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Homomorphisms of constant stretch between Möbius groups

PEKKA TUKIA

A. Introduction

A Möbius transformation g of \bar{R}^n is *loxodromic* if it can be conjugated by another Möbius transformation to the form

$$x \mapsto \lambda\beta(x) \quad (x \in R^n) \tag{A1}$$

where $\lambda > 1$ and β is an orthogonal linear map. The number $\lambda > 1$ does not depend on how the conjugacy is chosen and it is the *multiplier* $\text{mul } g$ of g ; for non-loxodromic g we set $\text{mul } g = 1$.

If G and H are two groups of Möbius transformations of \bar{R}^n and $\varphi : G \rightarrow H$ is a homomorphism between them, φ is said to be of *constant stretch* if there is $d > 0$ such that

$$\text{mul } \varphi(g) = (\text{mul } g)^d \tag{A2}$$

for all $g \in G$; more precisely we can say that φ is of constant stretch d . Note that d is well-defined if there are loxodromic elements in G and that g is loxodromic if and only if $\varphi(g)$ is. If $d = 1$, then we say that φ is *multiplier preserving*.

Our main Theorem C says that a homomorphism φ of non-elementary groups is of constant stretch if and only if it is multiplier preserving. Furthermore, such a φ comes very near to being a conjugation by a Möbius transformation. If the limit set $L(G)$ of G “fills” \bar{R}^n , that is, $h(L(G)) \not\subset \bar{R}^k$ for no $k < n$ and no Möbius transformation h , then we can actually show that φ is a conjugation by a Möbius transformation.

A consequence of Theorem C is that if a map $f : A \rightarrow \bar{R}^n$ is compatible with a homomorphism $\varphi : G \rightarrow H$ of non-elementary Möbius groups, that is for every $g \in G$, $gA = A$ and

$$fg(x) = \varphi(g)f(x) \tag{A3}$$

when $x \in A$, then φ is a conjugation by a Möbius transformation as soon as f satisfies a bilipschitz property and the above mentioned condition for the limit set is satisfied (Theorem D).

Originally, we needed Theorem C in [T2] but after we found a simpler method for [T2], we separated these results into the present paper. In [T1] we have already treated the case of a multiplier preserving φ . The present arrangement of the proof seems to be slightly simpler also for the multiplier preserving case. In our proof of Theorem C we will first show that a homomorphism of constant stretch is multiplier preserving and then sketch the remaining part for completeness although we could here refer to [T1].

DEFINITIONS AND NOTATIONS. We denote the group of all Möbius transformations of \bar{R}^n by $M(\bar{R}^n)$. Each $g \in M(\bar{R}^n)$ has a unique extension to a Möbius transformation of \bar{R}^{n+1} such that $g(H^{n+1}) = H^{n+1}$ when H^{n+1} is the $(n+1)$ -dimensional hyperbolic space

$$H^{n+1} = \{x \in R^{n+1} : x = (x_1, \dots, x_{n+1}) \text{ where } x_{n+1} > 0\}.$$

We identify g and this extension of g to \bar{R}^{n+1} ; thus $M(\bar{R}^n) \subset M(\bar{R}^{n+1})$.

A loxodromic $g \in M(\bar{R}^n)$ has two fixed points denoted by $P_g = P(g)$ and $N_g = N(g)$ so that P_g is the attracting fixed point and N_g the repelling fixed point; these names are self-explanatory. A loxodromic map g is *hyperbolic* if it is conjugate in $M(\bar{R}^n)$ to a map as in (A1) where $\beta = \text{id}$. If $g \in M(\bar{R}^n)$ is not loxodromic, then it is either elliptic or parabolic. If g is *elliptic*, then it is conjugate in $M(\bar{R}^n)$, or in $M(\bar{R}^{n+1})$, to a map as in (A1) where $\lambda = 1$, and g is *parabolic* if it is conjugate in $M(\bar{R}^n)$ to a map of the form

$$x \mapsto \beta(x) + a \tag{A4}$$

where $a \in R^n$, $a \neq 0$, and β is an orthogonal linear map such that $\beta(a) = a$ (cf. [T3, p. 560]).

A *Möbius group* G is a subgroup of $M(\bar{R}^n)$ and such a group is discrete if it is discrete in the compact-open topology of \bar{R}^n . A set A is *G-invariant* if $gA = A$ for every $g \in G$.

The *limit set* $L(G)$ of G is

$$L(G) = \text{cl } Gz \cap \bar{R}^n \tag{A5}$$

where $z \in H^{n+1}$ (and where cl is the closure). This does not depend on the choice of $z \in H^{n+1}$ and is a reasonable definition of $L(G)$ also for non-discrete G .

We define that a Möbius group G is *non-elementary* if it contains two loxodromic elements with disjoint fixed point sets. If G is discrete, then it is well-known that G is non-elementary if and only if $L(G)$ contains more than two points (see e.g. [T3, Theorem B2].)

We usually work in \bar{R}^n but we find it more natural to formulate Theorem D for Möbius groups of the n -sphere $S^n = \{x \in R^{n+1} : |x| = 1\}$. We also use above definitions with appropriate modifications for Möbius groups of S^n .

B. Representation of Möbius transformations by matrices

The proof of our main theorem depends on matrix representations of Möbius transformations. Let $O(1, n+1)$ be the group of $(n+2) \times (n+2)$ -matrices which preserves the quadratic form $x_1^2 - x_2^2 - \cdots - x_{n+2}^2$ and let $O_+(1, n+1)$ be the subgroup of $O(1, n+1)$ which preserves

$$\{(x_1, \dots, x_{n+2}) \in R^{n+2} : x_1^2 - x_2^2 - \cdots - x_{n+2}^2 = 1 \text{ and } x_1 > 0\}.$$

Then, as is well-known [W], every $g \in M(\bar{R}^n)$ can be represented by a unique matrix $A \in O_+(1, n+1)$.

If $n = 2$, then we identify R^2 and the complex plane C . If $g \in M(\bar{R}^2)$ is orientation preserving, then it can be represented by a matrix of $SL(2, C)$, that is, by a complex 2×2 -matrix with determinant 1.

We will now give two simple formulas that relate the multiplier of $g \in M(\bar{R}^n)$ and the trace $\text{tr } A$ of the matrix $A \in O_+(1, n+1)$ or $A \in SL(2, C)$ representing g . If $g \in M(\bar{R}^n)$ is represented by a matrix $A \in O_+(1, n+1)$, then

$$\text{mul } g = \text{tr } A + M(A) \tag{B1}$$

where $|M(A)| \leq n+2$. This follows from explicit matrix representations for loxodromic, elliptic or parabolic Möbius transformations, see Wielenberg [W, Section 5] and the classification of a Möbius transformation as loxodromic, elliptic or parabolic mentioned above. Recall that $\text{mul } g = 1$ for non-loxodromic g . Note that if g is elliptic, then we possibly need to extend g to a Möbius transformation of \bar{R}^{n+1} in order to obtain that g is conjugate to an orthogonal linear map.

If $g \in M(\bar{R}^2)$ is orientation preserving, then g can be represented by $B \in SL(2, C)$, and

$$\text{mul } g = |\text{tr } B|^2 + M'(B) \tag{B2}$$

where $|M'(B)| \leq 3$ as a simple calculation shows. The next lemma is based on these estimates.

LEMMA B. *Let $g, h \in M(\bar{R}^n)$ be loxodromic and let $\gamma = (\text{mul } g)^{1/2}$ and $\chi = (\text{mul } h)^{1/2}$. Then, for $m, k \in \mathbb{Z}$,*

$$\begin{aligned} \text{mul } g^m h^k &= |a(\gamma^m \chi^k + \gamma^{-m} \chi^{-k}) + b(\gamma^m \chi^{-k} + \gamma^{-m} \chi^k)|^2 \\ &\quad + c_{1m} \gamma^{2m} + c_{2m} \gamma^{-2m} + d_{1k} \chi^{2k} + d_{2k} \chi^{-2k} + e_{mk} \end{aligned} \quad (\text{B3})$$

where a, b are complex numbers such that $a + b = 1$ and depending only on the quadruple (P_g, N_g, P_h, N_h) of the fixed points. The constants c_{ik} , d_{im} , and e_{mk} are bounded and, furthermore,

- (a) $a \neq 0 \neq b$ if and only if g and h do not have common fixed points,
- (b) if g and h are hyperbolic, then $c_{mk} = d_{mk} = 0$ and $|e_{mk}| \leq 3$ for all m, k ; in the general loxodromic case there is a sequence $r_1 < r_2 < \dots$ such that as $j \rightarrow \infty$,

$$c_{i, \pm r_j} \rightarrow 0 \quad \text{and} \quad d_{i, \pm r_j} \rightarrow 0 \quad (i = 1, 2),$$

- (c) if the fixed fixed points of g and h are in $\bar{R}^2 = \bar{C}$ and $N_g = 0$, $P_g = \infty$, $N_h = 1$, and $P_h = p$, then there are the following relations between the numbers p , a and b :

$$p = -\frac{a}{b}, \quad a = \frac{p}{p-1} \quad \text{and} \quad b = \frac{1}{1-p}.$$

Proof. The fixed points of g and h lie in a 2-dimensional sphere and hence we may assume that their fixed points lie in \bar{C} and that

$$P_g = \infty \quad \text{and} \quad N_g = 0.$$

Let \bar{g} and \bar{h} be the corresponding hyperbolic Möbius transformations, i.e. they have the same multiplier and the same repelling and attractive fixed points. Then \bar{g} and \bar{h} preserve \bar{C} and can be represented by matrices $\tilde{A}, \tilde{B} \in SL(2, C)$, respectively. We can conjugate in $SL(2, C)$ to obtain

$$\tilde{A} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} s & t \\ u & v \end{pmatrix} \begin{pmatrix} \chi & 0 \\ 0 & \chi^{-1} \end{pmatrix} \begin{pmatrix} v & -t \\ -u & s \end{pmatrix}$$

where s, t, u and v are complex numbers such that $sv - tu = 1$. A simple calculation shows that if

$$a = sv \quad \text{and} \quad b = -tu,$$

then

$$\text{tr } \tilde{A}^m \tilde{B}^k = a(\gamma^m \chi^k + \gamma^{-m} \chi^{-k}) + b(\gamma^m \gamma^{-k} + \gamma^{-m} \chi^k). \quad (\text{B4})$$

Obviously, a and b depend only on the fixed points and $a + b = 1$. Since $P_h = s/u$ and $N_h = t/v$, we have that $a \neq 0 \neq b$ if and only if g and h do not have common fixed points. Remembering that $a + b = 1$, the formulas in (c) also follow.

We then represent g, h, \bar{g}, \bar{h} by matrices A, B, \bar{A}, \bar{B} of $O_+(1, n+1)$, respectively, and perform similar calculations. The matrix A has eigenvalues $\alpha_1, \dots, \alpha_{n+2}$ which we can enumerate so that $\alpha_1 = \gamma^2 = \text{mul } g$ and $\alpha_2 = \gamma^{-2}$ and that $\alpha_i, i > 2$ are complex numbers of modulus 1 as follows from the canonical forms for matrices of $O_+(1, n+1)$ representing loxodromic Möbius transformation [W, Section 5]. Similarly, B has eigenvalues $\beta_1 = \chi^2, \beta_2 = \chi^{-2}, \dots, \beta_{n+2}$. We can assume that A is diagonal (so that the diagonal entries are the eigenvalues) and that

$$B = EDE^{-1} \quad (\text{B5})$$

for some matrices E, D where D is diagonal (they need not be matrices of $O(1, n+1)$.) A calculation shows that there are constants $a_{ij}, i, j \leq n+2$ such that

$$\text{tr } A^m B^k = \sum_{i,j} a_{ij} \alpha_i^m \beta_j^k. \quad (\text{B6})$$

Thus if we set

$$c'_{ik} = \sum_{j>2} a_{ij} \beta_j^k, \quad (i = 1, 2, k \in \mathbb{Z}),$$

$$d'_{im} = \sum_{j>2} a_{ji} \alpha_j^m, \quad (i = 1, 2, m \in \mathbb{Z}),$$

$$e'_{mk} = \sum_{i>2, j>2} a_{ij} \alpha_i^m \beta_j^k, \quad (m, k \in \mathbb{Z}),$$

we obtain bounded numbers (since $|\alpha_j| = |\beta_j| = 1$ if $j > 2$) such that

$$\begin{aligned} \operatorname{tr} A^m B^k &= a_{11} \gamma^{2m} \chi^{2k} + a_{12} \gamma^{2m} \chi^{-2k} + a_{21} \gamma^{-2m} \chi^{2k} + a_{22} \gamma^{-2m} \chi^{-2k} \\ &\quad + c'_{1k} \gamma^{2m} + c'_{2k} \gamma^{-2m} + d'_{1m} \chi^{2k} + d'_{2m} \chi^{-2k} + e'_{mk}. \end{aligned} \quad (\text{B7})$$

We obtain the matrix \bar{A} from that of A by substituting 1 for α_i if $i > 2$ (and leaving α_1 and α_2 unchanged). Similarly, substituting 1 for β_j in D if $j > 2$, we obtain \bar{B} from the right hand side of (B5). With these substitutions (B6) gives $\operatorname{tr} \bar{A}^m \bar{B}^k$ with a_{ij} unchanged but with the new α_i and β_j , and (B7) is valid if we substitute in it for c'_{im} , d'_{im} and e'_{mk} the numbers

$$\begin{aligned} \bar{c}_i &= \sum_{j>2} a_{ij}, \\ \bar{d}_i &= \sum_{j>2} a_{ji}, \\ \bar{e} &= \sum_{i>2, j>2} a_{ij}. \end{aligned}$$

(They do not depend on m and k and so we have not marked them.) It follows by Kronecker's theorem ([A, Theorem 7.10] or [C, p. 53]) that there is a sequence $r_1 < r_2 < \dots$ such that, as $i \rightarrow \infty$,

$$c'_{i, \pm r_i} \rightarrow \bar{c}_i \quad \text{and} \quad d'_{i, \pm r_i} \rightarrow \bar{d}_i \quad (\text{B8})$$

for $i = 1, 2$.

Applying (B1) and (B2) to $\operatorname{mul} \bar{g}^m \bar{h}^k$ we obtain that

$$|\operatorname{tr} \bar{A}^m \bar{B}^k - |\operatorname{tr} \tilde{A}^m \tilde{B}^k|^2| \leq n + 5. \quad (\text{B9})$$

Set

$$\begin{aligned} c_{im} &= c'_{im} - \bar{c}_i, \\ d_{ik} &= d'_{ik} - \bar{d}_i, \\ e''_{mk} &= e'_{mk} - \bar{e}. \end{aligned} \quad (\text{B10})$$

These numbers are bounded and satisfy (b) with respect to the r_i in (B8). Write

$$\begin{aligned} \operatorname{mul} g^m h^k &= (\operatorname{mul} g^m h^k - \operatorname{tr} A^m B^k) + (\operatorname{tr} A^m B^k - \operatorname{tr} \bar{A}^m \bar{B}^k) \\ &\quad + (\operatorname{tr} \bar{A}^m \bar{B}^k - |\operatorname{tr} \tilde{A}^m \tilde{B}^k|^2) + |\operatorname{tr} \tilde{A}^m \tilde{B}^k|^2. \end{aligned}$$

On the right-hand sum the first parenthesis is bounded by (B1) and the third parenthesis by (B9). The second parenthesis can be estimated by (B7) when it is applied to $\text{tr } A^m B^k$ and to $\text{tr } \bar{A}^m \bar{B}^k$, and the last term is given by (B4). Combining all this, we have

$$\begin{aligned} \text{mul } g^m h^k &= |a(\gamma^m \chi^k + \gamma^{-m} \chi^{-k}) + b(\gamma^m \chi^{-k} + \gamma^{-m} \chi^k)|^2 \\ &\quad + c_{1m} \gamma^{2m} + c_{2m} \gamma^{-2m} + d_{1k} \chi^{2k} + d_{2k} \chi^{-2k} + e_{mk}. \end{aligned}$$

Here e_{mk} is the sum of e''_{mk} in (B10) and of the first and third parenthesis. They are bounded since e''_{mk} are bounded, and by what has been said above, and so are the numbers c_{im} and d_{ik} .

If g and h are hyperbolic, then we can use (B4) and (B2) to conclude that (B3) is true with $c_{im} = d_{jk} = 0$ and $|e_{mk}| \leq 3$.

Finally, (b) follows from (B8) and (B10).

Remark. If we have two (or, in fact, any number of) pairs g, h and \tilde{g}, \tilde{h} of loxodromic Möbius transformations and if \tilde{c}_{ij} and \tilde{d}_{ij} are the numbers in the expression for $\text{mul } \tilde{g}^m \tilde{h}^k$, and if \tilde{r}_i is the corresponding sequence in (b), then exactly as in (b), by Kronecker's theorem, one can choose these sequences so that $r_i = \tilde{r}_i$.

C. The main theorem

We can now prove our main

THEOREM C. *Let $\varphi : G \rightarrow H$ be a surjective homomorphism of two Möbius groups of $\bar{\mathbb{R}}^n$ such that one of the groups G and H is non-elementary. Then φ is multiplier preserving if it is of constant stretch $d > 0$.*

Furthermore, let S be the k -sphere of smallest dimension k such that $S = g\bar{R}^k$ for some $g \in M(\bar{\mathbb{R}}^n)$ and that $S \supset L(G)$, where $L(G)$ is the limit set of G (see (A5)). Then S is G -invariant and there is $h \in M(\bar{\mathbb{R}}^n)$ such that

$$hg(x) = \varphi(g)h(x) \tag{C0}$$

for $x \in S$ and $g \in G$.

In particular, if $S = \bar{\mathbb{R}}^n$, then φ is a conjugation by a Möbius transformation.

Remark. Actually, it would suffice to assume that (A2) is true for all $g \in G$ such that g is loxodromic (if G is non-elementary) or such that $\varphi(g)$ is loxodromic (if H is non-elementary).

The sphere S in the theorem is well-defined if $L(G)$ contains at least two points. Since either G or H is non-elementary, and φ is of constant stretch, G has loxodromics and hence S is well-defined (the proof shows that both groups are non-elementary).

Proof. We first assume that G is non-elementary and that φ is an isomorphism.

We first prove that $d = 1$. Since G is non-elementary, there are two loxodromic elements $g, h \in G$ without common fixed points. Then also $\varphi(g)$ and $\varphi(h)$ are loxodromic by (A2).

Let $\bar{g} = \varphi(g)$ and $\bar{h} = \varphi(h)$. Then $\text{mul } \bar{g} = (\text{mul } g)^d = \gamma^{2d}$ and $\text{mul } \bar{h} = (\text{mul } h)^d = \chi^{2d}$ and hence if $\bar{a}, \bar{b}, \bar{c}_{im}, \bar{d}_{ik}$ and \bar{e}_{mk} are numbers as in Lemma B, we have that

$$\begin{aligned} \text{mul } \bar{g}^m \bar{h}^k &= |\bar{a}(\gamma^{dm} \chi^{dk} + \gamma^{-dm} \chi^{-dk}) + \bar{b}(\gamma^{dm} \chi^{-dk} + \gamma^{-dm} \chi^{dk})|^2 \\ &\quad + \bar{c}_{1m} \gamma^{2dm} + \bar{c}_{2m} \gamma^{-2dm} + \bar{d}_{1k} \chi^{2dk} + \bar{d}_{2k} \chi^{-2dk} + \bar{e}_{mk}. \end{aligned} \quad (\text{C1})$$

We now use the equality

$$\text{mul } \bar{g}^m \bar{h}^k = (\text{mul } g^m h^k)^d \quad (\text{C2})$$

together with (B3) and (C1) and let m, k tend to $+\infty$ or to $-\infty$. Since g and h do not have common fixed points, $a \neq 0 \neq b$ by Lemma B (a) and it follows that

$$|\bar{a}| = |a|^d \quad \text{and} \quad |\bar{b}| = |b|^d. \quad (\text{C3})$$

In particular, it follows that $\bar{a} \neq 0 \neq \bar{b}$ and hence \bar{g} and \bar{h} do not have common fixed points by Lemma B (a). Thus H is also non-elementary and, if necessary, we can replace φ by φ^{-1} and thereby assume that

$$d \geq 1. \quad (\text{C4})$$

Next, substitute again (B3) and (C1) into (C2) and divide both sides of the resulting equation by $|a|^d \gamma^{2dm} \chi^{2dk} = |\bar{a}| \gamma^{2dm} \chi^{2dk}$. Keep k fixed and let m assume the values r_i of Lemma B (b) (see also the Remark following Lemma B) and let $i \rightarrow \infty$. We obtain

$$|1 + \bar{b} \chi^{-2kd} / \bar{a}|^2 = |1 + b \chi^{-2k} / a|^{2d} \quad (\text{C5})$$

which is valid for every $k \in \mathbb{Z}$.

Let

$$r = |b|/|a|, \quad \alpha = \arg b/a, \quad \bar{\alpha} = \arg \bar{b}/\bar{a},$$

so that $r^d = |\bar{b}|/|\bar{a}|$. Substituting this into (C5) and using elementary trigonometry, we obtain for all $k \in \mathbb{Z}$

$$1 + 2r^d \chi^{-2kd} \cos \bar{\alpha} + r^{2d} \chi^{-4kd} = (1 + 2r \chi^{-2k} \cos \alpha + r^2 \chi^{-4k})^d.$$

We develop the right hand side into a power series for $t = \chi^{-2k}$ and compare it to the left side. When $\cos \alpha \neq 0$, we obtain immediately a contradiction if $d > 1$.

We look at the geometric situation when $\cos \alpha = 0$, that is, $\alpha = \pm \pi/2$. Suppose that g fixes 0 and ∞ and that h fixes 1 so that 0 and 1 are the repelling fixed points. By Lemma B (c), h fixes also the point $-a/b$ and we know that $\arg -a/b = -\alpha = \pm \pi/2$ and hence this point lies on the imaginary axis.

We state this in terms independent of normalization. Let S_1 and S_2 be the two circles through the fixed points of g and through one fixed point of h . Then S_1 and S_2 intersect orthogonally.

But this is absurd since g and h can be any two loxodromic elements of G without common fixed points. If we have chosen $g, h \in G$ and S_1 and S_2 are orthogonal for these g and h , we can replace h by $h^k g h^{-k}$, k big, in such a way that they are no more orthogonal. This contradiction concludes the proof that $d = 1$.

We now assume that $d = 1$ and prove the remaining part of the theorem. As we have already done this in [T1, pp. 338–339] in more detail, we present only the main points.

We continue from the preceding situation with $g, h \in G$ loxodromic without common fixed points. Since $d = 1$, (C3) becomes $|\bar{a}| = |a|$, $|\bar{b}| = |b|$ and in addition we know that $a + b = 1$ and $\bar{a} + \bar{b} = 1$. Hence the two triangles with vertices 0, 1, a and 0, 1, \bar{a} , respectively, have the same sidelengths and consequently either

$$\bar{a} = a \quad \text{and} \quad \bar{b} = b, \quad \text{or} \quad \bar{a} = a^* \quad \text{and} \quad \bar{b} = b^*$$

where $*$ is the complex conjugation. Conjugating by a Möbius transformation we obtain that the fixed points are

$$P_g = P_{\bar{g}} = \infty, \quad N_g = N_{\bar{g}} = 0, \quad N_h = N_{\bar{h}} = 1 \quad P_h = -a/b, \quad P_{\bar{h}} = -\bar{a}/\bar{b}.$$

Here P and N denote the attractive and repelling fixed points (see Section A) and we have also used (c) of Lemma B for P_h and $P_{\bar{h}}$. Hence at least we can conjugate the fixed points of g and h to the fixed points of \bar{g} and \bar{h} . In particular, if

$N_h = N_{\bar{h}} = \infty$, then the two triangles with vertices P_g, N_g, P_h and $P_{\bar{g}}, N_{\bar{g}}, P_{\bar{h}}$, respectively, are similar.

Now, g and h can be any two loxodromic elements in G without common fixed points. Using this and the fact that every distinct point-pair of $L(G) \times L(G)$ can be approximated arbitrarily closely by the fixed points of a loxodromic map in G (this follows from [T3, Theorem B1]), we can show that the map defined by

$$P_g \mapsto P_{\varphi(g)} \tag{C6}$$

($g \in G$ loxodromic) is the restriction of a Möbius transformation f . It follows that $fg|L(G) = \varphi(g)f|L(G)$ for $g \in G$ from which fact the rest of Theorem C follows.

Finally, we remove the assumptions that φ was an isomorphism and G non-elementary. If φ is not an isomorphism but G is non-elementary, we pick as above loxodromic $g, h \in G$ without common fixed points. Then for big enough k , the group G' generated by g^k and h^k is a Schottky group which is a free group such that every element of $G' \setminus \{\text{id}\}$ is loxodromic (e.g. [T, p. 333] contains the simple argument). Then every $\varphi(g'), g' \in G' \setminus \{\text{id}\}$ is loxodromic by (A2) and hence $\varphi|G'$ is an isomorphism onto $\varphi(G')$ and we can apply above reasoning with G replaced by G' and H by $\varphi(G')$. Since replacing g by g^k and h by h^k does not affect the attractive and repelling fixed points, the reasoning leading to (C6) is still valid.

If G is elementary, then H is non-elementary. Thus there are loxodromic $g, h \in H$ without common fixed points. As above, for big enough k , the group H' generated by g^k and h^k is a Schottky group. Find $g_0, h_0 \in G$ such that $\varphi(g_0) = g^k$ and $\varphi(h_0) = h^k$ and let G' be the group generated by them. Since H' is free, $\varphi|G'$ is an isomorphism onto H' and we can apply the above reasoning to H', G' and $\varphi^{-1}|H'$ and show that g_0 and h_0 are loxodromic and without common fixed points and hence G was in fact non-elementary, contrary to the assumption.

Remark. It is clear that there are non-trivial situations in which Theorem C is not true. For instance, let G be generated by $g : x \mapsto 2x$ and H by $h : x \mapsto 4x$ which are Möbius groups of \bar{R}^n . Then the isomorphism mapping g onto h is of constant stretch 2.

Another example is given by the group G whose elements are of the form $x \mapsto \lambda x + a$ where $\lambda > 0$ and $a \in R^n$. Let α be an affine homeomorphism of R^n and let $\varphi(g) = \alpha g \alpha^{-1}$. Then φ is an isomorphism $G \rightarrow G$ which preserves multipliers but is not a conjugation by a Möbius transformation if α is not a similarity.

Thus it is necessary to assume something on the groups G and H although it might be, as is suggested by the last example, that if the groups contain two loxodromic elements with different fixed point sets (but which may have a common fixed point), then, if φ preserves multipliers, (A3) might be true for some affine h

(i.e. h is a map such that $h_0 h h_1 |R^n$ is affine for some Möbius transformations h_0 and h_1).

D. Bilipschitz maps and rigidity

In this last section we note a consequence of Theorem C. Roughly, it says that if φ is induced by f , i.e. (A3) is true, and f is a bilipschitz map, then φ preserves multipliers and hence is, or almost is, a conjugation by a Möbius transformation. Since we use the euclidean metric which is not a metric of whole \bar{R}^n , we transfer the situation to the n -sphere $S^n \subset R^{n+1}$. It turns out that the bilipschitz condition need not be satisfied everywhere, and taking account that in Theorem C we actually considered homomorphisms of constant stretch, we can generalize this as

THEOREM D. *Let $\varphi : G \rightarrow H$ be a homomorphism of two Möbius groups of S^n such that G is non-elementary. Let $A \subset S^n$ be a non-empty G -invariant set and let $f : A \rightarrow S^n$ be a map inducing φ . Suppose that there are an open set $U \subset S^n$ and numbers $L \geq 1$ and $d > 0$ such that $U \cap L(G) \neq \emptyset$ and that*

$$|x - y|^d / L \leq |f(x) - f(y)| \leq L|x - y|^d \quad (\text{D1})$$

for $x, y \in U \cap A$. Then $d = 1$, φ preserves multipliers and, if in addition, $L(G) \subset h(S^k)$ for no Möbius transformation h and no $k < n$, φ is a conjugation by a Möbius transformation.

Proof. Pick $z \in L(G) \cap U$. Thus there are $g_i \in G$ and $w \in S^n$ such that

$$g_i |S^n \setminus \{w\} \rightarrow z$$

locally uniformly, as follows easily from the definition of the limit set (cf. (A5)) and the convergence property of Möbius groups (see [GM, Theorem 3.2]). This fact has two consequences. The first is that if $\text{acc } A$ denotes the accumulation points of A , then

$$\text{acc } A \supset L(G), \quad (\text{D2})$$

(for (D2) we remark that A is in any case actually infinite by non-elementariness) and the second is that

$$\{g_i^{-1}(U)\} \text{ is a cover of } L(G) \setminus \{w\}. \quad (\text{D3})$$

It follows that if $g \in G$ is loxodromic, then there is $h \in G$ which is conjugate to g in G such that at least one of the fixed points of h is in U . Consequently, if we can prove that

$$\text{mul } \varphi(g) = (\text{mul } g)^d \tag{D4}$$

for all loxodromic $g \in G$ with one fixed point in U , then this is actually valid for all loxodromic $g \in G$.

So suppose that $g \in G$ is loxodromic and fixes $u \in U$. We can assume that u is the attractive fixed point of g . Then $u \in L(G)$ and hence, by (D2), there are distinct $x, y \in U \cap A$ not fixed by g . Under these circumstances we have, as can be seen from (A1),

$$\text{mul } g = \lim_{k \rightarrow \infty} |g^k(x) - g^k(y)|^{-1/k}. \tag{D5}$$

We observe that (D5) gives $\text{mul } g$ for any Möbius transformation g as follows from the representations (A1) and (A4), provided that x and y are not fixed by g . By (D1) and G -compatibility, $f(x)$ and $f(y)$ are not fixed by $\varphi(g)$. Hence (D5) and (D1) imply that

$$\begin{aligned} (\text{mul } g)^d &= \lim_{k \rightarrow \infty} |fg^k(x) - fg^k(y)|^{-1/k} \\ &= \lim_{k \rightarrow \infty} |\varphi(g)^k f(x) - \varphi(g)^k f(y)|^{-1/k} \\ &= \text{mul } \varphi(g) \end{aligned}$$

and (D4) is valid for all loxodromic $g \in G$.

This is all that is needed for the validity of Theorem C if G is non-elementary (see the Remark after it). Theorem C implies the rest of the present theorem, for instance that φ preserves multipliers for all $g \in G$.

Remarks. 1. We needed the assumption that G is non-elementary in order to apply Theorem C but to obtain (D4) for loxodromic g , this assumption was not used (though we must assume that A contains at least three points if G is elementary). In fact, if g is parabolic such that g is conjugate to some h with a fixed point in U or if g is elliptic, then basically as above one obtains that (D4) is valid for g ; in the non-elementary case a parabolic g is always conjugate to such h . It is valid even if g is parabolic and not conjugate to such a map h but then a more complicated reasoning, given below, is necessary. Thus even if G is elementary φ is still of constant stretch d , provided that A contains at least 3 points.

Suppose that g is parabolic with the fixed point v and $h(v) \in U$ for no $h \in G$. It follows that v must be the point w in (D3) and that v is fixed by every $g \in G$. We cannot have that $\{v\} = L(G)$ (since then $v \in U$) and hence there are at least two points in $L(G)$. Consequently, there is loxodromic $h \in G$ [T3, proof of Theorem E]. Then h fixes v and we assume that v is the repelling fixed point. It follows from (A1) and (A4) (transform the situation to \bar{R}^n so that $v = \infty$) that there are $k_i > 0$ and $n_i > 0$ such that if

$$g_i = h^{k_i} g^{n_i} h^{-k_i},$$

then $g_i(x) \rightarrow x$ for all $x \in S^n$.

It follows that also $\varphi(g_i)(x) \rightarrow x$ for all $x \in f(U \cap A)$. Since $f(U \cap A)$ is infinite, it follows by the convergence property [GM, 3.2] that we can pass to a subsequence in such a way that $\varphi(g_i) \rightarrow \bar{g}$ where \bar{g} is a Möbius transformation such that $\bar{g}|_{f(U \cap A)} = \text{id}$. hence \bar{g} is elliptic and $\text{mul } \bar{g} = 1 = \lim_{i \rightarrow \infty} \text{mul } \varphi(g_i)$. However, $\varphi(g_i)$ is conjugate to $\varphi(g)^{n_i}$. Consequently, $\text{mul } \varphi(g) = (\text{mul } \varphi(g_i))^{1/n_i} = 1$.

2. Actually, we need not assume that (D1) is true for all $x, y \in U \cap A$, only that for each loxodromic $g \in G$ there are distinct points $u, v \in A$, not both of them fixed by g , such that (D1) is valid for $x = g^k(u)$ and $y = g^k(v)$ when $k > 0$ with some $L \geq 1$ which may depend on g and some $d > 0$ which does not.

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