**Zeitschrift:** Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

**Band:** 66 (1991)

**Artikel:** A balanced proper modification of P3.

Autor: Alessandrini, L. / Bassanelli, G.

**DOI:** https://doi.org/10.5169/seals-50412

## Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. Mehr erfahren

### **Conditions d'utilisation**

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. En savoir plus

#### Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. Find out more

**Download PDF: 29.11.2025** 

ETH-Bibliothek Zürich, E-Periodica, https://www.e-periodica.ch

# A balanced proper modification of $P_3$

Lucia Alessandrini and Giovanni Bassanelli\*

### 1. Introduction

Since a proper modification M of a Kähler manifold is not necessarily Kähler, it is natural to ask what kind of weaker geometrical properties are preserved in this situation. In particular we are interested in the existence of a balanced metric on M, i.e. a metric such that the trace of the torsion of the Chern connection vanishes.

We point out that balanced manifolds are, in some sense, dual to Kählerian ones: in fact, a metric is balanced if its Kähler form is "co-closed" (see Definition 3.1); moreover, balanced n-dimensional manifolds are characterized by means of their (n-1, n-1)-currents as well as Kähler manifolds can be characterized by (1, 1)-currents (see [Mi] and [HL]).

The classical example of Hironaka ([Hi] or [S]) of a Moishezon non algebraic manifold X is given by a proper modification of  $P_3$ ; in this note we prove that X is balanced. From a geometrical point of view, the obstruction for X to be Kähler is a curve homologous to zero on the exceptional divisor E: in a dual manner, our proof lies in showing that if E is not homologous to zero then M is balanced. This result is a first confirmation of the conjecture: "Every proper modification of a Kähler manifold is balanced."

About techniques, since positive currents which are (p, p)-components of boundaries are not necessarily closed (see Question on p. 170 in [HL]), we cannot apply the well known results on (locally) flat currents, and therefore we must develope a parallel technique for positive  $\partial \bar{\partial}$ -closed currents.

<sup>\*</sup>This work is partially supported by MPI, 40%. The second author thanks the Department of Mathematics of the University of Trento for its kind hospitality during the preparation of this paper.

### 2. Preliminaries

Let M be a connected complex manifold of dimension n; we denote as usually by  $\Omega^p$  the sheaf of germs of holomorphic p-forms and by  $\mathscr{E}^p$  the sheaf of germs of  $\mathscr{E}^\infty$  p-forms. Moreover, let  $\mathscr{E}^{p,p}(M)_{\mathbb{R}}$  denote the Fréchet space of real (p,p)-forms, while  $\mathscr{E}'_{p,p}(M)_{\mathbb{R}}$  denotes its dual space of real currents of bidimension (p,p).

We will say that  $T \in \mathscr{E}'_{p,p}(M)_{\mathbb{R}}$  is the (p,p)-component of a boundary if there exists a current S such that for every  $\varphi \in \mathscr{E}^{p,p}(M)$ ,  $T(\varphi) = S(d\varphi)$ , i.e. there exists a (p,p+1)-current S such that  $T = \bar{\partial}S + \partial\bar{S}$ .

Let V be a complex vector space of dimension n, p an integer and  $\sigma_p := i^{p^2} 2^{-p}$ . An element of  $\Lambda^{p,0}$  that can be expressed as  $v_1 \wedge \cdots \wedge v_p$  with  $v_j \in V$  is called *simple*.

- 2.1 DEFINITION. ([AA] and [AB])  $\Omega \in \mathscr{E}^{p,p}(M)_{\mathbb{R}}$  is called *transverse* if  $\forall x \in M, \forall v \in \Lambda^p(T'_xM), v \neq 0$  and simple, it holds  $\Omega_x(\sigma_p^{-1}v \wedge \bar{v}) > 0$ .  $T \in \mathscr{E}'_{p,p}(M)_{\mathbb{R}}$  is called *positive* if  $T(\Omega) \geq 0$  for every transverse (p, p)-form  $\Omega$ .
- 2.2 REMARK. ([H1] p. 325-326) A positive (p, p)-current T has measure coefficients, and moreover, if ||T|| denotes the mass of T, there exists a ||T||-measurable function T such that  $||T_x|| = 1$  for ||T||-a.e.  $x \in M$ , and

$$T(\varphi) = \int \varphi_x(\vec{T}_x) ||T||$$
 for all test forms  $\varphi$ .

Let us prove now our main technical tool.

- 2.3 THEOREM. Let M be a complex n-dimensional manifold, and let T be a positive (p, p)-current on M such that  $\partial \bar{\partial} T = 0$  and the Hausdorff 2p-measure of supp T,  $H^{2p}(T)$ , vanishes. Then T = 0.
- 2.4 REMARKS. In this statement, M is not assumed to be compact. If moreover T is closed, the result is well known (see for instance [F]).

*Proof of Theorem* 2.3. Let  $x_0 \in \text{supp } T$ ; as in [H2] p. 72 we can choose a coordinate neighbourhood  $(U, z_1, \ldots, z_n)$  centered at  $x_0$  such that for every increasing multiindex I, |I| = p, it holds locally:

- (a) if  $\pi_I$  is the projection defined by  $\pi_I(z_1, \ldots, z_n) = (z_{i_1}, \ldots, z_{i_p})$ , supp  $T \cap \text{Ker } \pi_I = \{x_0\}$
- (b) there exist two balls  $\Delta'_I \subseteq \mathbb{C}^p$ ,  $\Delta''_I \subseteq \mathbb{C}^{n-p}$  such that  $(\Delta'_I \times b\Delta''_I) \cap \sup T = \emptyset$ .

This implies that  $\pi_I|_{\text{supp }T}$ : supp  $T \cap (\Delta_I' \times \Delta_I'') \to \Delta_I'$  is a proper map.

Fix an I, |I| = p, and call  $T_I := (\pi_I|_{\text{supp }T})_*(T)$ ;  $T_I$  is a  $\partial \bar{\partial}$ -closed (p, p)-current on  $\Delta_I'$ , hence it can be identified with a  $\partial \bar{\partial}$ -closed distribution  $f_I$  on  $\Delta_I'$ ; since  $f_I$  is pluriharmonic, it is smooth.

But  $\pi_I$  is a Lipschitzian map, and therefore  $H^{2p}(\text{supp } f_I) = 0$ , i.e.  $f_I = 0$ . Let

$$\omega := \sigma_1 \sum_{\alpha} dz_{\alpha} \wedge d\bar{z}_{\alpha} \text{ in } B := \bigcap_{I} (\Delta'_{I} \times \Delta''_{I}).$$

Since T is positive, in order to prove T=0 it is enough to verify that  $T(\chi_B(\omega^p/p!))=0$ . But in B

$$\frac{\omega^p}{p!} = \sum_I \sigma_p \, dz_I \wedge d\bar{z}_I = \sum_I (\pi_I)^* (\sigma_p \, dz_I \wedge d\bar{z}_I),$$

therefore

$$T\left(\chi_B \frac{\omega^p}{p!}\right) \leq \sum_I T(\chi_{\Delta_I \times \Delta_I}(\pi_I)^*(\sigma_p \, dz_I \wedge d\bar{z}_I)) = \sum_I \int_{\Delta_I} f_I \sigma_p \, dz_I \wedge d\bar{z}_I = 0.$$

# 3. Hironaka's example

Let us recall the definition of balanced manifold, and the characterization of them by means of positive currents.

- 3.1 DEFINITION. ([Mi]) A balanced manifold M is a compact complex n-dimensional manifold which satisfies one of the following equivalent conditions:
  - (i) M admits a hermitian metric h such that, if  $\omega$  is the Kähler form of h,  $d\omega^{n-1} = 0$ .
  - (ii) M admits a hermitian metric h such that, if  $\omega$  is the Kähler form of h and  $\delta$  is the formal adjoint of d in the metric h, it holds  $\delta \omega = 0$ .
  - (iii) M admits a hermitian metric h such that the torsion 1-form  $\tau_h$  of the canonical hermitian connection of h is zero.
  - (iv) there are no non trivial positive (n-1, n-1)-currents on M which are (n-1, n-1)-components of boundaries.

Let us recall also briefly the construction of Hironaka's example, which can be found in his Thesis [Hi] or, for instance, in [S].

3.2 Consider the projective space  $P_3$  with coordinates (x, y, z), and in it the curve C of equation

$$\begin{cases} y^2 = x^2 + x^3 \\ z = 0 \end{cases}$$

In a little ball near zero, blow up one branch of C first, then the other; outside of the origin, just blow up  $C - \{O\}$ . Then glue together to obtain the compact complex manifold X and the holomorphic map

$$f: X \to \mathbf{P}_3;$$

there exists an analytic subvariety E of X such that

$$f|_{X-E}: X-E \to \mathbf{P}_3 - C$$
 is biholomorphism.

3.3 THEOREM. The 3-dimensional Hironaka's manifold X described in 3.2 is balanced.

*Proof.* Let T be a positive (2, 2)-current on X which is the (2, 2)-component of a boundary, and let  $\omega$  be a Kähler form for  $\mathbf{P}_3$ .  $f^*\omega$  is a closed real (1, 1)-form which is positive semidefinite, so that from

$$0 = T(f^*\omega) = \int (f^*\omega)_x(\vec{T}_x) \|T\|$$

we get  $E \supseteq \text{supp } T$ .

We need now the following lemma, that we shall prove later.

3.4 LEMMA. Let  $f: X \to \mathbf{P}_3$ , C, E, T as before. Then there exists  $k \in \mathbb{C}$  such that T = k[E]. ([E] denotes the current defined by  $[E](\phi) = \int_{\text{Reg } E} \phi$  for all test forms  $\phi$ .)

The previous lemma tells us in particular that dT = 0, and therefore

$$T = \overline{\partial}S^{2,3} + \partial\overline{S^{2,3}}$$

for a (2, 3)-current  $S^{2,3}$  with  $\partial \bar{\partial} S^{2,3} = 0$ .  $\partial S^{2,3}$  becomes a holomorphic 2-form on X, hence the equality

$$\int \partial S^{2,3} \wedge \overline{\partial S^{2,3}} \wedge f^*\omega = \int d(S^{2,3} \wedge \overline{\partial S^{2,3}} \wedge f^*\omega) = 0$$

proves that  $\partial S^{2,3}$  vanishes, and we get

$$T = d(S^{2,3} + \overline{S^{2,3}}).$$

If k were different from zero, we should conclude that the exceptional set E is homologous to zero, which is not the case (for a proof of this fact, use a Mayer-Vietoris argument, as is done for example in [Mü]). In this way we conclude T = 0, i.e. X is balanced.

*Proof of Lemma* 3.4. Since dim E = 2, for every  $x \in \text{Reg } E$  we can choose a coordinate neighbourhood  $(U, z_1, z_2, z_3)$  of x such that

$$E \cap U = \{z \in U/z_3 = 0\}.$$

As in ([HL] Lemma 32), we get in U

$$T = \sigma_1 \sum_{\alpha,\beta=1}^{3} t_{\alpha\beta}(z) dz_{\alpha} \wedge d\bar{z}_{\beta}$$

where the measure  $t_{\alpha B}$  can be written as

$$t_{\alpha\beta}(z) = r_{\alpha\beta}(z_1, z_2) \otimes \delta_0(z_3).$$

Since T is a (2, 2)-component of a boundary,  $0 = \partial \bar{\partial} T =$ 

$$\sigma_{1} \sum_{\alpha < \delta, \beta < \gamma} \left( -\partial_{\delta} \partial_{\bar{\gamma}} t_{\alpha \bar{\beta}} + \partial_{\delta} \partial_{\beta} t_{\alpha \bar{\gamma}} + \partial_{\alpha} \partial_{\bar{\gamma}} t_{\delta \bar{\beta}} - \partial_{\alpha} \partial_{\beta} t_{\delta \bar{\gamma}} \right) dz_{\alpha} \wedge dz_{\delta} \wedge d\bar{z}_{\beta} \wedge d\bar{z}_{\gamma}.$$

For  $\gamma = \delta = 3$  we get

$$\begin{cases} r_{\alpha\beta} = 0 & \text{for } \alpha, \beta \in \{1, 2\} \\ r_{\alpha\bar{3}} & \text{is holomorphic for } \alpha \in \{1, 2\} \\ r_{3\bar{3}} & \text{is pluriharmonic} \end{cases}$$

As  $(r_{\alpha\beta})$  is a non negative matrix, it follows  $r_{\alpha\bar{3}} = 0$  for  $\alpha \in \{1, 2\}$  and therefore

$$T = \sigma_1(r_{3\overline{3}}(z_1, z_2) \otimes \delta_0(z_3)) dz_3 \wedge d\overline{z}_3 \text{ in } U.$$

In this way we have proved that there exists a non negative pluriharmonic map  $h: \text{Reg } E \to \mathbf{R}$  such that

$$T|_{X-\operatorname{Sing} E} = h[E]|_{X-\operatorname{Sing} E}.$$

The next step is to prove that h is constant.

For every  $x_0 \in C$ ,  $x_0 \neq O$ ,  $h|_{f^{-1}(x_0)}$  is harmonic and therefore constant, so that we can project h to a non negative harmonic map  $h_1: C - \{O\} \to \mathbb{R}$ .

The following parametrisation of C

$$\begin{cases} x(t) = t^2 - 1 \\ y(t) = t(t^2 - 1) \end{cases} \quad z = 0$$

induces biholomorphisms

$$C - \{O\} \cong \mathbf{P}_1 - \{a, b\} \cong \mathbf{C} - \{O\}$$

which allow us to use the following remark (see [B] p. 402-403): "Let B(O, b) be the ball of center O and radius b > 0 in C, and let  $g: B(O, b) - \{O\} \rightarrow \mathbb{R}$  be harmonic and non negative. Then there exist a constant A and a holomorphic function F in B(O, b) such that

$$g(z) = \text{Re } F(z) + A \log |z| \text{ for } z \in B(O, b) - \{O\}.$$
"

Let  $h_1(z) = \text{Re } F(z) + A \log |z|$ ; F and A are independent on the radius of the chosen ball and moreover, replacing z by 1/w, we get F(z) = constant. Since  $h_1$  is non negative, it must be A = 0, hence  $h_1$  is constant and h is constant too.

Now, the two currents T and h[E] are  $\partial \bar{\partial}$ -closed and coincide on  $X - \operatorname{Sing} E$ ; if we prove that T - h[E] is positive, the statement will follow from Theorem 2.3.

To do this, let  $\Omega$  be a transverse (2, 2)-form, let  $x \in \text{Sing } E$ , let K be a compact neighbourhood of x, let  $\{V_{\varepsilon}\}$  be a fundamental family of neighbourhoods of  $K \cap \text{Sing } E$ . Since

$$h\int_{E\cap(K-V_L)}\Omega=T(\chi_{K-V_L}\Omega)\uparrow T(\chi_{K-\operatorname{Sing}E}\Omega)$$

we get

$$h\int_{E\cap K}\Omega\leq T(\chi_{K-\operatorname{Sing}E}\Omega)\leq T(\chi_{K}\Omega).$$

#### **REFERENCES**

- [AA] L. ALESSANDRINI and M. ANDREATTA, Closed transverse (p, p)-forms on compact complex manifolds, Compositio Math. 61 (1987) 181-200. Erratum ibidem 63 (1987) 143.
- [AB] L. ALESSANDRINI and G. BASSANELLI, Compact p-Kähler manifolds, Geom. Ded. 38 (1991) 199-210.
- [B] R. B. Burckel, An Introduction to classical Complex Analysis, vol. 1 Pure and Applied Mathematics, Academic Press, New York-San Francisco, 1979.
- [F] H. FEDERER, Geometric Measure Theory, Springer Verlag, New York, 1969.
- [H1] R. HARVEY, Holomorphic chains and their boundaries, Proc. Symp. Pure Math., 30 Part 1, AMS Providence R.I., (1977) 309-382.
- [H2] R. HARVEY, Removable Singularities for positive currents, Amer. J. Math. 96 (1974) 67-78.
- [HL] R. HARVEY and J. R. LAWSON, An intrinsic characterization of Kähler manifolds, Inv. Math. 74 (1983) 169-198.
- [Hi] H. HIRONAKA, On the theory of birational blowing up, Thesis, Harward 1960, unpublished.
- [Mi] M. L. MICHELSON, On the existence of special metrics in complex geometry, Acta Math. 143 (1983) 261-295.
- [Mü] S. MÜLLER-STACH, Bimeromorphe Geometrie von dreidimensionalen Moishezonmannigfaltigkeiten, Diplomarbeit, Bayreuth 1987.
- [S] I. R. SHAFAREVICH, Basic Algebraic Geometry, Springer Verlag, Berlin Heidelberg New York, 1977.

Dipartimento di Matematica Università di Trento I-38050 POVO (Trento) (Italy)

and

Dipartimento di Matematica e Fisica Università di Camerino I-62032 CAMERINO (Macerata) (Italy)

Received February 5, 1990