

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 66 (1991)

Artikel: Perturbatively unstable eigenvalues of a periodic Schrödinger operator.
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DOI: <https://doi.org/10.5169/seals-50416>

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Perturbatively unstable eigenvalues of a periodic Schrödinger operator

JOEL FELDMAN, HORST KNÖRRER AND EUGENE TRUBOWITZ

1. Introduction

Let Γ be a lattice of maximal rank in \mathbf{R}^d , $d \leq 3$, and

$$\Gamma^* = \{b \in \mathbf{R}^d \mid \langle b, \gamma \rangle \in 2\pi\mathbf{Z} \text{ for all } \gamma \in \Gamma\}$$

the lattice dual to Γ . For $q \in L^2(\mathbf{R}^d/\Gamma)$ and $k \in \mathbf{R}^d$ the spectrum of $-\Delta + q$ acting on the space

$$\mathcal{F}_k = \{\psi \in H_{\text{loc}}^2(\mathbf{R}^d) \mid \psi(x + \gamma) = e^{i\langle k, \gamma \rangle} \psi(x) \text{ for all } \gamma \in \Gamma\},$$

or equivalently, the spectrum of $-\Delta_k + q$, where

$$\Delta_k = \Delta + 2ik \cdot \nabla - k^2$$

acting on

$$L^2(\mathbf{R}^d/\Gamma)$$

is called the Floquet spectrum of q with crystal momentum k . For example, the Floquet spectrum with crystal momentum k when $q = 0$, is the set

$$\{(k + b)^2 \mid b \in \Gamma^*\}.$$

The corresponding eigenfunctions are

$$e^{i\langle k + b, x \rangle}, \quad b \in \Gamma^*.$$

It is shown in [FKT] that for almost every $k \in \mathbf{R}^d$, and any sufficiently regular q , there is a density zero subset $S(k)$ of $k + \Gamma^*$ such that for all $l \in (k + \Gamma^*) - S(k)$

there is exactly one point in the spectrum of $-\Delta_k + q$ lying in the interval

$$\left[l^2 + \int_{\mathbb{R}^d/\Gamma} q \, dx - \frac{1}{|l|^{2-\epsilon}}, l^2 + \int_{\mathbb{R}^d/\Gamma} q \, dx + \frac{1}{|l|^{2-\epsilon}} \right].$$

Moreover, the corresponding eigenfunctions are close to $e^{i\langle l, x \rangle}$. We called the eigenvalues l^2 , $l \in (k + \Gamma^*) - S(k)$ of $-\Delta_k + q$ stable under the perturbation q . The purpose of this paper is to discuss some of the Floquet eigenvalues l^2 , $l \in S(k)$ that are *unstable* under the perturbation q .

We now recall the construction of [ERT] Section 3.b. It yields a class of unstable eigenvalues. Let $\gamma \in \Gamma - \{0\}$, and set

$$\begin{aligned} q_\gamma(x) &= \int_0^1 q(x + s\gamma) \, ds \\ &= \sum_{\substack{b \in \Gamma^* \\ \langle b, \gamma \rangle = 0}} \hat{q}(b) e^{i\langle b, x \rangle}, \end{aligned}$$

where

$$\hat{q}(b) = \frac{1}{|\mathbb{R}^d/\Gamma|} \int_{\mathbb{R}^d/\Gamma} q(x) e^{-i\langle b, x \rangle} \, dx$$

is the “ b ’th” Fourier coefficient of q . The averaged potential $q_\gamma(x)$ is constant on all translates of the line $\mathbf{R} \cdot \gamma$.

Fix $k' \in \mathbf{R}^d$. Let ϕ be an eigenfunction of $-\Delta + q_\gamma(x)$ with crystal momentum k' and eigenvalue μ that is constant on all translates of the line $\mathbf{R} \cdot \gamma$. Then,

$$\psi(x) = e^{it\langle \gamma, x \rangle} \phi(x)$$

is in the space $\mathcal{F}_{(t\gamma + k')}$ and satisfies

$$\frac{1}{\|\psi\|} \|(-\Delta + q)\psi - (t^2\gamma^2 + \mu)\psi\| = O(t^{-2}).$$

The last estimate, combined with the spectral theorem, guarantees that there is a genuine Floquet eigenvalue λ of q with crystal momentum $t\gamma + k'$ close to $t^2\gamma^2 + \mu$. Consequently, the unperturbed eigenvalues l^2 , l near the line $\mathbf{R} \cdot \gamma$, may be moved

far out of the interval

$$\left[l^2 + \int_{\mathbb{R}^d/\Gamma} q \, dx - \frac{1}{|l|^{2-\epsilon}}, l^2 + \int_{\mathbb{R}^d/\Gamma} q \, dx + \frac{1}{|l|^{2-\epsilon}} \right]$$

by μ and are therefore unstable in the sense of [FKT]. This phenomenon is consistent with the observation made in [FKT], Section 4, that points of $k + \Gamma^*$ close to a line $\mathbf{R} \cdot \gamma$ for some $\gamma \in \Gamma$ lie in $S(k)$.

The main object of this paper is to show that for each primitive $\gamma \in \Gamma$ and almost every k' satisfying $\langle k', \gamma \rangle = 0$ and almost every sufficiently large t the “WKB” Floquet eigenvalue λ produced in the last paragraph is bounded away from all other points of the Floquet spectrum of q with crystal momentum $t\gamma + k'$, and that the corresponding eigenfunction is close to the quasimode

$$\psi(x) = e^{it\langle \gamma, x \rangle} \phi(x).$$

We first, using the techniques of [FKT], make the WKB construction above more quantitative, giving estimates for the allowed values of t and the accuracy with which Floquet eigenvalues of q are determined. See, (i) of the Theorem below for a precise statement.

Next, for $d = 2$, counting carefully, it is shown ((ii) of the Theorem) that there is a constant Q , depending only on a norm of q , such that for all k lying in a density one subset of the line $k' + \mathbf{R} \cdot \gamma$ the eigenvalues of q with crystal momentum k in the interval

$$[k^2 - Q, k^2 + Q]$$

are all accounted for by stable eigenvalues of $-\Delta$ and eigenvalues constructed as above from $-\Delta + q_\gamma$.

Finally (part (iii)) for most k , the eigenvalues in the interval $[k^2 - Q, k^2 + Q]$ accounted for by $-\Delta$ are effectively separated from those accounted for by $-\Delta + q_\gamma$. This allows us to estimate how well the true eigenfunctions are approximated by the quasi-modes $\psi(x) = e^{it\langle \gamma, x \rangle} \phi(x)$.

2. Construction of eigenvalues and eigenfunctions

As in [FKT] we introduce a monotonically increasing function $f \geq 1$ on \mathbf{R}_+ such that $f(s)f(t) \geq f(s+t)$ and use the f -weighted l_1 -norm $\|q\|_f = \sum_{b \in \Gamma^*} f(|b|)|\hat{q}(b)|$. Furthermore choose constants $p < \frac{1}{2}$, $Q > 0$. We restrict ourselves to potentials q with mean zero and $\|q\|_f \leq Q$.

THEOREM. *Let γ be a primitive vector of Γ and $k' \in \mathbf{R}^d$ with $\langle k', \gamma \rangle = 0$. Let q be a function on \mathbf{R}^d/Γ with mean zero and $\|q\|_f \leq Q$.*

(i) Let t_0 obey $t_0 \geq 2^{1/2p}$, $|t_0\gamma|^p \geq 1/(2\sqrt{Q})|k'|$ and $|t_0\gamma| \geq ((72Q|\gamma|/\pi) + 12\sqrt{Q}) \cdot ((1+k'^2)^p + |t_0\gamma|^{2p})$. Let μ be any Floquet eigenvalue of $-\Delta + q_\gamma$ (acting on functions on the hyperplane $\{x \in \mathbf{R}^d \mid \langle x, \gamma \rangle = 0\}$) with multiplier k' of finite multiplicity m fulfilling $|\mu - k'^2| \leq Q - \tau$ where

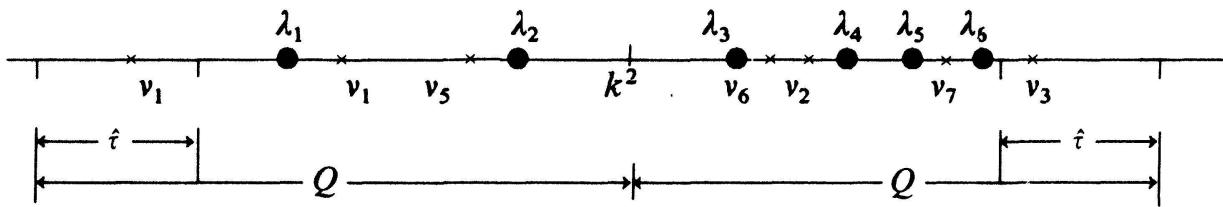
$$\tau := 4Q \left(\frac{1}{|t_0\gamma|^{2p}} + \frac{1}{f(6\sqrt{Q}|t_0\gamma|^p)} \right).$$

Then there are at least m Floquet eigenvalues λ (counted with multiplicity) of $-\Delta + q$ with multiplier $k' + t_0\gamma$ satisfying $|\mu + t_0^2\gamma^2 - \lambda| \leq \tau$.

(ii) Suppose in addition that $d = 2$, $p < 1/2$. Let $h(t) = 1 + \min(t^{1/2(1/2-p)}, t^{2p})$. Then there is a subset $K = K(k', \gamma, Q, p, h)$ of density one in $k' + \mathbf{R}\gamma$ such that for any $k = k' + t_0\gamma \in K$ the following holds. Let $\lambda_1, \dots, \lambda_r$ be Floquet eigenvalues of $-\Delta + q$ with multiplier k in the interval $[k^2 - Q + \hat{\tau}, k^2 + Q - \hat{\tau}]$ where

$$\hat{\tau}(k) = 4Q \left(\frac{1}{|t_0 \gamma|^{2p}} + \frac{1}{f(6\sqrt{Q}|t_0 \gamma|^p)} + \frac{1}{f(h(|k|))} \right).$$

Let μ_1, \dots, μ_m be the Floquet eigenvalues of $-\Delta + q_\gamma$ with multiplier k' in the interval $[k'^2 - Q, k'^2 + Q]$, $v_i := \mu_i + k^2 - k'^2$, and v_{m+1}, \dots, v_n the numbers $(k + b)^2$, $b \in \Gamma^*$ with $\langle b, \gamma \rangle \neq 0$ and $(k + b)^2 \in [k^2 - Q, k^2 + Q]$. All these numbers



are counted with multiplicity. Then there is an injection $\sigma : \{1, \dots, r\} \rightarrow \{1, \dots, n\}$ such that for $i = 1, \dots, r$

$$|\lambda_i - v_{\sigma(i)}| \leq \hat{\tau}.$$

Furthermore v_i is in the image of σ whenever $|v_i - k^2| \leq Q - \hat{\tau}$.

(iii) Suppose that for large t

$$f(6\sqrt{Q} |t\gamma|^p) \geq |t\gamma|^{2p} \quad \text{and} \quad f(h(\sqrt{t})) \geq |t\gamma|^{2p}.$$

Then for any $0 < \delta < 2p$ there is $K' \subset K$ of density one such that for every $k \in K'$ the sets $\bigcup_{j=m+1}^n [v_j - \hat{\tau}, v_j + \hat{\tau}]$ and $[\mu_{i_1} + k^2 - k'^2 - \hat{\tau}, \mu_{i_1} + k^2 - k'^2 + \hat{\tau}]$, $[\mu_{i_2} + k^2 - k'^2 - \hat{\tau}, \mu_{i_2} + k^2 - k'^2 + \hat{\tau}]$, \dots , $[\mu_{i_s} + k^2 - k'^2 - \hat{\tau}, \mu_{i_s} + k^2 - k'^2 + \hat{\tau}]$, where $\mu_{i_1}, \dots, \mu_{i_s}$ runs over the different Floquet eigenvalues of $-\Delta + q_\gamma$ to the multiplier k' , are mutually disjoint and have distance at least $1/|k|^{2p-\delta}$ from each other. If for some $i = 1, \dots, m$ one takes a Floquet eigenvalue λ of $-\Delta + q$ in $[\mu_i + k^2 - k'^2 - \hat{\tau}, \mu_i + k^2 - k'^2 + \hat{\tau}]$ with multiplier k and if ψ is a normalized eigenfunction for that eigenvalue then there is a Floquet eigenfunction ϕ of $-\Delta + q_\gamma$ for the eigenvalue μ_i and multiplier k' that is constant in γ -direction such that $\|\psi - e^{i\langle k - k', x \rangle} \phi\| \leq \text{const. } Q/|t_0 \gamma|^\delta$.

Remarks

(1) In [ERT] and also [KT] it was shown that the Floquet spectrum of $-\Delta + q$ determines that of $-\Delta + q_\gamma$. The proofs given there were non-constructive. For $d = 2$ the theorem above gives a constructive way of determining the Floquet spectrum of $-\Delta + q$ from that of $-\Delta + q_\gamma$. Suppose you want to determine the Floquet eigenvalues of $-\Delta + q_\gamma$ with multiplier k' ($\langle k', \gamma \rangle = 0$) up to accuracy ϵ . By minimax they are contained in $\bigcup_{b \in \Gamma^*, \langle b, \gamma \rangle = 0} [(k' + b)^2 - Q, (k' + b)^2 + Q]$. We show how one determines the desired spectrum up to accuracy ϵ in one of these intervals. Without loss of generality we may assume that this is the interval $[k'^2 - Q, k'^2 + Q]$. Choose R so big that

- (a) the set $\{k' + t\gamma \mid |t||\gamma| \leq R\} \cap K(k', \gamma, Q, p, h)$ has measure at least $3R/2$ in $k' + \mathbf{R}\gamma$.
- (b) For each $\mu \in [-Q, Q]$ the set

$$\{k' + t\gamma \mid |t||\gamma| \leq R, \text{ there is } b \in \Gamma^* \text{ with } \langle b, \gamma \rangle \neq 0 \text{ such that}$$

$$|(k' + t\gamma + b)^2 - (k' + t\gamma)^2 - \mu| \leq 2\hat{\tau}|k' + t\gamma| \}$$

has measure at most $R/2$ in $k' + \mathbf{R}\gamma$.

- (c) $\hat{\tau}|k' + R\gamma| < \epsilon/2$.

It is possible to find such an R by part (ii) of the Theorem above and Proposition 2 of Section 3. We will see that bounded pieces of the set K can be determined by finitely many operations. Similarly the constants involved in Proposition 2 of Section 3 can be estimated in terms of k' , γ and the lattice. So the choice of R is constructive.

Now choose $k_0 \in (k' + \mathbf{R}\gamma) \cap K$ with $|k_0 - k'| \geq R$. By part (ii) of the Theorem the Floquet spectrum of $-\Delta + q_\gamma$ in $[k'^2 - Q, k'^2 + Q]$ is contained in the union of

the intervals of length ϵ around the points $\lambda + k'^2 - k_0^2$, where λ runs over all points of the Floquet spectrum of $-\Delta + q_\gamma$ with multiplier k_0 in $[k_0^2 - Q, k_0^2 + Q]$. To test whether the interval around such a point $\lambda + k'^2 - k_0^2$ actually contains a point of the Floquet spectrum of $-\Delta + q_\gamma$ we proceed as follows. Put $\mu = \lambda - k_0^2$. By (a) and (b) there is $k_1 \in (k' + \mathbf{R}\gamma) \cap K$ with $|k_1 - k'| \leq R$ such that for all $b \in \Gamma^*$ with $\langle b, \gamma \rangle \neq 0$ one has $|(k_1 + b)^2 - k_1^2 - \mu| > 2\hat{\tau}(k_1)$. Again k_1 can be found by finitely many operations. By part (ii) of the Theorem the interval around $\lambda + k'^2 - k_0^2$ of length ϵ contains a point of the Floquet spectrum of $-\Delta + q_\gamma$ if and only if the interval of length $2\hat{\tau}(k_1)$ around the point $(\lambda + k' - k_0^2) + k_1^2 - k'^2 = k_1^2 + \mu$ contains a point of the spectrum of $-\Delta + q$ with multiplier k_1 .

(2) If q is sufficiently regular then the higher terms in the asymptotic expansion for the eigenvalues generated by the WKB-Ansatz (cf. [ERT2]) can also be determined by this method.

(3) With some extra work it should be possible to put all the sets $K(k', \gamma, Q, p)$ together in a subset of full density in a set of the form $\{k' + t\gamma \mid \langle k', \gamma \rangle = 0, |t| \geq C_\gamma \cdot |k'|^N\}$ for some $C_\gamma, N > 0$.

In the proof of the Theorem we use the techniques and results of [FKT]. For $k_0 \in \mathbf{R}^d$ we put $\Delta_{k_0} := \Delta + 2ik_0 \cdot \nabla - k_0^2$. Then $\psi(x)$ is a periodic eigenfunction of $-\Delta_{k_0} + q$ for the eigenvalue λ if and only if the function $e^{i\langle k_0, x \rangle} \psi(x)$ is a Floquet eigenfunction for the eigenvalue λ with multiplier k_0 . We showed in [FKT] that the eigenvalues of $-\Delta_{k_0} + q$ in a neighborhood of k_0^2 are the zeroes of the second regularized determinant of a certain infinite matrix. Precisely for $r > 0$ put

$$G = G_r := \{(k_0 + b) \mid b \in \Gamma^*, |(k_0 + b)^2 - k_0^2| \leq r\}$$

$$R_r := (k^2 \delta_{kl} + \hat{q}(k - l))_{k, l \in G_r}.$$

If r is sufficieintly big then the eigenvalues of $-\Delta_{k_0} + q$ in the interval $[k_0^2 - Q, k_0^2 + Q]$ are the zeroes of \det_2 of

$$\begin{array}{ccc} & \xrightarrow{l \in k_0 + \Gamma^*} & \\ & G_r & \\ \downarrow & G_r \left[\begin{array}{cc} R_r - \lambda & \hat{q}(k - l) \\ \frac{\hat{q}(k - l)}{k^2 - \lambda} & \delta_{kl} + \frac{\hat{q}(k - l)}{k^2 - \lambda} \end{array} \right] & \\ k \in k_0 + \Gamma^* & & \end{array} \quad (1)$$

Furthermore if $(v_k)_{k \in k_0 + \Gamma^*}$ lies in the kernel of this matrix then $\sum_{k \in k_0 + \Gamma^*} v_k e^{i\langle k - k_0, x \rangle}$ is in the kernel of $-\Delta_k + q - \lambda$. As $r \rightarrow \infty$ the eigenvalues and eigenfunctions of R_r approximate those of the whole infinite matrix above.

PROPOSITION. *Assume that $\|q\|_f \leq Q \leq \frac{1}{6}r$.*

(i) *Let $\lambda_1, \dots, \lambda_n$ be eigenvalues (allowing multiplicities) of $-\Delta_{k_0} + q$ that obey $|\lambda_j - k_0^2| \leq Q - 3Q^2/(r - Q)$. Then R_r has at least n eigenvalues (counting multiplicity) in $\bigcup_{j=1}^n [\lambda_j - 3Q^2/(r - Q), \lambda_j + 3Q^2/(r - Q)]$.*

(ii) *Let $\lambda'_1, \dots, \lambda'_n$ be eigenvalues (allowing multiplicities) of R_r that obey $|\lambda'_j - k_0^2| \leq Q - 3Q^2/(r - Q)$. Then $\Delta_{k_0} + q$ has at least n eigenvalues (counting multiplicity) in $\bigcup_{j=1}^n [\lambda'_j - 3Q^2/(r - Q), \lambda'_j + 3Q^2/(r - Q)]$.*

(iii) *Let $I \subset [k_0^2 - Q + 3Q^2/(r - Q), k_0^2 + Q - 3Q^2/(r - Q)]$ be an interval of length ϵ , such that all eigenvalues of $-\Delta_{k_0} + q$ and of R_r , either lie in I or have distance at least ρ from I . Let π resp. π' be the orthogonal projections to $\mathfrak{g} := \bigoplus_{\lambda \in I} \ker(-\Delta_{k_0} + q - \lambda)$ resp. $\mathfrak{g}' := \bigoplus_{\lambda' \in I} \{\sum_{k \in G_r} v_k e^{i\langle k - k_0, x \rangle} \mid v \in \ker(R_r - \lambda')\}$. Then for any $\Psi \in \mathfrak{g}$, $\Psi' \in \mathfrak{g}'$*

$$\frac{\|\Psi' - \pi(\Psi')\|}{\|\Psi'\|} \leq \frac{1}{\rho} \left(\epsilon + \frac{2Q^2}{r - Q} \right) + \frac{2Q}{r - Q},$$

$$\frac{\|\Psi - \pi'(\Psi)\|}{\|\Psi\|} \leq \frac{1}{\rho} \left(\epsilon + \frac{2Q^2}{r - Q} \right) + \frac{2Q}{r - Q}.$$

Proof. We put $W(\lambda) := (\hat{q}(k - l)/(k^2 - \lambda))_{k, l \in (k_0 + \Gamma^*) \setminus G_r}$. Since $|k^2 - \lambda| \geq r - Q$ for all $\lambda \in \Lambda := [k_0^2 - Q, k_0^2 + Q]$ and $k \in k_0 + \Gamma^* \setminus G_r$, one has

$$\|W(\lambda)\|_f \leq \frac{Q}{r - Q} \leq \frac{1}{5}, \quad \left\| \frac{d}{d\lambda} W(\lambda) \right\|_f \leq \frac{Q}{(r - Q)^2} \quad \text{for } \lambda \in \Lambda. \quad (2)$$

(The operator norm $\|\cdot\|_f$ and its properties are introduced in [FKT eq. (3.4)].) In particular $\mathbf{1} + W(\lambda)$ is invertible for $\lambda \in \Lambda$. So the eigenvalues of $-\Delta_{k_0} + q$ in Λ are the zeroes of

$$\det(R_r - \lambda \mathbf{1} - VU),$$

where

$$U := \left(\sum_{k' \in (k_0 + \Gamma^*) \setminus G_r} (\mathbf{1} + W)^{-1}_{k, k'} \cdot \frac{\hat{q}(k' - l)}{k'^2 - \lambda} \right)_{k \in G_r, l \in G_r},$$

$$V := (\hat{q}(k - l))_{k \in G_r, l \in G_r}.$$

Furthermore, for a vector y in the kernel of $R_r - \lambda \mathbf{1} - VU$ the vector $\begin{bmatrix} y \\ -Uy \end{bmatrix}$ lies in the kernel of the matrix (1).

Similar to [FKT], Lemma 3.2, one gets the bounds

$$\begin{aligned} \|U\| &\leq \frac{2Q}{r-Q}, \quad \|VU\| \leq \frac{2Q^2}{r-Q}, \\ \left\| \frac{d}{d\lambda} (VU) \right\| &\leq \frac{Q^3}{(r-Q)^3} + \frac{2Q^2}{(r-Q)^2} \leq \frac{1}{4}. \end{aligned} \tag{3}$$

As in the proof of [FKT], Theorem 3.3, we define the matrix $\tilde{R}(\lambda, v) := R - \lambda \mathbf{1} + vVU$ and call the eigenvalues of this matrix

$$\rho_1(\lambda, v) \leq \rho_2(\lambda, v) \leq \dots \leq \rho_k(\lambda, v).$$

Then

$$\begin{aligned} |\rho_i(\lambda, v) - \rho_i(\lambda, v')| &\leq \frac{2Q^2}{r-Q} |v - v'| \quad \text{for } \lambda \in \Lambda; \quad v, v' \in [0, 1], \\ \rho_i(\lambda, v) - \rho_i(\lambda', v) &\leq -\frac{3}{4}(\lambda - \lambda') \quad \text{for } \lambda \geq \lambda'; \quad \lambda, \lambda' \in \Lambda, \quad v \in [0, 1]. \end{aligned}$$

The zeroes of $\rho_i(\lambda, 0)$ are the eigenvalues of R while the zeroes of $\rho_i(\lambda, 1)$ in Λ are the eigenvalues of $-\Delta_{k_0} + q$ in this interval. The estimates above show that for all $v \in [0, 1]$ the function $\rho_i(\cdot, v)$ has at most one zero in Λ , and that this zero moves with speed at most $\frac{8}{3}(Q^2/(r-Q))$ with v . This proves part (i) and (ii) of the Proposition.

To prove (iii) let $\tilde{\pi}$ resp. $\tilde{\pi}'$ be the orthogonal projections onto $\mathfrak{F} := \bigoplus_{\lambda \in I} \ker(R_r - \lambda \mathbf{1} - V(\lambda)U(\lambda))$ resp. $\mathfrak{F}' := \bigoplus_{\lambda \in I} \ker(R_r - \lambda \mathbf{1})$. First we show that for all $v \in \mathfrak{F}$, $v' \in \mathfrak{F}'$

$$\frac{\|v' - \tilde{\pi}(v')\|}{\|v'\|} \leq \frac{1}{\rho} \left(\epsilon + \frac{2Q^2}{r-Q} \right), \quad \frac{\|v - \tilde{\pi}'(v)\|}{\|v\|} \leq \frac{1}{\rho} \left(\epsilon + \frac{2Q^2}{r-Q} \right). \tag{4}$$

Let for example $v \in \ker(R_r - \lambda \mathbf{1} - V(\lambda)U(\lambda))$ with $\lambda \in I$. Then

$$\|(R_r - \lambda \mathbf{1})(\tilde{\pi}'(v) - v + v)\| \leq \epsilon \|\tilde{\pi}'(v)\|,$$

hence

$$\begin{aligned} \|(R_r - \lambda \mathbf{1})(\tilde{\pi}'(v) - v)\| &\leq \epsilon \|\tilde{\pi}'(v)\| + \|(R_r - \lambda \mathbf{1}) \cdot v\| \\ &\leq \epsilon \|v\| + \|V(\lambda)U(\lambda) \cdot v\| \leq \left(\epsilon + \frac{2Q^2}{r-Q} \right) \|v\| \end{aligned}$$

by (3). Since $v - \tilde{\pi}'(v)$ is orthogonal to \mathfrak{J}' and the norm of $(R_r - \lambda \mathbf{1})^{-1}$ on \mathfrak{J}'^\perp is at most ρ^{-1} this gives the estimate (4).

Since for all $v \in \ker(R_r - \lambda \mathbf{1} - VU)$ the vector $\begin{bmatrix} v \\ -Uv \end{bmatrix}$ lies in the kernel of the matrix (1) and $\|Uv\| \leq (2Q/(r - Q))\|v\|$ by (3) we get the estimates stated in part (iii) of the Proposition. \square

We now proceed to the *proof of the theorem*. Let

$$k = k' + t_0 \gamma \quad \text{in } \mathbf{R}^d.$$

We will apply the Proposition with $k_0 = k$, $r = 4Q(1 + k^2)^p$. Split G_r into the union of

$$\mathcal{L}_1 := \{l \in G_r \mid \langle k - l, \gamma \rangle = 0\},$$

$$\mathcal{L}_2 := \{l \in G_r \mid \langle k - l, \gamma \rangle \neq 0\}.$$

Let $B_i := (l^2 \delta_{lm} + \hat{q}(l - m))_{l, m \in \mathcal{L}_i}$ be the subblock of R_r corresponding to \mathcal{L}_i . The key observation is that $B_1 - (k^2 - k'^2)\mathbf{1}$ is equal to a subblock of the matrix describing $-\Delta_{k'} + q_\gamma$. Precisely put

$$G'_{r'} := \{(k' + b) \mid b \in \Gamma^*, \langle b, \gamma \rangle = 0, |(k' + b)^2 - k'^2| \leq r'\},$$

$$R'_{r'} := (l'^2 \delta_{l'm'} + \hat{q}(l' - m'))_{l', m' \in G'_{r'}}.$$

Then

$$B_1 - (k^2 - k'^2)\mathbf{1} = R'_{r'}$$

and the proposition above also applies to the operator $-\Delta_{k'} + q_\gamma$ and r' . Thus eigenvalues and eigenvectors of B_1 are related to those of $-\Delta_{k'} + q_\gamma$. In order to also relate them to eigenvalues and eigenvectors of R_r (and then of $-\Delta_k + q$) we use that the entries $\hat{q}(l - l')$ of R_r with $l \in \mathcal{L}_1$, $l' \in \mathcal{L}_2$ are small. This will be a consequence of

LEMMA. *Assume that $|k'| \leq 2\sqrt{Q}(1 + k^2)^{p/2}$, and*

$$|t_0 \gamma| \geq 12\sqrt{Q}(1 + k^2)^{p/2} + 72|\gamma|Q(1 + k^2)^p/\pi.$$

Then for all $b \in \Gamma^*$ with $|(k+b)^2 - k^2| \leq 4Q(1+k^2)^p$ one has either

$$\langle b, \gamma \rangle = 0 \quad \text{and} \quad |b| \leq 5\sqrt{Q}(1+k^2)^{p/2}$$

or

$$\langle b, \gamma \rangle \neq 0 \quad \text{and} \quad |b| \geq 6\sqrt{Q}(1+k^2)^{p/2}.$$

In particular for any $l \in \mathcal{L}_1$, $l' \in \mathcal{L}_2$ one has $|l - l'| \geq \sqrt{Q}(1+k^2)^{p/2}$.

Proof. Let $b \in \Gamma^*$ such that $|(k+b)^2 - k^2| \leq 4Q(1+k^2)^p$. First assume that $\langle b, \gamma \rangle = 0$. Then $(k+b)^2 - k^2 = (k'+b)^2 - k'^2$ so that $(k'+b)^2 \leq 4Q(1+k^2)^p + k'^2 \leq 9Q(1+k^2)^p$ so $|b| \leq 3\sqrt{Q}(1+k^2)^{p/2} + |k'| \leq 5\sqrt{Q}(1+k^2)^{p/2}$.

Now assume that $\langle b, \gamma \rangle \neq 0$. Write $b = b' + s\gamma$ with $\langle b', \gamma \rangle = 0$. Since γ is primitive $|s\gamma| \geq 2\pi/|\gamma|$. If $|s\gamma| \geq 6\sqrt{Q}(1+k^2)^{p/2}$ then there is nothing to prove, so assume that $|s\gamma| \leq 6\sqrt{Q}(1+k^2)^{p/2}$. Then

$$(k+b^2) - k^2 = (k'+b')^2 - k'^2 + (t_0 + s)^2\gamma^2 - t_0^2\gamma^2,$$

so

$$\begin{aligned} (k'+b')^2 &\geq |(t_0 + s)^2 - t_0^2|\gamma^2 - k'^2 - 4Q(1+k^2)^p \\ &\geq |2t_0 + s||s|\gamma^2 - 8Q(1+k^2)^p \\ &\geq 2\pi|2t_0 + s| - 8Q(1+k^2)^p \\ &\geq \pi|t_0| - \frac{12\pi\sqrt{Q}}{|\gamma|}(1+k^2)^{p/2} - 8Q(1+k^2)^p \geq 64Q(1+k^2)^p. \end{aligned}$$

Therefore

$$|b'| \geq 8\sqrt{Q}(1+k^2)^{p/2} - |k'| \geq 6\sqrt{Q}(1+k^2)^{p/2}. \quad \square$$

From now on we assume that t_0 fulfills the hypotheses of part (i) of the theorem. Then the lemma above applies.

Put

$$g(t) := 6\sqrt{Q}(1+t)^{p/2}.$$

The lemma above implies that

$$\left\| R_r - \begin{pmatrix} B_1 & 0 \\ 0 & B_2 \end{pmatrix} \right\| \leq \frac{Q}{f(g(k^2))}. \quad (5)$$

Now let μ be a Floquet eigenvalue of $-\Delta_{k'} + q_\gamma$ of multiplicity m fulfilling $|\mu - k'^2| \leq \tau$. By the proposition applied to $-\Delta + q_\gamma$ there are at least m eigenvalues of R'_r in the interval $[\mu - 3Q^2/(r - Q), \mu + 3Q^2/(r - Q)]$. So there are at least m eigenvalues of B_1 in the interval around $\mu + k^2 - k'^2$ of length $3Q^2/(r - Q)$. By (5) there are then at least m eigenvalues of R_r in the interval around $\mu + k^2 - k'^2$ of length $3Q^2/(r - Q) + Q/f(g(k^2))$. Applying the proposition to $-\Delta + q_\gamma$ we see that there are at least m eigenvalues of $-\Delta_{k_0} + q$ satisfying

$$\begin{aligned} |\mu + k^2 - k'^2 - \lambda| &\leq \frac{6Q^2}{r - Q} + \frac{Q}{f(g(k^2))} \\ &\leq 4Q \left(\frac{1}{|t_0\gamma|^{2p}} + \frac{1}{f(6\sqrt{Q}|t_0\gamma|^p)} \right) = \tau. \end{aligned}$$

This proves part (i) of the theorem.

For part (ii) we put

$$M := \{k \in \mathbf{R}^d \mid \text{there are } c \neq c' \text{ in } \Gamma^* \text{ with } \langle c, \gamma \rangle \neq 0, \langle c', \gamma \rangle \neq 0 \text{ such that}$$

$$|(k + c)^2 - k^2| \leq 2Q, |(k + c')^2 - k^2| \leq 4Q(1 + k^2)^p \text{ and } |c - c'| \leq h(|k|)\}.$$

Then we define K as the intersection of $\{k = k' + t\gamma \mid |t| \geq 2^{1/2p}, |t\gamma|^p \geq 1/(2\sqrt{Q})|k'|, |t\gamma| \geq ((72/\pi)|\gamma| + 12\sqrt{Q})((1 + k'^2)^p + |t\gamma|^{2p})\}$ with $\mathbf{R}^d \setminus M$. In Section 3, Proposition 1, we show that

$$|\{k \in (k' + \mathbf{R}\gamma) \cap M \mid |k - k'| \leq s\}| = O(s^{1-\epsilon})$$

for some $\epsilon > 0$, so K is of density one in $k' + \mathbf{R}\gamma$.

Now assume that $k = k' + t_0\gamma$ lies in K . We keep the notation used in the proof of part (i) of the theorem. Put

$$\mathcal{L}'_2 := \{l \in \mathcal{L}_2 \mid |l^2 - k^2| \leq 2Q\}, \quad \mathcal{L}''_2 := \mathcal{L}_2 \setminus \mathcal{L}'_2,$$

and let B'_2 resp. B''_2 be the subblocks of B_2 corresponding to \mathcal{L}'_2 resp. \mathcal{L}''_2 . Furthermore let D be the diagonal part of B'_2 . Since for all $l \in \mathcal{L}_2, l' \in \mathcal{L}'_2$ one has $|l - l'| \geq h(|k|)$, by the definition of M

$$\left\| \begin{pmatrix} \mathcal{L}'_2 & \mathcal{L}''_2 \\ D & 0 \\ 0 & B''_2 \end{pmatrix} - B_2 \right\| \leq \frac{Q}{f(h(|k|))},$$

hence

$$\left\| R_r - \begin{pmatrix} B_1 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & B''_2 \end{pmatrix} \right\| \leq \frac{Q}{f(g(k^2))} + \frac{Q}{f(h(|k|))}. \quad (6)$$

By minimax B_2'' has no eigenvalues in $[k^2 - Q, k^2 + Q]$. Therefore the eigenvalues of $\begin{pmatrix} B_1 & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & B_2'' \end{pmatrix}$ in the interval $[k^2 - Q, k^2 + Q]$ are the eigenvalues of B_1 in this

interval, and the numbers $(k + b)^2$, $b \in \Gamma^*$, $\langle b, \gamma \rangle \neq 0$ that lie in this interval. We already know that the eigenvalues of B_1 in the interval under consideration are obtained from the eigenvalues of $-\Delta + q_\gamma$ by adding $k^2 - k'^2$ and shifting by at most $3Q^2/(r - Q)$. Similarly the eigenvalues of R_r are obtained from those of $-\Delta + q$ by shifting by at most $3Q^2/(r - Q)$. This yields part (ii) of the theorem.

To prove part (iii) put

$$K'_i := \left\{ k \in k' + \mathbf{R}\gamma \mid \text{for all } b \in \Gamma^* \text{ with } \langle b, \gamma \rangle \neq 0 \text{ one has} \right.$$

$$\left. |(k + b)^2 - k^2 + k'^2 - \mu_i| \geq \frac{1}{|k|^{2p-\delta}} + 2\hat{\tau}(k) \right\},$$

$$K' := \bigcap_{i=1}^m K'_i \cap K.$$

In Section 3 we will show that each K'_i and hence also K' has density one in $k' + \mathbf{R}\gamma$ (Proposition 2 of Section 3). Now suppose that $k \in K'$ is big enough that $1/|k|^{2p-\delta} + 2\hat{\tau}(k) \leq |\mu_i - \mu_j|$ for all i, j such that $\mu_i \neq \mu_j$. Then the first statement of part (iii) of the Theorem is trivially true.

Now let $\lambda \in [\mu_i + k^2 - k'^2 - \hat{\tau}, \mu_i + k^2 - k'^2 + \hat{\tau}]$ and $\tilde{\Psi}(x) = \sum_{l \in k + \Gamma^*} v_l e^{i\langle l - k, x \rangle}$ be a unit vector in $\ker(-\Delta_k + q - \lambda)$. Put $I := [\mu_i - \hat{\tau}, \mu_i + \hat{\tau}]$ and $\mathfrak{J}' := \bigoplus_{\mu' \in I} \ker(R'_r - \mu')$. Then $v := (v_l)_{l \in G_r}$ is an eigenvector of $R_r - VU$ to the eigenvalue λ and $\|(v_l)_{l \in (k + \Gamma^*) \setminus G_r}\| = \|Uv\| \leq \hat{\tau}$. Put $I := [\mu_i - \hat{\tau}, \mu_i + \hat{\tau}]$ and $\mathfrak{J} := \bigoplus_{\mu \in I} \ker(R'_r - \mu)$. Let w be the projection of v onto \mathfrak{J} . Then

$$\left\| \left(\begin{pmatrix} B_1 & D & B_2'' \\ & & \end{pmatrix} - v_i \right) w \right\| \leq \hat{\tau} \|w\|.$$

Since all eigenvalues of $\begin{pmatrix} B_1 & D & B_2'' \\ & & \end{pmatrix}$ that lie in $[k^2 - Q, k^2 + Q]$ are actually contained in $\bigcup_{i=1}^n [\mu_i - \hat{\tau}, \mu_i + \hat{\tau}]$

$$\left\| \left(\begin{pmatrix} B_1 & D & B_2'' \\ & & \end{pmatrix} - v_i \right) (v - w) \right\| \geq \frac{1}{|k|^{2p-\delta}} \|(v - w)\|.$$

As in the proof of part (iii) of the proposition we get (using (3) and (6)) that

$$\|v - w\| \leq 4\hat{\tau}|k|^{2p-\delta}.$$

After part (iii) of the proposition there is $\tilde{\phi} \in \ker(-\Delta_{k'} + q_\gamma - \mu_i)$ such that $\|\tilde{\phi} - \sum_{l \in G_r} w_l e^{i\langle l - k', x \rangle}\| \leq 4\hat{\tau}|k|^{2p-\delta} + \hat{\tau}$. Hence

$$\|\tilde{\phi} - \tilde{\Psi}\| \leq 8\hat{\tau}|k|^{2p-\delta} + 2\hat{\tau}.$$

Under the hypotheses in the theorem $\hat{\tau} \leq 12Q(1/|t_0\gamma|^{2p})$. So if t_0 was chosen big enough we get the claimed estimate. \square

3. Lattice properties

In Section 2 we used two purely lattice theoretic results, which we are going to prove now. As before fix $0 \leq p \leq \frac{1}{2}$ and $Q > 0$, and choose a monotonically increasing function $h(t) \geq 1$. With this notation put

$$\begin{aligned} M(P, Q, h) := \{k \in \mathbf{R}^2 \mid \exists b, c \in \Gamma^* \text{ with } \langle c, \gamma \rangle \neq 0, \langle b + c, \gamma \rangle \neq 0 \text{ such that} \\ |(k + c)^2 - k^2| \leq 2Q, b \neq 0, |(k + b + c)^2 - k^2| \leq 4Q(1 + k^2)^p \\ \text{and } |b| \leq h(|k|)\}. \end{aligned}$$

PROPOSITION 1. *Assume that $p < \frac{1}{2}$, $h(t) = O(\min(t^{1/2(1/2-p)}, t^{2p}))$. Then for each $k' \in \mathbf{R}^2$*

$$|\{k \in k' + \mathbf{R}\gamma \mid |k| \leq r\} \cap M(p, Q, h)| = O(r^{1-\epsilon})$$

for some $\epsilon > 0$.

The other result we needed can be phrased as follows. For any $0 \leq \alpha < 1$ and $\mu \in \mathbf{R}$ put

$$\begin{aligned} M'(\alpha, \mu) := \left\{ k \in \mathbf{R}^2 \mid \text{there is } b \in \Gamma^* \text{ with } \langle b, \gamma \rangle \neq 0 \text{ such that} \right. \\ \left. |(k + b)^2 - k^2 - \mu| \leq \frac{1}{|k|^\alpha} \right\}. \end{aligned}$$

PROPOSITION 2. *Let $k' \in \mathbf{R}^2$ and $m > 0$. Then there is a constant $C > 0$ such that for all $\mu \in \mathbf{R}$ with $|\mu| \leq m$*

$$|\{k \in k' + \mathbf{R}\gamma \mid |k| \leq r\} \cap M'(\alpha, \mu)| \leq C \cdot (1 + r^{1-\alpha}).$$

Remark. The proofs given below are constructive, i.e. each bounded piece of the sets $M(p, Q, h)$ resp. $M'(\alpha, \mu)$ can be determined by finitely many operations.

For the proof of Proposition 1 and Proposition 2 we may, after rotating and scaling the lattice, assume that $\gamma = (0, 2\pi)$. We prove the propositions in the case $k' = 0$, the general case is similar. To simplify notation write $B_r := \{x \in \mathbf{R}^2 \mid |x| \leq r\}$.

Proof of Proposition 1. Split $M(p, Q, h)$ into the union of

$$M_1(p, Q, h) := \{k \in \mathbf{R}^2 \mid \exists b, c \in \Gamma^* \text{ with } b_2 = 0, b \neq 0, c_2 \neq 0 \text{ and } |b| \leq h(|k|), \\ |(k + c)^2 - k^2| \leq 2Q, |(k + b + c)^2 - k^2| \leq 4Q(1 + |k|^2)^p\},$$

and

$$M_2(p, Q, h) := \{k \in \mathbf{R}^2 \mid \exists b, c \in \Gamma^* \text{ with } b_2 \neq 0, c_2 \neq 0, b_2 + c_2 \neq 0 \text{ and } |b| \leq h(|k|), \\ |(k + c)^2 - k^2| \leq 2Q, |(k + b + c)^2 - k^2| \leq 4Q(1 + |k|^2)^p\}.$$

LEMMA 1. *Suppose that $h(t) \leq t^{2p}$. Then for any $\epsilon > 0$*

$$|\mathbf{R}\gamma \cap M_1(p, Q, h) \cap B_r| = O(r^{2p+\epsilon}).$$

Proof. Take $\epsilon > 0$ and put

$$N := \{k \in \mathbf{R}\gamma \mid \exists c \in \Gamma^* \setminus \{0\} \text{ such that } |(k + c)^2 - k^2| \leq 2Q \text{ and} \\ |(k_2 + c_2)^2 - k_2^2| \leq |k|^{4p+2\epsilon}\}.$$

Below we show that $|\{k \in \mathbf{N} \mid |k| \leq r\}| = O(r^{2p+\epsilon})$. We claim that there is an $R > 0$ such that $M_1 \cap \{k \in \mathbf{R}\gamma \mid |k| \geq R\} \subset N$. So suppose that $k \in \mathbf{R}\gamma \cap M_1$ but $k \notin N$. By definition there are $b, c \in \Gamma^*$ with $b_2 = 0, c_2 \neq 0$ and $|b| \leq h(|k|)$ such that with $l := k + c$

$$|l^2 - k^2| \leq 2Q \quad \text{and} \quad |(l + b)^2 - l^2| \leq 6Q(1 + k^2)^p.$$

Since $k \notin N$ this implies

$$|l_2^2 - k_2^2| \geq |k|^{4p+2\epsilon}$$

and therefore

$$l_1^2 \geq |k|^{4p+2\epsilon} - 2Q. \quad (1)$$

On the other hand, the inequality $|(l+b)^2 - l^2| \leq 6Q(1+k^2)^p$ gives

$$|2l_1 + b_1| \leq \frac{6Q}{|b_1|} (1+k^2)^p.$$

Since $|b_1| \leq h(|k|) \leq |k|^{2p}$ we get

$$|2l_1| \leq \frac{6Q}{|b_1|} (1+k^2)^p + |k|^{2p},$$

which is a contradiction to (1) whenever k is big enough.

It remains to prove the estimate for N . For each $c \in \Gamma^* \setminus \{0\}$ the intersection of $\{k \in \mathbf{R}^2 \mid |(k+c)^2 - k^2| \leq 2Q\}$ with the line $\mathbf{R}y$ is contained in the interval J_c of length $2Q/|c_2|$ around the point $(0, -\frac{1}{2}(|c|^2/c_2))$. The inequalities $|(k_2 + c_2)^2 - k_2^2| \leq |k|^{4p+2\epsilon}$ and $|(k+c)^2 - k^2| \leq 2Q$ imply $c_1^2 \leq |k|^{4p+2\epsilon} + 2Q$. Therefore there is a compact subset C of N such that for all $r > 0$

$$\{k \in N \setminus C \mid |k| \leq r\} \subset \bigcup_{\substack{c \in \Gamma^* \\ c_1^2 \leq 2r^{4p+2\epsilon} \\ |c_2| \leq r+1}} J_c.$$

The measure of the latter set is bounded by

$$4 \sum_{c_2=1}^r \left(\sqrt{2} \frac{r^{2p+\epsilon}}{L} + 2 \right) \frac{4Q}{c_2},$$

where L is the length of the shortest non-zero vector in Γ . This proves Lemma 1. \square

We now discuss the set M_2 . Again for $c \in \Gamma^*$ the intersection of $\{k \in \mathbf{R}^2 \mid |(k+c)^2 - k^2| \leq 2Q\}$ with the line $\mathbf{R}y$ is contained in the interval J_c of length $2Q/|c_2|$ around $(0, -\frac{1}{2}(|c|^2/c_2))$. If $c^2/|c_2|$ is big enough then for any $b \in \Gamma^*$

with $b_2 + c_2 \neq 0$ this interval meets

$$\{k \in \mathbf{R}^2 \mid |(k + b + c)^2 - k^2| \leq 4Q(1 + k^2)^p\}$$

only if

$$\left| \frac{c^2}{c_2} - \frac{(c + b)^2}{c_2 + b_2} \right| \leq 6Q \frac{|c + b|^{4p}}{|c_2 + b_2|^{1+2p}}.$$

So up to a finite interval $\mathbf{R}\gamma \cap M_2$ is contained in the union of the intervals J_c over all c in the set

$$P := \left\{ c \in \Gamma^* \mid c_2 \neq 0 \text{ and there is } b \in \Gamma^* \text{ with } b_2 \neq 0, b_2 + c_2 \neq 0 \text{ and} \right. \\ \left. \left| \frac{c^2}{c_2} - \frac{(c + b)^2}{c_2 + b_2} \right| \leq 6Q \frac{|c + b|^{4p}}{|c_2 + b_2|^{1+p}}, |b| \leq h\left(\frac{c^2}{|c_2|}\right) + 1 \right\}.$$

Therefore we put for each $b \in \Gamma^*$

$$P_b := \left\{ x \in \mathbf{R}^2 \mid \left| \frac{x^2}{x_2} - \frac{(x + b)^2}{x_2 + b_2} \right| \leq 6Q \frac{|x + b|^{4p}}{|x_2 + b_2|^{1+2p}}, |x_2| \geq 1, |x_2 + b_2| \geq 1, \right. \\ \left. x^2 \geq |x_2|(h^{-1}(|b|) - 1) \right\}.$$

Then

$$P = \bigcup_{\substack{b \in \Gamma^* \\ b_2 \neq 0}} (P_b \cap \Gamma^*).$$

By elementary computation

$$P_b = \left\{ x \in \mathbf{R}^2 \mid \left| \left(x + \frac{b}{2} \right) \begin{pmatrix} b_2 & -b_1 \\ -b_1 & -b_2 \end{pmatrix} \left(x + \frac{b}{2} \right) + \frac{1}{4} b_2 |b|^2 \right| \right. \\ \left. \leq 6Q \frac{|x + b|^{4p}}{|x_2 + b_2|^{2p}} |x_2|, x^2 \geq |x_2|(h^{-1}(|b|) - 1), |x_2| \geq 1, |x_2 + b_2| \geq 1 \right\}.$$

LEMMA 2. Suppose that $p < \frac{1}{2}$, $\lim_{t \rightarrow \infty} (h^{-1}(t)/t^2) = \infty$.

(i) There is a constant A such that for all but finitely many $b \in \Gamma^*$ with $b_2 \neq 0$,

$$P_b \cap \left\{ x \in \mathbf{R}^2 \mid |x_2| \leq A \frac{h^{-1}(|b|)}{|b|^2} \right\} = \emptyset.$$

(ii) There is a constant μ such that for all $b \in \Gamma^*$ with $b_2 \neq 0$ and all $\eta \in \mathbf{R}$ with $|\eta| \geq \mu|b_2|$ the intersection of P_b with the line $\{x \in \mathbf{R}^2 \mid x_2 = \eta\}$ is contained in the union of at most two intervals, each of length at most $\text{const.} (|b|^{4p-1}/b_2^{4p})|\eta|^{2p}$. Here const. is a constant independent of b and η .

Let us first explain how Lemma 1 and Lemma 2 imply Proposition 1. By Lemma 2 and the assumption on h there is a finite set $S \subset \Gamma^*$ such that for all $b \in \Gamma^* \setminus S$ with $b_2 \neq 0$,

$$P_b \cap \left\{ x \in \mathbf{R}^2 \mid |x_2| \leq \frac{h^{-1}(|b|)}{|b|^2} \right\} = \emptyset$$

and

$$\mu|b_2| \geq A \frac{h^{-1}(|b|)}{|b|^2}.$$

Put $\rho := \max \{ \mu|b_2| \mid b \in S \}$, and for $b \in \Gamma^*$ with $b_2 \neq 0$,

$$\tilde{P}_b := \left\{ x \in P_b \mid |x_2| \geq \max \left(\mu|b_2|, A \frac{h^{-1}(|b|)}{|b|^2} \right) \right\}.$$

Then

$$P \subset \{x \in \mathbf{R}^2 \mid |x_2| \leq \rho\} \cup \bigcup_{\substack{b \in \Gamma \\ b_2 \neq 0}} \tilde{P}_b.$$

Now by Lemma 2 for each $|\eta| \geq \rho$ and each $b \in \Gamma^*$ with $b_2 \neq 0$, $\tilde{P}_b \cap \{x \in \mathbf{R}^2 \mid x_2 = \eta\}$ contains at most $\text{const.} (1 + |b|^{4p-1}/b_2^{4p})|\eta|^{2p}$ points of Γ^* .

Let $l(t)$ be the inverse function of $Ah^{-1}(t)/t^2$. The assumptions on h imply that $l(t) = O(t^{1/2-p-\epsilon})$ for some $\epsilon > 0$.

Then for sufficiently large r

$$\sum_{\substack{c \in P \\ J_c \cap B_r \neq \emptyset}} |J_c| \leq \text{const.} \sum_{\substack{c \in \Gamma^* \\ 1 \leq |c_2| \leq \rho \\ |c|^2 \leq 2r|c_2|}} \frac{1}{|c_2|} + \text{const.} \sum_{\substack{b \in \Gamma^* \\ b_2 \neq 0}} \sum_{\substack{c \in \Gamma^* \\ |c|^2 \leq 2r|c_2|}} \frac{1}{|c_2|}.$$

The first sum clearly is $O(r^{1/2})$. By what we said above the second sum is bounded by

$$\begin{aligned} \text{const.} \sum_{c_2=1}^{2r} \sum_{\substack{b \in \Gamma^* \\ |b| \leq l(c_2)}} \left(1 + \frac{|b|^{4p-1}}{|b_2|^{4p}} \right) c_2^{2p-1} &\leq \\ \leq \text{const.} \sum_{c_2=1}^{2r} c_2^{2p-1} l(c_2)^2 &= O(r^{1-\varepsilon}) \end{aligned}$$

So $|M_2 \cap B_r| = O(r^{1-\varepsilon})$. This, together with Lemma 1, implies Proposition 1. \square

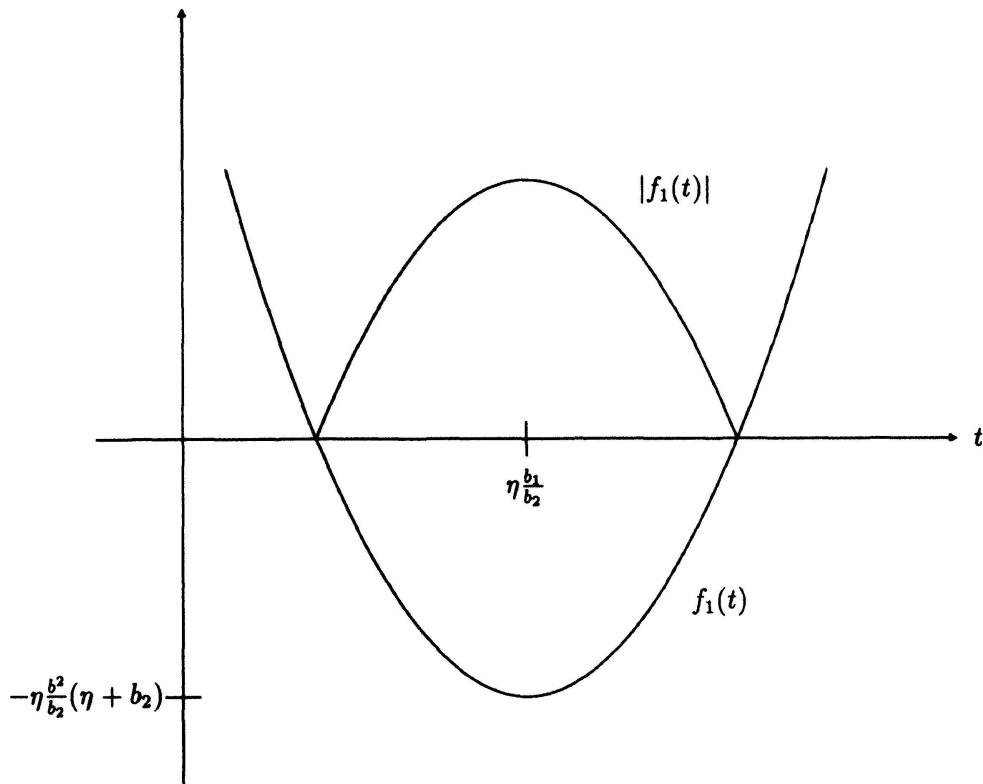
We now prove Lemma 2. Fix any $\eta \in \mathbf{R}$, $b \in \Gamma^*$ with $b_2 \neq 0$. Without loss of generality we may assume that $b_2 > 0$. Parametrise the line $\{x \in \mathbf{R}^2 | x_2 = \eta\}$ by Φ : $t \rightarrow (t, \eta)$, and denote by $f_1(t)$, resp. $f_2(t)$, the restriction of the functions $\left(x + \frac{b}{2}\right) \begin{pmatrix} b_2 & -b_1 \\ -b_2 & -b_2 \end{pmatrix} \left(x + \frac{b}{2}\right) + \frac{1}{4} b_2 b^2$, resp. $6Q|x + b|^{4p} \frac{|\eta|}{|\eta + b_2|^{2p}}$, to this line.

Then

$$\{t \in \mathbf{R} | \Phi(t) \in P_b\} = \{t \in \mathbf{R} | |f_1(t)| \leq f_2(t) \text{ and } t^2 \geq \eta(h^{-1}(|b|) - 1) - \eta^2\}$$

The matrix $\begin{pmatrix} b_2 & -b_1 \\ -b_2 & -b_2 \end{pmatrix}$ has $\pm |b|$ as eigenvalues. Its isotropic subspaces are spanned by the vectors $(b_1 \pm |b|, b_2)$. The zeros of the restriction of $\left(x + \frac{b}{2}\right) \begin{pmatrix} b_2 & -b_1 \\ -b_1 & -b_2 \end{pmatrix} \left(x + \frac{b}{2}\right)$ to $\{x \in \mathbf{R}^2 | x_2 = \eta\}$ are at $t = \eta \frac{b_1}{b_2} \pm |b| \left(\frac{\eta}{b_2} + \frac{1}{2}\right)$. The restriction of $\left(x + \frac{b}{2}\right) \begin{pmatrix} b_2 & -b_1 \\ -b_1 & -b_2 \end{pmatrix} \left(x + \frac{b}{2}\right)$ to the line $\{x \in \mathbf{R}^2 | x_2 = \eta\}$ is a quadratic polynomial in t with leading coefficient b_2 and the zeroes described above, so it equals $b_2(t - \eta(b_1/b_2))^2 - b_2 b^2(\eta/b_2 + \frac{1}{2})^2$. Therefore

$$f_1(t) = b_2 \left(t - \eta \frac{b_1}{b_2} \right)^2 - \frac{\eta b^2}{b_2} (\eta + b_2).$$



The function

$$f_2(t) = 6Q[(t + b_1)^2 + (\eta + b_2)^2]^{2p} \frac{|\eta|}{(\eta + b_2)^{2p}}$$

is symmetric about $t = -b_1$ and increasing monotonically but slower than quadratically in $|t + b_1|$.

We now show that any intersection point T with $f_1(T) = f_2(T)$ obeys

$$\left| T - \frac{b_1}{b_2} \eta \right| \leq \text{const.} \begin{cases} |b|, & |\eta| \leq 2b_2, \\ \frac{|b|}{b_2} |\eta|, & |\eta| \geq 2b_2. \end{cases} \quad (2)$$

To prove (2) we introduce $\tau = T - (b_1/b_2)\eta$ and observe that the equation

$$f_1\left(\tau + \eta \frac{b_1}{b_2}\right) = f_2\left(\tau + \eta \frac{b_1}{b_2}\right),$$

i.e.

$$b_2 \tau^2 = \eta \frac{b^2}{b_2} (\eta + b_2) + 6Q \left[\tau^2 + 2 \frac{b_1}{b_2} \tau (\eta + b_2) + \frac{|b|^2}{b_2^2} (\eta + b_2)^2 \right]^{2p} \frac{|\eta|}{(\eta + b_2)^{2p}}$$

implies

$$\tau^2 \leq \frac{\text{const.}}{b_2} \max \left\{ \frac{|\eta|}{(\eta + b_2)^{2p}} \tau^{4p}, |\eta| \left| \frac{b_1}{b_2} \tau \right|^{2p}, \eta \frac{b^2}{b_2} (\eta + b_2), \frac{|b|^{4p}}{b_2^{4p}} |\eta| (\eta + b_2)^{2p} \right\}.$$

When $|\eta| \geq 2b_2$ we get

$$\tau^2 \leq \text{const.} \max \left\{ \frac{|\eta|^{1-2p}}{b_2} \tau^{4p}, \frac{|b_1|^{2p}}{|b_2|^{1+2p}} |\eta| |\tau|^{2p}, \frac{b^2}{b_2^2} \eta^2, \frac{|b|^{4p}}{b_2^{1+4p}} |\eta|^{1+2p} \right\},$$

which yields

$$\begin{aligned} |\tau| &\leq \text{const.} \max \left\{ |\eta|^{1/2}, \left| \frac{b_1}{b_2} \right|^{p/(1-p)} |\eta|^{1/(2-2p)}, \frac{|b|}{b_2} |\eta|, \left(\frac{|b|}{b_2} \right)^{2p} |\eta|^{1/2+p} \right\} \\ &\leq \text{const.} \frac{|b|}{b_2} |\eta|. \end{aligned}$$

The case $|\eta| \leq 2b_2$ is treated similarly.

The inequality (2) implies that for any $t \in P_b \setminus \{x_2 = \eta\}$

$$|\tau| \leq |\tau| + \left| \eta \frac{b_1}{b_2} \right| \leq C \begin{cases} \frac{|b|}{b_2} & \text{if } |\eta| \leq 2b_2, \\ \frac{|b|}{b_2} |\eta| & \text{if } |\eta| \geq 2b_2, \end{cases}$$

where the constant C is independent of b and η . So if $P_b \cap \{x \in \mathbf{R}^2 \mid x_2 = \eta\}$ is not empty then

$$|\eta| (h^{-1}(|b|) - 1) - \eta^2 \leq C^2 \begin{cases} \frac{|b|^2}{b_2^2} & \text{if } |\eta| \leq 2b_2, \\ \frac{|b|^2}{b_2^2} \eta^2 & \text{if } |\eta| \geq 2b_2. \end{cases}$$

When $1 \leq |\eta| \leq 2b_2$

$$h^{-1}(|b|) \leq (C^2 + 4)|b|^2 + 1,$$

which is satisfied only by finitely many b 's since $h^{-1}(|b|)/|b|^2$ tends to infinity with $|b|$. When $2b_2 \leq |\eta| \leq A(h^{-1}(|b|)/|b|^2)$ with $A = 1/(2C^2)$ this would imply

$$h^{-1}(|b|) - 1 - |\eta| \leq \frac{\eta^{-1}(|b|)}{2b_2^2},$$

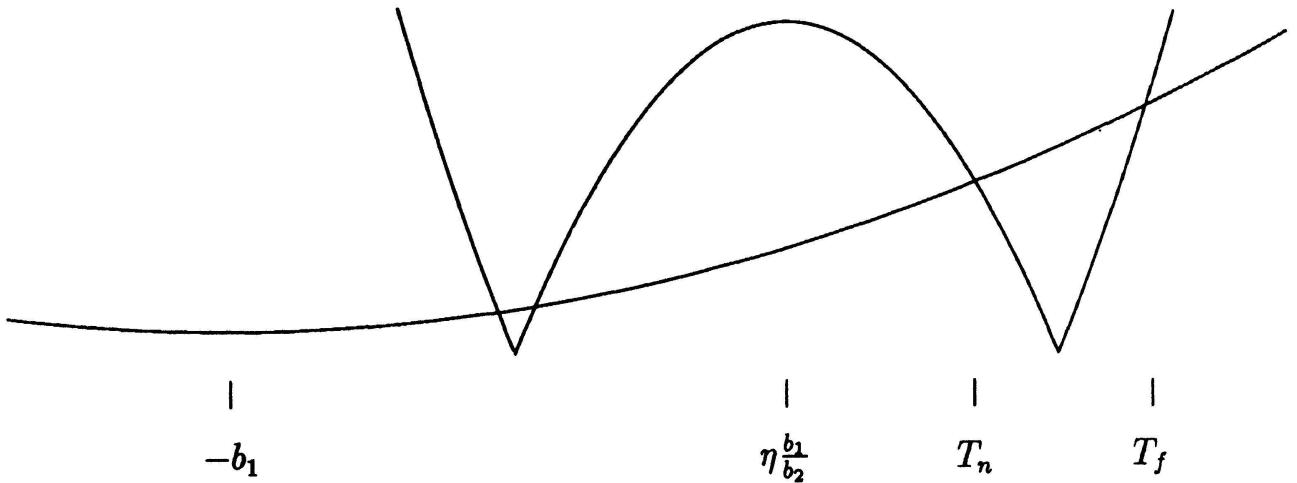
which is impossible. We have thus shown part (i) of Lemma 2.

We now prove part (ii). Assume that $|\eta| \geq \mu b_2$. Observe that

$$f_1\left(\eta \frac{b_1}{b_2}\right) = -\eta \frac{b^2}{b_2}(\eta + b_2) < 0,$$

$$f_2\left(\eta \frac{b_1}{b_2}\right) \leq \left|f_1\left(\eta \frac{b_1}{b_2}\right)\right|$$

provided μ is chosen sufficiently large. (Consequently $P_b \cap \{\eta \in \mathbf{R}^2 \mid x_2 = \eta\}$ is contained in the union of two intervals, one to the left and one to the right of $\eta(b_1/b_2)$. The longer of these two intervals is that on the side of $\eta(b_1/b_2)$ opposite to $-b_1$. See the figure. Define the end points T_n , resp. T_f , of this interval to be the solution of $|f_1(t)| = f_2(t)$ nearest to, resp. farthest, from $\eta(b_1/b_2)$ on the side of



$\eta(b_1/b_2)$ opposite $-b_1$. To bound $|T_f - T_n|$ observe that

$$f_1(T_f) = f_2(T_f),$$

$$f_1(T_n) = -f_2(T_n),$$

$$\Rightarrow f_1(T_f) - f_1(T_n) = f_2(T_f) + f_2(T_n),$$

$$\Rightarrow b_2(T_f - T_n) \left(T_f + T_n - 2\eta \frac{b_1}{b_2} \right) = f_2(T_f) + f_2(T_n),$$

$$\Rightarrow |T_f - T_n| \leq \frac{2}{b_2} \frac{f_2(T_f)}{\left| T_f - \eta \frac{b_1}{b_2} \right|}.$$

Setting $\tau = T_f - \eta \frac{b_1}{b_2}$ we have that

$$\text{const. } \frac{|b|}{b_2} |\eta| \leq |\tau| \leq \text{const. } \frac{|b|}{b_2} |\eta|,$$

with the upper bound coming from (2) and the lower bound coming from the fact that T_f is farther from $\eta \frac{b_1}{b_2}$ than the zeroes $\eta \frac{b_1}{b_2} \pm \left[\eta \frac{b^2}{b_2^2} (\eta + b_2) \right]^{1/2}$ of $f_1(t)$.

Consequently

$$\begin{aligned} |T_f - T_n| &\leq \text{const. } \frac{1}{b_2} \frac{\left[\left(\tau + \eta \frac{b_1}{b_2} + b_1 \right)^2 + (\eta + b_2)^2 \right]^{2p}}{|\tau|} \frac{|\eta|}{(n + b_2)^{2p}} \\ &\leq \text{const. } \frac{1}{b_2} \tau^{4p-1} |\eta|^{1-2p} \\ &\leq \text{const. } \frac{|b|^{4p-1}}{b_2^{4p}} |\eta|^{2p} \end{aligned} \quad \square$$

Proof of Proposition 2

Choose a finite set $S \subset \Gamma^*$ such that for all $b \in \Gamma^* \setminus S$ with $b_2 \neq 0$

$$(i) \quad b^2 \geq 2m, 4 \frac{|b_2|^{\alpha-1}}{(b^2 - \mu)^\alpha} \leq m$$

(ii) for all $\mu \in [-m, m]$ the intersection of $\{k \in \mathbf{R}^2 \mid (k+b)^2 - k^2 - \mu\} \leq \frac{1}{|k|^\alpha}$ with

$\mathbf{R}\gamma$ is contained in the interval on this axis around this point

$$\left(0, \frac{-b^2 - \mu}{2b_2} \right) \text{ of radius } 4 \frac{|b_2|^{\alpha-1}}{(b^2 - \mu)^\alpha}.$$

Then it suffices to show that there is a constant C and that for all $\mu \in [-m, m]$

$$\sum_{\substack{b \in \Gamma^* \setminus S, b_2 \neq 0 \\ I_b \cap B_r \neq \emptyset}} |I_b| \leq Cr^{1-\alpha}.$$

The sum under consideration is bounded by

$$8 \sum_{\substack{b \in \Gamma^* \\ \frac{b^2 - \mu}{4|b_2|} \leq r + m}} \frac{|b_2|^{\alpha-1}}{(b^2 - \mu)^\alpha} \leq 16 \sum_{\substack{b \in \Gamma^* \\ b^2 \leq 4(r+2m)|b_2|}} \frac{|b_2|^{\alpha-1}}{|b|^\alpha} \leq O((r+2m)^{1-\alpha}), \quad \square$$

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Received July 2, 1990