$

The zeroes of $\rho_i(\lambda, 0)$ are the eigenvalues of R while the zeroes of $\rho_i(\lambda, 1)$ in Λ are the eigenvalues of $-\Delta_{k_0} + q$ in this interval. The estimates above show that for all $\nu \in [0, 1]$ the function $\rho_i(-, \nu)$ has at most one zero in Λ, and that this zero moves with speed at most $\frac{8}{3}(Q^2/(r-Q))$ with ν. This proves part (i) and (ii) of the Proposition.

To prove (iii) let $\tilde{\pi}$ resp. $\tilde{\pi}'$ be the orthogonal projections onto $\mathfrak{I} := \bigoplus_{\lambda \in I} \ker(R_r - \lambda\mathbf{1} - V(\lambda)U(\lambda))$ resp. $\mathfrak{I}' := \bigoplus_{\lambda \in I} \ker(R_r - \lambda\mathbf{1})$. First we show that for all $v \in \mathfrak{I}$, $v' \in \mathfrak{I}'$

$$\frac{\|v' - \tilde{\pi}(v')\|}{\|v'\|} \leq \frac{1}{\rho}\left(\epsilon + \frac{2Q^2}{r-Q}\right), \qquad \frac{\|v - \tilde{\pi}'(v)\|}{\|v\|} \leq \frac{1}{\rho}\left(\epsilon + \frac{2Q^2}{r-Q}\right). \tag{4}$$

Let for example $v \in \ker(R_r - \lambda\mathbf{1} - V(\lambda)U(\lambda))$ with $\lambda \in I$. Then

$$\|(R_r - \lambda\mathbf{1})(\tilde{\pi}'(v) - v + v)\| \leq \epsilon\|\tilde{\pi}'(v)\|,$$

hence

$$\|(R_r - \lambda\mathbf{1})(\tilde{\pi}'(v) - v)\| \leq \epsilon\|\tilde{\pi}'(v)\| + \|(R_r - \lambda\mathbf{1}) \cdot v\|$$
$$\leq \epsilon\|v\| + \|V(\lambda)U(\lambda) \cdot v\| \leq \left(\epsilon + \frac{2Q^2}{r-Q}\right)\|v\|$$

by (3). Since $v - \tilde{\pi}'(v)$ is orthogonal to $\mathfrak{F}'$ and the norm of $(R_r - \lambda \mathbf{1})^{-1}$ on $\mathfrak{F}'^{\perp}$ is at most ρ^{-1} this gives the estimate (4).

Since for all $v \in \ker(R_r - \lambda\mathbf{1} - VU)$ the vector $\begin{bmatrix} v \\ -Uv \end{bmatrix}$ lies in the kernel of the matrix (1) and $\|Uv\| \le (2Q/(r-Q))\|v\|$ by (3) we get the estimates stated in part (iii) of the Proposition. □

We now proceed to the *proof of the theorem*. Let

$$k = k' + t_0\gamma \qquad \text{in } \mathbf{R}^d.$$

We will apply the Proposition with $k_0 = k$, $r = 4Q(1+k^2)^p$. Split G_r into the union of

$$\mathscr{L}_1 := \{l \in G_r \mid \langle k - l, \gamma\rangle = 0\},$$

$$\mathscr{L}_2 := \{l \in G_r \mid \langle k - l, \gamma\rangle \ne 0\}.$$

Let $B_i := (l^2\delta_{lm} + \hat{q}(l-m))_{l,m \in \mathscr{L}_i}$ be the subblock of R_r corresponding to $\mathscr{L}_i$. The key observation is that $B_1 - (k^2 - k'^2)\mathbf{1}$ is equal to a subblock of the matrix describing $-\Delta_{k'} + q_\gamma$. Precisely put

$$G'_{r'} := \{(k'+b) \mid b \in \Gamma^{\#}, \langle b, \gamma\rangle = 0, |(k'+b)^2 - k'^2| \le r'\},$$

$$R'_{r'} := (l'^2\delta_{l'm'} + \hat{q}(l'-m'))_{l',m' \in G'_{r'}}.$$

Then

$$B_1 - (k^2 - k'^2)\mathbf{1} = R'_{r'}$$

and the proposition above also applies to the operator $-\Delta_{k'} + q_\gamma$ and r'. Thus eigenvalues and eigenvectors of B_1 are related to those of $-\Delta_{k'} + q_\gamma$. In order to also relate them to eigenvalues and eigenvectors of R_r (and then of $-\Delta_k + q$) we use that the entries $\hat{q}(l - l')$ of R_r with $l \in \mathscr{L}_1$, $l' \in \mathscr{L}_2$ are small. This will be a consequence of

LEMMA. *Assume that* $|k'| \le 2\sqrt{Q}(1+k^2)^{p/2}$, *and*

$$|t_0\gamma| \ge 12\sqrt{Q}(1+k^2)^{p/2} + 72|\gamma|Q(1+k^2)^p/\pi.$$

Then for all $b \in \Gamma^{\#}$ *with* $|(k+b)^2 - k^2| \leq 4Q(1+k^2)^p$ *one has either*

$$\langle b, \gamma \rangle = 0 \quad \textit{and} \quad |b| \leq 5\sqrt{Q}(1+k^2)^{p/2}$$

or

$$\langle b, \gamma \rangle \neq 0 \quad \textit{and} \quad |b| \geq 6\sqrt{Q}(1+k^2)^{p/2}.$$

In particular for any $l \in \mathscr{L}_1$, $l' \in \mathscr{L}_2$ *one has* $|l - l'| \geq \sqrt{Q}(1+k^2)^{p/2}$.

Proof. Let $b \in \Gamma^{\#}$ such that $|(k+b)^2 - k^2| \leq 4Q(1+k^2)^p$. First assume that $\langle b, \gamma \rangle = 0$. Then $(k+b)^2 - k^2 = (k'+b)^2 - k'^2$ so that $(k'+b)^2 \leq 4Q(1+k^2)^p + k'^2 \leq 9Q(1+k^2)^p$ so $|b| \leq 3\sqrt{Q}(1+k^2)^{p/2} + |k'| \leq 5\sqrt{Q}(1+k^2)^{p/2}$.

Now assume that $\langle b, \gamma \rangle \neq 0$. Write $b = b' + s\gamma$ with $\langle b', \gamma \rangle = 0$. Since γ is primitive $|s\gamma| \geq 2\pi/|\gamma|$. If $|s\gamma| \geq 6\sqrt{Q}(1+k^2)^{p/2}$ then there is nothing to prove, so assume that $|s\gamma| \leq 6\sqrt{Q}(1+k^2)^{p/2}$. Then

$$(k + b^2) - k^2 = (k'+b')^2 - k'^2 + (t_0+s)^2\gamma^2 - t_0^2\gamma^2,$$

so

$$\begin{aligned}
(k'+b')^2 &\geq |(t_0+s)^2 - t_0^2|\gamma^2 - k'^2 - 4Q(1+k^2)^p \\
&\geq |2t_0+s||s|\gamma^2 - 8Q(1+k^2)^p \\
&\geq 2\pi|2t_0+s| - 8Q(1+k^2)^p \\
&\geq \pi|t_0| - \frac{12\pi\sqrt{Q}}{|\gamma|}(1+k^2)^{p/2} - 8Q(1+k^2)^p \geq 64Q(1+k^2)^p.
\end{aligned}$$

Therefore

$$|b'| \geq 8\sqrt{Q}(1+k^2)^{p/2} - |k'| \geq 6\sqrt{Q}(1+k^2)^{p/2}. \qquad \square$$

From now on we assume that t_0 fulfills the hypotheses of part (i) of the theorem. Then the lemma above applies.

Put

$$g(t) := 6\sqrt{Q}(1+t)^{p/2}.$$

The lemma above implies that

$$\left\| R_r - \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \right\| \leq \frac{Q}{f(g(k^2))}. \tag{5}$$

Now let μ be a Floquet eigenvalue of $-\Delta_{k'}+q_\gamma$ of multiplicity m fulfilling $|\mu-k'^2|\le\tau$. By the proposition applied to $-\Delta+q_\gamma$ there are at least m eigenvalues of R'_r in the interval $[\mu-3Q^2/(r-Q), \mu+3Q^2/(r-Q)]$. So there are at least m eigenvalues of B_1 in the interval around $\mu+k^2-k'^2$ of length $3Q^2/(r-Q)$. By (5) there are then at least m eigenvalues of R_r in the interval around $\mu+k^2-k'^2$ of length $3Q^2/(r-Q)+Q/f(g(k^2))$. Applying the proposition to $-\Delta+q_\gamma$ we see that there are at least m eigenvalues of $-\Delta_{k_0}+q$ satisfying

$$\begin{aligned}|\mu+k^2-k'^2-\lambda| &\le \frac{6Q^2}{r-Q}+\frac{Q}{f(g(k^2))}\\ &\le 4Q\left(\frac{1}{|t_0\gamma|^{2p}}+\frac{1}{f(6\sqrt{Q}\,|t_0\gamma|^p)}\right)=\tau.\end{aligned}$$

This proves part (i) of the theorem.

For part (ii) we put

$$\begin{aligned}M:=\{k\in\mathbf{R}^d \mid &\text{ there are } c\ne c' \text{ in } \Gamma^\# \text{ with } \langle c,\gamma\rangle\ne0,\ \langle c',\gamma\rangle\ne0 \text{ such that}\\ &|(k+c)^2-k^2|\le 2Q,\ |(k+c')^2-k^2|\le 4Q(1+k^2)^p \text{ and } |c-c'|\le h(|k|)\}.\end{aligned}$$

Then we define K as the intersection of $\{k=k'+t\gamma \mid |t|\ge 2^{1/2p}, |t\gamma|^p\ge 1/(2\sqrt{Q})|k'|, |t\gamma|\ge((72/\pi)|\gamma|+12\sqrt{Q})((1+k'^2)^p+|t\gamma^{2p}|)\}$ with $\mathbf{R}^d\backslash M$. In Section 3, Proposition 1, we show that

$$|\{k\in(k'+\mathbf{R}\gamma)\cap M \mid |k-k'|\le s\}|=O(s^{1-\epsilon})$$

for some $\epsilon>0$, so K is of density one in $k'+\mathbf{R}\gamma$.

Now assume that $k=k'+t_0\gamma$ lies in K. We keep the notation used in the proof of part (i) of the theorem. Put

$$\mathscr{L}'_2:=\{l\in\mathscr{L}_2 \mid |l^2-k^2|\le 2Q\},\qquad \mathscr{L}''_2:=\mathscr{L}_2\backslash\mathscr{L}'_2,$$

and let B'_2 resp. B''_2 be the subblocks of B_2 corresponding to $\mathscr{L}'_2$ resp. $\mathscr{L}''_2$. Furthermore let D be the diagonal part of B'_2. Since for all $l\in\mathscr{L}_2$, $l'\in\mathscr{L}'_2$ one has $|l-l'|\ge h(|k|)$, by the definition of M

$$\left\|\begin{array}{c}\begin{array}{cc}\mathscr{L}'_2 & \mathscr{L}''_2\end{array}\\ \begin{pmatrix}D & 0\\ 0 & B''_2\end{pmatrix}\end{array}-B_2\right\|\le\frac{Q}{f(h(|k|))},$$

hence

$$\left\|R_r-\begin{pmatrix}B_1 & 0 & 0\\ 0 & D & 0\\ 0 & 0 & B''_2\end{pmatrix}\right\|\le\frac{Q}{f(g(k^2))}+\frac{Q}{f(h(|k|))}. \tag{6}$$

By minimax B_2'' has no eigenvalues in $[k^2-Q, k^2+Q]$. Therefore the eigenvalues of $\begin{pmatrix} B_1 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & B_2'' \end{pmatrix}$ in the interval $[k^2-Q, k^2+Q]$ are the eigenvalues of B_1 in this interval, and the numbers $(k+b)^2$, $b \in \Gamma^{\#}$, $\langle b, \gamma \rangle \neq 0$ that lie in this interval. We already know that the eigenvalues of B_1 in the interval under consideration are obtained from the eigenvalues of $-\Delta + q_\gamma$ by adding $k^2 - k'^2$ and shifting by at most $3Q^2/(r-Q)$. Similarly the eigenvalues of R_r are obtained from those of $-\Delta + q$ by shifting by at most $3Q^2/(r-Q)$. This yields part (ii) of the theorem.

To prove part (iii) put

$$K_i' := \Big\{ k \in k' + \mathbf{R}\gamma \,\big|\, \text{for all } b \in \Gamma^{\#} \text{ with } \langle b, \gamma \rangle \neq 0 \text{ one has}$$

$$|(k+b)^2 - k^2 + k'^2 - \mu_i| \geq \frac{1}{|k|^{2p-\delta}} + 2\hat{\tau}(k) \Big\},$$

$$K' := \bigcap_{i=1}^{m} K_i' \cap K.$$

In Section 3 we will show that each K_i' and hence also K' has density one in $k' + \mathbf{R}\gamma$ (Proposition 2 of Section 3). Now suppose that $k \in K'$ is big enough that $1/|k|^{2p-\delta} + 2\hat{\tau}(k) \leq |\mu_i - \mu_j|$ for all i, j such that $\mu_i \neq \mu_j$. Then the first statement of part (iii) of the Theorem is trivially true.

Now let $\lambda \in [\mu_i + k^2 - k'^2 - \hat{\tau}, \mu_i + k^2 - k'^2 + \hat{\tau}]$ and $\tilde{\Psi}(x) = \Sigma_{l \in k + \Gamma^{\#}} v_l e^{i\langle l - k, x\rangle}$ be a unit vector in $\ker(-\Delta_k + q - \lambda)$. Put $I := [\mu_i - \hat{\tau}, \mu_i + \hat{\tau}]$ and $\mathfrak{H}' := \bigoplus_{\mu' \in I} \ker(R_r' - \mu')$. Then $v := (v_l)_{l \in G_r}$ is an eigenvector of $R_r - VU$ to the eigenvalue λ and $\|(v_l)_{l \in (k+\Gamma^{\#}) \setminus G_r}\| = \|Uv\| \leq \hat{\tau}$. Put $I := [\mu_i - \hat{\tau}, \mu_i + \hat{\tau}]$ and $\mathfrak{H} := \bigoplus_{\mu \in I} \ker(R_r' - \mu)$. Let w be the projection of v onto $\mathfrak{H}$. Then

$$\left\| \left(\begin{pmatrix} B_1 & & \\ & D & \\ & & B_2'' \end{pmatrix} - \nu_i \right) w \right\| \leq \hat{\tau} \|w\|.$$

Since all eigenvalues of $\begin{pmatrix} B_1 & & \\ & D & \\ & & B_2'' \end{pmatrix}$ that lie in $[k^2-Q, k^2+Q]$ are actually contained in $\bigcup_{i=1}^{n} [\nu_i - \hat{\tau}, \nu_i + \hat{\tau}]$

$$\left\| \left(\begin{pmatrix} B_1 & & \\ & D & \\ & & B_2'' \end{pmatrix} - \nu_i \right) (v - w) \right\| \geq \frac{1}{|k|^{2p-\delta}} \|(v - w)\|.$$

As in the proof of part (iii) of the proposition we get (using (3) and (6)) that

$$\|v - w\| \le 4\hat{\tau}|k|^{2p-\delta}.$$

After part (iii) of the proposition there is $\tilde{\Phi} \in \ker(-\Delta_{k'} + q_\gamma - \mu_i)$ such that $\|\tilde{\Phi} - \Sigma_{l \in G_r} w_l e^{i\langle l-k', x\rangle}\| \le 4\hat{\tau}|k|^{2p-\delta} + \hat{\tau}$. Hence

$$\|\tilde{\Phi} - \tilde{\Psi}\| \le 8\hat{\tau}|k|^{2p-\delta} + 2\hat{\tau}.$$

Under the hypotheses in the theorem $\hat{\tau} \le 12Q(1/|t_0\gamma|^{2p})$. So if t_0 was chosen big enough we get the claimed estimate. □

3. Lattice properties

In Section 2 we used two purely lattice theoretic results, which we are going to prove now. As before fix $0 \le p \le \frac{1}{2}$ and $Q > 0$, and choose a monotonically increasing function $h(t) \ge 1$. With this notation put

$$\begin{aligned} M(P, Q, h) := \{k \in \mathbf{R}^2 \mid \exists b, c \in \Gamma^\# \text{ with } \langle c, \gamma\rangle \ne 0,\ \langle b + c, \gamma\rangle \ne 0 \text{ such that } \\ |(k+c)^2 - k^2| \le 2Q,\ b \ne 0,\ |(k+b+c)^2 - k^2| \le 4Q(1+k^2)^p \\ \text{and } |b| \le h(|k|)\}. \end{aligned}$$

PROPOSITION 1. *Assume that $p < \frac{1}{2}$, $h(t) = O(\min(t^{1/2(1/2-p)}, t^{2p})$. Then for each $k' \in \mathbf{R}^2$*

$$|\{k \in k' + \mathbf{R}\gamma \mid |k| \le r\} \cap M(p, Q, h)| = O(r^{1-\epsilon})$$

for some $\epsilon > 0$.

The other result we needed can be phrased as follows. For any $0 \le \alpha < 1$ and $\mu \in \mathbf{R}$ put

$$M'(\alpha, \mu) := \left\{k \in \mathbf{R}^2 \mid \text{there is } b \in \Gamma^\# \text{ with } \langle b, \gamma\rangle \ne 0 \text{ such that } |(k+b)^2 - k^2 - \mu| \le \frac{1}{|k|^\alpha}\right\}.$$

PROPOSITION 2. *Let $k' \in \mathbf{R}^2$ and $m > 0$. Then there is a constant $C > 0$ such that for all $\mu \in \mathbf{R}$ with $|\mu| \le m$*

$$|\{k \in k' + \mathbf{R}\gamma \mid |k| \le r\} \cap M'(\alpha, \mu)| \le C \cdot (1 + r^{1-\alpha}).$$

Remark. The proofs given below are constructive, i.e. each bounded piece of the sets $M(p, Q, h)$ resp. $M'(\alpha, \mu)$ can be determined by finitely many operations.

For the proof of Proposition 1 and Proposition 2 we may, after rotating and scaling the lattice, assume that $\gamma = (0, 2\pi)$. We prove the propositions in the case $k' = 0$, the general case is similar. To simplify notation write $B_r := \{x \in \mathbf{R}^2 \mid |x| \le r\}$.

Proof of Proposition 1. Split $M(p, Q, h)$ into the union of

$$M_1(p, Q, h) := \{k \in \mathbf{R}^2 \mid \exists b, c \in \Gamma^{\#} \text{ with } b_2 = 0,\ b \ne 0,\ c_2 \ne 0 \text{ and } |b| \le h(|k|),$$
$$|(k + c)^2 - k^2| \le 2Q,\ |(k + b + c)^2 - k^2| \le 4Q(1 + |k|^2)^p\},$$

and

$$M_2(p, Q, h) := \{k \in \mathbf{R}^2 \mid \exists b, c \in \Gamma^{\#} \text{ with } b_2 \ne 0,\ c_2 \ne 0,\ b_2 + c_2 \ne 0 \text{ and } |b| \le h(|k|),$$
$$|(k + c)^2 - k^2| \le 2Q,\ |(k + b + c)^2 - k^2| \le 4Q(1 + |k|^2)^p\}.$$

LEMMA 1. *Suppose that $h(t) \le t^{2p}$. Then for any $\epsilon > 0$*

$$|\mathbf{R}\gamma \cap M_1(p, Q, h) \cap B_r| = O(r^{2p+\epsilon}).$$

Proof. Take $\epsilon > 0$ and put

$$N := \{k \in \mathbf{R}\gamma \mid \exists c \in \Gamma^{\#} \backslash \{0\} \text{ such that } |(k + c)^2 - k^2| \le 2Q \text{ and}$$
$$|(k_2 + c_2)^2 - k_2^2| \le |k|^{4p+2\epsilon}\}.$$

Below we show that $|\{k \in \mathbf{N} \mid |k| \le r\}| = O(r^{2p+\epsilon})$. We claim that there is an $R > 0$ such that $M_1 \cap \{k \in \mathbf{R}\gamma \mid |k| \ge R\} \subset N$. So suppose that $k \in \mathbf{R}\gamma \cap M_1$ but $k \notin N$. By definition there are $b, c \in \Gamma^{\#}$ with $b_2 = 0$, $c_2 \ne 0$ and $|b| \le h(|k|)$ such that with $l := k + c$

$$|l^2 - k^2| \le 2Q \quad \text{and} \quad |(l + b)^2 - l^2| \le 6Q(1 + k^2)^p.$$

Since $k \notin N$ this implies

$$|l_2^2 - k_2^2| \geq |k|^{4p+2\epsilon}$$

and therefore

$$l_1^2 \geq |k|^{4p+2\epsilon} - 2Q. \tag{1}$$

On the other hand, the inequality $|(l+b)^2 - l^2| \leq 6Q(1+k^2)^p$ gives

$$|2l_1 + b_1| \leq \frac{6Q}{|b_1|}(1+k^2)^p.$$

Since $|b_1| \leq h(|k|) \leq |k|^{2p}$ we get

$$|2l_1| \leq \frac{6Q}{|b_1|}(1+k^2)^p + |k|^{2p},$$

which is a contradiction to (1) whenever k is big enough.

It remains to prove the estimate for N. For each $c \in \Gamma^{\#} \backslash \{0\}$ the intersection of $\{k \in \mathbf{R}^2 \mid |(k+c)^2 - k^2| \leq 2Q\}$ with the line $\mathbf{R}\gamma$ is contained in the interval J_c of length $2Q/|c_2|$ around the point $(0, -\frac{1}{2}(|c|^2/c_2))$. The inequalities $|(k_2+c_2)^2 - k_2^2| \leq |k|^{4p+2\epsilon}$ and $|(k+c)^2 - k^2| \leq 2Q$ imply $c_1^2 \leq |k|^{4p+2\epsilon} + 2Q$. Therefore there is a compact subset C of N such that for all $r > 0$

$$\{k \in N \backslash C \mid |k| \leq r\} \subset \bigcup_{\substack{c \in \Gamma^{\#} \\ c_1^2 \leq 2r^{4p+2\epsilon} \\ |c_2| \leq r+1}} J_c.$$

The measure of the latter set is bounded by

$$4 \sum_{c_2=1}^{r} \left(\sqrt{2}\,\frac{r^{2p+\epsilon}}{L} + 2\right) \frac{4Q}{c_2},$$

where L is the length of the shortest non-zero vector in Γ. This proves Lemma 1. □

We now discuss the set M_2. Again for $c \in \Gamma^{\#}$ the intersection of $\{k \in \mathbf{R}^2 \mid |(k+c)^2 - k^2| \leq 2Q\}$ with the line $\mathbf{R}\gamma$ is contained in the interval J_c of length $2Q/|c_2|$ around $(0, -\frac{1}{2}(|c|^2/c_2))$. If $c^2/|c_2|$ is big enough then for any $b \in \Gamma^{\#}$

with $b_2 + c_2 \neq 0$ this interval meets

$$\{k \in \mathbf{R}^2 \mid |(k + b + c)^2 - k^2| \leq 4Q(1 + k^2)^p\}$$

only if

$$\left|\frac{c^2}{c_2} - \frac{(c+b)^2}{c_2+b_2}\right| \leq 6Q \frac{|c+b|^{4p}}{|c_2+b_2|^{1+2p}}.$$

So up to a finite interval $\mathbf{R}\gamma \cap M_2$ is contained in the union of the intervals J_c over all c in the set

$$P := \left\{c \in \Gamma^\# \mid c_2 \neq 0 \text{ and there is } b \in \Gamma^\# \text{ with } b_2 \neq 0,\ b_2 + c_2 \neq 0 \text{ and}\right.$$

$$\left.\left|\frac{c^2}{c_2} - \frac{(c+b)^2}{c_2+b_2}\right| \leq 6Q \frac{|c+b|^{4p}}{|c_2+b_2|^{1+p}},\ |b| \leq h\left(\frac{c^2}{|c_2|}\right) + 1\right\}.$$

Therefore we put for each $b \in \Gamma^\#$

$$P_b := \left\{x \in \mathbf{R}^2 \,\middle|\, \left|\frac{x^2}{x_2} - \frac{(x+b)^2}{x_2+b_2}\right| \leq 6Q \frac{|x+b|^{4p}}{|x_2+b_2|^{1+2p}},\ |x_2| \geq 1,\ |x_2+b_2| \geq 1,\right.$$

$$\left. x^2 \geq |x_2|(h^{-1}(|b|) - 1)\right\}.$$

Then

$$P = \bigcup_{\substack{b \in \Gamma^\# \\ b_2 \neq 0}} (P_b \cap \Gamma^\#).$$

By elementary computation

$$P_b = \left\{x \in \mathbf{R}^2 \,\middle|\, \left|\left(x + \frac{b}{2}\right)\begin{pmatrix} b_2 & -b_1 \\ -b_1 & -b_2 \end{pmatrix}\left(x + \frac{b}{2}\right) + \frac{1}{4} b_2 |b|^2\right|\right.$$

$$\left. \leq 6Q \frac{|x+b|^{4p}}{|x_2+b_2|^{2p}} |x_2|,\ x^2 \geq |x_2|(h^{-1}(|b|) - 1),\ |x_2| \geq 1,\ |x_2+b_2| \geq 1\right\}.$$

LEMMA 2. *Suppose that* $p < \frac{1}{2}$, $\lim_{t\to\infty} (h^{-1}(t)/t^2) = \infty$.

(i) *There is a constant* A *such that for all but finitely many* $b \in \Gamma^{\#}$ *with* $b_2 \neq 0$,

$$P_b \cap \left\{ x \in \mathbf{R}^2 \,\middle|\, |x_2| \leq A \frac{h^{-1}(|b|)}{|b|^2} \right\} = \emptyset.$$

(ii) *There is a constant* μ *such that for all* $b \in \Gamma^{\#}$ *with* $b_2 \neq 0$ *and all* $\eta \in \mathbf{R}$ *with* $|\eta| \geq \mu |b_2|$ *the intersection of* P_b *with the line* $\{x \in \mathbf{R}^2 \mid x_2 = \eta\}$ *is contained in the union of at most two intervals, each of length at most const.* $(|b|^{4p-1}/b_2^{4p})|\eta|^{2p}$. *Here const. is a constant independent of* b *and* η.

Let us first explain how Lemma 1 and Lemma 2 imply Proposition 1. By Lemma 2 and the assumption on h there is a finite set $S \subset \Gamma^{\#}$ such that for all $b \in \Gamma^{\#} \backslash S$ with $b_2 \neq 0$,

$$P_b \cap \left\{ x \in \mathbf{R}^2 \,\middle|\, |x_2| \leq \frac{h^{-1}(|b|)}{|b|^2} \right\} = \emptyset$$

and

$$\mu |b_2| \geq A \frac{h^{-1}(|b|)}{|b|^2}.$$

Put $\rho := \max \{ \mu |b_2| \mid b \in S\}$, and for $b \in \Gamma^{\#}$ with $b_2 \neq 0$,

$$\tilde{P}_b := \left\{ x \in P_b \,\middle|\, |x_2| \geq \max \left(\mu |b_2|, A \frac{h^{-1}(|b|)}{|b|^2} \right) \right\}.$$

Then

$$P \subset \{ x \in \mathbf{R}^2 \mid |x_2| \leq \rho \} \cup \bigcup_{\substack{b \in \Gamma \\ b_2 \neq 0}} \tilde{P}_b.$$

Now by Lemma 2 for each $|\eta| \geq \rho$ and each $b \in \Gamma^{\#}$ with $b_2 \neq 0$, $\tilde{P}_b \cap \{x \in \mathbf{R}^2 \mid x_2 = \eta\}$ contains at most const. $(1 + |b|^{4p-1}/b_2^{4p})|\eta|^{2p}$ points of $\Gamma^{\#}$.

Let $l(t)$ be the inverse function of $Ah^{-1}(t)/t^2$. The assumptions on h imply that $l(t) = O(t^{1/2 - p - \epsilon})$ for some $\epsilon > 0$.

Then for sufficiently large r

$$\sum_{\substack{c \in P \\ J_c \cap B_r \neq \varnothing}} |J_c| \leq \text{const.} \sum_{\substack{c \in \Gamma^{\#} \\ 1 \leq |c_2| \leq \rho \\ |c|^2 \leq 2r|c_2|}} \frac{1}{|c_2|} + \text{const.} \sum_{\substack{b \in \Gamma^{\#} \\ b_2 \neq 0}} \sum_{\substack{c \in \Gamma^{\#} \cap P_b \\ |c|^2 \leq 2r|c_2|}} \frac{1}{|c_2|}.$$

The first sum clearly is $O(r^{1/2})$. By what we said above the second sum is bounded by

$$\text{const.} \sum_{c_2 = 1}^{2r} \sum_{\substack{b \in \Gamma^{\#} \\ |b| \leq l(c_2)}} \left(1 + \frac{|b|^{4p-1}}{|b_2|^{4p}}\right) c_2^{2p-1} \leq$$

$$\leq \text{const.} \sum_{c_2 = 1}^{2r} c_c^{2p-1} l(c_2)^2 = 0(r^{1-\varepsilon})$$

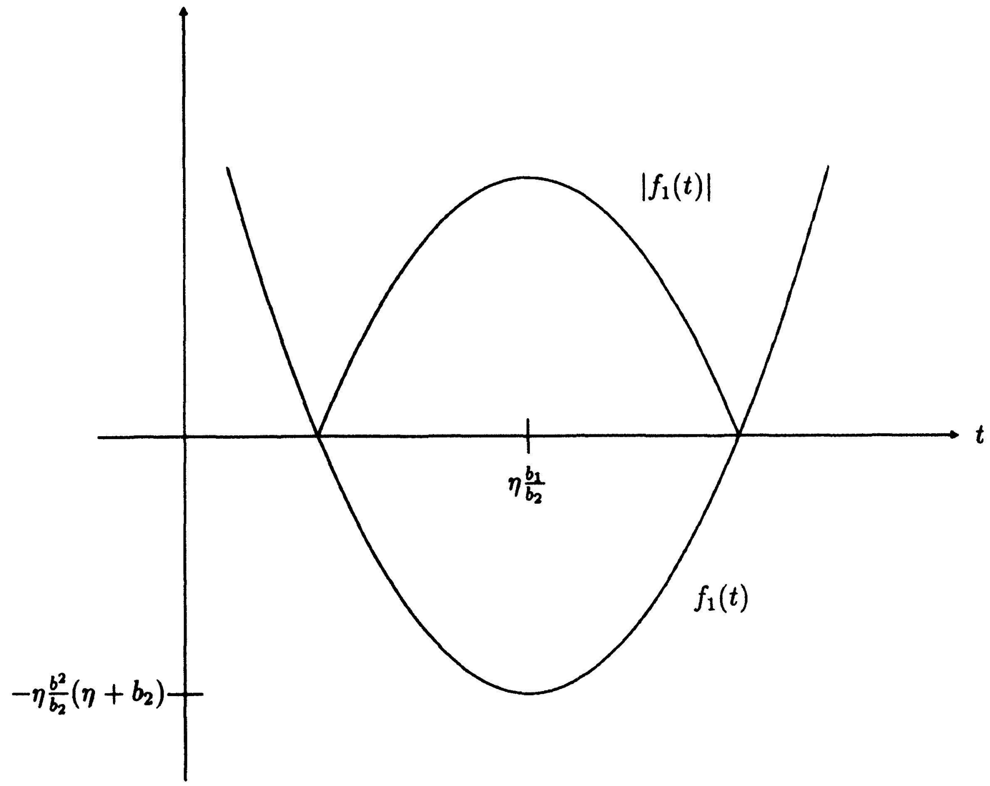
So $|M_2 \cap B_r| = O(r^{1-\varepsilon})$. This, together with Lemma 1, implies Proposition 1. □

We now *prove Lemma 2.* Fix any $\eta \in \mathbf{R}$, $b \in \Gamma^{\#}$ with $b_2 \neq 0$. Without loss of generality we may assume that $b_2 > 0$. Parametrise the line $\{x \in R^2 | x_2 = \eta\}$ by Φ: $t \to (t, \eta)$, and denote by $f_1(t)$, resp. $f_2(t)$, the restriction of the functions $\left(x + \frac{b}{2}\right)\begin{pmatrix} b_2 & -b_1 \\ -b_2 & -b_2 \end{pmatrix}\left(x + \frac{b}{2}\right) + \frac{1}{4} b_2 b^2$, resp. $6Q|x + b|^{4p} \frac{|\eta|}{|\eta + b_2|^{2p}}$, to this line. Then

$$\{t \in \mathbf{R} | \Phi(t) \in P_b\} = \{t \in \mathbf{R} \| f_1(t)| \leq f_2(t) \quad \text{and} \quad t^2 \geq \eta(h^{-1}(|b|) - 1) - \eta^2\}$$

The matrix $\begin{pmatrix} b_2 & -b_1 \\ -b_2 & -b_2 \end{pmatrix}$ has $\pm |b|$ as eigenvalues. Its isotropic subspaces are spanned by the vectors $(b_1 \pm |b|, b_2)$. The zeros of the restriction of $\left(x + \frac{b}{2}\right)\begin{pmatrix} b_2 & -b_1 \\ -b_1 & -b_2 \end{pmatrix}\left(x + \frac{b}{2}\right)$ to $\{x \in \mathbf{R}^2 | x_2 = \eta\}$ are at $t = \eta \frac{b_1}{b_2} \pm |b|\left(\frac{\eta}{b_2} + \frac{1}{2}\right)$. The restriction of $\left(x + \frac{b}{2}\right)\begin{pmatrix} b_2 & -b_1 \\ -b_1 & -b_2 \end{pmatrix}\left(x + \frac{b}{2}\right)$ to the line $\{x \in \mathbf{R}^2 \mid x_2 = \eta\}$ is a quadratic polynomial in t with leading coefficient b_2 and the zeroes described above, so it equals $b_2(t - \eta(b_1/b_2))^2 - b_2 b^2(\eta/b_2 + \frac{1}{2})^2$. Therefore

$$f_1(t) = b_2\left(t - \eta \frac{b_1}{b_2}\right)^2 - \frac{\eta b^2}{b_2}(\eta + b_2).$$



The function

$$f_2(t) = 6Q[(t+b_1)^2 + (\eta+b_2)^2]^{2p} \frac{|\eta|}{(\eta+b_2)^{2p}}$$

is symmetric about $t = -b_1$ and increasing monotonically but slower than quadratically in $|t+b_1|$.

We now show that any intersection point T with $f_1(T) = f_2(T)$ obeys

$$\left|T - \frac{b_1}{b_2}\eta\right| \leq \text{const.} \begin{cases} |b|, & |\eta| \leq 2b_2, \\ \dfrac{|b|}{b_2}|\eta|, & |\eta| \geq 2b_2. \end{cases} \tag{2}$$

To prove (2) we introduce $\tau = T - (b_1/b_2)\eta$ and observe that the equation

$$f_1\left(\tau + \eta\frac{b_1}{b_2}\right) = f_2\left(\tau + \eta\frac{b_1}{b_2}\right),$$

i.e.

$$b_2\tau^2 = \eta\frac{b^2}{b_2}(\eta+b_2) + 6Q\left[\tau^2 + 2\frac{b_1}{b_2}\tau(\eta+b_2) + \frac{|b|^2}{b_2^2}(\eta+b_2)^2\right]^{2p} \frac{|\eta|}{(\eta+b_2)^{2p}}$$

implies

$$\tau^2 \le \frac{\text{const.}}{b_2} \max\left\{\frac{|\eta|}{(\eta+b_2)^{2p}}\tau^{4p},\ |\eta|\left|\frac{b_1}{b_2}\tau\right|^{2p},\ \eta\frac{b^2}{b_2}(\eta+b_2),\ \frac{|b|^{4p}}{b_2^{4p}}|\eta|(\eta+b_2)^{2p}\right\}.$$

When $|\eta| \ge 2b_2$ we get

$$\tau^2 \le \text{const.} \max\left\{\frac{|\eta|^{1-2p}}{b_2}\tau^{4p},\ \frac{|b_1|^{2p}}{|b_2|^{1+2p}}|\eta|\,|\tau|^{2p},\ \frac{b^2}{b_2^2}\eta^2,\ \frac{|b|^{4p}}{b_2^{1+4p}}|\eta|^{1+2p}\right\},$$

which yields

$$\begin{aligned}|\tau| &\le \text{const.} \max\left\{|\eta|^{1/2},\ \left|\frac{b_1}{b_2}\right|^{p/(1-p)}|\eta|^{1/(2-2p)},\ \frac{|b|}{b_2}|\eta|,\ \left(\frac{|b|}{b_2}\right)^{2p}|\eta|^{1/2+p}\right\}\\ &\le \text{const.}\frac{|b|}{b_2}|\eta|.\end{aligned}$$

The case $|\eta| \le 2b_2$ is treated similarly.

The inequality (2) implies that for any $t \in P_b$ $\{x_2 = \eta\}$

$$|\tau| \le |\tau| + \left|\eta\frac{b_1}{b_2}\right| \le C\begin{cases}|b| & \text{if } |\eta| \le 2b_2,\\ \dfrac{|b|}{b_2}|\eta| & \text{if } |\eta| \ge 2b,\end{cases}$$

where the constant C is independent of b and η. So if $P_b \cap \{x \in \mathbf{R}^2 \mid x_2 = \eta\}$ is not empty then

$$|\eta|(h^{-1}(|b|)-1)-\eta^2 \le C^2\begin{cases}|b|^2 & \text{if } |\eta| \le 2b_2,\\ \dfrac{|b|^2}{b_2^2}\eta^2 & \text{if } |\eta| \ge 2b_2.\end{cases}$$

When $1 \le |\eta| \le 2b_2$

$$h^{-1}(|b|) \le (C^2+4)|b|^2+1,$$

which is satisfied only by finitely many b's since $h^{-1}(|b|)/|b|^2$ tends to infinity with $|b|$. When $2b_2 \le |\eta| \le A(h^{-1}(|b|)/|b|^2)$ with $A = 1/(2C^2)$ this would imply

$$h^{-1}(|b|) - 1 - |\eta| \le \frac{\eta^{-1}(|b|)}{2b_2^2},$$

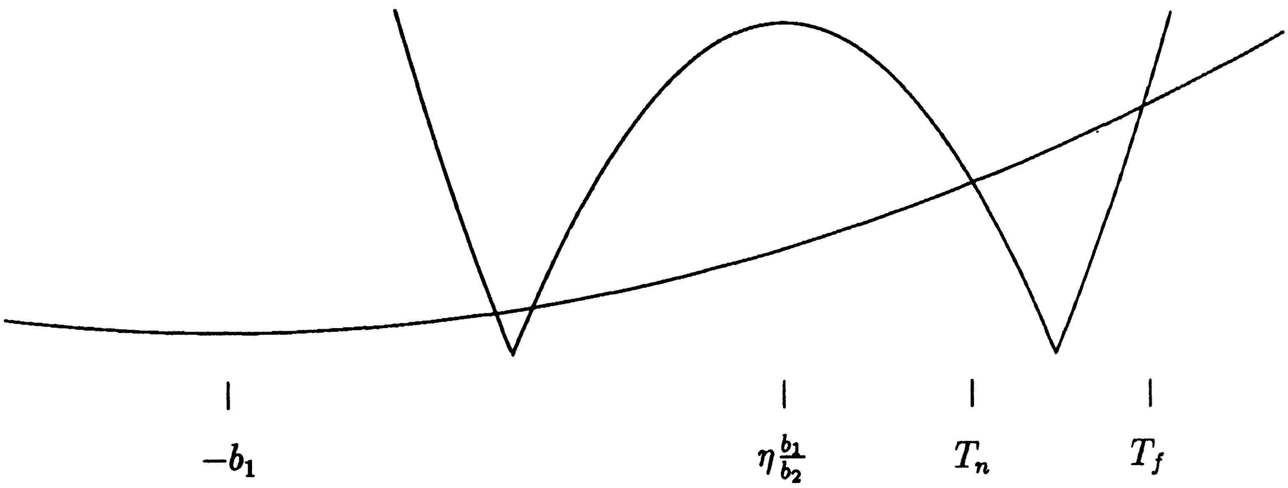
which is impossible. We have thus shown part (i) of Lemma 2.

We now prove part (ii). Assume that $|\eta| \geq \mu b_2$. Observe that

$$f_1\left(\eta\frac{b_1}{b_2}\right) = -\eta\frac{b^2}{b_2}(\eta + b_2) < 0,$$

$$f_2\left(\eta\frac{b_1}{b_2}\right) \leq \left|f_1\left(\eta\frac{b_1}{b_2}\right)\right|$$

provided μ is chosen sufficiently large. (Consequently $P_b \cap \{\eta \in \mathbf{R}^2 \mid x_2 = \eta\}$ is contained in the union of two intervals, one to the left and one to the right of $\eta(b_1/b_2)$. The longer of these two intervals is that on the side of $\eta(b_1/b_2)$ opposite to $-b_1$. See the figure. Define the end points T_n, resp. T_f, of this interval to be the solution of $|f_1(t)| = f_2(t)$ nearest to, resp. farthest, from $\eta(b_1/b_2)$ on the side of



$\eta(b_1/b_2)$ opposite $-b_1$. To bound $|T_f - T_n|$ observe that

$$f_1(T_f) = f_2(T_f),$$

$$f_1(T_n) = -f_2(T_n),$$

$$\Rightarrow f_1(T_f) - f_1(T_n) = f_2(T_f) + f_2(T_n),$$

$$\Rightarrow b_2(T_f - T_n)\left(T_f + T_n - 2\eta\frac{b_1}{b_2}\right) = f_2(T_f) + f_2(T_n),$$

$$\Rightarrow |T_f - T_n| \leq \frac{2}{b_2}\frac{f_2(T_f)}{\left|T_f - \eta\dfrac{b_1}{b_2}\right|}.$$

Setting $\tau = T_f - \eta \frac{b_1}{b_2}$ we have that

$$\text{const.} \frac{|b|}{b_2} |\eta| \leq |\tau| \leq \text{const.} \frac{|b|}{b_2} |\eta|,$$

with the upper bound coming from (2) and the lower bound coming from the fact that T_f is farther from $\eta \frac{b_1}{b_2}$ than the zeroes $\eta \frac{b_1}{b_2} \pm \left[\eta \frac{b^2}{b_2^2}(\eta + b_2)\right]^{1/2}$ of $f_1(t)$. Consequently

$$\begin{aligned} |T_f - T_n| &\leq \text{const} \frac{1}{b_2} \frac{\left[\left(\tau + \eta \frac{b_1}{b_2} + b_1\right)^2 + (\eta + b_2)^2\right]^{2p}}{|\tau|} \frac{|\eta|}{(n + b_2)^{2p}} \\ &\leq \text{const} \frac{1}{b_2} \tau^{4p-1} |\eta|^{1-2p} \\ &\leq \text{const} \frac{|b|^{4p-1}}{b_2^{4p}} |\eta|^{2p} \end{aligned}$$

□

Proof of Proposition 2

Choose a finite set $S \subset \Gamma^*$ such that for all $b \in \Gamma^* \backslash S$ with $b_2 \neq 0$

(i) $b^2 \geq 2m,\ 4 \frac{|b_2|^{\alpha-1}}{(b^2-\mu)^\alpha} \leq m$

(ii) for all $\mu \in [-m, m]$ the intersection of $\{k \in \mathbf{R}^2 \| (k+b)^2 - k^2 - \mu| \leq \frac{1}{|k|^\alpha}$ with $\mathbf{R}\gamma$ is contained in the interval on this axis around this point

$\left(0, \frac{-b^2-\mu}{2b_2}\right)$ of radius $4 \frac{|b_2|^{\alpha-1}}{(b^2-\mu)^\alpha}$.

Then it suffices to show that there is a constant C and that for all $\mu \in [-m, m]$

$$\sum_{\substack{b \in \Gamma^* \backslash S, b_2 \neq 0 \\ I_b \cap B_r \neq \varnothing}} |I_b| \leq C r^{1-\alpha}.$$

The sum under consideration is bounded by

$$8 \sum_{\substack{b \in \Gamma^* \\ \frac{b^2-\mu}{4|b_2|} \leq r+m,}} \frac{|b_2|^{\alpha-1}}{(b^2-\mu)^\alpha} \leq 16 \sum_{\substack{b \in \Gamma^* \\ b^2 \leq 4(r+2m)|b_2|.}} \frac{|b_2|^{\alpha-1}}{|b|^\alpha} \leq O((r+2m)^{1-\alpha}),$$

□

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