Homomorphisms of constant stretch between Möbius groups.

Comment. Math. Helvetici 66 (1991) 155-167 0010-2571/91/010155-13\$1.50+0.20/0

Homomorphisms of constant stretch between Möbius groups

PEKKA TUKIA

A. Introduction

A Möbius transformation g of $\bar{R}^n$ is *loxodromic* if it can be conjugated by another Möbius transformation to the form

$$x \mapsto \lambda\beta(x) \qquad (x \in R^n) \tag{A1}$$

where $\lambda > 1$ and β is an orthogonal linear map. The number $\lambda > 1$ does not depend on how the conjugacy is chosen and it is the *multiplier* $\operatorname{mul} g$ of g; for non-loxodromic g we set $\operatorname{mul} g = 1$.

If G and H are two groups of Möbius transformations of $\bar{R}^n$ and $\varphi : G \to H$ is a homomorphism between them, φ is said to be of *constant stretch* if there is $d > 0$ such that

$$\operatorname{mul} \varphi(g) = (\operatorname{mul} g)^d \tag{A2}$$

for all $g \in G$; more precisely we can say that φ is of constant stretch d. Note that d is well-defined if there are loxodromic elements in G and that g is loxodromic if and only if $\varphi(g)$ is. If $d = 1$, then we say that φ is *multiplier preserving*.

Our main Theorem C says that a homomorphism φ of non-elementary groups is of constant stretch if and only if it is multiplier preserving. Furthermore, such a φ comes very near to being a conjugation by a Möbius transformation. If the limit set $L(G)$ of G "fills" $\bar{R}^n$, that is, $h(L(G)) \not\subset \bar{R}^k$ for no $k < n$ and no Möbius transformation h, then we can actually show that φ is a conjugation by a Möbius transformation.

A consequence of Theorem C is that if a map $f : A \to \bar{R}^n$ is compatible with a homomorphism $\varphi : G \to H$ of non-elementary Möbius groups, that is for every $g \in G$, $gA = A$ and

$$fg(x) = \varphi(g)f(x) \tag{A3}$$

when $x \in A$, then φ is a conjugation by a Möbius transformation as soon as f satisfies a bilipschitz property and the above mentioned condition for the limit set is satisfied (Theorem D).

Originally, we needed Theorem C in [T2] but after we found a simpler method for [T2], we separated these results into the present paper. In [T1] we have already treated the case of a multiplier preserving φ. The present arrangement of the proof seems to be slightly simpler also for the multiplier preserving case. In our proof of Theorem C we will first show that a homomorphism of constant stretch is multiplier preserving and then sketch the remaining part for completeness although we could here refer to [T1].

DEFINITIONS AND NOTATIONS. We denote the group of all Möbius transformations of $\bar{R}^n$ by $M(\bar{R}^n)$. Each $g \in M(\bar{R}^n)$ has a unique extension to a Möbius transformation of $\bar{R}^{n+1}$ such that $g(H^{n+1}) = H^{n+1}$ when H^{n+1} is the $(n+1)$-dimensional hyperbolic space

$$H^{n+1} = \{x \in R^{n+1} : x = (x_1, \ldots, x_{n+1}) \text{ where } x_{n+1} > 0\}.$$

We identify g and this extension of g to $\bar{R}^{n+1}$; thus $M(\bar{R}^n) \subset M(\bar{R}^{n+1})$.

A loxodromic $g \in M(\bar{R}^n)$ has two fixed points denoted by $P_g = P(g)$ and $N_g = N(g)$ so that P_g is the attracting fixed point and N_g the repelling fixed point; these names are self-explanatory. A loxodromic map g is *hyperbolic* if it is conjugate in $M(\bar{R}^n)$ to a map as in (A1) where $\beta = \text{id}$. If $g \in M(\bar{R}^n)$ is not loxodromic, then it is either elliptic or parabolic. If g is *elliptic*, then it is conjugate in $M(\bar{R}^n)$, or in $M(\bar{R}^{n+1})$, to a map as in (A1) where $\lambda = 1$, and g is *parabolic* if it is conjugate in $M(\bar{R}^n)$ to a map of the form

$$x \mapsto \beta(x) + a \tag{A4}$$

where $a \in R^n$, $a \neq 0$, and β is an orthogonal linear map such that $\beta(a) = a$ (cf. [T3, p. 560]).

A *Möbius group* G is a subgroup of $M(\bar{R}^n)$ and such a group is discrete if it is discrete in the compact-open topology of $\bar{R}^n$. A set A is *G-invariant* if $gA = A$ for every $g \in G$.

The *limit set* $L(G)$ of G is

$$L(G) = \text{cl}\, Gz \cap \bar{R}^n \tag{A5}$$

where $z \in H^{n+1}$ (and where cl is the closure). This does not depend on the choice of $z \in H^{n+1}$ and is a reasonable definition of $L(G)$ also for non-discrete G.

We define that a Möbius group G is *non-elementary* if it contains two loxodromic elements with disjoint fixed point sets. If G is discrete, then it is well-known that G is non-elementary if and only if $L(G)$ contains more than two points (see e.g. [T3, Theorem B2].)

We usually work in $\bar{R}^n$ but we find it more natural to formulate Theorem D for Möbius groups of the n-sphere $S^n = \{x \in R^{n+1} : |x| = 1\}$. We also use above definitions with appropriate modifications for Möbius groups of S^n.

B. Representation of Möbius transformations by matrices

The proof of our main theorem depends on matrix representations of Möbius transformations. Let $O(1, n+1)$ be the group of $(n+2) \times (n+2)$-matrices which preserves the quadratic form $x_1^2 - x_2^2 - \cdots - x_{n+2}^2$ and let $O_+(1, n+1)$ be the subgroup of $O(1, n+1)$ which preserves

$$\{(x_1, \ldots, x_{n+2}) \in R^{n+2} : x_1^2 - x_2^2 - \cdots - x_{n+2}^2 = 1 \text{ and } x_1 > 0\}.$$

Then, as is well-known [W], every $g \in M(\bar{R}^n)$ can be represented by a unique matrix $A \in O_+(1, n+1)$.

If $n = 2$, then we identify R^2 and the complex plane C. If $g \in M(\bar{R}^2)$ is orientation preserving, then it can be represented by a matrix of $SL(2, C)$, that is, by a complex 2×2-matrix with determinant 1.

We will now give two simple formulas that relate the multiplier of $g \in M(\bar{R}^n)$ and the trace $\operatorname{tr} A$ of the matrix $A \in O_+(1, n+1)$ or $A \in SL(2, C)$ representing g. If $g \in M(\bar{R}^n)$ is represented by a matrix $A \in O_+(1, n+1)$, then

$$\operatorname{mul} g = \operatorname{tr} A + M(A) \tag{B1}$$

where $|M(A)| \le n + 2$. This follows from explicit matrix representations for loxodromic, elliptic or parabolic Möbius transformations, see Wielenberg [W, Section 5] and the classification of a Möbius transformation as loxodromic, elliptic or parabolic mentioned above. Recall that $\operatorname{mul} g = 1$ for non-loxodromic g. Note that if g is elliptic, then we possibly need to extend g to a Möbius transformation of $\bar{R}^{n+1}$ in order to obtain that g is conjugate to an orthogonal linear map.

If $g \in M(\bar{R}^2)$ is orientation preserving, then g can be represented by $B \in SL(2, C)$, and

$$\operatorname{mul} g = |\operatorname{tr} B|^2 + M'(B) \tag{B2}$$

where $|M'(B)| \leq 3$ as a simple calculation shows. The next lemma is based on these estimates.

LEMMA B. *Let* $g, h \in M(\bar{R}^n)$ *be loxodromic and let* $\gamma = (\operatorname{mul} g)^{1/2}$ *and* $\chi = (\operatorname{mul} h)^{1/2}$. *Then, for* $m, k \in Z$,

$$\operatorname{mul} g^m h^k = |a(\gamma^m \chi^k + \gamma^{-m}\chi^{-k}) + b(\gamma^m \chi^{-k} + \gamma^{-m}\chi^k)|^2$$
$$+ c_{1m}\gamma^{2m} + c_{2m}\gamma^{-2m} + d_{1k}\chi^{2k} + d_{2k}\chi^{-2k} + e_{mk} \qquad \text{(B3)}$$

where a, b *are complex numbers such that* $a + b = 1$ *and depending only on the quadruple* (P_g, N_g, P_h, N_h) *of the fixed points. The constants* c_{ik}, d_{im}, *and* e_{mk} *are bounded and, furthermore,*

(a) $a \neq 0 \neq b$ *if and only if* g *and* h *do not have common fixed points,*

(b) *if* g *and* h *are hyperbolic, then* $c_{mk} = d_{mk} = 0$ *and* $|e_{mk}| \leq 3$ *for all* m, k; *in the general loxodromic case there is a sequence* $r_1 < r_2 < \cdots$ *such that as* $j \to \infty$,

$$c_{i,\pm r_j} \to 0 \quad \textit{and} \quad d_{i,\pm r_j} \to 0 \qquad (i = 1, 2),$$

(c) *if the fixedfixed points of* g *and* h *are in* $\bar{R}^2 = \bar{C}$ *and* $N_g = 0$, $P_g = \infty$, $N_h = 1$, *and* $P_h = p$, *then there are the following relations between the numbers* p, a *and* b:

$$p = -\frac{a}{b}, \qquad a = \frac{p}{p-1} \quad \textit{and} \quad b = \frac{1}{1-p}.$$

Proof. The fixed points of g and h lie in a 2-dimensional sphere and hence we may assume that their fixed points lie in $\bar{C}$ and that

$$P_g = \infty \quad \text{and} \quad N_g = 0.$$

Let $\bar{g}$ and $\bar{h}$ be the corresponding hyperbolic Möbius transformations, i.e. they have the same multiplier and the same repelling and attractive fixed points. Then $\bar{g}$ and $\bar{h}$ preserve $\bar{C}$ and can be represented by matrices $\tilde{A}, \tilde{B} \in SL(2, C)$, respectively. We can conjugate in $SL(2, C)$ to obtain

$$\tilde{A} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} s & t \\ u & v \end{pmatrix}\begin{pmatrix} \chi & 0 \\ 0 & \chi^{-1} \end{pmatrix}\begin{pmatrix} v & -t \\ -u & s \end{pmatrix}$$

where s, t, u and v are complex numbers such that $sv - tu = 1$. A simple calculation shows that if

$$a = sv \quad \text{and} \quad b = -tu,$$

then

$$\operatorname{tr} \tilde{A}^m \tilde{B}^k = a(\gamma^m \chi^k + \gamma^{-m} \chi^{-k}) + b(\gamma^m \gamma^{-k} + \gamma^{-m} \chi^k). \tag{B4}$$

Obviously, a and b depend only on the fixed points and $a + b = 1$. Since $P_h = s/u$ and $N_h = t/v$, we have that $a \neq 0 \neq b$ if and only if g and h do not have common fixed points. Remembering that $a + b = 1$, the formulas in (c) also follow.

We then represent $g, h, \bar{g}, \bar{h}$ by matrices $A, B, \bar{A}, \bar{B}$ of $O_+(1, n+1)$, respectively, and perform similar calculations. The matrix A has eigenvalues $\alpha_1, \ldots, \alpha_{n+2}$ which we can enumerate so that $\alpha_1 = \gamma^2 = \operatorname{mul} g$ and $\alpha_2 = \gamma^{-2}$ and that α_i, $i > 2$ are complex numbers of modulus 1 as follows from the canonical forms for matrices of $O_+(1, n+1)$ representing loxodromic Möbius transformation [W, Section 5]. Similarly, B has eigenvalues $\beta_1 = \chi^2, \beta_2 = \chi^{-2}, \ldots, \beta_{n+2}$. We can assume that A is diagonal (so that the diagonal entries are the eigenvalues) and that

$$B = EDE^{-1} \tag{B5}$$

for some matrices E, D where D is diagonal (they need not be matrices of $O(1, n+1)$.) A calculation shows that there are constants a_{ij}, $i, j \leq n+2$ such that

$$\operatorname{tr} A^m B^k = \sum_{i,j} a_{ij} \alpha_i^m \beta_j^k. \tag{B6}$$

Thus if we set

$$c'_{ik} = \sum_{j>2} a_{ij} \beta_j^k, \qquad (i = 1, 2, k \in Z),$$

$$d'_{im} = \sum_{j>2} a_{ji} \alpha_j^m, \qquad (i = 1, 2, m \in Z),$$

$$e'_{mk} = \sum_{i>2, j>2} a_{ij} \alpha_i^m \beta_j^k, \qquad (m, k \in Z),$$

we obtain bounded numbers (since $|\alpha_j| = |\beta_j| = 1$ if $j > 2$) such that

$$\operatorname{tr} A^m B^k = a_{11}\gamma^{2m}\chi^{2k} + a_{12}\gamma^{2m}\chi^{-2k} + a_{21}\gamma^{-2m}\chi^{2k} + a_{22}\gamma^{-2m}\chi^{-2k} + c'_{1k}\gamma^{2m} + c'_{2k}\gamma^{-2m} + d'_{1m}\chi^{2k} + d'_{2m}\chi^{-2k} + e'_{mk}. \tag{B7}$$

We obtain the matrix $\bar{A}$ from that of A by substituting 1 for α_i if $i > 2$ (and leaving α_1 and α_2 unchanged). Similarly, substituting 1 for β_j in D if $j > 2$, we obtain $\bar{B}$ from the right hand side of (B5). With these substitutions (B6) gives $\operatorname{tr} \bar{A}^m \bar{B}^k$ with a_{ij} unchanged but with the new α_i and β_j, and (B7) is valid if we substitute in it for c'_{im}, d'_{im} and e'_{mk} the numbers

$$\bar{c}_i = \sum_{j>2} a_{ij},$$

$$\bar{d}_i = \sum_{j>2} a_{ji},$$

$$\bar{e} = \sum_{i>2, j>2} a_{ij}.$$

(They do not depend on m and k and so we have not marked them.) It follows by Kronecker's theorem ([A, Theorem 7.10] or [C, p. 53]) that there is a sequence $r_1 < r_2 < \cdots$ such that, as $i \to \infty$,

$$c'_{i,\pm r_i} \to \bar{c}_i \quad \text{and} \quad d'_{i,\pm r_i} \to \bar{d}_i \tag{B8}$$

for $i = 1, 2$.

Applying (B1) and (B2) to $\operatorname{mul} \bar{g}^m \bar{h}^k$ we obtain that

$$\left|\operatorname{tr} \bar{A}^m \bar{B}^k - |\operatorname{tr} \tilde{A}^m \tilde{B}^k|^2\right| \leqslant n + 5. \tag{B9}$$

Set

$$\begin{aligned} c_{im} &= c'_{im} - \bar{c}_i, \\ d_{ik} &= d'_{ik} - \bar{d}_i, \\ e''_{mk} &= e'_{mk} - \bar{e}. \end{aligned} \tag{B10}$$

These numbers are bounded and satisfy (b) with respect to the r_i in (B8). Write

$$\operatorname{mul} g^m h^k = (\operatorname{mul} g^m h^k - \operatorname{tr} A^m B^k) + (\operatorname{tr} A^m B^k - \operatorname{tr} \bar{A}^m \bar{B}^k) + (\operatorname{tr} \bar{A}^m \bar{B}^k - |\operatorname{tr} \tilde{A}^m \tilde{B}^k|^2) + |\operatorname{tr} \tilde{A}^m \tilde{B}^k|^2.$$

On the right-hand sum the first parenthesis is bounded by (B1) and the third parenthesis by (B9). The second parenthesis can be estimated by (B7) when it is applied to tr $A^m B^k$ and to tr $\bar{A}^m \bar{B}^k$, and the last term is given by (B4). Combining all this, we have

$$\begin{aligned}\operatorname{mul} g^m h^k = &|a(\gamma^m \chi^k + \gamma^{-m} \chi^{-k}) + b(\gamma^m \chi^{-k} + \gamma^{-m} \chi^k)|^2 \\ &+ c_{1m} \gamma^{2m} + c_{2m} \gamma^{-2m} + d_{1k} \chi^{2k} + d_{2k} \chi^{-2k} + e_{mk}.\end{aligned}$$

Here e_{mk} is the sum of e''_{mk} in (B10) and of the first and third parenthesis. They are bounded since e''_{mk} are bounded, and by what has been said above, and so are the numbers c_{im} and d_{ik}.

If g and h are hyperbolic, then we can use (B4) and (B2) to conclude that (B3) is true with $c_{im} = d_{jk} = 0$ and $|e_{mk}| \leq 3$.

Finally, (b) follows from (B8) and (B10).

Remark. If we have two (or, in fact, any number of) pairs g, h and $\tilde{g}, \tilde{h}$ of loxodromic Möbius transformations and if $\tilde{c}_{ij}$ and $\tilde{d}_{ij}$ are the numbers in the expression for mul $\tilde{g}^m \tilde{h}^k$, and if $\tilde{r}_i$ is the corresponding sequence in (b), then exactly as in (b), by Kronecker's theorem, one can choose these sequences so that $r_i = \tilde{r}_i$.

C. The main theorem

We can now prove our main

THEOREM C. *Let $\varphi : G \to H$ be a surjective homomorphism of two Möbius groups of $\bar{R}^n$ such that one of the groups G and H is non-elementary. Then φ is multiplier preserving if it is of constant stretch $d > 0$.*

Furthermore, let S be the k-sphere of smallest dimension k such that $S = g\bar{R}^k$ for some $g \in M(\bar{R}^n)$ and that $S \supset L(G)$, where $L(G)$ is the limit set of G (see (A5)). Then S is G-invariant and there is $h \in M(\bar{R}^n)$ such that

$$hg(x) = \varphi(g)h(x) \tag{C0}$$

for $x \in S$ and $g \in G$.

In particular, if $S = \bar{R}^n$, then φ is a conjugation by a Möbius transformation.

Remark. Actually, it would suffice to assume that (A2) is true for all $g \in G$ such that g is loxodromic (if G is non-elementary) or such that $\varphi(g)$ is loxodromic (if H is non-elementary).

The sphere S in the theorem is well-defined if $L(G)$ contains at least two points. Since either G or H is non-elementary, and φ is of constant stretch, G has loxodromics and hence S is well-defined (the proof shows that both groups are non-elementary).

Proof. We first assume that G is non-elementary and that φ is an isomorphism.

We first prove that $d = 1$. Since G is non-elementary, there are two loxodromic elements $g, h \in G$ without common fixed points. Then also $\varphi(g)$ and $\varphi(h)$ are loxodromic by (A2).

Let $\bar{g} = \varphi(g)$ and $\bar{h} = \varphi(h)$. Then $\operatorname{mul} \bar{g} = (\operatorname{mul} g)^d = \gamma^{2d}$ and $\operatorname{mul} \bar{h} = (\operatorname{mul} h)^d = \chi^{2d}$ and hence if $\bar{a}$, $\bar{b}$, $\bar{c}_{im}$, $\bar{d}_{ik}$ and $\bar{e}_{mk}$ are numbers as in Lemma B, we have that

$$\begin{aligned} \operatorname{mul} \bar{g}^m \bar{h}^k = {} & |\bar{a}(\gamma^{dm}\chi^{dk} + \gamma^{-dm}\chi^{-dk}) + \bar{b}(\gamma^{dm}\chi^{-dk} + \gamma^{-dm}\chi^{dk})|^2 \\ & + \bar{c}_{1m}\gamma^{2dm} + \bar{c}_{2m}\gamma^{-2dm} + \bar{d}_{1k}\chi^{2dk} + \bar{d}_{2k}\chi^{-2dk} + \bar{e}_{mk}. \end{aligned} \tag{C1}$$

We now use the equality

$$\operatorname{mul} \bar{g}^m \bar{h}^k = (\operatorname{mul} g^m h^k)^d \tag{C2}$$

together with (B3) and (C1) and let m, k tend to $+\infty$ or to $-\infty$. Since g and h do not have common fixed points, $a \neq 0 \neq b$ by Lemma B (a) and it follows that

$$|\bar{a}| = |a|^d \quad \text{and} \quad |\bar{b}| = |b|^d. \tag{C3}$$

In particular, it follows that $\bar{a} \neq 0 \neq \bar{b}$ and hence $\bar{g}$ and $\bar{h}$ do not have common fixed points by Lemma B (a). Thus H is also non-elementary and, if necessary, we can replace φ by φ^{-1} and thereby assume that

$$d \geq 1. \tag{C4}$$

Next, substitute again (B3) and (C1) into (C2) and divide both sides of the resulting equation by $|a|^d\gamma^{2dm}\chi^{2dk} = |\bar{a}|\gamma^{2dm}\chi^{2dk}$. Keep k fixed and let m assume the values r_i of Lemma B (b) (see also the Remark following Lemma B) and let $i \to \infty$. We obtain

$$|1 + \bar{b}\chi^{-2kd}/\bar{a}|^2 = |1 + b\chi^{-2k}/a|^{2d} \tag{C5}$$

which is valid for every $k \in Z$.

Let

$$r = |b|/|a|, \qquad \alpha = \arg b/a, \qquad \bar{\alpha} = \arg \bar{b}/\bar{a},$$

so that $r^d = |\bar{b}|/|\bar{a}|$. Substituting this into (C5) and using elementary trigonometry, we obtain for all $k \in Z$

$$1 + 2r^d \chi^{-2kd} \cos \bar{\alpha} + r^{2d} \chi^{-4kd} = (1 + 2r\chi^{-2k} \cos \alpha + r^2 \chi^{-4k})^d.$$

We develop the right hand side into a power series for $t = \chi^{-2k}$ and compare it to the left side. When $\cos \alpha \neq 0$, we obtain immediately a contradiction if $d > 1$.

We look at the geometric situation when $\cos \alpha = 0$, that is, $\alpha = \pm \pi/2$. Suppose that g fixes 0 and ∞ and that h fixes 1 so that 0 and 1 are the repelling fixed points. By Lemma B (c), h fixes also the point $-a/b$ and we know that $\arg -a/b = -\alpha = \pm \pi/2$ and hence this point lies on the imaginary axis.

We state this in terms independent of normalization. Let S_1 and S_2 be the two circles through the fixed points of g and through one fixed point of h. Then S_1 and S_2 intersect orthogonally.

But this is absurd since g and h can be any two loxodromic elements of G without common fixed points. If we have chosen $g, h \in G$ and S_1 and S_2 are orthogonal for these g and h, we can replace h by $h^k g h^{-k}$, k big, in such a way that they are no more orthogonal. This contradiction concludes the proof that $d = 1$.

We now assume that $d = 1$ and prove the remaining part of the theorem. As we have already done this in [T1, pp. 338–339] in more detail, we present only the main points.

We continue from the preceding situation with $g, h \in G$ loxodromic without common fixed points. Since $d = 1$, (C3) becomes $|\bar{a}| = |a|$, $|\bar{b}| = |b|$ and in addition we know that $a + b = 1$ and $\bar{a} + \bar{b} = 1$. Hence the two triangles with vertices $0, 1, a$ and $0, 1, \bar{a}$, respectively, have the same sidelengths and consequently either

$$\bar{a} = a \quad \text{and} \quad \bar{b} = b, \quad \text{or} \quad \bar{a} = a^* \quad \text{and} \quad \bar{b} = b^*$$

where * is the complex conjugation. Conjugating by a Möbius transformation we obtain that the fixed points are

$$P_g = P_{\bar{g}} = \infty, \quad N_g = N_{\bar{g}} = 0, \quad N_h = N_{\bar{h}} = 1 \quad P_h = -a/b, \quad P_{\bar{h}} = -\bar{a}/\bar{b}.$$

Here P and N denote the attractive and repelling fixed points (see Section A) and we have also used (c) of Lemma B for P_h and $P_{\bar{h}}$. Hence at least we can conjugate the fixed points of g and h to the fixed points of $\bar{g}$ and $\bar{h}$. In particular, if

$N_h = N_{\bar{h}} = \infty$, then the two triangles with vertices P_g, N_g, P_h and $P_{\bar{g}}, N_{\bar{g}}, P_{\bar{h}}$, respectively, are similar.

Now, g and h can be any two loxodromic elements in G without common fixed points. Using this and the fact that every distinct point-pair of $L(G) \times L(G)$ can be approximated arbitrarily closely by the fixed points of a loxodromic map in G (this follows from [T3, Theorem B1]), we can show that the map defined by

$$P_g \mapsto P_{\varphi(g)} \tag{C6}$$

($g \in G$ loxodromic) is the restriction of a Möbius transformation f. It follows that $fg|L(G) = \varphi(g)f|L(G)$ for $g \in G$ from which fact the rest of Theorem C follows.

Finally, we remove the assumptions that φ was an isomorphism and G non-elementary. If φ is not an isomorphism but G is non-elementary, we pick as above loxodromic $g, h \in G$ without common fixed points. Then for big enough k, the group G' generated by g^k and h^k is a Schottky group which is a free group such that every element of $G' \setminus \{\mathrm{id}\}$ is loxodromic (e.g. [T, p. 333] contains the simple argument). Then every $\varphi(g')$, $g' \in G' \setminus \{\mathrm{id}\}$ is loxodromic by (A2) and hence $\varphi|G'$ is an isomorphism onto $\varphi(G')$ and we can apply above reasoning with G replaced by G' and H by $\varphi(G')$. Since replacing g by g^k and h by h^k does not affect the attractive and repelling fixed points, the reasoning leading to (C6) is still valid.

If G is elementary, then H is non-elementary. Thus there are loxodromic $g, h \in H$ without common fixed points. As above, for big enough k, the group H' generated by g^k and h^k is a Schottky group. Find $g_0, h_0 \in G$ such that $\varphi(g_0) = g^k$ and $\varphi(h_0) = h^k$ and let G' be the group generated by them. Since H' is free, $\varphi|G'$ is an isomorphism onto H' and we can apply the above reasoning to H', G' and $\varphi^{-1}|H'$ and show that g_0 and h_0 are loxodromic and without common fixed points and hence G was in fact non-elementary, contrary to the assumption.

Remark. It is clear that there are non-trivial situations in which Theorem C is not true. For instance, let G be generated by $g : x \mapsto 2x$ and H by $h : x \mapsto 4x$ which are Möbius groups of $\bar{R}^n$. Then the isomorphism mapping g onto h is of constant stretch 2.

Another example is given by the group G whose elements are of the form $x \mapsto \lambda x + a$ where $\lambda > 0$ and $a \in R^n$. Let α be an affine homeomorphism of R^n and let $\varphi(g) = \alpha g \alpha^{-1}$. Then φ is an isomorphism $G \to G$ which preserves multipliers but is not a conjugation by a Möbius transformation if α is not a similarity.

Thus it is necessary to assume something on the groups G and H although it might be, as is suggested by the last example, that if the groups contain two loxodromic elements with different fixed point sets (but which may have a common fixed point), then, if φ preserves multipliers, (A3) might be true for some affine h

(i.e. h is a map such that $h_0hh_1|R^n$ is affine for some Möbius transformations h_0 and h_1).

D. Bilipschitz maps and rigidity

In this last section we note a consequence of Theorem C. Roughly, it says that if φ is induced by f, i.e. (A3) is true, and f is a bilipschitz map, then φ preserves multipliers and hence is, or almost is, a conjugation by a Möbius transformation. Since we use the euclidean metric which is not a metric of whole $\bar{R}^n$, we transfer the situation to the n-sphere $S^n \subset R^{n+1}$. It turns out that the bilipschitz condition need not be satisfied everywhere, and taking account that in Theorem C we actually considered homomorphisms of constant stretch, we can generalize this as

THEOREM D. *Let $\varphi : G \to H$ be a homomorphism of two Möbius groups of S^n such that G is non-elementary. Let $A \subset S^n$ be a non-empty G-invariant set and let $f : A \to S^n$ be a map inducing φ. Suppose that there are an open set $U \subset S^n$ and numbers $L \geq 1$ and $d > 0$ such that $U \cap L(G) \neq \phi$ and that*

$$|x-y|^d/L \leq |f(x)-f(y)| \leq L|x-y|^d \tag{D1}$$

for $x, y \in U \cap A$. Then $d = 1$, φ preserves multipliers and, if in addition, $L(G) \subset h(S^k)$ for no Möbius transformation h and no $k < n$, φ is a conjugation by a Möbius transformation.

Proof. Pick $z \in L(G) \cap U$. Thus there are $g_i \in G$ and $w \in S^n$ such that

$$g_i|S^n \backslash \{w\} \to z$$

locally uniformly, as follows easily from the definition of the limit set (cf. (A5)) and the convergence property of Möbius groups (see [GM, Theorem 3.2]). This fact has two consequences. The first is that if acc A denotes the accumulation points of A, then

$$\operatorname{acc} A \supset L(G), \tag{D2}$$

(for (D2) we remark that A is in any case actually infinite by non-elementariness) and the second is that

$$\{g_i^{-1}(U)\} \text{ is a cover of } L(G)\backslash\{w\}. \tag{D3}$$

It follows that if $g \in G$ is loxodromic, then there is $h \in G$ which is conjugate to g in G such that at least one of the fixed points of h is in U. Consequently, if we can prove that

$$\operatorname{mul} \varphi(g) = (\operatorname{mul} g)^d \tag{D4}$$

for all loxodromic $g \in G$ with one fixed point in U, then this is actually valid for all loxodromic $g \in G$.

So suppose that $g \in G$ is loxodromic and fixes $u \in U$. We can assume that u is the attractive fixed point of g. Then $u \in L(G)$ and hence, by (D2), there are distinct $x, y \in U \cap A$ not fixed by g. Under these circumstances we have, as can be seen from (A1),

$$\operatorname{mul} g = \lim_{k \to \infty} |g^k(x) - g^k(y)|^{-1/k}. \tag{D5}$$

We observe that (D5) gives $\operatorname{mul} g$ for any Möbius transformation g as follows from the representations (A1) and (A4), provided that x and y are not fixed by g. By (D1) and G-compatibility, $f(x)$ and $f(y)$ are not fixed by $\varphi(g)$. Hence (D5) and (D1) imply that

$$\begin{aligned}(\operatorname{mul} g)^d &= \lim_{k \to \infty} |fg^k(x) - fg^k(y)|^{-1/k} \\ &= \lim_{k \to \infty} |\varphi(g)^k f(x) - \varphi(g)^k f(y)|^{-1/k} \\ &= \operatorname{mul} \varphi(g)\end{aligned}$$

and (D4) is valid for all loxodromic $g \in G$.

This is all that is needed for the validity of Theorem C if G is non-elementary (see the Remark after it). Theorem C implies the rest of the present theorem, for instance that φ preserves multipliers for all $g \in G$.

Remarks. 1. We needed the assumption that G is non-elementary in order to apply Theorem C but to obtain (D4) for loxodromic g, this assumption was not used (though we must assume that A contains at least three points if G is elementary). In fact, if g is parabolic such that g is conjugate to some h with a fixed point in U or if g is elliptic, then basically as above one obtains that (D4) is valid for g; in the non-elementary case a parabolic g is always conjugate to such h. It is valid even if g is parabolic and not conjugate to such a map h but then a more complicated reasoning, given below, is necessary. Thus even if G is elementary φ is still of constant stretch d, provided that A contains at least 3 points.

Suppose that g is parabolic with the fixed point v and $h(v) \in U$ for no $h \in G$. It follows that v must be the point w in (D3) and that v is fixed by every $g \in G$. We cannot have that $\{v\} = L(G)$ (since then $v \in U$) and hence there are at least two points in $L(G)$. Consequently, there is loxodromic $h \in G$ [T3, proof of Theorem E]. Then h fixes v and we assume that v is the repelling fixed point. It follows from (A1) and (A4) (transform the situation to $\bar{R}^n$ so that $v = \infty$) that there are $k_i > 0$ and $n_i > 0$ such that if

$$g_i = h^{k_i} g^{n_i} h^{-k_i},$$

then $g_i(x) \to x$ for all $x \in S^n$.

It follows that also $\varphi(g_i)(x) \to x$ for all $x \in f(U \cap A)$. Since $f(U \cap A)$ is infinite, it follows by the convergence property [GM, 3.2] that we can pass to a subsequence in such a way that $\varphi(g_i) \to \bar{g}$ where g is a Möbius transformation such that $\bar{g} | f(U \cap A) = \mathrm{id}$. hence $\bar{g}$ is elliptic and $\mathrm{mul}\, \bar{g} = 1 = \lim_{i \to \infty} \mathrm{mul}\, \varphi(g_i)$. However, $\varphi(g_i)$ is conjugate to $\varphi(g)^{n_i}$. Consequently, $\mathrm{mul}\, \varphi(g) = (\mathrm{mul}\, \varphi(g_i))^{1/n_i} = 1$.

2. Actually, we need not assume that (D1) is true for all $x, y \in U \cap A$, only that for each loxodromic $g \in G$ there are distinct points $u, v \in A$, not both of them fixed by g, such that (D1) is valid for $x = g^k(u)$ and $y = g^k(v)$ when $k > 0$ with some $L \geq 1$ which may depend on g and some $d > 0$ which does not.

REFERENCES

[A] T. M. Apostol, *Modular functions and Dirichlet series in number theory*. Graduate Texts in Mathematics *41*. Springer 1976.

[C] J. W. S. Cassels, *An introduction to diophantine approximation*. Cambridge U.P. 1957.

[GM] F. W. Gehring and G. J. Martin, *Discrete quasiconformal groups I*. Proc. London Math. Soc. *55*, 331–358 (1987).

[T1] P. Tukia, *Multiplier preserving isomorphisms between Möbius groups*. Ann. Acad. Sci. Fenn. Ser A I *1*, 327–341 (1975).

[T2] P. Tukia, *A rigidity theorem for Möbius groups*. Inventiones math. *97*, 405–431 (1989).

[T3] P. Tukia, *Differentiability and rigidity of Möbius groups*. Inventiones math. *82*, 557–578 (1985).

[W] N. J. Wielenberg, *Discrete Moebius groups: Fundamental polyhedra and convergence*. Amer. J. Math. *99*, 861–877 (1977).

Department of Mathematics
University of Helsinki
Hallituskatu 15
SF-00100 Helsinki
Finland

Received February 26, 1990

Buchanzeigen

MICHAL KAROŃSKI, JERZY JAWORSKI, ANDRZEJ RUCIŃSKI, **Random Graphs,** John Wiley & Sons, 1990, 368 pages, £40.00.

Preface – List of Participants – Bollobás B., Sharp concentration of measure phenomena in the theory of random graphs – Buckley F. and Z. Palka, Property preserving spanning trees in random graphs – Cooper C. and A. M. Frieze, Pancyclic random graphs – Frieze A. M. and L. Kučera. Parallel colouring of random graphs – Frieze A. M. and T. Łuczak, Hamiltonian cycles in a class of random graphs: one step further – Godehardt E., Connectivity of random graphs of small order and statistical testing – Janson S., T. Łuczak and A. Ruciński, An exponential bound for the probability of nonexistence of a specified subgraph of a random graph – Jaworski J., Random mapping with independent choices of images – Kemp R., Further results on leftist trees – Kordecki W., normal approximation and isolated vertices in random graphs – Kratochvíl M., J. Malý and J. Matoušek, on the existence of perfect codes in a random graph – Łuczak T., On the equivalence of two basic models of random graphs – Maehara H., On the intersection graph of random arcs on a circle – Matula D. and L. Kučera, An expose-and-merge algorithm and the chromatic number of a random graph – Mutafchiev L., Large components and cycles in a random mapping pattern – Nowicki K., Asymptotic distributions of graph statistics for colored graphs – Pittel B., W. A. Woyczynski and J. A. Mann, Random tree-type partitions as a model for acyclic polymerization: Gaussian behavior of the subcritical sol phase – Prömel H. J., Almost bipartite-making graphs – Ruciński A., Small subgraphs of random graphs – a survey – Spencer J., Undecidable probabilities – Szymański J., On the maximum degree and the height of a random recursive tree – Tomescu I., Almost all digraphs have a kernel – Vahidi-Asl M. Q. and J. C. Wierman, First-passage percolation on the Voronoi tessellation and Delaunay triangulation – Weber K., Random spread of information, random graph processes and random walks – Random graphs '87: Open problems.

FREDERIC BLEN, **D-Modules and Spherical Representations,** Princeton University Press, 1990, 131 pages, $22.50.

Introduction – I. Localization Theory – II. Spherical D. Modules – III. Microlocalization and Irreducibility – IV. Singularities and Multiplicities.

JONATHAN D. ROGAWSKI, **Automorphic Representations of Unitary Groups in Three Variables,** Princeton University Press, 1990, 257 pages, $60.00 Cloth, $19.95 Paper.

Introduction – 1. Preliminary definitions and notation – 2. The trace formula – 3. Stable conjugacy – 4. Orbital integrals and endoscopic groups – 5. Stabilization – 6. Weighted orbital integrals – 7. Elliptic singular terms – 8. Germ expansions and limit formulas – 9. Singularities – 10. The stable trace formula – 11. The unitary group in two variables – 12. Representation theory – 13. Automorphic representations – 14. Comparison of inner forms – 15. Additional results.

NICHOLAS M. KATZ, **Exponential Sums and Differential Equations,** Princeton University Press, 1990, 430 pages, $65.00 Cloth, $22.50 Paper.

1. Results from Representation Theory – 2. D.E.'s and $\mathcal{D}$-Modules – 3. The generalized hypergeometric equation – 4. Detailed analysis of the exceptional cases – 5. Convolution of $\mathcal{D}$-modules – 6. Fourier transforms of Kummer pullbacks of Hypergeometrics – 7. The ℓ-adic theory – 8. ℓ-adic hypergeometrics – 9. G_2 examples, Fourier transforms and hypergeometrics – 10. ℓ-adic exceptional cases – 11. Reductive tannakian categories – 12. Fourier universality – 13. Stratifications and convolution – 14. The fundamental comparison theorems.