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A splitting principle for group representations

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§1. Introduction

In the theory of complex vector bundles it is known that for every vector bundle $\xi : E \to B$ there exists a space S and a map $s : S \to B$ such that the pullback bundle $s^*\xi$ decomposes as a sum of line bundles and such that $s^* : H^*(B) \to H^*(S)$ is injective. This reduces many questions about characteristic classes of vector bundles to questions dealing only with line bundles.

In representation theory the corresponding device is to use Brauer's theorem that any representation V of a finite group G can be expressed as a virtual sum with integer coefficients, $V = \Sigma\, a_i \operatorname{ind}_{H_i}^G \varphi_i$, where $H_i \leq G$ and φ_i is a 1-dimensional representation of H_i. Complications arise because such an expression can be made in many ways and because, in the calculation of characteristic classes, one still has to evaluate the class on each $\operatorname{ind}_H^G \varphi_i$. An expression of this form for all finite G and which is natural in G was given by Snaith [Sn1] under the name of explicit Brauer induction and its properties were exploited with applications in number theory [Sn2]. Another natural expression and other applications were given by Boltje [B1, 2].

In this paper we shall define a sum $L(V) = \Sigma\, \alpha_{(J,\varphi)}(V) \operatorname{ind}_J^G \varphi$ such that $L(V) = V$ and where the coefficients are defined in terms of the action of G on the projective space $\mathbb{P}(V)$. This $L(V)$ has good naturality properties, and the sum involves only proper subgroups of G provided that V contains no 1-dimensional summands. These naturality properties of the coefficients can be exploited in other contexts. Let c denote the total Chern class, considered as taking values in the multiplicative group $H^{**}(G; \mathbb{Z})$ (of infinite sums in $H^{ev}(G; \mathbb{Z})$). Then $c(V) = \Pi\, \mathcal{N}_J^G c(\varphi)^{\alpha_{(J,\varphi)}(V)}$, where $\mathcal{N}$ is the multiplicative transfer [E1]. This formula can be considered either as a splitting principle for Chern classes of group representations, in which case it should be compared with [E2, E-K, T], or as a construction of them which does not involve vector bundles.

In §2 we give a sketch of this theory for finite groups, using the most elementary framework where the methods are not obscured by technical details. However some of the proofs for characteristic classes are awkward and unnatural in this context, so the rest of the paper is devoted to the case of compact Lie groups, where we can use the universal example of the canonical representation of $U(n)$.

§3 deals with functors with Mackey structure, i.e. with transfers that satisfy a double coset formula. These yield the groups in which the characteristic classes will take their values. §4 and §5 consider the complex representations of compact Lie groups and their characteristic classes. §6 and §7 treat the real case, which is considerably more complicated because it is necessary to induce from 2-dimensional representations as well as 1-dimensional ones.

§2. The case of finite groups

Let G be a finite group and let V be a complex representation of G. We shall consider the projective space $\mathbb{P}(V)$ together with its canonical line bundle γ. Now G acts on γ and we can give $\mathbb{P}(V)$ the structure of an equivariant CW-complex with admissible action, i.e. such that the stabiliser of a cell as a set also stabilises it pointwise.

We define a Lefschetz element $L(V)$ of the representation ring $R(G)$ of G by

$$L(V) = \sum_{\sigma \in G\backslash\mathbb{P}(V)} (-1)^{\dim \sigma} \operatorname{ind}^G_{\operatorname{stab}_G \tilde{\sigma}} \varphi_{\tilde{\sigma}}$$

where the sum is over the cells σ of $G\backslash\mathbb{P}(V)$, $\tilde{\sigma}$ is a cell of $\mathbb{P}(V)$ that lies above σ and $\varphi_{\tilde{\sigma}}$ is the representation of $\operatorname{stab}_G \tilde{\sigma}$ on a fibre of γ above $\tilde{\sigma}$. Note that if all the $\varphi_{\tilde{\sigma}}$ are replaced by the trivial representation, then $L(V)$ becomes the usual Euler characteristic or Lefschetz element, and just as in that case it will turn out that $L(V)$ does not depend on the CW-structure chosen.

Since $\operatorname{ind}^G_J \varphi$ depends only on the conjugacy class of the pair (J, φ) in G, the expression $L(V)$ is independent of the choice of $\tilde{\sigma}$. Now we can group together all the terms with conjugate pairs $(\operatorname{stab}_G \tilde{\sigma}, \varphi_{\tilde{\sigma}})$:

$$L(V) = \sum_{(J,\varphi)} \alpha_{(J,\alpha)}(V) \operatorname{ind}^G_J \varphi,$$

where (J, φ) runs through the conjugacy classes of pairs $J \leq G$ and φ a 1-dimensional representation of J.

The coefficient $\alpha_{(J,\varphi)}(V)$ can be calculated as follows: given $x \in G\backslash\mathbb{P}(V)$ let $\tilde{x}$ be a point of $\mathbb{P}(V)$ above x; then $(\operatorname{stab}_G \tilde{x}, \varphi_{\tilde{x}})$ is well defined up to conjugation, where $\varphi_{\tilde{x}}$ is the representation of $\operatorname{stab}_G \tilde{x}$ on the fibre of γ above $\tilde{x}$. Let $G\backslash\mathbb{P}(V)_{(J,\varphi)}$

represent the subspace of points $x \in G\backslash\mathbb{P}(V)$ for which $(\mathrm{stab}_G\,\tilde{x},\ \varphi_{\tilde{x}})$ is conjugate to (J, φ). This will be called the stratum of $G\backslash\mathbb{P}(V)$ of type (J, φ). Now by definition,

$$\alpha_{(J,\varphi)}(V) = \sum_{\mathrm{int}\,\sigma \subset G\backslash P(V)_{(J,\varphi)}} (-1)^{\dim \sigma}.$$

But this is the Euler number in the compactly supported cohomology of $G\backslash\mathbb{P}(V)_{(J,\varphi)}$, i.e.

$$\alpha_{(J,\varphi)}(V) = \chi_c(G\backslash P(V)_{(J,\varphi)}).$$

See (Sn2) for a discussion of χ_c in this context. In particular this shows that the coefficients $\alpha_{(J,\varphi)}(V)$ are independent of the CW-structure chosen. It is these numbers which will be important, not just the sum $L(V)$, and in the following proposition we regard $L(V)$ as a formal sum indexed by the (J, φ).

PROPOSITION 2.1. (a) *$L(V)$ is natural with respect to homomorphisms of groups: $L(f^*V) = f^*L(V)$. In particular, in order to obtain $L(\mathrm{res}^G_H V)$ one can apply the double coset formula,*

$$\mathrm{res}^G_H\, \mathrm{ind}^G_J\, \varphi = \sum_{g \in H\backslash G/J} \mathrm{ind}^H_{H \cap {}^gJ}\, \mathrm{res}^{{}^gJ}_{H \cap {}^gJ}\, {}^g\varphi,$$

to each of the terms in the expression for $L(V)$ and then collect the resulting terms with conjugate (J, φ).

(b) *$L(V)$ is additive: $\alpha_{(J,\varphi)}(V \oplus W) = \alpha_{(J,\varphi)}(V) + \alpha_{(J,\varphi)}(W)$. In particular $L(V \oplus W) = L(V) + L(W)$.*

(c) *For a 1-dimensional representation V, $L(V)$ has just one term, V.*

Proof. (a) The way in which the orbits of cells decompose under restriction corresponds precisely to the double cosets.

(b) Let the complex numbers of unit length act on $V \oplus W$ by acting trivially on V and by scalar multiplication on W. This leads to an action of the circle group S^1 on $\mathbb{P}(V \oplus W)$ and hence on $G\backslash\mathbb{P}(V \oplus W)$. This action respects the decomposition into strata and its fixed point set is precisely $G\backslash\mathbb{P}(V) \cup G\backslash\mathbb{P}(W)$. The part of each stratum that lies outside $\mathbb{P}(V)$ and $\mathbb{P}(W)$ therefore has zero Euler number ((tD) p. 220).

(c) $\mathbb{P}(V)$ consists of just one point. □

THEOREM 2.2. *$L(V) = V$ in $R(G)$.*

Proof. It is sufficient to check the characters on each element of G, hence to check res_C^G for each cyclic subgroup C of G. By naturality, $\mathrm{res}_C^G L(V) = L(\mathrm{res}_C^G V)$, so it is enough to prove the statement for cyclic groups. But then V is a sum of 1-dimensional representations, so by additivity we only have to give a proof for 1-dimensional representations. This is part (c) of Proposition 2.1. □

We now have the formula

$$V = \sum_{(J,\varphi)} \alpha_{(J,\varphi)}(V) \, \mathrm{ind}_J^G \varphi.$$

Note that J runs only through proper subgroups of G provided that V contains no 1-dimensional summand. The same method of proof shows that we can introduce Adams operations,

$$\psi^r V = \sum_{(J,\varphi)} \alpha_{(J,\varphi)}(V) \, \mathrm{ind}_J^G \varphi^r,$$

even though ψ^r does not commute with ind.

We can also consider the φ as elements of Hom $(J, \mathbb{C}^*) \cong H^2(J; \mathbb{Z})$. Then for the representation given by the determinant of V we obtain

$$\det V = \sum_{(J,\varphi)} \alpha_{(J,\varphi)}(V) \, \mathrm{tr}_J^G \varphi \quad \text{in } H^2(G; \mathbb{Z}),$$

where tr_J^G is the transfer in $H^2(-, \mathbb{Z})$. Again, the fact that $\mathrm{res}_K^G \mathrm{tr}_H^G$ satisfies a double coset formula implies that the right-hand side is natural, and the proof proceeds as before.

In the context of Chern classes, this lead us to consider the expression

$$\tilde{c}(V) = \prod_{(J,\varphi)} \mathcal{N}_J^G c(\varphi)^{\alpha_{(J,\varphi)}(V)} \quad \text{in } H^{**}(G; \mathbb{Z}),$$

where $H^{**}(G, \mathbb{Z})$ is the multiplicative group of infinite series $1 + a_2 + a_4 + \cdots$, $a_{2i} \in H^{2i}(G; \mathbb{Z})$. Here c denotes the total Chern class and $\mathcal{N}$ is the multiplicative transfer of Evens [Ev1]. $\mathcal{N}$ satisfies a double coset formula so we obtain:

PROPOSITION 2.3. (a) *$\tilde{c}$ is natural with respect to homomorphisms of groups: $\tilde{c}(f^* V) = f^* \tilde{c}(V)$.*

(b) $\tilde{c}$ *is 'additive'*: $\tilde{c}(V \oplus W) = \tilde{c}(V)\tilde{c}(W)$.
(c) $\tilde{c}(V) = c(V)$ *for* 1*-dimensional* V. □

It is not immediately clear that $\tilde{c}(V) = c(V)$, however, since it is no longer sufficient to check only on cyclic subgroups.

PROPOSITION 2.4. $\tilde{c}(V) = c(V)$.

Proof. Kroll [Kr] shows that c is uniquely determined by the properties stated for $\tilde{c}$ in Proposition 2.3. □

REMARKS 2.5. (a) Our formula for $L(V)$ is the same as that of Boltje [B1, 2], who defines the coefficients algebraically. This is because the expression of [B1, 2] is shown to be uniquely determined by certain properties which are shared by $L(V)$. Alternatively, note that $\mathbb{P}(V)_{(J,\varphi)}$ (the part with stabiliser equal to (J, φ)) has as closure a sub-projective-space of dimension equal to the multiplicity of φ as a summand of $\mathrm{res}^G_J V$. To get $\mathbb{P}(V)_{(J,\varphi)}$ one then removes the parts with larger stabiliser: this leads to a Möbius function on the poset of the (J, φ) and so to formulas 2.35 of [B1] and 2.5b of [B2]. The formula of Snaith [Sn1, 2] is different (see Remark 4.5).

(b) In §5 we shall give an independent proof of Proposition 2.4 which is more natural in this context.

(c) This formula for $\tilde{c}(V)$ should be compared with others that express $c(V)$ in terms of transfers from subgroups [E2, E-K, T].

§3. Functors with Mackey structure

In order to be precise about the double coset formula and other properties of the functors that we use, we introduce the notion of a functor with Mackey structure.

NOTATION. If G is a compact Lie group and $H \leq G, g \in G$, then ${}^gH = gHg^{-1}$ and $c_g : {}^gH \to H$ is the homomorphism $c_g(x) = g^{-1}xg, g \in H$. $W_G(H) = N_G(H)/H$ is the Weyl group of H in G. Let M be a contravariant functor from compact Lie groups to abelian groups. If $f: H \to G$ is a homomorphism then we write f^* for $M(f)$. If f is an inclusion then we may also write res^G_H for $M(f)$ (of course we can only consider closed subgroups). M is said to be invariant under conjugation if $c_g^* : M(G) \to M(G)$ is the identity whenever $g \in G$.

DEFINITION 3.1. If M is invariant under conjugation then a Mackey structure for M consists of a homomorphism $\mathrm{tr}_H^G : M(H) \to M(G)$ for each $H \leq G$ such that:

(a) If $K \leq H \leq G$ then $\mathrm{tr}_H^G \, \mathrm{tr}_K^H = \mathrm{tr}_K^G$.

(b) If

$$\begin{array}{ccc} H & \xrightarrow{f'} & H' \\ \downarrow & & \downarrow \\ G & \xrightarrow{f} & G' \end{array}$$

is a commutative diagram of groups in which the horizontal arrows are surjective, the vertical ones are inclusions and $H = f^{-1}(H')$, then $f^* \, \mathrm{tr}_{H'}^{G'} = \mathrm{tr}_H^G f'^*$.

(c) If $H, K \leq G$ then

$$\mathrm{res}_K^G \, \mathrm{tr}_H^G = \sum_{K \backslash G/H} \chi_g \, \mathrm{tr}_{K \cap {}^gH}^K \, \mathrm{res}_{K \cap {}^gH}^{{}^gH} \, c_g^*,$$

where the sum is over the connected components of the strata of $K \backslash G/H$. The symbol χ_g, $g \in G$, represents χ_c of the component of a stratum which contains KgH.

This is how a Mackey structure is usually given in practice, however there is a more elegant description which we now give.

PROPOSITION 3.2. *If M is invariant under conjugation there is a one-to-one correspondence between Mackey structures for M and covariant functors $M_!$ from the category of compact Lie groups and monomorphisms to abelian groups which agree with M on objects and which satisfy the following condition, (where $f_! = M_!(f)$, and if $H \leq G$ then tr_H^G represents $M_!$ of the inclusion map). If $f : K \to G$ is any homomorphism and $i : H \to G$ is a monomorphism then*

$$f^* i_! = \sum_{f(K) \backslash G/i(H)} \chi_g \, \mathrm{tr}_{f^{-1}({}^g(i(H)))}^K \, (i^{-1} c_g f : f^{-1}({}^g(i(H))) \to H)^*,$$

where the sum is over the connected components of the strata, as before.

Proof. Given $M_!$ we can define tr_H^G for an inclusion $i : H \to G$ as $M_!(i)$. Conversely, given a Mackey structure and a monomorphism $i : H \to G$, let $i' : H \to i(H)$

be the isomorphism induced by i; then we can define $i : M(H) \to M(G)$ by $i_! = \mathrm{tr}^G_{i(H)}(i'^*)^{-1}$. The details are left to the reader. □

If we just look at one finite group and its subgroups then M and $M_!$ restrict to give a Mackey functor in the sense of [Dr].

EXAMPLES. (i) The representation ring $R_F(G)(F = \mathbb{R}, \mathbb{C}, \mathbb{H})$ has a Mackey structure, using the induced representation of Segal [Se1] as transfer. The double coset formula is proved in [Sn2]. A virtual representation is determined by the values of its character χ on regular elements $g \in G$ (i.e. those g that generate a sub-group of G with finite Weyl group) and, for such g, $(\mathrm{ind}^G_H \chi)(g) = \Sigma\chi(x^{-1}gx)$, where xH runs through the fixed points of g on G/H.

(ii) If h^* is any generalised cohomology theory then $h^i(BG)$ (for any i) has a Mackey structure using Dold's transfer [Do]. The double coset formula is proved in [F].

(iii) Given any contravariant functor M from compact Lie groups to abelian groups we can construct a functor M'_+ with Mackey structure as follows, (cf. [B1, 2, De]). Given G, consider $\oplus_{H \leq G} M(H)$ (sum over closed subgroups): the image of $\varphi \in M(H)$ will be denoted by (H, φ). Let X be the subgroup generated by elements of the form $(H, \varphi) - ({}^gH, c_g^*\varphi)$, $g \in G$. We define $M'_+(G) = \oplus_{H \leq G} M(H)/X$ and denote the image of (H, φ) by $(H, \varphi)^G$. If $f : K \to G$ is a homomorphism then $M'_+(f)$ is defined by

$$M'_+(f)(H, \varphi)^G = \sum_{f(K)\backslash G/H} \chi_g(f^{-1}({}^gH), M(c_g f)\varphi)^K.$$

If $K \leq H \leq G$ then $\mathrm{tr}^G_H(K, \varphi)^H = (K, \varphi)^G$.

If M originally had a Mackey structure then there is a natural transformation $a : M'_+ \to M$ given by $a(H, \varphi)^G = \mathrm{tr}^G_H \varphi$. It is also a natural transformation of the functors $(M'_+)_!$ and $M_!$. Most of these assertions are clear except for the following.

LEMMA 3.3. *M'_+ is a contravariant functor.*

Proof. We need to prove that $(ef)^* = f^*e^*$. Now every homomorphism factorises as the composition of an epimorphism and an inclusion. If $d : K \to G$ is an epimorphism then $d^*(H, \varphi)^G = (d^{-1}(H), d^*\varphi)^K$, and the proof is clear if either e or f is an epimorphism, so we shall concentrate on the case of two inclusions.

Suppose $L \leq K \leq G \geq H$: then

$$\operatorname{res}^G_L (H, \varphi)^G = \sum_{L\backslash G/H} \chi_c(L\backslash G/H_f)(L_\cap {}^f H, c^*_f \varphi)^L$$

(where $L\backslash G/H_f, f \in G$, denotes the component of a stratum of $L\backslash G/H$ which contains LfH), whilst

$$\operatorname{res}^K_L \operatorname{res}^G_K (H, \varphi)^G = \sum_{K\backslash G/H} \sum_{L\backslash K/K_\cap {}^g H} \chi_c(L\backslash G/H_g)\chi_c(L\backslash K/K_\cap {}^g H_k)(L_\cap {}^{kg}H, c^*_{kg}\varphi)^L.$$

We shall show that each $L\backslash G/H_f$ can be subdivided into connected parts $\bar{Z}_{k,g}$, each of which contains $LkgH$ and such that

$$\chi_c(\bar{Z}_{k,g}) = \chi_c(L\backslash G/H_g)\chi_c(L\backslash K/K_\cap {}^g H_k).$$

Let $\bar{X}_g$ denote $K\backslash G/H_g$ and let X_g denote the set of points of G/H above $\bar{X}_g$ which have stabiliser $K_\cap {}^g H$. Then $\bar{X}_g = W_K(K_\cap {}^g H)\backslash X_g$. The space G/H is a disjoint union of its subspaces KX_g. Similarly let $\bar{Y}_k = L\backslash K/K_\cap {}^g H_k$: it is covered by the set $Y_k \subset K/K_\cap {}^g H$ of points with stabiliser $L_\cap {}^{kg}H$, and $\bar{Y}_k = W_L(L_\cap {}^{kg}H)\backslash Y_k$.

Let

$$Z_{k,g} = \{y\alpha \in KX_g, y \in Y_k, \alpha \in X_g\} \cong Y_k \times_{W_K(K_\cap {}^g H)} X_g.$$

Then $kgH \in Z_{k,g}$, $\operatorname{stab}_L(kgH \in Z_{k,g}) = L_\cap {}^{kg}H$ and G/H is a disjoint union of pieces $Z_{k,g}$. Let $\bar{Z}_{k,g} \subset L\backslash G/H$ be the image of $Z_{k,g}$, so that $\bar{Z}_{k,g} \cong W_L(L_\cap {}^{kg}H)\backslash Z_{k,g}$. But now

$$\bar{Z}_{k,g} \cong \bar{Y}_k \times_{W_K(K_\cap {}^g H)} X_g,$$

and as

$$\bar{X}_g \cong * \times_{W_K(K_\cap {}^g H)} X_g$$

we see that collapsing $\bar{Y}_k$ to a point induces a fibration

$$\bar{Y}_k \to \bar{Z}_{k,g} \to \bar{X}_g,$$

and hence $\chi_c(\bar{Z}_{k,g}) = \chi_c(\bar{X}_g)\chi_c(\bar{Y}_k)$, as required. □

(iv) We shall usually deal with the functor M_+ which is obtained from M'_+ by quotienting out from $M'_+(G)$ the submodule generated by the terms $(H, \varphi)^G$ for which $W_G(H)$ is infinite. We need to check that if $f: K \to G$ is a homomorphism and $W_G(H)$ is infinite then $f^*(H, \varphi)^G$ is contained in this submodule. But

$$f^*(H, \varphi)^G = \sum_{f(K)\backslash G/H} \chi_g(f^{-1}({}^gH), f^*c_g^*\varphi)^K,$$

and $W_G(H)$ acts on the right of $f(K)\backslash G/H$ preserving the strata. The stabiliser of $f(K)gH$ under this action is $S = (({}^{g^{-1}}f(K))H \cap N_G(H))/H$. If S is finite then the stratum has an S^1-action without fixed points so $\chi_g = 0$. If S is infinite then so is $W_K(f^{-1}({}^gH))$, because $c_g f$ induces a surjection from it to S.

If M has a Mackey structure for which $\mathrm{tr}_H^G = 0$ whenever $W_G(H)$ is infinite then there is a natural transformation $a_M: M_+ \to M$ as before. Such a structure will be termed regular. The Mackey structures in examples (i) and (ii) are all regular.

(v) Let $\hat{R}_F(G)$ denote the submodule of $R_F(G)$ that is generated by 1-dimensional representations. The functor $\hat{R}_{F+}$ will be of importance later. Notice that $\hat{R}_{F+}(G)$ can be regarded as a kind of Burnside group of G-equivariant line bundles by associating to $(H, \varphi)^G$ the induced line bundle over G/H. The transformation $\hat{R}_{F+} \to R_{F+} \xrightarrow{a} R$ will also be denoted by a. Note that some authors use the symbol R_+ for what we call $\hat{R}_{\mathbb{C}+}$.

It is clear that in the definition of $L(V)$ in §2 we could have defined instead $\hat{L}_+(V) \in \hat{R}_{\mathbb{C}+}(G)$ by $\hat{L}_+(V) = \Sigma\, \alpha_{(J,\varphi)}(V)(J, \varphi)^G$. Then $L(V) = a\hat{L}_+(V)$.

§4. The complex case for compact Lie groups

In this section and the following one we shall write R instead of $R_{\mathbb{C}}$. Let $L(n)$ denote the canonical n-dimensional representation of the unitary group $U(n)$. Let $U(r_1, r_2, r_3, \ldots)$ denote the subgroup $U(r_1) \times U(r_2) \times U(r_3) \cdots \subset U(r_1 + r_2 + r_3 \cdots)$ and let $\pi_i: U(r_1, r_2, r_3, \ldots) \to U(r_i)$ be projection onto the ith factor.

LEMMA 4.1. $\mathrm{ind}_{U(n-1,1)}^{U(n)} \pi_2^* L(1) \cong L(n)$.

Proof. Check the characters on regular elements, using the formula of (Se2) for the character of the induced representation. □

We want to construct a natural transformation $b: R \to \hat{R}_+$ such that $ab = id_R$. The lemma shows that $a_{U(n)}(U(n-1, 1), \pi_2^* L(1))^{U(n)} = L(n)$: this suggests that we set $b_{U(n)} L(n) = (U(n-1, 1), \pi_2^* L(1))^{U(n)}$. We can now extend by naturality, for if V

is an n-dimensional representation of G then there exists a homomorphism $f: G \to U(n)$ such that $V = f^*L(n)$ and we can set

$$b_G(V) = f^*(U(n-1,1), \pi_2^* L(1))^{U(n)}$$

$$= \sum_{f(G)\backslash U(n)/U(n-1,1)} \chi_g(f^{-1}({}^gU(n-1,1), f^*c_g^*\pi_2^*L(1)))^G.$$

A different choice of f would differ only by an inner automorphism of $U(n)$ so would yield the same $b_G(V)$.

Now $U(n)/U(n-1,1) \cong \mathbb{P}(V)$ as a G-space so we regain the formula of §2:

$$b_G(V) = \sum_{(J,\varphi)} \alpha_{(J,\varphi)}(V)(J,\varphi)^G,$$

where $\alpha_{(J,\varphi)}$ is the Euler number in compactly supported cohomology of the stratum of $G\backslash\mathbb{P}(V)$ of type (J,φ).

To show that b determines a natural transformation there remains only to check additivity.

LEMMA 4.2.

$$\operatorname{res}^{U(r+s)}_{U(r,s)}(U(r+s-1,1), \pi_2^*L(1))^{U(r+s)} = \pi_1^*(U(r-1,1), \pi_2^*L(1))^{U(r)}$$

$$+ \pi_2^*(U(s-1,1), \pi_2^*L(1))^{U(s)}.$$

Proof. The formula for restriction gives a sum over the strata of $U(r,s)\backslash\mathbb{P}(L(r+s))$. This space is homeomorphic to a closed line interval and has three strata: the endpoints corresponding to stabilisers $U(r-1,1,s)$ and $U(r,s-1,1)$ with representation on the fibre by projection onto the $U(1)$ factor; and the interior corresponding to stabiliser $U(r-1,*,s-1,*)$, i.e. the subgroup of $U(r-1,1,s-1,1)$ in which the two $U(1)$ entries are equal. But $(U(r-1,1,s), \pi_2^*L(1))^{U(r,s)} = \pi_1^*(U(r-1,1), \pi_2^*L(1))^{U(r)}$, and similarly for the other point stratum. $W_{U(r,s)}(U(r-1,*,s-1,*)) \cong U(1)$ is infinite, so the third term disappears. □

PROPOSITION 4.3. *b is additive.*

Proof. Suppose that G has representations U and V of dimensions r and s respectively and that $u: G \to U(r)$ and $v: G \to U(s)$ are such that $U = u^*L(s)$ and

$V = v^* L(r)$. We get a homomorphism $(u, v) : G \to U(r, s)$ and

$$\begin{aligned} b_G(U \oplus V) &= (u, v)^* \operatorname{res}^{U(r+s)}_{U(r,s)} (U(r+s-1, 1), \pi_2^* L(1))^{U(r+s)} \\ &= (u, v)^* \pi_1^* (U(r-1, 1), \pi_2^* L(1))^{U(r)} \\ &\quad + (u, v)^* \pi_2^* (U(s-1, 1), \pi_2^* L(1))^{U(s)}. \end{aligned}$$

But $\pi_1(u, v) = u$ and $\pi_2(u, v) = v$, so the right hand side is equal to $b_G(U) + b_G(V)$, as required. □

To sum up we have

PROPOSITION 4.4. *There is a natural transformation $b : R \to \hat{R}_+$ such that $ab = id_R$ and such that when b is restricted to $\hat{R}$, $b_G \varphi = (G, \varphi)^G$.* □

REMARK 4.5. Snaith [Sn2] uses a different formula for a map b defined on n-dimensional representations such that $ab = id_R$. It is not additive so it does not extend to a natural transformation. In our context it arises from the fact that Lemma 4.1 is also true in the form $L(n) = \operatorname{ind}^{U(n)}_{NT(n-1) \times U(1)} \pi_2^* L(1)$, where $NT(n-1)$ is the normaliser of a maximal torus in $U(n-1)$. The construction can then proceed as before.

§5. Complex characteristic classes

DEFINITION 5.1. A complex characteristic class with values in a functor with regular Mackey structure, M, is a natural transformation of contravariant functors $c : R \to M$.

Any natural transformation $t : \hat{R} \to M$ induces a transformation $t_+ : \hat{R}_+ \to M_+$, so we can use it to construct a characteristic class $\tilde{t} : R \to M$ by letting $\tilde{t} = a_M t_+ b$. In particular if $c : R \to M$ is already a characteristic class then it induces a transformation of functors with Mackey structure (i.e. a transformation $R_! \to M_!$ too), $c_+ : \hat{R}_+ \to M_+$. We consider $\tilde{c} = a_M c_+ b : R \to M$ and ask if it is equal to c.

PROPOSITION 5.2. *$\tilde{c} = c$ provided that $\operatorname{res}^{U(n)}_{T(n)} : M(U(n)) \to M(T(n))$ is injective for all $n \geq 1$. Here $T(n)$ denotes the maximal torus $U(1)^n$ of $U(n)$.*

Proof. By naturality, we only need to check for $L(n)$. By the injectivity condition this reduces to $\operatorname{res}^{U(n)}_{T(n)} L(n) = \eta$, say. The latter is a sum of 1-dimensional representations, so $b\eta = (T(n), \eta)^{T(n)}$ and $\tilde{c}\eta = a_M c_+ (T(n), \eta)^{T(n)} = a_M (T(n), c\eta)^{T(n)} = c\eta$, as required. □

EXAMPLES. (i) b itself is a characteristic class.

(ii) The Adams operations are characteristic classes $\psi^r : R \to R$. In this case $\psi^r_+ (H, \varphi)^G = (H, \varphi^r)^G$, where φ is a 1-dimensional representation of H.

(iii) Taking the total Chern class of a representation yields a characteristic class $c : R \to H^{**}(-; \mathbb{Z})$. Segal showed that $H^{**}(G, \mathbb{Z})$ is the zeroth term of a generalised cohomology theory applied to BG [Se2], so it has a regular Mackey structure by §3 example (ii). Proposition 5.2 applies [Bor], so $\tilde{c} = c$.

(iv) Taking the Newton polynomials in the Chern classes (see [T] for example) yields a characteristic class $N : R \to H^*(-, \mathbb{Z})$ (additive group and transfer) and again $\tilde{N} = N$ by Proposition 5.2.

§6. The real and symplectic cases

The symplectic case presents no new difficulties; one just replaces complex projective space by the quaternionic version. The real case is more complicated and it is well known that it is necessary to consider 2-dimensional as well as 1-dimensional representations.

Let $M(n)$ denote the canonical real representation of the orthogonal group $O(n)$. The groups $O(r, s)$ etc. are defined in the same way as the $U(r, s)$ in §4. Let $R_{\mathbb{R},i}$ denote the free $\mathbb{Z}$-module with the isomorphism classes of i-dimensional representations of G as basis. $R_{\mathbb{R},i}$ is a contravariant functor with a canonical transformation $R_{\mathbb{R},i} \to R_{\mathbb{R}}$, which is not injective in general for $i \geq 2$.

The role of $\hat{R}_+$ in the complex case will be performed by a functor S which we shall now define. Consider $(R_{\mathbb{R},1} \oplus R_{\mathbb{R},2})_+$ formed as in §3. The image of $(H, \eta)^G \in R_{\mathbb{R},i}$ will be written $(H, \eta)^G_i$. Let N be the smallest subfunctor with Mackey structure (i.e. closed under both f^* and $f_!$) such that $N(O(1, 1))$ contains $(O(*, *), \pi_1^* M(1))_1^{O(1,1)} - (O(1, 1), \mathrm{res}^{O(2)}_{O(1,1)} M(2))_2^{O(1,1)}$. We set $S = (R_{\mathbb{R},1} \oplus R_{\mathbb{R},2})_+ / N$. The point is that $\mathrm{ind}^{O(1,1)}_{O(*,*)} \pi_1^* M(1) = \mathrm{res}^{O(2)}_{O(1,1)} M(2)$ so that we still get a well-defined homomorphism $a : S \to R_{\mathbb{R}}$ by setting $a_G(H, \eta)^G_i = \mathrm{ind}^G_H \eta$, $i = 1, 2$.

LEMMA 6.1. $M(n) = \mathrm{ind}^{O(n)}_{O(n-1,1)} \pi_2^* M(1) + \mathrm{ind}^{O(n)}_{O(n-2,2)} \pi_2^* M(2)$, $n \geq 2$.

Proof. Check characters. □

This suggests that we should define a map $b : R_{\mathbb{R}} \to S$ such that $ab = id_{R_{\mathbb{R}}}$ by letting

$$b_{O(n)}M(n) = (O(n-1, 1), \pi_2^* M(1))_1^{O(n)} + (O(n-2, 2), \pi_2^* M(2))_2^{O(n)}, n \geq 2$$
$$b_{O(1)}M(1) = (O(1), M(1))^{O(1)}$$

and then extending by naturality.

Let

$$A_n = (O(n-1, 1), \pi_2^* M(1))_1^{O(n)}$$

and

$$B_n = (O(n-2, 2), \pi_2^* M(2))_2^{O(n)}, \quad n \geq 2$$
$$B_1 = 0$$

so that $b_{O(n)}M(n) = A_n + B_n$.

Geometrically this corresponds to

$$b_G(V) = \sum_{(J,\varphi)} \chi_c(G \backslash \mathbb{P}_{\mathbb{R}}(V)_{(J,\varphi)})(J, \varphi)_1^G + \sum_{(K,\theta)} \chi_c(G \backslash Gr_2(V)_{(K,\theta)})(K, \theta)_2^G,$$

where $Gr_2(V)$ is the Grassman variety of 2-planes in V and θ is the representation of K on a fibre of the canonical plane bundle above a point stabilised by K.

We still need to check additivity: as in the complex case it follows immediately from the next lemma.

LEMMA 6.2. $\mathrm{res}_{O(r,s)}^{O(r+s)} (A_{r+s} + B_{r+s}) = \pi_1^*(A_r + B_r) + \pi_2^*(A_s + B_s)$.

Proof. $\mathrm{res}_{O(r,s)}^{O(r+s)} A_{r+s}$ involves a sum over the strata of $O(r, s)\backslash \mathbb{P}_{\mathbb{R}}(\mathrm{res}_{O(r,s)}^{O(r+s)} M(r+s))$. This space is homeomorphic to a closed line interval. The endpoints consist of one of type $(O(r-1, 1, s), \pi_2^* M(1))_1^{O(r,s)}$ and its pair, which is obtained by reversing the roles of r and s. These correspond to the A-terms on the right hand side. The interior has Euler number -1 and is of type $(O(r-1, *, s-1, *), \pi_2^* M(1))_1^{O(r,s)}$, which in contrast to the complex case has finite Weyl group and does not disappear.

$\mathrm{res}_{O(r,s)}^{O(r+s)} B_{r+s}$ involves a sum over the strata of $O(r, s)\backslash Gr_2(\pi_1^* M(r) \oplus \pi_2^* M(s))$. The stratum containing the image of a plane $X \subset \pi_1^* M(r) \oplus \pi_2^* M(s)$ is determined by the pair of numbers $(\dim (X \cap \pi_1^* M(r)), \dim (X \cap \pi_2^* M(s)))$.

Stratum $(2, 0)$ consists of one point and is of type $(O(r-2, 2, s), \pi_2^* M(2))_2^{O(r,s)}$. This together with its pair $(0, 2)$ give the B-terms on the right hand side. Stratum

$(1, 1)$ also consists of one point and is of type $(O(r-1, 1, s-1, 1), \pi_2^* M(1) + \pi_4^* M(1))_2^{O(r,s)}$. This cancels with the unwanted term in res A, modulo N. The remaining strata are stratum $(1, 0)$, of type $(O(r-2, 1, *, s-1, *), \pi_2^* M(1) + \pi_3^* M(1))_2^{O(r,s)}$, which is equivalent modulo N to $(O(r-2, *, *, s-1, *), \pi_2^* M(1))_1^{O(r,s)}$. But this has infinite Weyl group. The case of stratum $(0, 1)$ is similar, which leaves only stratum $(0, 0)$. This has type $(O(r-2, *, *, s-2, *, *), 2\pi_2^* M(1))_2^{O(r,s)}$, which again has infinite Weyl group. □

§7. Real characteristic classes

Suppose that M is a functor with regular Mackey structure and that we are given transformations $w^1 : R_{\mathbb{R},1} \to M$ and $w^2 : R_{\mathbb{R},2} \to M$ such that

$$\mathrm{tr}_{O(*,*)}^{O(1,1)} w^1(O(*,*), \pi_1^* M(1)) = w^2(O(1, 1), \mathrm{res}_{O(1,1)}^{O(2)} M(2)).$$

These combine to give $w_+ : S \to M$ and hence $w = w_+ b : R_{\mathbb{R}} \to M$. Note that w^1 and w^2 are completely determined by their values $v_1 = w^1(O(1), M(1))$ and $v_2 = w^2(O(2), M(2))$, and that we require that

$$\mathrm{res}_{O(1,1)}^{O(2)} v_2 = \mathrm{tr}_{O(*,*)}^{O(1,1)} \pi_1^* v_1 \qquad (*)$$

Conversely, given a natural transformation w: $R_{\mathbb{R}} \to M$, we can define w^1 and w^2 by specifying v_1 and v_2 thus:

$$v_1 = w(O(1), M(1))$$

$$v_2 = w(O(2), M(2)) - \mathrm{tr}_{O(1,1)}^{O(2)} \pi_1^* v_1.$$

Then

$$\mathrm{res}_{O(1,1)}^{O(2)} v_2 = w(O(1, 1), \mathrm{res}_{O(1,1)}^{O(2)} M(2)) - \mathrm{res}_{O(1,1)}^{O(2)} \mathrm{tr}_{O(1,1)}^{O(2)} \pi_1^* v_1$$

We apply the double coset formula to the last term. Now $O(1, 1)\backslash O(2)/O(1, 1) = O(1, 1)\backslash \mathbb{P}_{\mathbb{R}}(\mathrm{res}_{O(1,1)}^{O(2)} M(2))$ is a closed interval with three strata: the two endpoints have stabiliser $O(1, 1)$, one with representation $\pi_1^* M(1)$ on the fibre, the other with $\pi_2^* M(1)$. The interior has type $(O(_{*,*}), \pi_1^* M(1))$ and Euler number -1. Hence

$$\begin{aligned}\mathrm{res}_{O(1,1)}^{O(2)} v_2 &= w(O(1, 1), \mathrm{res}_{O(1,1)}^{O(2)} M(2) - \pi_1^* M(1) - \pi_2^* M(1)) + \mathrm{tr}_{O(*,*)}^{O(1,1)} \pi_1^* v_1 \\ &= \mathrm{tr}_{O(*,*)}^{O(1,1)} \pi_1^* v_1, \quad \text{as required by condition } (*).\end{aligned}$$

Now let $w = 1 + w_1 + w_2 + \cdots$ be the total Stiefel–Whitney class, considered as taking values in $H^{**}(G; \mathbb{Z}/2)$, i.e. the multiplicative group of infinite sums $1 + a_1 + a_2 + \cdots$, $a_i \in H^i(G, \mathbb{Z}/2)$. Again, the Mackey structure arises from [Se2] and §4 example (ii).

Recall that $H^*(O(1), \mathbb{Z}/2) \cong \mathbb{Z}[x]$, $x \in H^1(G; \mathbb{Z}/2)$, hence $H^*(O(1)^n, \mathbb{Z}/2) \cong \mathbb{Z}[x_1, \ldots, x_n]$, $x_i = \pi_i^* x$, and that $\mathrm{res}^{O(n)}_{O(1)^n} : H^*(O(n), \mathbb{Z}/2) \to H^*(O(1)^n, \mathbb{Z}/2)$ is injective with image consisting of the symmetric expressions in the x_i.

For the Stiefel–Whitney class we find that $v_1 = 1 + x_1$ and $v_2 = 1 + x_1 x_2$ and thus

$$w^1\varphi = 1 + w_1\varphi, \quad \text{for } \varphi \text{ 1-dimensional,}$$

$$w^2\theta = 1 + w_2\theta, \quad \text{for } \theta \text{ 2-dimensional.}$$

PROPOSITION 7.1. $\tilde{w} = w$, *i.e.*

$$w(V) = \prod_{(J,\varphi)} \mathcal{N}^G_J(1 + w_1\varphi)^{\chi_c(G\backslash \mathbb{P}_{\mathbb{R}}(V)_{(J,\varphi)})} \prod_{(K,\theta)} \mathcal{N}^G_K[1 + w_2\theta]^{\chi_c(G\backslash Gr_2(V)_{(K,\theta)})}.$$

Proof. Because $\mathrm{res}^{O(n)}_{O(1)^n}$ is injective, we only need to check that $\tilde{w} = w$ for $\mathrm{res}^{O(n)}_{O(1)^n} M(n)$.

Now

$$\begin{aligned}\tilde{w}(\mathrm{res}^{O(n)}_{O(1)^n} M(n)) &= \tilde{w}\left(\sum \pi_i^* M(1)\right)\\ &= \prod \tilde{w}\pi_i^* M(1)\\ &= \prod \pi_i^* \tilde{w} M(1)\\ &= \prod \pi_i^*(1 + x),\end{aligned}$$

and the same is true with $\tilde{w}$ replaced by w. □

For general w when $M = H^{**}(-, \mathbb{Z}/2)$, note that in condition (*), $\mathcal{N}^{O(1,1)}_{O(*,*)} \pi_1^* v_1$ is symmetric with respect to interchanging the two $O(1)$s, so it is in the image of $H^{**}(O(2), \mathbb{Z}/2)$. Therefore v_1 can be chosen arbitrarily and it uniquely determines v_2.

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