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Deforming representations of knot groups in $SU(2)$

CHARLES D. FROHMAN[1] AND ERIC P. KLASSEN[2]

1. Introduction

In this paper we address the following question: When is an abelian representation of a knot group in $SU(2)$ a limit point of non-abelian representations? We provide a sufficient condition involving the Alexander polynomial of the knot.

Suppose $K \subset S^3$ is a smooth knot, and let $\Gamma = \pi_1(S^3 - K)$. For $0 \leq \gamma \leq \pi$, define an element $g_\gamma \in SU(2)$ by

$$g_\gamma = \begin{pmatrix} e^{i\gamma} & 0 \\ 0 & e^{-i\gamma} \end{pmatrix}$$

The set $\{g_\gamma\}_{0 \leq \gamma \leq \pi}$ contains a unique representative of each conjugacy class in $SU(2)$. Let $\mu \in \Gamma$ represent a meridian of K. Define a family $\{\rho_\gamma\}$ of abelian homomorphisms from $\Gamma \to SU(2)$ by $\rho_\gamma(\mu) = g_\gamma$. This family contains exactly one representative of each conjugacy class of abelian homomorphisms from $\Gamma \to SU(2)$. A linearization argument (see [K], theorem 19) shows that if ρ_γ is a limit of non-abelian homomorphisms, then $P_K(g_\gamma^2) = 0$, where P_K is the Alexander polynomial of K. The main result in this paper is a partial converse to this fact.

THEOREM 1.1. *If g_γ^2 is a simple root of P_K, then ρ_γ is an endpoint of an arc of non-abelian representations $\Gamma \to SU(2)$. Furthermore g_γ is conjugate in $SU(2)$ to the matrix*

$$\tilde{g}_\gamma = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \in SL(2, \mathbf{R}).$$

Still under the hypothesis that g_γ^2 is a simple root of P_K, the abelian representation

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$\tilde{\rho} : \Gamma \to SL(2, \mathbf{R})$ defined by $\tilde{\rho}(\mu) = \tilde{g}_\gamma$ is the endpoint of an arc of non-abelian representations $\Gamma \to SL(2, \mathbf{R})$.

We outline here the strategy of our proof. First, using the double covers $SU(2) \to SO(3, \mathbf{R})$ and $SL(2, \mathbf{R}) \to PSL(2, \mathbf{R})$, we reduce the problem to finding arcs of representations in the isometries of the 2-sphere or hyperbolic plane, respectively. We then recall an observation of de Rham (see [D]) that the hypothesis of our theorem implies that there is a non-abelian representation of the knot group in the isometries of the Euclidean plane, sending meridians to rotations of angle 2γ. Next, we put a family of metrics of constant curvature ranging from negative to positive on the Euclidean plane and show that the non-abelian representation of Γ in Euclidean isometries can be smoothly deformed to give an arc of representations in the isometries of our deformed metrics; this gives rise to our representations in $SO(3, \mathbf{R})$ and $PSL(2, \mathbf{R})$.

The idea of "smashing" a knot, or fusing it at a crossing, is introduced in Section 5, and may at first appear to the reader to be unmotivated. Smashing is simply an adaptation of a key idea in Thurston's proof that the space of irreducible characters of a knot group in $SL(2, \mathbf{C})$ has dimension at least one (see [T] or [C-S]). Thurston describes this idea in terms of boring out an additional tunnel in the knot complement, thereby increasing the genus of its boundary to 2.

From our point of view, the smashed knot group is simply a group extension of the knot group, whose representation space is less singular than that of the knot group in a neighborhood of the relevant abelian representation. This larger, less singular representation space provides a convenient ambient space from which to carve out the representation space of the knot group which we are studying.

Finally, it is natural to ask (and the referee did so!) what are the prospects of proving an analogous result at an abelian representation which corresponds to a multiple root of the Alexander polynomial. As yet the authors have been unable to prove such a theorem; however we have studied some relevant examples.

First, suppose K is a knot and $\rho_0 : \pi_1(S^3 - K) \to SU(2)$ is an abelian representation with $\rho_0(\mu)^2$ a simple root of the Alexander polynomial P_K. Then $\rho_0(\mu)^2$ is a double root of $P_{K \# K} = P_K^2$. In this case it is not hard to see that there is a 2-dimensional set of conjugacy classes of irreducible representations of $\pi_1(S^3 - K \# K)$ whose closure contains the abelian representation $\rho_0^{\#}$ defined by $\rho_0^{\#}(\mu^{\#}) = \rho_0(\mu)$.

Secondly, in an unpublished analysis of "Riley's favorite knot" (which is a Montesinos knot with eleven crossings; see [R]), the second author has ascertained that a small neighborhood of an abelian representation corresponding to a double root of the Alexander polynomial is made up of two arcs of non-abelian representations intersecting each other transversely and intersecting the arc of abelian representations transversely in the given abelian representation.

It is interesting to note that in both of these examples there does not exist a basis of the Zariski tangent space (at the abelian representations) consisting entirely of integrable tangent vectors. In other words, the linear span of the tangent cone is not the entire Zariski tangent space at these points. This remark will be elaborated on in a future paper.

2. Lifting paths of projective representations

Our main objective in this paper is to construct certain arcs of representations of a group G in $SL(2, \mathbf{C})$. In this section we prove a lemma enabling us to lift arcs of representations in $PSL(2, \mathbf{C})$ to arcs of representations in $SL(2, \mathbf{C})$. This will reduce our problem to constructing arcs of representations in $PSL(2, \mathbf{C})$. Let $q : SL(2, \mathbf{C}) \to PSL(2, \mathbf{C})$ be the canonical double cover.

LEMMA 2.1. *Let $\bar{\rho}_t : G \to PSL(2, \mathbf{C})$ be an arc of representations. Let $\rho : G \to SL(2, \mathbf{C})$ be a representation satisfying $\bar{\rho}_0 = q \circ \rho$. Then there is a unique (continuous) arc of representations $\rho_t : G \to SL(2, \mathbf{C})$ satisfying the two conditions $\rho_0 = \rho$ and $\bar{\rho}_t = q \circ \rho_t$ for all t.*

Proof. Suppose G has a presentation with generators $\{x_i\}$ and relators $\{w_j\}$, where each w_j is a word in the x_i. Then since q is a covering map, we define $\rho_t(x_i)$ to be the unique lift of the path $\bar{\rho}_t(x_i)$ starting at the point $\rho(x_i)$ in $SL(2, \mathbf{C})$. To see that this choice of $\rho_t(x_i)$ does indeed define an arc of representations of G, we need to check that $\rho_t(w_j) = 1$ for all t and j. This follows from the facts that $\rho_t(w_j)$ is a continuous path in the set $\{1, -1\}$ and $\rho_0(w_j) = \rho(w_j) = 1$.

3. Fox differentiation

Let F_n be the free group generated by the set $\{x_1, \ldots, x_n\}$. For $l = 1, \ldots, n$, we define a map $d_l : F_n \to \mathbf{Z}[t, t^{-1}]$ as follows. Let $w = \Pi_{i=1}^{m} x_{j_i}^{\epsilon_i}$ be a word in $\{x_1, \ldots, x_n\}$, where $\epsilon_i = \pm 1$ and $j_i \in \{1, \ldots, n\}$ for all i. Then define

$$d_l(w) = \sum_{k=1}^{n} v_k,$$

where

$$v_k = \begin{cases} 0 & \text{if } j_k \neq l \\ t^{u_k} \text{ where } u_k = \Sigma_{j=1}^{k-1} \epsilon_j & \text{if } j_k = l \text{ and } \epsilon_k = 1 \\ -t^{u_k} \text{ where } u_k = \Sigma_{j=1}^{k} \epsilon_j & \text{if } j_k = l \text{ and } \epsilon_k = -1. \end{cases}$$

We call d_l the *Fox derivative* with respect to x_l.

Note that in the literature the Fox derivative is often defined to take its value in the ring $\mathbf{Z}[F_n]$. The Fox derivative as we define it is the image of the other definition under the map taking x_i to t for all i.

We define the *Fox gradient* of w to be the vector $(d_1 w, \ldots, d_n w) \in \mathbf{Z}[t, t^{-1}]^n$.

4. Euclidean representations of knot groups and the Fox Jacobian

Let $K \subset S^3$ be a knot (which we will confuse with its projection). Orient K and label the bridges $x_1, \ldots, x_n$. Note that each bridge of K corresponds to a Wirtinger generator x_i of Γ.

We will number the bridges of K "in order," i.e., as we walk along K in the direction of its orientation we follow the bridge corresponding to x_1, then x_2, etc. on up to x_n, after which we arrive back at x_1. We will number the crossings, which correspond to relators w_k, in the same way, so that for $1 \leq k \leq n-1$, we have

$$w_k = x_{j_k}^{\epsilon_k} x_k x_{j_k}^{-\epsilon_k} x_{k+1}^{-1}$$

and

$$w_n = x_{j_n}^{\epsilon_n} x_n x_{j_n}^{-\epsilon_n} x_1^{-1}.$$

We will refer to the crossing corresponding to w_k as the "k-th crossing."

Label with integers the regions into which the projection of the knot separates the plane as follows. Label the outside region 0. To label a region R, draw an oriented arc which starts in the outside region, ends in R, and is transverse to the projection of K. Using the orientation of K, compute the algebraic intersection number of this arc with the knot projection in the plane, and label the region R with that intersection number. In figure 1, we indicate the result of this process for the trefoil knot.

Draw a small circle around each crossing and put a dot in one of the regions at the crossing. Read off the Wirtinger relator in the clockwise direction starting at the dot. In figure 1, we show the result of doing this for the trefoil. The relators are

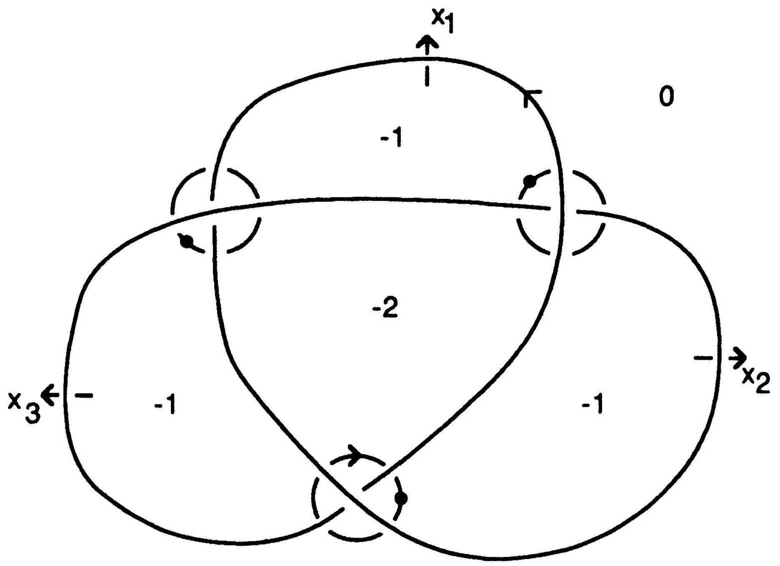

Figure 1

$w_1 = x_3x_1^{-1}x_3^{-1}x_2$, $w_2 = x_1x_2^{-1}x_1^{-1}x_3$, $w_3 = x_2x_3^{-1}x_2^{-1}x_1$. We now define the normalized Fox Jacobian matrix $J_K(t)$, an $n \times n$ matrix over the ring $\mathbf{Z}[t, t^{-1}]$. Let the i-th row of $J_K(t)$, which we denote by $R_i(t)$, be the Fox gradient of w_i with respect to $x_1, \ldots, x_n$ multiplied by $t^{n(i)}$, where $n(i)$ is the label assigned to the region where the dot was placed to read off the i-th relator. The normalization was chosen so that

$$\sum_{i=1}^{n} R_i(t) = 0.$$

Below we show $J_K(t)$, for K the trefoil knot.

$$J_K(t) = \begin{pmatrix} -t^{-1} & t^{-2} & t^{-1} - t^{-2} \\ t^{-1} - t^{-2} & -t^{-1} & t^{-2} \\ t^{-2} & t^{-1} - t^{-2} & -t^{-1} \end{pmatrix}$$

To form the Alexander polynomial of the knot K, which we denote by $P_K(t)$, erase one row and one column of $J_K(t)$ and then take the determinant. In the case of the trefoil, after eliminating excess powers of t we obtain $P_K(t) = t^2 - t + 1$.

Though what follows is entirely self-contained, we will pause here to give a few brief remarks about the relevance of the Fox Jacobian to representation spaces. The space of representations of a finitely presented group into an algebraic group can be given the structure of an algebraic set using the images of the generators as coordinates and the relations as the defining polynomials. (See, for example, [L-M].) When one differentiates the relation map at a particular representation, one obtains essentially the Fox Jacobian. Thus, it is natural to identify the kernel of the Fox Jacobian with the Zariski tangent space of the representation variety. The reason why we substitute a single variable t for all of the x_i is that we are working at an abelian representation, which must take all the x_i to a single group element.

We now review some material due to de Rham [D]. Define $\mathrm{Isom}_+(\mathbf{E}^2)$ to be the group of orientation preserving isometries of the Euclidean plane. In complex coordinates, these are the maps of the form $z \mapsto \omega z + b$, where $\omega \in S^1$ and $b \in \mathbf{C}$. Fix $\omega \in S^1$. Denote by B_ω the set of all n-tuples $(b_1, \ldots, b_n) \in \mathbf{C}^n$ such that the assignment

$$\sigma : \{x_1, \ldots, x_n\} \to \mathrm{Isom}_+(\mathbf{E}^2)$$

defined by $\sigma(x_i)(z) = \omega z + b_i$ determines a group homomorphism $\Gamma \to \mathrm{Isom}_+(\mathbf{E}^2)$.

PROPOSITION 4.1. (*de Rham*, [*D*])

$$B_\omega = \text{kernel}\,\{J_K(\omega) : \mathbf{C}^n \to \mathbf{C}^n\}.$$

Proof. Each relation for Γ can be written in the form $x_j x_i = x_k x_j$. Clearly $(b_1, \ldots, b_n) \in B_\omega$ if and only if all relations hold among the $\sigma(x_i)$. We compute

$$\sigma(x_j)\sigma(x_i) = \sigma(x_k)\sigma(x_j)$$

if and only if

$$\omega(\omega z + b_i) + b_j = \omega(\omega z + b_j) + b_k \qquad \text{for all } z \in \mathbf{C}$$

if and only if

$$\omega b_i + (1 - \omega)b_j + (-1)b_k = 0.$$

The resulting linear transformation is just $J_K(\omega)$, up to multiplication of each row by some power of ω, which doesn't affect its kernel. □

NOTE. Define $\tilde{\sigma}(x_i) \in \mathrm{Isom}_+(\mathbf{E}^2)$ to be the rotation of angle ω about the point b_i. It is easy to verify that $\tilde{\sigma}$ defines a homomorphism $\Gamma \to \mathrm{Isom}_+(\mathbf{E}^2)$ if and only if σ does.

From the above proof, it is clear that the sum of the columns of $J_K(t)$ is 0, and it follows that B_ω contains the diagonal of $\mathbf{C}^n$. Clearly this subset of B_ω corresponds exactly to the abelian representations of Γ in $\mathrm{Isom}_+(\mathbf{E}^2)$. Since the first $n - 1$

relations imply the last, deleting the last row of $J_K(t)$ will not affect its kernel. Let $\tilde{J}_K(t)$ denote $J_K(t)$ with its last row and last column removed. The kernel of $\tilde{J}_K(\omega)$ corresponds to those representations taking x_n to a rotation about the origin of $\mathbf{E}^2$. Clearly every representation is conjugate to one of this form. It is now clear that there exists a non-abelian representation of Γ in $\mathrm{Isom}_+(\mathbf{E}^2)$ taking each meridian to a rotation of angle ω if and only if $P_K(\omega) = \det(\tilde{J}_K(\omega)) = 0$, i.e. if and only if ω is a root of $P_K(t)$. Furthermore, if we assume that ω is a simple root of $P_K(t)$, then it follows (by putting $\tilde{J}_K(t)$ in rational canonical form) that the dimension of kernel $(\tilde{J}_K(\omega))$ is one. Choose j between 1 and $n-1$ so that the projection of kernel $(\tilde{J}_K(t))$ onto the j-th coordinate is non-trivial. It follows that if we fix the centers of rotation of x_j and x_n, then we fix a unique non-abelian representation of Γ in $\mathrm{Isom}_+(\mathbf{E}^2)$ taking meridians to rotations of angle ω.

5. Smashed knot groups and the Fox Jacobian

We introduce an operation on knots which we call smashing. This operation has its origin in Thurston's proof that the variety of irreducible characters of a knot group in $SL(2, \mathbf{C})$ has dimension at least one (see [T] or [C-S] for an exposition of this proof). Consider a crossing whose corresponding Wirtinger relator is $x_m^{\epsilon_k} x_k x_m^{-\epsilon_k} x_{k+1}^{-1} = 1$. (This is just the k-th crossing, with $m = j_k$.) At this crossing we bore out a vertical tube running from the overcrosser x_m to the undercrosser. We call the union of the knot with this tube the *smashed knot*, and the fundamental group of its complement the *smashed knot group*, which we denote by Λ. To calculate a presentation for Λ, we must relabel the portion of x_m lying on one side of the smashed crossing; we will call it x_{n+1}. A presentation for Λ may be obtained from a presentation for Γ by making the following two changes: (1) Add to the original set of generators the new generator x_{n+1}. (2) Any appearance of x_m in a relator that corresponds to the portion of x_m that was relabelled must be changed to x_{n+1}. All the new relators will have the same form as the old $(x_j x_i x_j^{-1} = x_l)$ except for the relator corresponding to the smashed crossing, which will now involve four different generators. As before, any one relation is a consequence of the other $n-1$. Also, note that if we adjoin the relation $x_m = x_{n+1}$ to our presentation for Λ, we obtain a presentation for the original knot group Γ. In figure 2, we show the result of smashing the trefoil at its first crossing.

We define the normalized Fox Jacobian of the smashed knot group, $J_K^s(t)$, by the same algorithm introduced for the original knot group: let the i-th row of $J_K^s(t)$, which we denote by $R_i^s(t)$, be the Fox gradient of the i-th relation multiplied by the same power of t used in the corresponding relation of Γ. Below we give the smashed

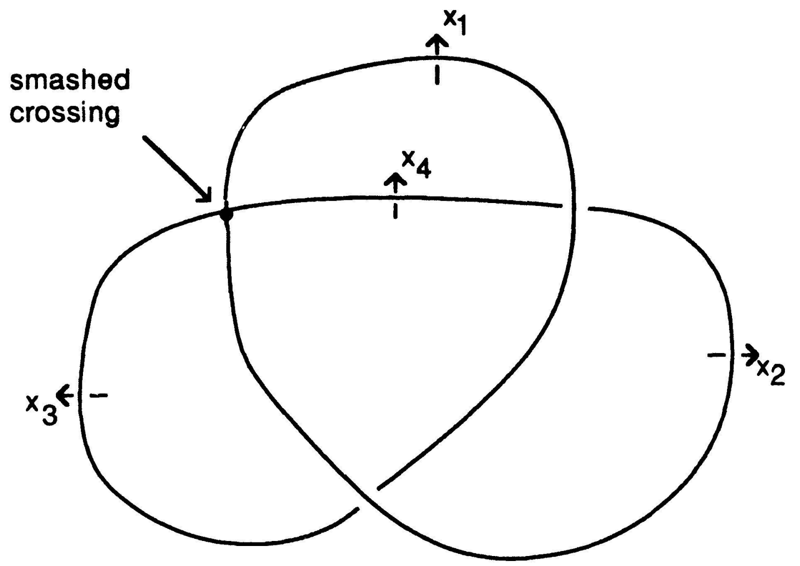

Figure 2

Jacobian of the trefoil:

$$J_K^s(t) = \begin{pmatrix} -t^{-1} & t^{-2} & t^{-1} & -t^{-2} \\ t^{-1}-t^{-2} & -t^{-1} & 0 & t^{-2} \\ t^{-2} & t^{-1}-t^{-2} & -t^{-1} & 0 \end{pmatrix}$$

Define L to be the set of all $l \in \{1, \ldots, n\}$ such that $l \neq k$ and the l-th crossing of K involves that portion of x_m which was relabelled x_{n+1}. We now describe how $J_K^s(t)$, which is an $n \times (n+1)$ matrix over $\mathbf{Z}[t, t^{-1}]$, may be obtained by modifying $J_K(t)$. Suppose $i \in L$. To obtain $R_i^s(t)$, we move the m-th entry of $R_i(t)$ into the newly created $(n+1)$-st spot, leaving behind a 0 as the m-th entry of $R_i^s(t)$. In all other entries, $R_i^s(t)$ is identical to $R_i(t)$. In the k-th row, which corresponds to the smashed crossing, the m-th entry of $R_k(t)$ is of the form $(-1)^\epsilon t^{v+1} - (-1)^\epsilon t^v$. In forming $R_k^s(t)$, one of these two summands will be moved to the $(n+1)$-st spot, and the other will remain in column m. In its other entries, $R_k^s(t)$ is identical to $R_k(t)$. If $i \neq k$ and $i \notin L$, then $R_i^s(t)$ is obtained from $R_i(t)$ simply by adjoining a zero in the $(n+1)$-st entry.

Assume that $\alpha \in S^1$ is a simple root of $P_K(t)$. It follows that the space of relations that hold between the $R_j(\alpha)$ is precisely 2-dimensional with basis consisting of the relations

$$\sum_{i=1}^{n} R_i(\alpha) = 0 \quad \text{and} \quad \sum_{i=1}^{n} c_i R_i(\alpha) = 0$$

where the c_i are complex numbers and not all of them are equal. We will also assume, by subtracting off a multiple of the first relation if necessary, that $c_k = 0$. Of course the dimension of the space of relations holding among the columns is also two; this space has as a basis

$$\sum_{i=1}^{n} S_i(\alpha) = 0 \quad \text{and} \quad \sum_{i=1}^{n} b_i S_i(\alpha) = 0$$

where S_i is the i-th column and the $b_i \in \mathbf{C}$ are not all equal.

Define

$$C = \sum_{l \in L} c_l f_{lm}$$

where f_{lm} denotes the (l, m)-th entry of $J_K(\alpha)$.

LEMMA 5.1. *If* $C \neq 0$, *then* $J^s_K(\alpha)$ *has* rank $n - 1$.

Proof. Note that since $\Sigma_{i=1}^{n} R^s_i(\alpha) = 0$, it follows that rank $J^s_K(\alpha) \leq n - 1$. Suppose rank $J^s_K(\alpha) \neq n - 1$. Then there exists another relation among the rows, independent of the first, of the form $\Sigma_{i=1}^{n} h_i R^s_i(\alpha)$, where the $h_i \in \mathbf{C}$ are not all equal, and we may assume (by subtracting off a multiple of the first relation) that $h_k = 0$. Because $S_m = S^s_m + S^s_{n+1}$, and $S_i = S^s_i$ for $1 \leq i \leq n$ and $i \neq m$, it follows that

$$\sum_{i=1}^{n} h_i R_i(\alpha) = 0.$$

Hence $\{h_i\}$ is a constant non-zero multiple of $\{c_i\}$. It follows that

$$C = \sum_{l \in L} c_l f_{lk} = 0,$$

which is a contradiction, and proves the lemma. □

Fix $\omega \in S^1$. Define

$$\tau : \{x_1, \ldots, x_{n+1}\} \to \mathrm{Isom}_+ (\mathbf{E}^2)$$

by $\tau(x_i)(z) = \omega z + b_i$, where $(b_1, \ldots, b_{n+1}) \in \mathbf{C}^{n+1}$. The following proposition is analogous to proposition 4.1:

PROPOSITION 5.2. *Denote by B_ω the set of all $(n+1)$-tuples $(b_1, \ldots, b_{n+1})$ such that τ determines a homomorphism from Λ to* $\mathrm{Isom}_+(\mathbf{E}^2)$. *Then*

$$B_\omega = \ker\,(J^s_K(\omega) : \mathbf{C}^{n+1} \to \mathbf{C}^n).$$

We omit the proof because it is identical to the proof of proposition 4.1.

6. Good smashes

In this section we prove that given a simple root of the Alexander polynomial, we can find a projection of the knot and a crossing in that projection such that the knot group smashed at that crossing, and its Jacobian, satisfy certain technical conditions. Such a smash will be called a good smash.

As usual, let $K \subset S^3$ be a knot, Γ its fundamental group, and assume that $\alpha \in S^1$ is a simple root of P_K. Recall that the space of relations holding among the rows R_i of the normalized Fox Jacobian $J_K(\alpha)$ is precisely 2-dimensional with basis

$$\sum_i R_i(\alpha) = 0 \quad \text{and} \quad \sum_i c_i R_i(\alpha) = 0$$

where the c_i are complex numbers and not all of them are equal. Hence the Jacobian evaluated at α has rank $n-2$. Of course the dimension of the space of relations holding among the columns is also two; this space has a basis

$$\sum_i S_i(\alpha) = 0 \quad \text{and} \quad \sum_i b_i S_i(\alpha) = 0$$

where S_i is the i-th column and the $b_i \in \mathbf{C}$ are not all equal.

DEFINITION 6.1. A crossing in a projection of K, and the smash at that crossing, will be called *good* if they satisfy the following two properties:

G1: The three b_i's corresponding to the bridges which meet at the crossing are not all equal.

G2: The Fox Jacobian of the smashed knot group resulting from smashing at this crossing has rank $n-1$.

NOTE. The verification of G1 is independent of which second basis element we chose for the relation space of the columns.

THEOREM 6.2. *If $K \subset S^3$ is a knot and $\alpha \in S^1$ is a simple root of its Alexander polynomial, then there exists a projection of K containing a good crossing.*

Proof. It will suffice to produce a projection of K with a crossing satisfying $G1$ and the hypothesis of lemma 5.1. Our main technique is an operation on knot projections which we call threading, and which the reader may recognize as the second Reidemeister move. The strategy of the proof is to show that if the original projection contains no good crossings, it may be altered by a finite number of threadings such that the resulting projection contains a good crossing. The threading operation is pictured in figure 3. Assume that the bridge that is broken into three pieces was originally labelled x_l and that the other bridge was labelled x_p. The three portions of x_l resulting from the threading must be renumbered. Continue to denote one of its ends by x_l and label its other two pieces x_{n+1} and x_{n+2}, with x_{n+1} being the new bridge both of whose endpoints lie near the bridge x_p that crossed over. The Fox Jacobian of the threaded projection, which we will call $J'_K(t)$, is an $(n+2) \times (n+2)$ matrix. We will denote the i-th row and column of $J'_K(t)$ by $R'_i(t)$ and $S'_i(t)$, respectively. Let $V \subset \{1, \ldots, n\}$ be the set of integers indexing relators that come from crossings (in the original projection) involving that part of the bridge x_l which has been relabelled x_{n+2}. If $i \notin V$ and $1 \leq i \leq n$, then $R'_i(t)$ is obtained from $R_i(t)$ by adjoining two zeroes in the $(n+1)$-st and $(n+2)$-nd entries. If $i \in V$ then $R'_i(t)$ is obtained from $R_i(t)$ by moving the l-th entry over to the $(n+2)$-nd entry, leaving behind a zero in the l-th entry, and putting a 0 in the $(n+1)$-st entry.

To obtain $R'_{n+1}(t)$ and $R'_{n+2}(t)$, first write the relators corresponding to the two new crossings:

$$w_{n+1} = x_p^{-1} x_{n+1} x_p x_l^{-1} \quad \text{and} \quad w_{n+2} = x_{n+2} x_p^{-1} x_{n+1}^{-1} x_p$$

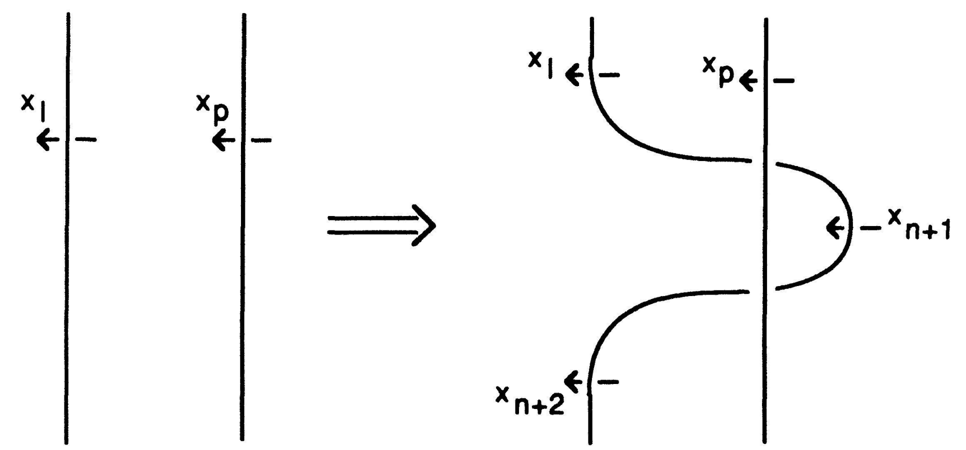

Figure 3

The Fox gradients of these words are:

column index:	other	l	p	$n+1$	$n+2$
dw_{n+1}:	0	-1	$1-t^{-1}$	t^{-1}	0
dw_{n+2}:	0	0	$-1+t^{-1}$	$-t^{-1}$	1

The corresponding rows of $J'_K(t)$ are obtained by multiplying each of these gradients by t^ν, where ν is the intersection number referred to in the original definition of the Fox Jacobian.

Because of the normalization built into the Jacobian, the sum of the rows of the new Jacobian is zero. Suppose that the rows of the original Jacobian, with α plugged in, satisfy a relation of the form

$$\sum_{i=1}^{n} c_i R_i(\alpha) = 0.$$

Let f_{ij} denote the entry in the i-th row and j-th column of $J_K(t)$. Let

$$C = \sum_{j \in V} c_j f_{jl}.$$

Then, using the above description of the $R'_i(t)$, it is easy to see that for some $\epsilon = \pm 1$ the following relation holds betwen the rows of the threaded Jacobian:

$$(*) \quad \left\{\sum_{i=1}^{n} c_i R'_i(\alpha)\right\} + (-1)^\epsilon \alpha^{-\nu} C(R'_{n+1}(\alpha) + R'_{n+2}(\alpha)) = 0.$$

If not all the c_i in the original relation are equal, then not all coefficients occurring in the new relation are equal.

Let Q be the oriented planar 4-valent graph obtained be "smashing together" all the crossings of our original projection of K. Denote by E the set of edges of Q. Define two functions $b : E \to \mathbf{C}$ and $C : E \to \mathbf{C}$ as follows. Let $e \in E$, and let x_i be the bridge of K containing e. Let $b(e)$ be the b_i associated to x_i in the non-trivial relation on the $S_i(\alpha)$. To define $C(e)$, begin by letting q be a point on the interior of e. Let Y be the set of indices of all crossings of K which involve the bridge x_i at a point on x_i "ahead" of q with respect to the orientation of x_i. Define

$$C(e) = \sum_{y \in Y} c_y f_{yi}.$$

Note that the functions b and C depend on our particular choices of relations between the rows and columns of $J_K(\alpha)$.

LEMMA 6.3. *There exist two edges e and f of Q, and a relation* $\Sigma c_i R_i(\alpha) = 0$ *with not all* c_i *equal, such that* (1) $b(e) \neq b(f)$, (2) $C(e) = 0$, *and* (3) $C(f) \neq 0$.

Proof. Let e_1 be an edge which is at the end of a bridge x_i. Let j be the index of the crossing at that end of the bridge. By subtracting off a multiple of $\Sigma R_i(\alpha) = 0$, we obtain a relation $\Sigma c_i R_i(\alpha)$ such that $c_j = 0$ but there exists an i with $c_i \neq 0$. Define $C : E \to \mathbf{C}$ using this relation. Then $C(e_1) = 0$. Suppose $c_i \neq 0$. Then for one of the two edges, which we call e_2, adjacent to the i-th crossing and contained in the undercrosser, $C(e_2) \neq 0$. If $b(e_1) \neq b(e_2)$, we are done. So assume that $b(e_1) = b(e_2)$. Let e_3 be an edge such that $b(e_3) \neq b(e_1)$. If $C(e_3) = 0$, then (e_3, e_2) is our desired pair, while if $C(e_3) \neq 0$, then (e_1, e_3) is our desired pair. □

Define a distance function $d : E \times E \to \mathbf{Z}_+$ as follows. Suppose $e_1, e_2 \in E$. Let A be the set of all arcs in the plane which connect a point in e_1 to a point in e_2, which are transverse to all edges of Q, and which miss all vertices of Q. Define the length of such an arc to be the number of complementary regions the arc passes through. Define $d(e_1, e_2)$ to be the infinum of the lengths of all arcs $a \in A$. Note that arcs e and f occur in the boundary of the same complementary component of Q if and only if $d(e,f) = 1$. We will now prove theorem 6.2 by induction on the minimum value taken by $d(e,f)$, where (e,f) ranges over all pairs of edges satisfying the conclusion of lemma 6.3.

First, assume that e and f satisfy the conclusions of lemma 6.3 and $d(e,f) = 1$, so e and f occur in the boundary of the same complementary component of Q. Then simply thread e under f. Both of the crossings created by the threading are good crossings, for the following two reasons:

(1) Still using the same Euclidean representation of Γ, the b_i's corresponding to the bridges meeting at one of these crossings are $b(e)$, $b(f)$, and the image of $b(e)$ under a rotation of $\mathbf{C}$ by angle α about the point $b(f)$. These are distinct points, so $G1$ is satisfied.

(2) In the relation on the rows of $J'_K(\alpha)$ given by $(*)$ earlier in this section, the coefficients corresponding to the new crossings are multiples of $C(e)$ and hence they are 0. It follows that this relation can be used to compute the C, corresponding to one of these smashes, which occurs in the hypothesis of lemma 5.1. But this C is $\pm\alpha^v C(f)$, which is non-zero. It follows that $G2$ is satisfied, by lemma 5.1.

For the inductive step, assume the theorem has been proven for minimal value of $d(e,f) = n - 1$. Now assume that $d(e,f) = n$ for the "closest" pair (e,f) satisfying the conclusion of lemma 6.3. Let a be an arc from e to f, transverse to Q, which realizes this distance. Let e' be the first edge that a intersects on its way from e to f. We now consider four possibilities for $C(e')$ and $b(e')$.

(1) If $C(e') = 0$ and $b(e') = b(e)$, then the pair (e', f) contradicts the minimality of $d(e,f)$.

(2) If $C(e') \neq 0$ and $b(e') \neq b(e)$ then the pair (e, e') contradicts the same minimality.

(3) If $C(e') = 0$ and $b(e') \neq b(e)$, then alter the projection of K by threading e over e'. Denote by $\bar{e}$ the (short) new edge of the graph created by the threading. In the relation (*) which holds among the rows of the threaded Fox Jacobian, the coefficients of the rows corresponding to the new crossings are multiples of $C(e') = 0$. Using this new relation $C(\bar{e}) = 0$ and $b(\bar{e}) = b(e) \neq b(f)$. Also, $d(\bar{e}, f) = n - 1$. It follows from the induction hypothesis that K has a projection with a good crossing.

(4) If $C(e') \neq 0$ and $b(e') = b(e)$, then by threading e under e' we obtain a new edge $\bar{e}$ with $C(\bar{e}) = C(e) = 0$ and $b(\bar{e}) = b(e) \neq b(f)$ and with $d(\bar{e}, f) = n - 1$. Thus we again invoke the induction hypothesis, and the theorem is proved. □

7. A family of metrics on the unit disc

Let $\mathbf{H}$ denote the upper half plane in $\mathbf{C}$, and denote by $g = g_{-1}$ the usual hyperbolic metric on $\mathbf{H}$. For $c \in (0, 2)$, denote by $\mathbf{H}_{-c}$ the manifold $\mathbf{H}$ equipped with the metric $g_{-c} = \frac{1}{c}g$, of constant curvature $-c$. Note that $\mathbf{H}_{-1}$ is equipped with the usual hyperbolic metric. Recall that $PSL(2, \mathbf{R})$ acts on the underlying space $\mathbf{H}$ by Moebius transformations, and observe that these transformations are isometries no matter which of the metrics g_{-c} we put on $\mathbf{H}$. In other words,

$$\mathrm{Isom}_+ (\mathbf{H}_{-c}) = PSL(2, \mathbf{R})$$

for all c. Define $id_c : \mathbf{H}_{-c} \to \mathbf{H}_{-1}$ to be the identity on the underlying space $\mathbf{H}$.

Let $\Delta \subset \mathbf{C}$ be the open unit disc in $\mathbf{C}$. Define a family of metrics ds_c on Δ by

$$ds_c^2 = \frac{dx^2 + dy^2}{(1 + (c/4)(x^2 + y^2))^2}$$

where $c \in [-2, 2]$. Denote Δ equipped with the metric ds_c by Δ_c. The curvature of Δ_c is $-c$ (see, for example, Ahlfors [A], p. 12). Consider first the case $c > 0$: Note that for each such c, Δ_c embeds isometrically in $\mathbf{H}_{-c}$. For each $c > 0$, select an orientation preserving isometry $f_c : \Delta_c \to \mathbf{H}_{-c}$ by insisting that $f_c(0) = i$ and

$$(df_c)_0(1) = \sqrt{c}i,$$

where we are thinking of 1 as a (unit) tangent vector to Δ_c at 0, and i as a tangent vector to $\mathbf{H}_{-c}$ at i. Now define $h_c : \Delta_c \to \mathbf{H}_{-1} = \mathbf{H}$ by $h_c = id_c \circ f_c$.

LEMMA 7.1. *Given $\epsilon > 0$, there exists a $c_0 > 0$ such that if $0 < c < c_0$, then $h_c(\Delta_c)$ is contained in an ϵ-neighborhood of i in* $\mathbf{H}$ *(with the usual hyperbolic metric).*

Proof. Since

$$ds_c = \frac{\sqrt{dx^2 + dy^2}}{1 + (c/4)(x^2 + y^2)},$$

the radius of Δ_{-c} (with respect to ds_c) is less than 1. Since f_c preserves distance and id_c multiplies distance by $\sqrt{c}$, it follows that the radius of $h_c(\Delta_{-c})$ is less than $\sqrt{c}$. So choosing $c_0 \le \epsilon^2$ satisfies the conclusion of the lemma. □

In the case $c < 0$, we can mimic the preceding constructions using a family of metrics of positive curvature on S^2 in place of metrics of negative curvature on $\mathbf{H}$, and a family of maps $h_c : \Delta_c \to S^2$. Also, as above, the maps h_c for $c < 0$ have the property that

$$\lim_{c \to 0} h_c(\Delta_c) = (0, 0, 1) \in S^2,$$

i.e., as c approaches 0, h_c shrinks Δ_c to a single point in S^2.

8. The proof of theorem 1.1

Using theorem 6.2, we assume that we are working with a projection of our knot K in which there is a good crossing. Smash at this crossing, and denote by Λ the smashed knot group. We will now study representations of Λ into $\mathrm{Isom}_+(\mathbf{H})$, $\mathrm{Isom}_+(\mathbf{E}^2)$ and $\mathrm{Isom}_+(S^2)$. Let $\{x_1, \ldots, x_n\}$ be the Wirtinger generators of our original knot group Γ and $\{x_1, \ldots, x_n, x_{n+1}\}$ the corresponding generators of Λ. We assume that α is a simple root of $P_K(t)$, the Alexander polynomial of our knot. It follows from proposition 4.1 that there is a non-abelian representation ρ of Γ in $\mathrm{Isom}_+(\mathbf{E}^2)$ which takes each x_i to a rotation of $\mathbf{E}^2$ by the angle α.

Since this representation is non-abelian, we may assume that two generators, say x_1 and x_2, are taken under ρ to rotations about different points in $\mathbf{E}^2$. Because α is a simple root, we may specify a center of rotation for $\rho(x_1)$ and a different one for $\rho(x_2)$, and this completely determines the representation ρ. Choose $b \in \mathbf{C}$ small enough that the unique ρ taking x_1 to a rotation of angle α about 0, and x_2 to a rotation of angle α about b satisfies the condition that the center of rotation of $\rho(x_i)$ is contained in Δ for all i. Later in the proof we will need the following:

LEMMA 8.1. *Define R_0 to be the set of all $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Isom}_+(\mathbf{E}^2))$ which* *satisfy* (i) $\rho(x_1)$ *is a rotation about* 0 *and* (ii) $\rho(x_2)$ *is a rotation about* b. *Suppose* $\rho_0 \in R_0$ *is a representation taking each meridian to a rotation of angle* α. *Then* $T_{\rho_0}R_0 = 0$, *where* $T_{\rho_0}R_0$ *denotes the Zariski tangent space.*

Proof. Denote by $\tilde{J}_K(t)$ the Fox Jacobian of K with the first row and first column deleted. Define

$$R = \{\rho \in \mathrm{Hom}\,(\Gamma, \mathrm{Isom}_+\,(\mathbf{E}^2)) : \rho(x_1) \text{ is a rotation about } 0\}.$$

Then it is an immediate consequence of proposition 4.1 that

$$R \cong \{(\mathbf{z}, \omega) \in \mathbf{C}^{n-1} \times S^1 : \tilde{J}_K(\omega)\mathbf{z} = 0\},$$

where $\mathbf{z} = (z_2, \ldots, z_n)$. Clearly

$$R_0 = \{(\mathbf{z}, \omega) \in R : z_2 = b\}.$$

We are assuming that α is a simple root of $P_K(t) = \det \tilde{J}_K(t)$. Let $y_i =$ the center of rotation of $\rho_0(x_i)$ for all i, so that $y_1 = 0$ and $y_2 = b$. Clearly, an element of $T_{\rho_0}R$ is given by a path $(\mathbf{y}(s), \omega(s))$, where $\mathbf{y}(s) = (y_2(s), \ldots, y_n(s))$, with the property that

$$\frac{d}{ds}(\tilde{J}_K(\omega(s))\mathbf{y}(s))\,|_{s=0} = 0.$$

Since this path represents a tangent vector at ρ_0, we know that $\omega(0) = \alpha$ and $y_i(0) = y_i$ for $i = 2, \ldots, n$. An element of $T_{\rho_0}R_0$ must satisfy the additional condition that $dy_2(s)/ds\,|_{s=0} = 0$. Suppose $(\mathbf{y}(s), \omega(s))$ satisfies these conditions and, in addition, $(\mathbf{y}(s), \omega(s))$ represents a non-zero tangent vector, i.e. $d/ds(\mathbf{y}(s), \omega(s))\,|_{s=0} \neq 0$. Then there are two possibilities:

Case 1. Assume $\omega'(0) = 0$. In this case, it follows from the product and chain rules that

$$\tilde{J}_K(\alpha)\frac{d}{ds}\,\mathbf{y}(s)\,|_{s=0} = 0,$$

i.e.,

$$\frac{d}{ds}\,\mathbf{y}(s)\,|_{s=0} \in \ker \tilde{J}_K(\alpha).$$

Since $\ker \tilde{J}_K(\alpha)$ is 1-dimensional and we are assuming that projection onto its first coordinate (i.e., y_2) is a non-zero linear map, it follows that $dy_2/ds\,|_{s=0} = 0$ implies that $d/ds\,\mathbf{y}(s)\,|_{s=0} = 0$, contradicting the fact that $(\mathbf{y}(s), \omega(s))$ represents a non-zero tangent vector.

Case 2. Assume $\omega'(0) \neq 0$. Let $Y(s)$ be an $(n-1)$ by $(n-1)$ matrix whose first column is $\mathbf{y}(s)$, and whose other columns do not depend on s and, together with $\mathbf{y}(0)$, form a basis for $\mathbf{C}^{n-1}$. Since determinant is a linear function of each row, it follows that

$$\frac{d}{ds}\det(\mathcal{J}_K(\omega(s))Y(s))\,|_{s=0} = 0.$$

Because $(\mathbf{y}(s), \omega(s))$ is a tangent vector and $\det \mathcal{J}_K(\omega(0)) = \det \mathcal{J}_K(\alpha) = 0$, it follows that

$$\begin{aligned} 0 &= \frac{d}{ds}\det(\mathcal{J}_K(\omega(s))Y(s))\,|_{s=0} \\ &= \frac{d}{ds}\det(\mathcal{J}_K(\omega(s)))\,|_{s=0}\det(Y(0)) \\ &= \frac{d}{d\omega}\det(\mathcal{J}_K(\omega))\,|_{\omega=\alpha}\,\omega'(0)\det(Y(0)). \end{aligned}$$

The first of these three factors is non-zero because α is a simple root of $P_K(t)$. Since the other two factors are also non-zero, we have a contradiction, which proves the lemma. □

As usual, we identify $T_z\Delta$ with $\mathbf{C}$; then using our metric ds_c, we have, for each $z \in \Delta$, a well-defined exponential map $\exp_{c,z} : \mathbf{C} \to \Delta$. This map is only defined for those tangent vectors which don't take us outside of Δ, but that will not be a problem in the following. For each c, we now define a map $v_c : \Delta \times \Delta \to \mathbf{C}$ by the condition that

$$\exp_{c,z_1}(v_c(z_1, z_2)) = z_2.$$

Note that because $\exp_{c,z}$ is smooth and a diffeomorphism on the relevant domain, $v_c(z_1, z_2)$ is well-defined and smooth in all of its arguments.

In the following, define a *standard* relator to be one of the form

$$x_j^{\epsilon} x_k x_j^{-\epsilon} = x_l.$$

Note that we have a Wirtinger presentation of Λ consisting entirely of $n-1$ standard relators, since the one non-standard relator is a consequence of the standard ones. Number these standard relators 1 to $n-1$, and assume that the i-th one is of the form

$$x_{j_i}^{\epsilon_i} x_{k_i} x_{j_i}^{-\epsilon_i} = x_{l_i}.$$

where $\epsilon_i = \pm 1$.

We will now define a function

$$F : \Delta^{n-1} \times S^1 \times [-2, 2] \to \mathbf{C}^{n-1}.$$

Let F_i be the i-th component of F, and set $z_1 = 0$ and $z_2 = b$. Define

$$F_i(z_3, \ldots, z_{n+1}, \omega, c) = \omega^{\epsilon_i} v_c(z_{j_i}, z_{k_i}) - v_c(z_{j_i}, z_{l_i}).$$

It follows from the above that F is smooth. If we set $\omega = \alpha$ and $c = 0$, the function F is precisely the Jacobian matrix of the smashed knot (up to multiplication of each row by a non-zero constant) with α substituted for t, which has rank $n-1$. (This follows from the fact that $v_0(x, y) = y - x$.) It follows that $F|\Delta^{n-1} \times \{\alpha\} \times \{0\}$ is a linear isomorphism. Hence F is a submersion at $(p_0, \alpha, 0)$, where $p_0 \in \Delta^{n-1}$ is the $(n-1)$-tuple whose components are the centers of rotation of $\rho(x_3), \ldots, \rho(x_{n+1})$. As a consequence, there exists an open set $U \subset \Delta^{n-1} \times S^1 \times \mathbf{R}$, with $(p_0, \alpha, 0) \in U$, such that $F^{-1}(0) \cap U$ is a 2-dimensional submanifold of U. Let $\bar{W}$ be the component of $F^{-1}(0) \cap U$ containing $(p_0, \alpha, 0)$.

CLAIM 8.2. *Projection onto the last two factors of $\Delta^{n-1} \times S^1 \times \mathbf{R}$, which we denote by (ω, c), gives a smooth coordinate system on some neighborhood of $(p_0, \alpha, 0)$ in $\bar{W}$.*

Proof. This is an immediate consequence of the fact that $F|\Delta^{n-1} \times \{\alpha\} \times \{0\}$ is a linear isomorphism. □

Denote by W the neighborhood produced in the claim. The following lemma explains the meaning of the function F.

LEMMA 8.3. *Denote by $\mathbf{z}$ the element $(z_3, \ldots, z_{n+1}) \in \Delta^{n-1}$, and set $z_1 = 0$ and $z_2 = b$. Then $F(\mathbf{z}, \omega, c) = 0$ if and only if the map σ taking x_i to a rotation of $\{\mathbf{H}$ if $c > 0$, S^2 if $c < 0$, $\mathbf{E}^2$ if $c = 0\}$ about the point $h_c(z_i)$ by the angle ω for each i defines a representation of Λ into $\{\mathrm{Isom}_+ (\mathbf{H})$ or $\mathrm{Isom}_+ (S^2)$ or $\mathrm{Isom}_+ (\mathbf{E}^2)\}$.*

Proof. We need only verify that the $\sigma(x_i)$ satisfy the $n-1$ relations for Λ. Note that $F_i(\mathbf{z}, \omega, c) = 0$ if and only if a rotation about $h(z_{j_i})$ by the angle ω^{ϵ_i} takes $h(z_i)$ to $h(z_{i+1})$. This in turn is equivalent to the condition that

$$\sigma(x_{j_i})^{\epsilon_i}\sigma(x_{k_i})\sigma(x_{j_i})^{-\epsilon_i} = \sigma(x_{l_i}),$$

where $\sigma(x_i)$ is defined as in the statement of the lemma. Thus, $F(\mathbf{z}, \omega, c) = 0$ if and only if the $\sigma(x_i)$ satisfy all the relations for Λ, which proves the lemma. □

We now digress for a moment in order to establish some useful facts about homogeneous geometry. Let $X = \mathbf{H}$ or S^2 or $\mathbf{E}^2$. Suppose $x, y \in X$ are distinct and, in the case of S^2, non-antipodal. Let R_x^θ and R_y^θ be rotations of angle θ about x and y, respectively, where $0 \le \theta < \pi$. Denote by A the unique shortest length geodesic arc in X from x to y. Assuming x and y are sufficiently close, there is a (unique) point $z \in X$ such that

$$\angle zxy = \angle xyz = \frac{\theta}{2},$$

where all angles are oriented.

LEMMA 8.4. *Given the situation in the last paragraph, it follows that* $R_y^\theta R_x^\theta = R_z^{-\gamma}$, *where* $\gamma = 2\angle xzy$.

Proof. Let F_{xy} denote reflection in the unique geodesic determined by x and y. Then

$$R_y^\theta R_x^\theta = (F_{yz}F_{xy})(F_{xy}F_{xz}) = F_{yz}F_{xz} = R_z^{-\gamma}.$$

□

We now observe that in the cases $X = S^2$ or $\mathbf{H}$, as long as x and y are sufficiently close, the angle of rotation of the product $R_y^\theta R_x^\theta$ is a monotone function of the distance between x and y and hence determines this distance. To see this, use the classical fact that the sum of the angles in a triangle is a monotone function of the triangle's area.

The relation corresponding to the smashed crossing has the form $x_m x_k x_{n+1}^{-1} = x_l$, which we may rewrite $x_m x_k = x_l x_{n+1}$. We define $d_c : \Delta_c \times \Delta_c \to \mathbf{R}_+$ to be the distance between the two arguments with respect to the metric ds_c.

LEMMA 8.5. *As usual set* $z_1 = 0$ *and* $z_2 = b$ *and* $\mathbf{z} = (z_3, \ldots, z_{n+1})$. *If* $(\mathbf{z}, \omega, c) \in W$, *then* $d_c(z_m, z_k) = d_c(z_l, z_{n+1})$.

Proof. Define a function $g : W \to \mathbf{R}$ by

$$g(\mathbf{z}, \omega, c) = d_c(z_m, z_k) - d_c(z_l, z_{n+1}).$$

Clearly, g is continuous. Since (ω, c) is a coordinate system on W, it will suffice to prove that g vanishes on the set $\{(\mathbf{z}, \omega, c) \in W : c \neq 0\}$.

Assume $(\mathbf{z}, \omega, c) \in W$ and $c \neq 0$. Let σ be the corresponding representation of Λ in $\mathrm{Isom}_+(\mathbf{H})$ or $\mathrm{Isom}_+(S^2)$, as defined in Lemma 2.3, depending on whether $c > 0$ or $c < 0$. Recall that $\sigma(x_i)$ is a rotation of angle ω for all i. Since σ is a representation, we know that $\sigma(x_m)\sigma(x_k) = \sigma(x_l)\sigma(x_{n+1})$. However, in non-flat homogeneous geometry, whether hyperbolic or spherical, if we multiply two rotations of the same angle, the angle of the resulting rotation depends on the distance between the centers of rotation of the original two rotations. Since h_c multiplies distances by a constant factor (if we fix c), it follows that $g(\mathbf{z}, \omega, c) = 0$, proving the lemma. □

Next, we will define a function $\theta : W \to S^1$. Suppose we are at a fixed point $(\mathbf{z}, \omega, c) \in W$. This point determines a homomorphism σ from Λ to a group of isometries, as defined above. Let $\tilde{z}$ be the image under h_c^{-1} of the center of rotation of $\sigma(x_m)\sigma(x_k) = \sigma(x_l)\sigma(x_{n+1})$. Since our smash was done at a "good" crossing, and by shrinking the size of W if necessary, we may assume that $z_m \neq z_k$ and $z_l \neq z_{n+1}$, since this is true at the representation ρ. It follows that the angle $\theta = \angle z_m \tilde{z} z_{n+1}$ is a well-defined, smooth function on W, where this angle is with respect to geodesics in the metric ds_c. Note that since we have defined θ to take its values in S^1, the angle 0 corresponds to the value $\theta = 1$. Clearly $\theta(p_0, \alpha, 0) = 1$, because at this point the corresponding representation factors through to give a representation of Γ, so $z_m = z_{n+1}$. Consider the smooth curve on W defined by $c = 0$. By claim 8.2, we see that ω provides a local coordinate for this curve near the point $(p_0, \alpha, 0)$, so along this curve we may think of θ as a function of ω.

PROPOSITION 8.6. *$\partial\theta/\partial\omega \neq 0$ at the point $(p_0, \alpha, 0)$.*

Proof. Suppose $(\partial\theta/\partial\omega)_{(p_0,\alpha,0)} = 0$. In a neighborhood of the point $(p_0, \alpha, 0)$, θ is defined by the equation $v_c(\tilde{z}, z_{n+1}) = \theta v_c(\tilde{z}, z_m)$. Thus, we may conclude that $\partial v_c(\tilde{z}, z_{n+1})/\partial\omega = \partial v_c(\tilde{z}, z_m)/\partial\omega$ at the point $(p_0, \alpha, 0)$. Because the exponential is a local diffeomorphism, it follows that $\partial z_{n+1}/\partial\omega = \partial z_m/\partial\omega$. Note that the arc in W defined by $c = 0$ corresponds to an arc of representations $\rho_\omega : \Lambda \to \mathrm{Isom}_+(\mathbf{E}^2)$. It follows that the derivative (with respect to ω) of each relator for Λ is 0 along the arc ρ_ω.. Since $\rho_\omega(x_m)$ and $\rho_\omega(x_{n+1})$ are rotations of angle ω about the points z_m

and z_{n+1} respectively, and since we have established that $\partial z_{n+1}/\partial\omega = \partial z_m/\partial\omega$, it follows that

$$\frac{d}{d\omega}\rho_\omega(x_m) = \frac{d}{d\omega}\rho_\omega(x_{n+1})$$

at the point $\omega = \alpha$. Hence the derivative (with respect to ω) of the relator $\rho_\omega(x_m)\rho_\omega(x_{n+1})^{-1}$ is 0 at the point $\omega = \alpha$. Since this relator together with the relators for Λ give a complete set of relators for Γ, it follows that $d/d\omega$ is a non-zero Zariski tangent vector at the point $\rho = \rho_\alpha$ to the set of representations of Γ in $\mathrm{Isom}_+(\mathbf{E}^2)$ which fix the centers of rotation of x_1 and x_2 at 0 and b, respectively. This contradicts lemma 8.1 and thereby proves the proposition. □

It follows that $\theta^{-1}(1)$ is a smooth arc in W and is transverse to the arc $c = 0$. Note that a point $(\mathbf{z}, \omega, c) \in W$ corresponds to a representation of Γ in $\mathrm{Isom}_+(\mathbf{H})$ or $\mathrm{Isom}_+(S^2)$ or $\mathrm{Isom}_+(\mathbf{E}^2)$ if and only if $(\mathbf{z}, \omega, c) \in \theta^{-1}(1)$. Consider the half-arc of $\theta^{-1}(1)$ on which $c < 0$. This half arc gives us an arc of representations

$$\rho_c : \Gamma \to \mathrm{Isom}_+(S^2) = SO(3, \mathbf{R}).$$

Furthermore, by lemma 7.1, as $c \to 0$, the centers of rotation of $\rho_c(x_i)$ all approach a common point in S^2. In addition, the angle of rotation of $\rho_c(x_i)$ approaches α for all i. It follows that for $c \leq 0$, ρ_c is a continuous ray of representations, non-abelian for $c < 0$ (since $\rho_c(x_1) \neq \rho_c(x_2)$), with ρ_0 our original abelian representation in $SO(3, \mathbf{R})$ of angle α. By lemma 2.1, ρ_c lifts to an arc of representations in $SU(2)$.

Similarly, using the half-arc of $\theta^{-1}(1)$ on which $c \geq 0$, we obtain an arc of irreducible representations in $SL(2, \mathbf{R})$ converging to our original abelian representation of Γ. □

REFERENCES

[A] L. AHLFORS, *Conformal Invariants—Topics in Geometric Function Theory*, McGraw Hill, New York, N.Y., 1973.

[C-S] M. CULLER and P. SHALEN, *Varieties of group representations and splittings of 3-manifolds*, Annals of Math., *117* (1983), pp. 109–146.

[D] G. DE RHAM, *Introduction aux polynomes d'un noeud*, Enseignement Math. (2) *13* (1967), pp. 187–192.

[K] E. KLASSEN, *Representations of knot groups in SU*(2), Trans. A.M.S., to appear.

[L-M] A. LUBOTZKY and A. MAGID, *Varieties of representations of finitely generated groups*, Memoirs of the A.M.S. (58) 336 (1985).

[R] R. RILEY, *Parabolic representations of knot groups, I*, Proc. London Math. Soc. (3) *24* (1972), pp. 217–242.
[T] W. THURSTON, *The geometry and topology of 3-manifolds*, Princeton lecture notes.

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