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# Harmonic map heat flow with free boundary

MA LI

*Abstract.* In this paper, we study the harmonic map heat flow with free boundary from a Riemannian surface with smooth boundary into a compact Riemannian manifold. As a consequence, we get at least one disk-type minimal surface in a compact Riemannian manifold without minimal 2-sphere.

## 1. The motivation

Let  $(M, g)$  be a normalized compact Riemannian surface, i.e. a Riemannian surface whose boundary consists of a finite number of closed geodesics of  $(M, g)$ . Let  $(N, h)$  be a compact Riemannian manifold of dimension  $n$ . For any smooth map  $u : M \rightarrow N$ , we define an energy of  $u$  by  $E(u) = \int_M |du|^2 dM$  where  $|du|$  is the Hilbert-Schmidt norm of the operator  $du$ .

Let  $K$  be a  $k$ -dimensional closed submanifold of  $N$  and we define

$$C(K) = \{u \in C^2(M, N); u(\partial M) \subset K\}.$$

Then the critical point of  $E(\ )$  over  $C(K)$  is a harmonic map with free boundary. The motivation of this paper comes from following basic problem: WHEN DOES A GIVEN COMPONENT OF  $C(K)$  CONTAIN A HARMONIC MAP WITH FREE-BOUNDARY?

A natural way to attack this problem is to flow  $u_0$  in  $C(K)$  along the negative  $L^2$ -gradientline of  $E$ . That is, we want to find  $u : M \times (0, +\infty) \rightarrow N$  satisfying following equations:

$$\partial_t u(x, t) = \tau(u(x, t)), \quad \forall (x, t) \in M \times (0, \infty), \quad (1)$$

$$u(, 0) = u_0, \quad (2)$$

$$u(x, t) \in K, \quad a.a.x \in \partial M \quad \text{for } t \geq 0, \quad (3)$$

$$du_{(x,t)}(\vec{n}) \perp T_{u(x,t)}K, \quad \forall (x, t) \in \partial M \times (0, \infty), \quad (4)$$

where  $\tau(u)$  is the tension field of  $u$  in the sense of Eells-Sampson [ES],  $\vec{n}$  is the inward unit normal on  $M$ , and “ $\perp$ ” means orthogonal.

A global way to write (1) in one chart in the range was found in [ES]. We adapt a statement from [H]. We imbed  $N$  in a Euclidean space  $E^d$  by Nash's isometric imbedding theorem, but we prefer to change the ordinary metric on  $E^d$ . Let  $\mathbf{T}$  be a tubular neighborhood of  $N$  in  $E^d$ , define  $\iota : \mathbf{T} \rightarrow \mathbf{T}$  be an involution having precisely  $N$  for its fixed point set, take an extension of the metric  $h$  to  $\mathbf{T}$ , and average it under the action of  $\iota$ , then  $\iota$  is an isometry. Finally, let  $B$  be a large ball in  $E^d$  containing  $\mathbf{T}$ , and extend the metric on  $\mathbf{T}$  smoothly to all of  $E^d$  so as to equal the Euclidean metric outside of  $B$ , then we obtain a new metric vector space  $R^d$ . In  $R^d$ ,  $B$  is also convex. If we consider  $u$  as a map from  $M$  to  $R^d$ , then (1) can be written as:

$$\partial_t u^i = \Delta_M u^i + A_u^i(\nabla u, \nabla u), \quad 1 \leq i \leq d \quad (5)$$

where  $A_u : T_u N \times T_u N \rightarrow T_u R^d$  is the second fundamental form of  $N$  in  $R^d$ .

Since the natural space for  $E$  make sense is  $H^1(M, N)$ , we shall enlarge the initial values class of (2) to

$$H(K) = \{u \in H^1(M, N); u(x) \in K; a.a.x \in \partial M\},$$

By the argument of [SU], we know that  $C(K)$  is dense in  $H(K)$ . Our main result is the following:

**THEOREM 1.** *Let  $u_0 \in H(K)$ , then there exists a (distribution) global solution  $u$  of (1-4), which is regular except a finite point set  $\{(x^l, T^l)\}_{1 \leq l \leq L} \subset M \times (0, \infty)$  such that at each point  $(x^l, T^l)$ , there exist sequences  $x_j^l \rightarrow x^l$ ,  $t_j^l \nearrow T^l$ , and  $R_j^l \searrow 0$  such that (a) if  $x^l \in \text{int } M$ , then  $u(x^l + R_j^l \times x, t_j^l) \rightarrow u_l(x)$  in  $H^2_{\text{loc}}(R^2, N)$  where  $u_l : R^2 \rightarrow N$  can be extended to a minimal 2-sphere in  $N$ .*

(b) if  $x^l \in \partial M$ , then

$$u(x^l + R_j^l \times x, t_j^l) \rightarrow u_l(x) \quad \text{in } H^2_{\text{loc}}(R^2_+, N)$$

where  $u_l : R^2_+ \rightarrow N$  can be extended to minimal 2-disk in  $N$  with free boundary in  $K$ .

*The solution  $u$  is unique in this class and the singularities are characterized by the condition that  $\lim_{t_j^l \rightarrow T^l} E(u(t), B_R(x^l)) \geq \varepsilon_0$ ,  $\forall R > 0$  where  $\varepsilon_0$  is a uniform constant.*

*If  $u_0 \in C(K)$ , then the solution  $u$  is regular on  $M \times [0, \bar{T})$  where  $\bar{T} = \min \{T^l\}$ .*

As a direct consequence of Theorem 1, we obtain

**THEOREM 2.** *Let  $N$  be a compact Riemannian manifold with no minimal 2-sphere. Let  $K$  be a connected and simply connected closed submanifold of  $N$ . Then there exists at least one minimal 2-disk with free boundary in  $K$ .*

So our Theorem 2 is a generalization of the result of [St] and one may see [St1] for further references. The main difficulty in proving Theorem 1 is as follows. We cannot directly use implicit function theorem to obtain the small time existence of a solution of (1)–(4), but we work through by proving a uniqueness theorem and using fixed point theorem after a subtle geometrical construction. The large time behaviour of the solution was studied as in [St]. Our argument is similar to [St2]. I should mention the work of Kungching Chang [C] which is very important in relating harmonic map heat flow with minimax principle and Morse theory. In the following, without loss of generality, we will assume that the injectivity radius of  $(M, g)$  is one and the  $\text{sect}(N) \leq 1$ .

## Notations

For a domain  $\Omega \subset M$ ,  $-\infty \leq s < t \leq +\infty$  let

$$\Omega'_s = \Omega \times (s, t), \quad \Omega_s^0 = \Omega_s, \quad \Omega'_0 = \Omega'$$

and the space

$$V(\Omega'_s) = \{u \in C^0([s, t]; H^1(\Omega, N)); |\nabla^2 u|, |\partial u| \in L^2(\Omega'_s)\}$$

where the derivatives are taken in the distribution sense.

$\nabla$  denotes the covariant derivative on  $(M, g)$ . In particular, for a local frame  $\{e_1, e_2\}$  we write  $\nabla_i = \nabla_{e_i}$  for  $i = 1, 2$ .

At a local frame of  $x_0$ , we denote  $x_0 + x = \exp_{x_0} x$  for  $x \in \mathbb{R}^2$ .  $\zeta(x) \in C_0^3(B_{2R}^M(x_0))$  denotes a non-increasing function of the distance  $\text{dist}_M(x, x_0)$  such that  $\zeta = 1$  on  $B_R^M(x_0)$  and  $=0$  outside  $B_{2R}^M(x_0)$  and  $|\nabla \zeta| \leq 3R^{-1}$

$c$  denotes a generic constant depending only on  $M$ ,  $N$  and  $K$  sometimes numbered for clarity.

## 2. A-priori bounds

The following Sobolev-type equality is crucial in our estimates.

LEMMA 1. *There exist two constants  $c$  and  $R_0$  for any  $R \in (0, R_0)$ ,  $T \leq \infty$ , and any  $u \in V(M^T)$ , there holds the estimate*

$$\begin{aligned} \int_{M^T} |\nabla u|^4 dM dt &\leq c \operatorname{esssup}_{(x,t) \in M^T} \int_{B_R(x)} |\nabla u|^2 dM \\ &\times \left( \int_{M^T} |\nabla^2 u|^2 dM dt + R^{-2} \int_{M^T} |\nabla u|^2 dM dt \right). \end{aligned}$$

Moreover, for any  $x_0 \in M$ ,  $\zeta$  as before, we have estimate:

$$\begin{aligned} \int_{M^T} |\nabla u|^4 \zeta dM dt &\leq \operatorname{esssup}_{0 \leq t \leq T} \int_{B_R(x_0)} |\nabla u|^2 dM \\ &\times \left( \int_{M^T} |\nabla^2 u|^2 \zeta dM dt + R^{-2} \int_{M^T} |\nabla u|^2 \zeta dM dt \right). \end{aligned}$$

*Proof.* The same argument as that of Lemma 3.2 in [St].

Q.E.D

Now let us introduce a key quantity

$$\varepsilon(R) = \lim_{(x,t) \in M^T} E(u(\cdot, t); B_R(x)),$$

which describes the energy distribution of the map  $u$  over all balls of radius  $R$  in  $M$  and plays an important role in following estimates.

LEMMA 2. *If  $u \in V(M^T)$  be a solution of (1)–(4), then  $E(u(t))$  is absolutely continuous in  $t \in [0, T]$  and there holds the estimate*

$$E(u(t)) + \int_{M^t} |\partial_t u|^2 dM dt \leq E(u_0) \tag{6}$$

*Proof.* By Lemma 1 we may multiply (5) by  $u_t$  and integrate. On account of (3)–(4) this gives for any  $s, t \in [0, T]$ :

$$\int_s^t \frac{d}{dt} E(u(t)) dt + \int_{M_s^t} |\partial_t u|^2 dM dt = 0$$

and the claim follows.

LEMMA 3. *There exist constants  $c_1, \varepsilon_1 > 0$  depending only on  $M, K$  and  $N$  such that for any solution  $u \in V(M^T)$  of (1)–(4) and any  $R \in (0, 0.5]$  there holds the estimate*

$$\int_{B_R(x)} |\nabla u(T)|^2 + \int_{B_r(x)^T} |\nabla^2 u|^2 \leq c_1 E(u_0) + c_1 T R^{-2} \sup_{0 \leq t \leq T} E(u(t))$$

provided  $\varepsilon(2R) \leq \varepsilon_1$ .

*Proof.* Testing (5) with  $-\Delta_M u \zeta^2$  we obtain

$$\begin{aligned} & \int_{M^T} |\Delta u|^2 \zeta^2 \, dM \, dt + \int_{M^T} |\nabla u(T)|^2 \zeta^2 \, dM - \int_M |\nabla u_0|^2 \, dM \\ & \leq c \varepsilon_1 \int_{M^T} |\nabla^2 u|^2 \zeta^2 \, dM \, dt + c T R^{-2} \sup_{0 \leq t \leq T} E(u(t)). \end{aligned} \quad (7)$$

Without loss of generality, we assume  $x_0 \in \partial M$ . Let  $\{e_1, e_2\}$  be a local frame on  $B_R(x_0)$  such that  $e_1$  is the outward normal on  $\partial M$ . Testing (5) with  $-\nabla_2^2 u \zeta^2$  we find

$$\begin{aligned} & \int_{M^T} \partial_t \left( \frac{1}{2} |\nabla_2 u|^2 \right) \zeta^2 + \int_{M^T} |\nabla \nabla_2 u|^2 - c T R^{-2} \sup_{0 \leq t \leq T} E(u(t)) \\ & \leq c \left( \int_{M^T} |\nabla_2 u|^2 \zeta^2 + \int_{M^T} |\nabla u|^2 |\nabla_2^2 u| \right) + 1 \left| \int_{(\partial M)^T} \langle \nabla_1 u, \nabla_2^2 u \rangle \zeta^2 \right| + \int_{M^T} |\nabla u|^2 \zeta^2. \end{aligned} \quad (8)$$

For  $u \in K$  now let  $\{G_1(u), \dots, G_{n-k}(u)\} \subseteq TN$  be the outward normal frame to  $K$ . By (4),

$$\nabla_1 u = \sum_{i=1}^{n-k} \langle \nabla_1 u, G_i(u) \rangle G_i(u) = \sum_{i=1}^{n-k} (\nabla_1 u \cdot G_i(u)) G_i(u) \quad \text{on } \partial M.$$

By (3)

$$\langle \nabla_2 u, G_i(u) \rangle = 0 \quad \text{a.e. on } \partial M.$$

Differentiating this equality w.r.t  $e_2$  on  $\partial M$  yields that

$$\langle G_i(u), \nabla_2^2 u \rangle = -\langle \nabla G_i(u) \cdot \nabla_2 u, \nabla_2 u \rangle,$$

hence on  $\partial M$ ,

$$\langle \nabla_1 u, \nabla_2^2 u \rangle = - \sum_{i=1}^{n-k} (\nabla_1 u \cdot G_i(u)) (\nabla_2 u \cdot \nabla G_i(u) \cdot \nabla_2 u).$$

Smoothly extended  $G_i$  to  $B \subset \mathbb{R}^d$  and by Stokes' theorem,

$$\begin{aligned} \int_{\partial M} \nabla_1 u \cdot \nabla_2^2 u \zeta^2 &= \int_M \Delta u \cdot [G_i(u) (\nabla_2 u \cdot \nabla G_i(u) \cdot \nabla_2 u)] \zeta^2 \\ &\quad + \int_M \nabla u \cdot \nabla [G_i(u) (\nabla_2 u \cdot \nabla G_i(u) \cdot \nabla_2 u)] \zeta^2 \\ &\quad + \int_M \nabla u \cdot G_i(u) (\nabla_2 u \cdot \nabla G_i(u) \cdot \nabla_2 u) \nabla \zeta^2 \\ &\leq c \int_M (|\nabla^2 u| |\nabla u|^2 + |\nabla u|^4) \zeta^2 + |\nabla \zeta|^2 |\nabla u|^2. \end{aligned}$$

Hence for any  $\delta \in (0, 1)$  we know from (8):

$$\begin{aligned} &\frac{1}{2} \int_M |\nabla_2 u(T)|^2 \zeta^2 - \frac{1}{2} \int_M |\nabla_2 u_0|^2 \zeta^2 + \int_{M^T} |\nabla \nabla_2 u|^2 \zeta^2 \\ &\leq (c\varepsilon_1 + \delta) \int_{M^T} |\nabla^2 u|^2 \zeta^2 + c(\delta) T R^{-2} \sup_{0 \leq t \leq T} E(u(t)). \end{aligned}$$

Since

$$\begin{aligned} |\nabla^2 u|^2 &= |\nabla_1^2 u|^2 + |\nabla_2^2 u|^2 + |\nabla_1 \nabla_2 u|^2 \\ &= (\Delta u - \nabla_2^2 u)^2 + |\nabla_2^2 u|^2 + 2|\nabla_1 \nabla_2 u|^2 \\ &\leq 2(|\Delta u|^2 + 2|\nabla \nabla_2 u|^2), \end{aligned} \tag{9}$$

we obtain from (9) and (7):

$$\int_{M^T} |\nabla_2^2 u|^2 \zeta^2 \leq (c\varepsilon_1 + 4\delta) \int_{M^T} |\nabla^2 u|^2 \zeta^2 + cE(u_0) + c(\delta) T R^{-2} \sup_{0 \leq t \leq T} E(u(t)),$$

and the claim follows from  $\varepsilon_1, \delta > 0$  small enough.

Using

$$\int_{M^T} |\partial_t u|^2 dM dt \leq c \int_{M^T} |\nabla^2 u|^2 \zeta^2 dM dt + c \int_{M^T} |\nabla u|^4 \zeta^2 dM dt,$$

in the above argument we obtain the following:

**LEMMA 4.** *There exist constants  $c_2, \varepsilon_2 > 0$  depending only on  $M, K$  and  $N$  such that for any solution  $u \in V(M^T)$  of (1)–(4), any  $R \in (0, 0.5)$ , any  $x \in M$  there holds:*

$$E(u(T), B_r(x_0)) \leq 2E(u_0; B_{2R}(x_0)) + c_2 T R^{-2}$$

and

$$\int_{B_R(x_0)} |\nabla^2 u|^2 dM dt \leq 4c_2(1 + T R^{-2})$$

provided  $\varepsilon(2R) \leq \varepsilon_2$ .

### 3. Uniqueness

Let  $\hat{M}$  be the double of  $M$ , it is well-known that  $\hat{M}$  is a closed Riemannian surface. We will denote  $M_{(a)}$  the  $a$ -tubular neighbourhood of  $M$  in  $\hat{M}$  and  $M_{(-a)} = \{x \in M; \text{dist}(x, \partial M) \geq a\}$ . Then for  $a$  small enough and for any  $x \in M_{(-a)}$ , there exists a unique projection point  $x' \in \partial M$  and a unique point  $\bar{x} \in M_{(a)}/M$  that has also  $x'$  as the unique projection onto  $\partial M$  along the geodesic  $c(t) = \exp_{x'} \{-\exp_{x'}^{-1} x\}$ . So we may define an involution  $\Phi$  (that is  $\Phi^2 = \text{Id}$ ) by

$$\Phi(x) = \bar{x} \quad \text{if } x \in M_{(-a)} \quad \text{and} \quad \Phi(\bar{x}) = x \quad \text{for } \bar{x} \in M_{(a)}/M$$

Similarly, we can define an involution  $I$  in a  $b$ -neighbourhood  $K_b$  of  $K$  in  $B$ . So the restriction of  $I$  in  $N$  is an involution in the  $b$ -neighborhood of  $K$  in  $N$ . For a solution  $u \in V(M^T)$  to (1)–(4) now we consider the set

$$\tilde{\Omega} = \{(x, t) \in (M_{(a)}/M))^T, u(\Phi(x), t) \in K_b\}$$

and define an extension of  $u$  to  $\hat{\Omega} \equiv M^T \cup \tilde{\Omega}$

$$\hat{u}(x, t) = \begin{cases} u(x, t), & \text{if } x \in M; \\ I(u(\Phi(x), t)), & \text{if } (x, t) \in \tilde{\Omega}. \end{cases}$$

By the interior regularity results of (1) (see [C]),  $\tilde{\Omega}$  is open in  $M_{(a)} \times (0, T)$  and it is meaningful to consider

$$I(\hat{u}(\Phi(x), t)) = u(x, t)$$

for  $(x, t) \in M^T$  with  $(\Phi(x), t) \in \tilde{\Omega}$ . It can be verified that:

LEMMA 5. Suppose  $u \in V(M^T)$  solves (1)–(4), then  $\hat{u}$  satisfies

$$\nabla^2 \hat{u}, \partial_t \hat{u} \in L^2_{\text{loc}}(\hat{\Omega})$$

and  $\hat{u}$  is a weak solution to the system

$$\partial_t \hat{u} - \Delta_{\hat{M}} \hat{u} + \Gamma_{\hat{u}}(\nabla \hat{u}, \nabla \hat{u}) = 0 \quad (10)$$

on  $\hat{\Omega}$ , where  $\Gamma_{\hat{u}}(\cdot, \cdot)$  be a bounded bilinear form.

*Proof.* To obtain (10) note the pointwise estimates for all  $(x, t) \in \tilde{\Omega}$ .

$$|\partial_t \hat{u}(x, t)| \leq |DI(u) \cdot \partial_t u(\Phi(x), t)| \leq c |\partial_t u(\Phi(x), t)|, \text{ etc.} \quad (*)$$

Moreover, it is easy to see that  $\hat{u}(t)$  is of class  $H^2$  for a.e.  $t \in [0, T]$  on its domain  $\hat{\Omega}(t)$ . Indeed  $\hat{u}(t) \in H^2$  separately on  $M$  and  $\hat{\Omega}(t)/M =: \tilde{\Omega}(t)$  for a.e.t. By properties (3)–(4) we have that for any  $\phi \in C_0^\infty(\hat{\Omega}(t); R^d)$ :

$$\int_{\hat{\Omega}(t)} \hat{u} \nabla^2 \phi \, dM = \int_{\tilde{\Omega}(t)} \hat{u} \nabla^2 \phi \, dM + \int_M \hat{u} \nabla^2 \phi \, dM = \int_{\hat{\Omega}(t)} \nabla^2 u \phi \, dM + \int_M \nabla^2 u \phi \, dM,$$

while the boundary terms cancel. Hence the  $L^2$ -function  $\nabla^2 u(t)$  (defined on  $M \cup \tilde{\Omega}(t)$ ) is the second distributional derivative of  $\hat{u}(t) \in H^2(\hat{\Omega}(t), R^d)$  for a.e.t.

Since  $I^2 = \text{Id}$  on  $K_b$  and for  $v \in TK_b$ , we obtain that

$$DI_{I(y)} DI_y v = v$$

for  $y \in K_b$ . Let  $\bar{v} = DI(v)$ , then by differentiate the above equation we have that

$$D^2 I_{I(y)}(DI_y(v), DI_y(v)) + DI_{I(y)} D^2 I_y(v, v) = 0$$

here we assume  $D_v v = 0$  at the point  $y$ . Then

$$D^2 I_{I(y)}(\bar{v}, \bar{v}) = -DI_{\bar{y}} D^2 I_y(v, v)$$

where  $\tilde{y} = I(y)$ . Let  $e_i$  be a moving frame on  $M_{(a)}$ , by the relation

$$u(x, t) = I(\hat{u}(\Phi(x), t))$$

we obtain further that

$$\begin{aligned} \partial_t u(x, t) &= DI_{\hat{u}} \cdot \partial_t \hat{u}(x, t), \\ \nabla_i u(x, t) &= DI_{\hat{u}} \cdot \nabla \hat{u}_{(\bar{x}, t)}(\nabla_i \Phi(x), t), \\ \Delta_M u(x, t) &= \nabla_i \nabla_i u(x, t) \\ &= D^2 I_{\hat{u}}(\nabla_{\bar{e}_i} \hat{u}, \nabla_{\bar{e}_i} \hat{u}) + DI_{\hat{u}} \cdot \nabla^2 \hat{u}_{(x, t)}(\bar{e}_i, \bar{e}_i) \\ &= DI_{\hat{u}} \cdot \Delta_{M_{(a)}} \hat{u}_{(\bar{x}, t)} + DI_{\hat{u}} \cdot D^2 I_{\Phi(a)}(\nabla \Phi(\nabla_{\bar{e}_i} \hat{u}), \nabla \Phi(\nabla_{\bar{e}_i} \hat{u})), \end{aligned}$$

where  $\bar{e}_i = \nabla \Phi(e_i)$ . By (1) we have (10). Q.E.D

Later we will use another implication of (3)–(4). Define  $P : K_b \rightarrow K$  by  $\text{dist}(y, p(y)) = \text{dist}(y, K)$ . Define  $\phi \in C_0^\infty(R)$  satisfying  $0 \leq \phi \leq 1$ ,  $\phi(s) = 1$  if  $|s| < a/2$ , while  $\phi(s) = 0$ , if  $|s| > a$ . Then for any solution  $u \in V(M^T)$  with extension  $\hat{u}$  the function

$$\hat{\phi}(x, t) = \begin{cases} 1 & \text{if } x \in M \\ \phi(\text{dist}_N(\hat{u}(x, t), P(\hat{u}(x, t)))) & \text{if } (x, t) \in \tilde{\Omega} \\ 0 & \text{if } (x, t) \notin \tilde{\Omega} \end{cases} \quad (11)$$

belongs to  $H_{\text{loc}}^2(\tilde{\Omega}(t))$  for a.e.t. and satisfies a.e.

$$|\nabla \hat{\phi}| \leq c |\nabla \hat{u}|, \quad |\nabla^2 \hat{\phi}| \leq c |\nabla^2 \hat{u}|, \quad (11)$$

moreover, the distributional derivative  $\partial_t \hat{\phi} \in L_{\text{loc}}^2$  and

$$|\partial_t \hat{\phi}| \leq c |\partial_t \hat{u}| \quad \text{a.e.} \quad (12)$$

**LEMMA 6.** Suppose  $u_1, u_2 \in V(M^T)$  are twoo weak solutions to (1)–(4) with  $u_1(0) = u_2(0) = u_0$ , then  $u_1 = u_2$ .

*Proof.* Let  $\hat{u}_j$  be the extension of  $u_j$  defined in the above of Lemma 5 and  $\hat{\varphi}_j$  the associated truncation function  $j = 1, 2$ . Define  $\hat{\varphi} = \min\{\hat{\varphi}_1, \hat{\varphi}_2\}$  and  $|\nabla \hat{u}| = |\nabla \hat{u}_1| + |\nabla \hat{u}_2|$ , etc. Subtracting equations (5) for  $u_1, u_2$  and testing with  $(u_1 - u_2)\hat{\varphi}^2$  we obtain the following estimates for  $\hat{v} = \hat{u}_1 - \hat{u}_2$ :

$$\begin{aligned} |\partial_t \hat{v} - \Delta_{\hat{M}} \hat{v}| &\leq c(|\nabla \hat{v}| |\nabla \hat{v}| + |\nabla \hat{u}|^2 |\hat{v}|) \\ \int_{M^T} \left( \frac{1}{2} \partial_t (|\hat{v}|^2 \varphi^2) + |\nabla \hat{u}|^2 \hat{\varphi}^2 \right) dM dt &\leq c \int_{M^T} \{ |\hat{v}|^2 |\partial_t \hat{\varphi}| \hat{\varphi} + |\nabla \hat{v}| |\hat{v}| |\nabla \hat{\varphi}| \hat{\varphi} + |\nabla \hat{v}| |\hat{v}| \hat{\varphi}^2 \right. \\ &\quad \left. + |\nabla \hat{u}| |\nabla \hat{v}| |\hat{v}| \hat{\varphi}^2 + |\nabla \hat{u}|^2 \hat{\varphi}^2 \} dM dt. \end{aligned}$$

By (11)–(12) we may bound the above by

$$c \int_{M^T} [|v|^2 (|\partial_t u| + |\nabla u|^2) + |\nabla v| |v| (1 + |\nabla u|)] dM dt,$$

here we have used the mean value theorem:

$$\begin{aligned} \tilde{v}(x, t) &= (\hat{u}_1 - \hat{u}_2)(x, t) = (I \circ u_1 - I \circ u_2)(\Phi(x), t) \\ &= \int_0^1 DI(u_2 + \theta(u_1 - u_2)) \cdot (u_1 - u_2) d\theta|_{(\Phi(x), t)}, \end{aligned}$$

and the bounds

$$|\hat{v}(x, t)| \leq c |v(\Phi(x), t)|, \quad |\nabla \hat{v}(x, t)| \leq c |\nabla v(\Phi(x), t)|$$

for all  $x \notin M$ , and integral estimates for  $\hat{v}$  and  $\nabla \hat{v}$  can be obtained from estimates for  $v$  and  $\nabla v$  on  $M^T$  respectively.

From Holder's inequality we now obtain

$$\begin{aligned} \int_M |v(T)|^2 dM + \int_{M^T} |\nabla v|^2 dM dt &\leq c \\ &\times \left( \int_{M^T} (|\partial_t u|^2 + |\nabla u|^4) dM dt \left( \int_{M^T} |v|^4 dM dt \right)^{1/2} \right. \\ &\quad \left. + \frac{1}{2} \int_{M^T} |\nabla v|^2 dM dt + c \int_{M^T} |v|^2 dM dt. \right) \end{aligned}$$

Without loss of generality, we may assume that  $T$  is chosen such that

$$\int_M |v(T)|^2 dM = \sup_{0 \leq t \leq T} \int_M |v(t)|^2 dM.$$

Using Lemma 1 for the estimate of  $L^4$ -norm of  $v$ , we conclude that

$$\begin{aligned} \sup_{0 \leq t \leq T} \int_M |v(t)|^2 dM + \int_{M^T} |\nabla v|^2 dM dt &\leq cT \sup_{0 \leq t \leq T} \int_M |v(t)|^2 dM dt \\ &+ c \left( \int_{M^T} (|\partial_t u|^2 + |\nabla u|^4) dM dt \right)^{1/2} \left( \sup_{0 \leq t \leq T} \int_M |v(t)|^2 dM + \int_{M^T} |\nabla v|^2 dM dt \right) \end{aligned}$$

and by absolute continuity of the Lebesgue integral the above can be bounded by

$$\frac{1}{2} \left( \sup_{0 \leq t \leq T} \int_M |v(t)|^2 dM + \int_{M^T} |\nabla v|^2 dM dt \right)$$

for  $T > 0$  sufficiently small. Hence  $v = 0$  on  $M^T$  for this  $T$ . By iteration  $v = 0$  on  $M^T$  for any  $T > 0$  such that  $u_1, u_2$  are defined on  $[0, T]$ .

#### 4. Local existence

**THEOREM A.** *For  $u_0 \in C(K)$ , there exists a number  $T > 0$  depending only on  $M, K$  and  $N$  such that (1)–(4) admits a solution in  $V(M^T)$ .*

*Proof.* We want to prove this theorem by Schauder's fixed point principle. Let

$$\hat{u}_0(x) = \begin{cases} u_0(x) & \text{if } x \in M; \\ I(u_0(\Phi(x))) & \text{if } x \in M_{(a)}/M. \end{cases}$$

and for any  $T > 0$  and  $\sigma > 0$  sufficiently small let

$$\mathcal{M} = \left\{ \hat{u} \in V(M_a^T; B) \mid \hat{u}(0) = \hat{u}_0, \text{esssup}_{0 \leq t \leq T} \int_{M_{(a)}^T} |\nabla \hat{u}(t) - \nabla \hat{u}_0|^q dM \leq \sigma \right\}$$

where  $q$  be a number  $\geq 4$  to be determined later.  $\mathcal{M}$  is a closed convex set in  $V(M_{(a)}^T; \mathbb{R}^d)$ . To  $\hat{u} \in \mathcal{M}$  we now associate the unique solution  $\hat{v} = f(\hat{u}) \in V(M_{(a)}^T; \mathbb{R}^d)$

of the Cauchy-Dirichlet problem

$$\partial_t \hat{v} - \Delta_{\hat{M}} \hat{v} + \Gamma_{\hat{u}}(\nabla \hat{u}, \nabla \hat{u}) = 0 \quad \text{in } M_{(a)}^T, \quad (13)$$

$$\hat{v}(0) = \hat{u}_0, \quad (14)$$

$$\hat{v} = \hat{u} \quad \text{on } \partial M_{(a)} \times [0, T]. \quad (15)$$

It is a standard fact that we have the following uniform estimate

$$\int_{M^T} (|\partial_t \hat{v}|^{q/2} + |\nabla^2 \hat{v}|^{q/2}) dM dt \leq c(u_0).$$

(see [LSU], Theorem 4.9.1). By Lemma 4.1 of [C] we may bound the derivative of  $\hat{v}$  on  $M_{(-a)} \times [0, T]$  if  $u_0$  is sufficiently smooth on  $M$ . Here we followed the argument of [St2]. Hence for  $a > 0, T > 0, \sigma > 0$  small enough, the point set  $\{\hat{v}(\Phi(x), t); x \in \partial M_{(a)}\}$  will lie in a  $b$ -neighbourhood of  $K$  in  $B$  and their reflection in  $K$  will be defined. We may therefore define a map  $F: \mathcal{M} \rightarrow \mathcal{M}$  by letting  $\hat{w} = F(\hat{u})$  be the unique solution to the problem

$$\partial_t \hat{w} - \Delta_{\hat{M}} \hat{w} + \Gamma_{\hat{u}}(\nabla \hat{u}, \nabla \hat{u}) = 0 \quad \text{in } M_{(a)}^T, \quad (16)$$

$$\hat{w}(0) = \hat{u}_0, \quad (17)$$

$$\hat{w}(x, t) = I(\hat{v}(\Phi(x), t)) \quad \text{on } \partial M_{(a)} \times [0, T]. \quad (18)$$

Note that if  $q > 4$  is chosen sufficiently large from Lemma 4.1 of [C] and the regularity of  $\hat{v}$  on  $\partial M_{(-a)}$  we obtain uniform Holder estimates for  $\nabla \hat{w}$  in  $(x, t)$ . Therefore, for this small  $T$  we have  $\hat{w} \in \mathcal{M}$  and  $F: \mathcal{M} \rightarrow \mathcal{M}$  and  $F(\mathcal{M})$  is bounded in  $V(M_{(a)}^T; B)$ .

In fact,  $F$  is a compact operator. Let  $\mathcal{B}$  be a bounded subset of  $\mathcal{M}$ , then this  $\mathcal{B}$  is compact in  $L^2([0, T]; L_1^4(M_{(a)}; R^d))$  by the weak compactness of  $V(M_{(a)}^T; R^d)$  and uniform boundedness of  $\nabla \hat{u}(t)$  in  $L^q(M_{(a)}; R^d)$  for any  $t \in [0, T]$  and any  $\hat{u} \in \mathcal{M}$ . Moreover, the associated set of traces

$$\{\hat{v}|_{\partial M_{(a)}^T}; \hat{v} \in \mathcal{B}\}$$

is compact in  $W_2^{3/2, 3/4}$ . From (16)–(18) and Lemma 4.1 of [C] it follows that  $F$  is compact in  $V(M^T; R^d)$ . By Schauder's fixed point theorem  $F$  has a fixed point  $\hat{u} = \hat{w}$ . Necessarily

$$\hat{v}(x, t) = \hat{u}(x, t) = \hat{w}(x, t) = I(\hat{v}(\Phi(x), t)) \quad \text{on } \partial M_{(a)} \times [0, T],$$

i.e.  $\hat{v}$  is also a solution to (16)–(18). Hence  $\hat{u} = \hat{v} = \hat{w}$  and  $\hat{u}$  is a solution to (10). But by our construction also  $I(\hat{u}(\Phi(x), t))$  is a solution to (10) in  $(M_{(a)} / M_{(-a)})^T$  with the same initial and boundary values as  $\hat{u}$ . By an argument like the proof of uniqueness for (1)–(4) we get that  $\hat{u}(x, t) = I(\hat{u}(\Phi(x), t))$ . So (3) is satisfied. Now by the theorem in page 109 of [H] we know that  $\hat{u}(M \times [0, T]) \subset N$  satisfying (1). (4) should hold—otherwise  $\nabla^2 u$  would not be in  $L^2(M_{(a)}^T)$ . Q.E.D.

## 5. The large time behaviour

The following Lemma was obtained by M. Struwe [St] for harmonic map heat flow in closed Riemannian surface.

**LEMMA 7.** *There exists a constant  $\varepsilon_3 > 0$  depending only on  $M, K$  and  $N$  such that the following is true:*

*Any weak solution  $u \in V(M^T)$  to (1)–(4) with initial data  $u_0 \in H^1$  is Holder continuous on  $M \times (0, T)$ , and any subinterval  $[s, T]$ ,  $s > 0$ , the Holder norm of  $u$  is uniformly bounded in terms of  $T, S$  and the number*

$$\bar{R} = \sup\{R \in (0, \frac{1}{2}]; \varepsilon(2R) \leq \varepsilon_3\}.$$

*If  $u_0 \in C(K)$ , the solution  $u$  is Holder continuous on  $M \times [0, T]$  and its Holder norm is bounded in terms of  $T, \bar{R}$ , and the  $H^2$ -norm of  $u_0$ .*

*Proof.* First we derive uniform bounds for smooth solutions for the  $L^2$ -norm of  $\partial_t u(t)$  for a.e.  $t > 0$ .

Differentiate (10) with respect to  $t$  to obtain the equality

$$|\partial_t^2 \hat{u} - \Delta_{\hat{M}} \partial_t \hat{u}| \leq c |\nabla \partial_t \hat{u}| |\nabla \hat{u}| + c |\partial_t \hat{u}| |\nabla \hat{u}|^2$$

on  $\hat{\Omega}$ . Testing this inequality with the function  $\partial_t \hat{u} \hat{\phi}^2 \zeta^2$  and integrating over  $M_{(a)}$  and  $[t_1, t_2] \subset [0, T]$  we obtain that

$$\begin{aligned} & \frac{1}{2} \int_{M_{(a)}} |\partial_t \hat{u}(t)|^2 \hat{\phi} \zeta^2 \Big|_{t=t_1}^{t=t_2} + \int_{(M_{(a)})_{t_1}^{t_2}} |\nabla \partial_t \hat{u}| \hat{\phi} \zeta^2 \\ &= \int_{(M_{(a)})_{t_1}^{t_2}} \left( \frac{1}{2} \partial_t (|\partial_t \hat{u}|^2 \zeta^2) + |\nabla \partial_t \hat{u}| \hat{\phi} \zeta^2 \right) \\ &\leq c \int_{(M_{(a)})_{t_1}^{t_2}} \{ |\partial_t \hat{u}| \hat{\phi} \zeta^2 + |\nabla \partial_t \hat{u}| |\partial_t \hat{u}| |\partial_t \hat{u}| \hat{\phi}^2 \zeta^2 + |\nabla \partial_t \hat{u}| |\partial_t \hat{u}| |\nabla \hat{u}| \hat{\phi} \nabla \hat{u} | \hat{\phi}^2 \zeta^2 \\ &\quad + |\nabla \partial_t \hat{u}| |\partial_t \hat{u}| \hat{\phi}^2 \zeta | \nabla \zeta | + |\partial_t \hat{u}|^2 |\nabla \hat{u}|^2 \hat{\phi} \zeta^2 \} \, dM \, dt. \end{aligned}$$

Note that we have used the bounds of  $\hat{\phi}$ .

Next recall the integrals and derivatives of  $\hat{u}$  over  $\hat{\Omega}$  can be estimated by corresponding integrals of  $u$  over  $M$ ; see Lemma 5. Hence we may replace the domain  $M_{(a)}$  by  $M$  and omit  $\hat{\phi}$  in all integrals at the expense of enlarge our constants  $c$ . Applying Holder's inequality we obtain

$$\begin{aligned} \int_M |\partial_t u(t_2)|^2 \zeta^2 + \int_{M_{t_1}^{t_2}} |\nabla \partial_t u|^2 \zeta^2 &\leq c \int_M |\partial_t u(t_1)|^2 \zeta^2 dM + \frac{1}{2} \int_{M_{t_1}^{t_2}} |\nabla \partial_t u|^2 \zeta^2 dM dt \\ &+ c \left( \int_{M_{t_1}^{t_2}} (|\partial_t u|^2 \zeta^2 + |\nabla u|^4 \zeta^2) dM dt \right)^{1/2} \left( \int_{M_{t_1}^{t_2}} |\partial_t u|^4 \zeta^2 dM dt \right)^{1/2} \\ &+ c \int_{M_{t_1}^{t_2}} |\partial_t u|^2 (\zeta^2 + |\nabla \zeta|^2) dM dt. \end{aligned} \quad (19)$$

From (5) there holds

$$|\partial_t u| \leq |\nabla^2 u| + c |\nabla u|^2, \quad (20)$$

we now have the estimate

$$\begin{aligned} \int_{M_{t_1}^{t_2}} |\partial_t u|^2 \zeta^2 + |\nabla u|^2 \zeta^2 &\leq c \int_{M_{t_1}^{t_2}} |\nabla^2 u|^2 \zeta^2 + |\nabla u|^4 \zeta^2 \\ &\leq c \left( \int_{M_{t_1}^{t_2}} |\nabla^2 u|^2 \zeta^2 + |t_2 - t_1| R^{-2} \sup_{t_1 \leq t \leq t_2} E(u(t), B_{2R}^M(x_0)) \right) \\ &\leq c(t_2 - t_1) R^{-2} + \varepsilon_3. \end{aligned}$$

For the last inequality we have also used Lemma 1, Lemma 4, and our assumption that  $\varepsilon(2R) \leq \varepsilon_3$ . Moreover, note that we may also apply Lemma 1 to the term  $\int_{M_{t_1}^{t_2}} |\partial_t u|^4 \zeta^2$  appearing on the right of (19). Going back to (19) we now simply write

$$\begin{aligned} \int_M |\partial_t u(t_2)|^2 \zeta^2 + \int_{M_{t_1}^{t_2}} |\nabla \partial_t u|^2 \zeta^2 &\leq c \int_M |\partial_t u(t_1)|^2 \zeta^2 \\ &+ c[(1 + (t_2 - t_1) R^{-2}) \varepsilon_3]^{1/2} \left[ \sup_{t_1 \leq t \leq t_2} \int_M |\partial_t u(t)|^2 \zeta^2 \right. \\ &\quad \times \left. \left( \int_{M_{t_1}^{t_2}} |\nabla \partial_t u|^2 \zeta^2 + R^{-2} |\partial_t u|^2 \zeta^2 \right) \right] + c[(t_2 - t_1) R^{-2} + \varepsilon_3] \lim_{t \leq t \leq t_2} \int_{B_{2R}(x_0)} |\partial_t u(t)|^2 \\ &\leq c \int_M |\partial_t u(t_1)|^2 \zeta^2 + c[1 + (t_2 - t_1) R^{-1} \varepsilon_3]^{1/2} \int_{M_{t_1}^{t_2}} |\nabla \partial_t u|^2 \zeta^2 \\ &+ c(\varepsilon_3 + (t_2 - t_1) R^{-2}) \sup_{t_1 \leq t \leq t_2} \int_{B_{2R}(x_0)} |\partial_t u(t)|^2. \end{aligned}$$

i.e. for sufficiently small  $\varepsilon_3 > 0$ ,  $t_2 - t_1 \leq \varepsilon_3 R^2$  there holds

$$\int_{B_R(x_0)} |\partial_t u(t_2)|^2 \leq c \int_{B_{2R}(x_0)} |\partial_t u(t_1)|^2 + c\varepsilon_3 \sup_{t_1 \leq t \leq t_2} \int_{B_{2R}(x_0)} |\partial_t u(t)|^2.$$

This inequality will hold for any  $x_0 \in M$  and any  $t_1, t_2 \in [0, T]$  such that  $t_2 - t_1 \leq \varepsilon_3 R^2$ . Fix

$$0 \leq \bar{t}_1 \leq \bar{t}_3 \leq \bar{t}_1 + \varepsilon_3 R^2, \quad \bar{t}_2 = \frac{1}{2}(\bar{t}_1 + \bar{t}_3),$$

and for  $t_1 \in [\bar{t}_1, \bar{t}_2]$  let  $x_0 \in M$ ,  $t_2 \in [t_1, \bar{t}_3]$  be defined such that

$$2 \int_{B_R(x_0)} |\partial_t u(t_2)|^2 \geq \operatorname{esssup}_{(\bar{x}, \bar{t}) \in M_{t_1}^{\bar{t}_3}} \int_{B_R(\bar{x})} |\partial_t u(\bar{t})|^2.$$

Covering  $M$  with balls of radius  $R$ , for sufficiently small  $\varepsilon_3 > 0$  and suitably chosen  $t_i \in [\bar{t}_1, \bar{t}_2]$  we obtain that

$$\begin{aligned} c^{-1} R^2 \int_M |\partial_t u(\bar{t}_2)|^2 &\leq \int_{B_R(x_0)} |\partial_t u(t_2)|^2 \leq c \int_{B_{2R}(x_0)} |\partial_t u(t_1)|^2 \\ &\leq c \inf_{t_1 \leq t \leq \bar{t}_2} \int_M |\partial_t u(t)|^2 \leq \frac{c}{\bar{t}_2 - \bar{t}_1} \int_{M_{t_1}^{\bar{t}_2}} |\partial_t u|^2. \end{aligned} \quad (21)$$

By (20) and Lemma 3 we finally get

$$\int_M |\partial_t u(t_2)|^2 \leq \frac{c}{t_2 - t_1} \quad (22)$$

for all  $t_2, t_1 \in [0, T]$  such that  $0 < t_2 - t_1 \leq \varepsilon_3 R^2$ , with constant  $c = c(T, R)$ , i.e. for all  $t \in (0, T]$  there holds

$$\int_M |\partial_t u(t)|^2 \leq c(1 + t^{-1}). \quad (23)$$

If  $u_0 \in C(K)$  from (23)–(24) we obtain

$$\int_M |\partial_t u(t)|^2 \leq c \quad (24)$$

uniformly, with  $c$  depending in addition on  $H^2(M)$ -norm of  $u_0$ . Now we derive

pointwise estimates for

$$\int_M |\nabla^2 u(t)|^2$$

by using (10). Note that (10) implies that

$$|\Delta \hat{u}(t)| \leq c(|\partial_t \hat{u}(t)| + |\nabla \hat{u}(t)|^2).$$

Testing it with  $\Delta \hat{u}(t) \hat{\phi}^2$  we find that

$$\int_M |\nabla^2 \hat{u}|^2 \hat{\phi}^2 \leq c \int_M (|\partial_t \hat{u}|^2 + |\nabla \hat{u}|^2) \hat{\phi}^2 + c \int_M |\nabla^2 \hat{u}| |\nabla \hat{u}|^2 \hat{\phi}^2$$

i.e. by (\*) again

$$\int_M |\nabla^2 u(t)|^2 \leq c \int_M (|\partial_t u(t)|^2 + |\nabla u(t)|^4),$$

Lemma 1 and our assumption  $\varepsilon(2R) \leq \varepsilon_3$  now imply that

$$\int_M |\nabla^2 u(t)|^2 \leq c \int_M |\partial_t u(t)|^2 + cR^{-2},$$

and (24) yields the estimate

$$\int_M |\nabla^2 u(t)|^2 \leq c(T, s, R)$$

for all  $t \in [s, T]$ ,  $s > 0$  and the global bound

$$\int_M |\nabla^2 u(t)|^2 \leq c(T, R, \|u_0\|_{H^2(M)})$$

for all  $u_0 \in C(K)$ . By Sobolev's imbedding theorem  $H^2(M) \hookrightarrow C^0(M)$  and  $u(t)$  is locally uniformly continuous on  $(0, t]$  resp. on  $[0, T]$  for regular  $u_0$ . In particular,  $\hat{\Omega}$  contains a uniform neighbourhood of  $M \times (0, T]$ , resp. of  $M \times [0, T]$ , as we can use Lemma 4.2 of [C] to derive all required estimates for  $u$ .

For weak solution  $u \in V(M^T)$  the time derivative has to be replaced by difference quotients. One can proceed as above. Q.E.D.

*Remark.* Now we can prove the small time existence of  $V(M^T)$ -weak solution to (1)–(4) for  $u_0 \in H^1$ . To see this we approximate  $u_0$  by smooth data  $u_0^m \in C(K)$  and let  $u^m \in V(M^{T_m})$  be the corresponding solution of (1)–(4). By Lemma 3, 4 and 7, each  $u_m$  persists as a regular solution to (1)–(4) for at least  $\bar{T} = \bar{\varepsilon}R^2/2c_2 > 0$ . In fact, Lemma 3 guarantees the estimate  $\varepsilon(2\bar{R}) \leq \bar{\varepsilon}$  for all  $u^m$  on  $[0, \bar{T}]$ , and Lemmas 4 and 7 apply. Moreover, we have a uniform bound

$$\int_{M^T} (|\partial_t u^m|^2 + |\nabla^2 u^m|^2 + |\nabla u^m|^4) dM dt + \sup_{0 \leq t \leq T} E(u^m(t)) \leq c(\bar{R}),$$

and we may extract a subsequence that converges weakly in  $V(M^T)$  to a solution of (1)–(4).

Now we are ready to following

**LEMMA 8.** *Suppose  $u \in \cap_{T < \infty} V(M^T)$  is a solution to (1)–(4) and for some  $R > 0$  there holds*

$$\sup_{(x,t) \in M^\infty} E(u(t), B_R(x)) \leq \bar{\varepsilon}.$$

*Then  $u$  is globally regular and there exists a sequence  $t_m \rightarrow \infty$  such that  $u(t_m) \in C^2$  and*

$$u(t_m) \rightarrow \bar{u} \quad \text{in } H^2(M),$$

*where  $\bar{u}$  is a harmonic map from  $M$  into  $N$  with free boundary in  $K$ .*

*Proof.* Our assumption and Lemma 3 imply that for  $m \in \mathbb{N}$ ,

$$\begin{aligned} \int_{M_m^{m+1}} |\partial_t u|^2 dM dt &\rightarrow 0 \quad (m \rightarrow \infty), \\ \int_{M_m^{m+1}} |\nabla^2 u|^2 dM dt &\leq c \quad \text{uniformly in } m. \end{aligned}$$

By Fubini's theorem we may choose a sequence  $t_m \rightarrow \infty$  such that  $u_m = u(t_m) \in C^2$  satisfies

$$\begin{aligned} \partial_t u(t_m) &= \Delta_M u_m + A(u_m)(\nabla u_m, \nabla u_m) \rightarrow 0 \quad \text{in } L^2(M), \\ \partial_t u_m(x) &\perp T_{u_m(x)} K \quad \text{a.e. on } \partial M, \\ \sup_{x \in M} E(u_m, B_R(x)) &\leq \bar{\varepsilon}, \quad \text{for all } m. \end{aligned}$$

Now the claim follows from Theorem B below.

Q.E.D.

As in [St] we study singularities created by concentration of energy near some points.

LEMMA 9. *Suppose that for some  $\bar{T} \leq \infty$ ,  $u \in \cap_{T < \bar{T}} V(M^T)$  is a solution of (1)–(4). If for all  $R > 0$ ,*

$$\lim_{T \uparrow \bar{T}} \sup_{(x, t) \in M^T} E(u(t), B_r(x)) \geq \bar{\varepsilon},$$

*then there exist sequence  $t_m \nearrow \bar{T}$ ,  $x_m \in M$ ,  $R_m \searrow 0$  and a map  $\bar{u} \in H^2_{\text{loc}}$  such that  $u(t_m) \in C^2$  and the rescaled functions*

$$u_m(t) \equiv u(x_m + R_m x, t_m) \rightarrow \bar{u} \quad \text{in } H^2_{\text{loc}},$$

*after a possible rotation of coordinates. Moreover,  $E(\bar{u}) < \infty$ , and  $\bar{u}$  is conformal to either some minimal 2-disk with free boundary in  $K$  or some minimal 2-sphere in  $N$ .*

*Proof.* For any sequence  $R_m \searrow 0$  let  $t_m \leq \bar{T}$  be maximal with the property that for some  $x_m \in M$

$$E(u(t_m), B_{R_m}(x_m)) = \sup_{(x, t) \in M^{t_m}} E(u(t), B_{R_m}(x)) = \bar{\varepsilon}.$$

Clearly  $t_m \nearrow \bar{T}$  as  $m \rightarrow \infty$ . After possibly take a subsequence we assume that  $x_m \rightarrow \bar{x}$  in  $M$ . Let  $\mathcal{O}$  be a local chart of  $\bar{x}$ . By Lemma 4 there exists a constant  $c_3 = \bar{\varepsilon}(2c_2)^{-1}$  such that for all  $t \in [t_m - c_3 R_m^2, t_m]$

$$E(u(t), B_{2R_m}(x_m)) \geq \frac{\bar{\varepsilon}}{4} > 0.$$

Moreover, by lemma 3 and Lemma 1

$$\int_{M^{t_m - c_3 R_m^2}} |\nabla^2 u|^2 \leq c$$

uniformly for all  $m$ , while again by Lemma 3 and absolute continuity of the Legesgue integral

$$\int_{M^{t_m - c_3 R_m^2}} |\nabla^2 u| dM dt \rightarrow 0 \quad (m \rightarrow +\infty).$$

Finally, by Lemma 3 also

$$\text{esssup}_{t_m - c_3 R_m^2 \leq t \leq t_m} E(u(t)) \leq c < \infty$$

uniformly for all  $m$ . Hence if we let

$$u^m(x, t) = u(x_m + R_m x, t_m + R_m^2 t)$$

and

$$M^{(m)} = \{x \in R^2 \mid x_m + R_m x \in \mathcal{O}\},$$

then  $u^m \in V((M^{(m)})_{-c_3})$  will satisfy (1) on  $(M^{(m)})_{-c_3}$  and following estimates

$$\int_{(M^{(m)})_{-c_3}} |\partial_t u^m|^2 dM dt \rightarrow 0 \quad (m \rightarrow \infty),$$

$$\int_{(M^{(m)})_{-c_3}} |\nabla^2 u^m| dM dt \leq c,$$

$$\operatorname{esssup}_{-c_3 \leq t \leq 0} E(u^m(t), B_2(0) \cap M^m) > 0,$$

$$\operatorname{esssup}_{(x,t) \in (M^{(m)})_{-c_3}} E(u^m(t)) \leq c < \infty,$$

uniformly in  $m$ . As in [C], we can choose  $s_m \in [-c_3, 0]$  such that  $u_m = u^m(s_m)$  satisfies

$$u_m \in H^2(M^{(m)}, R^d),$$

$$\partial_t u^m(s_m) = \Delta_M u_m + A_{u_m}(\nabla u_m, \nabla u_m),$$

$$\partial_{\bar{n}} u_m(x) \perp T_{u_m(x)} K, \quad \text{a.e. on } \partial M^m \cap \partial R_+^2 \equiv \partial M_+^{(m)}$$

$$0 < c \leq E(u_m, B_2(0) \cap M^{(m)}) \leq E(u_m) \leq c' < \infty,$$

$$\sup_{x \in M^{(m)}} E(u_m, B_1(x) \cap M^{(m)}) \leq \bar{\varepsilon},$$

uniformly in  $m$ . After shifting time we assume  $s_m = 0$  and invoke theorem B below to conclude our lemma.

**THEOREM B.** Suppose  $M^{(m)} \subset R^2$ , is a sequence of bounded domains that converges to a limiting domain  $M^{(\infty)} \subset R^2$ ,  $u^m \in H^2(M^{(m)})$  satisfies that

$$u_m(\partial M_+^{(m)}) \subset K, \tag{26}$$

$$\partial_{\bar{n}} u_m(x) \perp T_{u_m(x)} K \quad \text{a.e. on } \partial M_+^{(m)}, \tag{27}$$

$$E(u_m) \leq c \leq \infty \quad \text{uniformly in } m, \tag{28}$$

$$\sup_{x \in M^{(m)}} E(u_m, B_1(0) \cap M^{(m)}) \leq \bar{\varepsilon} \quad \text{uniformly in } m, \quad (29)$$

$$\int_{M^{(m)}} |\Delta_{M^{(m)}} u_m + A_{u_m}(\nabla u_m, \nabla u_m)|^2 \rightarrow 0 \quad (m \rightarrow \infty). \quad (30)$$

Then there exists a map  $\bar{u} \in H^2_{\text{loc}}(M^{(\infty)}, \mathbb{R}^d)$  and a subsequence (we still denote)  $m \rightarrow \infty$  such that

$$u_m \rightarrow \bar{u} \quad \text{in } H^2_{\text{loc}}(M^{(\infty)}, \mathbb{R}^d),$$

where  $\bar{u} \in H^1(M^{(\infty)})$  solves

$$\Delta \bar{u} + A_{\bar{u}}(\nabla \bar{u}, \nabla \bar{u}) = 0 \quad \text{in } M^{(\infty)}, \quad (31)$$

$$|\bar{u}_{x_1}|^2 - |\bar{u}_{x_2}|^2 = 0 = \bar{u}_{x_1} \cdot \bar{u}_{x_2} \quad \text{in } M^{(\infty)}, \quad (32)$$

$$\bar{u}(\partial M_+^{(\infty)}) \subset K, \quad (33)$$

$$\partial_{\bar{n}} \bar{u}(x) \perp T_{\bar{u}(x)} K, \quad \text{on } \partial M_+^{(\infty)}, \quad (34)$$

and

$$E(\bar{u}) < \infty. \quad (35)$$

If  $M^{(\infty)} = \mathbb{R}_+^2$ , then  $\bar{u}$  is conformal to a minimal 2-disk,

If  $M^{(\infty)} = \mathbb{R}^2$ , then  $\bar{u}$  is conformal to minimal 2-sphere.

*Proof.* By (30) we obtain

$$\int_{M^{(m)}} |\Delta u_m|^2 \leq c \int_{M^{(m)}} |\nabla u_m|^2 |\Delta u_m| + 0(1) \left( \int_{M^{(m)}} |\Delta u_m|^2 \right)^{1/2}.$$

By Lemma 1 this gives

$$\begin{aligned} \int_{M^{(m)}} |\Delta u_m|^2 &\leq c \int_{M^{(m)}} |\nabla^2 u_m|^4 + 0(1) \\ &\leq c \bar{\varepsilon} \int_{M^{(m)}} |\nabla^2 u_m|^4 + c + 0(1). \end{aligned}$$

As in the proof of Lemma 3, we have

$$\int_{M^{(m)}} |\nabla^2 u_m| \leq c \quad \text{uniformly in } m, \quad (36)$$

By Sobolev's imbedding theorem,  $u_m$  are equicontinuous and hence may be extended to a  $r$ -neighbourhood  $\hat{M}^{(m)}$  of  $M^{(m)}$  by reflection in  $K$  as before, here  $r > 0$  independent of  $m$  (26)–(27) guarantee that these extensions  $\hat{u}_m \in H^2(\hat{M}^{(m)}; \mathbb{R}^d)$  and satisfy (10) on  $\hat{M}^{(m)}$ , as we verified in Lemma 5. Moreover, as in the proof of Lemma 5, for the extension  $\hat{u}_m$  of  $u_m$  we derive that

$$\int_{\hat{M}^{(m)}} |\Delta \hat{u}_m + A_{\hat{u}_m}(\nabla \hat{u}_m, \nabla \hat{u}_m)|^2 \rightarrow 0$$

for all  $m$ .

By (36) we may select a subsequence  $\hat{u}_m$  such that for any bounded domain  $\Omega \subset \mathbb{R}^2$

$$\hat{u}_m \rightarrow \bar{u} \quad \text{weakly in } H^2(M^{(\infty)} \cap \Omega; \mathbb{R}^d).$$

By Rellich's theorem we have

$$\hat{u}_m \rightarrow \bar{u} \quad \text{strongly in } L^{1,q}(M^{(\infty)} \cap \Omega; \mathbb{R}^d)$$

and

$$\hat{u}_m \rightarrow \bar{u} \quad \text{strongly in } L^{1,q}(M^{(\infty)} \cap \Omega; \mathbb{R}^d)$$

for any  $q < \infty$ . Hence we may pass to the limit  $m \rightarrow \infty$  in (30), (26), (27), and find that  $\bar{u}$  be a harmonic map as claimed.

Moreover, letting  $\hat{\phi}_m$  be the cutoff function associated with  $\hat{u}_m$  by (11). Given  $j, k$  let  $\hat{\phi} = \min \{\hat{\phi}_j, \hat{\phi}_k\}$ ,  $\xi \in C_0^3(\Omega)$  upon testing the difference of Eq. (5) for  $u_j, u_k$  with the function  $\Delta(u_j - u_k)\hat{\phi}\xi$  we obtain

$$\begin{aligned} \int_{\mathbb{R}^2} |\nabla^2(\hat{u}_j - \hat{u}_k)|^2 \hat{\phi} \xi &= \int_{\mathbb{R}^2} |\Delta(\hat{u}_j - \hat{u}_k)|^2 \hat{\phi} \xi + 0(1) \\ &\leq c \int_{\mathbb{R}^2} [A_{\hat{u}_j}(\nabla \hat{u}_j, \nabla \hat{u}_j) - A_{\hat{u}_k}(\nabla \hat{u}_k, \nabla \hat{u}_k)] \cdot \Delta(\hat{u}_j - \hat{u}_k) \hat{\phi} \xi + 0(1) \end{aligned}$$

as  $m \rightarrow \infty$ ,

and  $\hat{u}_m \rightarrow \hat{u}$  strongly in  $H^2(M^{(\infty)} \cap \Omega, \mathbb{R}^d)$  for any bounded  $\Omega \subset \mathbb{R}^2$ .

The remaining part of our claim are now easily verified. By (31) the complex valued function of  $z = x + iy \in M^{(\infty)} \subset \mathbb{R}^2 = \mathbb{C}$  is holomorphic and

$$H(z) \doteq |\bar{u}_x|^2 - |u_y|^2 - 2i\bar{u}_x \cdot \bar{u}_y$$

by (35) is integrable over  $M^{(\infty)}$ . In case  $M^{(\infty)} = \mathbb{R}^2$  from the mean value theorem for harmonic functions

$$H(z) = \frac{1}{2\pi R} \int_{\partial B_R(x)} H(z') dz',$$

upon letting  $R \rightarrow \infty$  we obtain that  $H(z) = 0$ , i.e. that  $\bar{u}$  is weak conformal. By [S],  $\bar{u}$  is smooth and hence a minimal 2-sphere.

Similarly, if  $M^{(\infty)} = \mathbb{R}_+^2$ , by (33)–(34)  $H$  is real on  $\partial M^{(\infty)}$ . By reflection and (35) the imaginary part of  $H$  may be extended to a harmonic function in  $L^1(\mathbb{R}^2)$ , hence it must vanish identically by the above argument. The Cauchy-Riemann equations now imply that  $H = \text{constant}$ . But  $H \in L^1((\mathbb{R}_+^2))$ , thus  $H = 0$ , and  $\bar{u}$  is weak conformal. By the conformal equivalence  $D^2 = \mathbb{R}_+^2$ , the map  $\bar{u}$  will be conformal to a weak harmonic map. By the regularity result of [S] again  $\bar{u}$  is smooth and hence a minimal 2-sphere.

Q.E.D.

## 6. The end

Now, the proof of Theorem 1 is a direct consequence of Theorem A, the remark, and Theorem B.

*Proof of Theorem 2.* First, suppose  $K$  is diffeomorphic to  $S^2$ . Let  $(r, \phi)$  be a polar coordinate on  $S^2$  and denote  $C = \partial D^2$ , then  $S^2 = C \times [0, \pi]$  with  $C \times \{0\}$ ,  $C \times \pi$  collapsed to points. Let  $D : K \rightarrow S^2$  be the diffeomorphism in our assumption. Then any continuous mapping  $p : [0, \pi] \rightarrow H(K)$  such that  $p(0), p(\pi)$  are constant maps induces a mapping  $\delta p : S^2 \rightarrow S^2$  by letting

$$\delta p(\gamma, \phi) = D(p(\phi)(e^{i\gamma})). \quad (37)$$

Endowing the space of mappings  $S^2 \rightarrow S^2$  with  $C^0$ -topology set

$$\begin{aligned} \mathcal{P} = \{p \in C^0([0, \pi]; H(K)) &| p(0) \equiv \text{const}, \\ \delta p \in C^0(S^2, S^2) &\text{ is homotopic to the identity on } S^2\}. \end{aligned}$$

Since (37) for  $\delta p = \text{id}|_{S^2}$  defines a path  $p \in C^1$ . Clearly  $\mathcal{P} \neq \emptyset$ .

Suppose the conclusion of Theorem 2 is false, then for any  $u_0 \in H(K)$ , there is a global regular solution of (1)–(4) which converges to constant at infinity. Hence  $\text{id}|_{S^2}$  is homotopic to constant mapping in  $C^0(S^2, S^2)$ , a contradiction.

The general case can be proved in the same spirit. More precisely, it goes as the same as the proof of Theorem 8.50 in the book of J. T. Schwartz: *Nonlinear Functional Analysis*, Gordon and Breach, New York, 1969. Q.E.D.

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