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Leafwise hyperbolicity; a correction

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In [1], a proof of the following theorem was proposed.

THEOREM 1. *Let* $(M, \mathscr{F})$ *be a* C^2*-foliated manifold of codimension* 1, *transversely orientable and such that* M *is compact, every leaf is proper, and* $\mathscr{F}$ *is tangent to* ∂M. *If no leaf of* $\mathscr{F}$ *is a torus or a sphere, then there is a Riemannian metric on* M *relative to which each leaf of* $\mathscr{F}$ *has constant curvature* -1.

This theorem is correct, but there was an erroneous step in the proof, namely [1, Lemma (2.2)]. We are grateful to S. Matsumoto and N. Tsuchiya for pointing this out to us.

We fix the hypotheses of Theorem 1. A metric g with the property in that theorem will be called leafwise hyperbolic.

Let $M_0 \subseteq M_1 \subseteq \cdots \subseteq M_k \subseteq M_{k+1} \subseteq \cdots$ denote the level filtration [2]. Each M_k is a compact, nonempty, $\mathscr{F}$-saturated set, the leaves in $M_k \backslash M_{k-1}$ being the leaves of $\mathscr{F}$ at level k. When all leaves are proper, it has become customary to use the term "depth" rather than "level". Since all leaves are proper and the foliation is of class C^2, every leaf of $\mathscr{F}$ has finite depth, hence $M = \bigcup_{k=0}^{\infty} M_k$.

PROPOSITION 1. *Let* M_k *denote the union of leaves at dephts at most* k. *Then there is a nest* $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_k \subseteq W_{k+1} \subseteq \cdots \subseteq M$, *where* W_k *is an open neighborhood of* M_k, *and there is a Riemannian metric* g_k *on* M *such that* $g_k \mid W_k$ *is leafwise hyperbolic for* $\mathscr{F} \mid W_k$, $\forall k \geq 0$.

Theorem 1 follows. Indeed, $\{W_k\}_{k=0}^{\infty}$ is an open, nested cover of the compact manifold M, hence passing to a finite subcover yields a value of k for which $W_k = M$. It remains, then, to prove Proposition 1.

We fix a smooth, 1-dimensional foliation $\mathscr{F}^{\perp}$, everywhere transverse to $\mathscr{F}$. Projections along the leaves of $\mathscr{F}^{\perp}$ can be used to define local diffeomorphisms between leaves of $\mathscr{F}$.

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If $U \subseteq M$ is an open, connected, $\mathscr{F}$-saturated set, we use the notations $\hat{U}, \hat{\imath} : \hat{U} \to M$, $\hat{\mathscr{F}} = \hat{\imath}^{-1}(\mathscr{F})$, and $\hat{\mathscr{F}}^{\perp} = \hat{\imath}^{-1}(\mathscr{F}^{\perp})$ from [1], [2], *et al.*, for the completion of U, its natural immersion into M, and the induced foliations of $\hat{U}$, respectively. Recall that U and $\hat{U}$ are called *foliated products* if $\hat{U}$ is diffeomorphic to $L \times [0, 1]$ in such a way that the leaves of $\hat{\mathscr{F}}^{\perp}$ are the $[0, 1]$-fibers. Recall that, if U is a foliated product, then $\hat{\imath}(\partial \hat{U})$ is either a single leaf or a pair of leaves of $\mathscr{F}$.

DEFINITION 1. A closed subset $X \subset M$ that is a finite union of leaves of $\mathscr{F}$ will be called a skeleton if each component of $M \backslash X$ is a foliated product. If k is the highest depth of the leaves in X, the skeleton has depth k. We will say that X (of depth k) is a full skeleton if, for each component U of $M \backslash X$, at least one of the following holds.

(1) Every leaf L of $\hat{\mathscr{F}}$ has image $\hat{\imath}(L)$ at the same depth $k_0 \leq k$.

(2) If $L \subset \partial \hat{U}$ is a boundary leaf, then $\hat{\imath}(L)$ is a leaf at depth k.

If X is a skeleton, it was proven in [1, (1.2)] that there is an open neighborhood $W \supset X$ and a Riemannian metric g on M such that $g \mid W$ is leafwise hyperbolic for $\mathscr{F} \mid W$. Furthermore, projection along the leaves of $\mathscr{F}^{\perp}$ defines local isometries between the leaves of $\mathscr{F} \mid W$. Finally, $\hat{U} \backslash \hat{\imath}^{-1}(W)$ is compact, for each component U of $M \backslash X$.

LEMMA 1. *If there is a full skeleton X of depth N, then there is a neighborhood $W_N \supset M_N$ and a Riemannian metric g_N on M which is leafwise hyperbolic on W_N.*

Proof. Let U be a component of $M \backslash X$. There are two cases, corresponding to possibilities (1) and (2) of Definition 1.

(1) In this case, the proof of [1, Lemma (2.1)] shows how to extend the metric smoothly over all of U so as to make the curvature of the leaves of $\mathscr{F} \mid U$ constantly -1. Indeed, the metric was already appropriately defined on all but a compact submanifold $A \times [0, 1] \subset \hat{U}$ and $\mathscr{F}$ induces the product foliation on this submanifold. A deformation argument, using the Teichmüller space of A, created the extension. (The error in [1] was to claim that, even in the second case, where the foliation of $A \times [0, 1]$ was not a product, the above metric on the product could be "tilted" to give a hyperbolic metric along the leaves.)

(2) We assume that the situation in (1) does not also occur. In this case, the argument is actually easier. Since M_N is compact [2, (4.6)], $\hat{\imath}^{-1}(M_N) \cap \hat{U} = L \times C$, where $C \subset [0, 1]$ is a closed subset containing $\{0, 1\}$. Since $U \backslash M_N \neq \varnothing$, $[0, 1] \backslash C$ has at least one component (a, b). Let $a < a' < b' < b$. The metric g is already defined on $W \cap \hat{\imath}(\hat{U})$ in such a way that projections along $\mathscr{F}^{\perp}$ are local isometries between leaves. Using the projections $p^{+} : L \times (b', 1] \to L \times \{1\}$ and $p^{-} : L \times [0, a') \to$

$L \times \{0\}$, one lifts this metric smoothly to $L \times [0, a') \cup L \times (b', 1]$. This metric agrees with g wherever both are defined.

Finite repetition of this argument, as U ranges over the components of $M \backslash X$, completes the proof. □

LEMMA 2. *For some integer $N \geq 0$, there exists a full skeleton of depth N.*

Proof. As in [1, (1.1)], one constructs a skeleton X. Let N be the depth of X. If X is not full, consider a component U of $M \backslash X$ with boundary component(s) at depth $k < N$. If every leaf of $\mathscr{F} \mid U$ is at depth k, there is nothing to do. Otherwise, there is a leaf $L \subset U$ at depth $k + 1 \leq N$. It is elementary that $X' = X \cup L$ is again a skeleton of depth N. If X' is not full repeat the process for X'. Finite repetition will ultimately produce a full skeleton of depth N. □

For $0 \leq k \leq N$, we set $W_k = W_N$ and $g_k = g_N$. We also set $X = X_N$.

Each component U_i of $M \backslash X_N$ that has not been engulfed by W_N must contain a leaf L_i at depth $N + 1$. Throwing these finitely many leaves in with X_N provides a full skeleton X_{N+1} of depth $N + 1$. An application of Lemma 1 produces W_{N+1} and g_{N+1} as desired. It is not hard to see that W_{N+1} can be chosen to engulf W_N. Proceeding in this way, we construct the nest of open sets and the metrics as in Proposition 1.

REMARK. Projection along the leaves of $\mathscr{F}^{\perp}$ does not always define local isometries between the leaves of $\mathscr{F}$. In the pieces $A \times [0, 1]$, where the metric is extended by a deformation in Teichmüller space, these projections will not be isometric. If it were possible to avoid introducing these regions, it would follow that the leafwise hyperbolic metric for $\mathscr{F}$ is a bundlelike metric for $\mathscr{F}^{\perp}$, hence that the leaves of $\mathscr{F}$ are totally geodesic in this metric. But totally geodesic foliations of compact 3-manifolds by surfaces are relatively rare.

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