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Homomorphisms of constant stretch between Möbius groups

PEKKA TUKIA

A. Introduction

A Möbius transformation g of \bar{R}^n is *loxodromic* if it can be conjugated by another Möbius transformation to the form

$$x \mapsto \lambda \beta(x) \qquad (x \in R^n)$$
 (A1)

where $\lambda > 1$ and β is an orthogonal linear map. The number $\lambda > 1$ does not depend on how the conjugacy is chosen and it is the *multiplier* mul g of g; for non-loxodromic g we set mul g = 1.

If G and H are two groups of Möbius transformations of \bar{R}^n and $\varphi: G \to H$ is a homomorphism between them, φ is said to be of *constant stretch* if there is d > 0 such that

$$\operatorname{mul} \varphi(g) = (\operatorname{mul} g)^d \tag{A2}$$

for all $g \in G$; more precisely we can say that φ is of constant stretch d. Note that d is well-defined if there are loxodromic elements in G and that g is loxodromic if and only if $\varphi(g)$ is. If d = 1, then we say that φ is multiplier preserving.

Our main Theorem C says that a homomorphism φ of non-elementary groups is of constant stretch if and only if it is multiplier preserving. Furthermore, such a φ comes very near to being a conjugation by a Möbius transformation. If the limit set L(G) of G "fills" \bar{R}^n , that is, $h(L(G)) \not\subset \bar{R}^k$ for no k < n and no Möbius transformation h, then we can actually show that φ is a conjugation by a Möbius transformation.

A consequence of Theorem C is that if a map $f: A \to \overline{R}^n$ is compatible with a homomorphism $\varphi: G \to H$ of non-elementary Möbius groups, that is for every $g \in G$, gA = A and

$$fg(x) = \varphi(g)f(x) \tag{A3}$$

when $x \in A$, then φ is a conjugation by a Möbius transformation as soon as f satisfies a bilipschitz property and the above mentioned condition for the limit set is satisfied (Theorem D).

Originally, we needed Theorem C in [T2] but after we found a simpler method for [T2], we separated these results into the present paper. In [T1] we have already treated the case of a multiplier preserving φ . The present arrangement of the proof seems to be slightly simpler also for the multiplier preserving case. In our proof of Theorem C we will first show that a homomorphism of constant stretch is multiplier preserving and then sketch the remaining part for completeness although we could here refer to [T1].

DEFINITIONS AND NOTATIONS. We denote the group of all Möbius transformations of \bar{R}^n by $M(\bar{R}^n)$. Each $g \in M(\bar{R}^n)$ has a unique extension to a Möbius transformation of \bar{R}^{n+1} such that $g(H^{n+1}) = H^{n+1}$ when H^{n+1} is the (n+1)-dimensional hyperbolic space

$$H^{n+1} = \{x \in \mathbb{R}^{n+1} : x = (x_1, \dots, x_{n+1}) \text{ where } x_{n+1} > 0\}.$$

We identify g and this extension of g to \bar{R}^{n+1} ; thus $M(\bar{R}^n) \subset M(\bar{R}^{n+1})$.

A loxodromic $g \in M(\bar{R}^n)$ has two fixed points denoted by $P_g = P(g)$ and $N_g = N(g)$ so that P_g is the attracting fixed point and N_g the repelling fixed point; these names are self-explanatory. A loxodromic map g is *hyperbolic* if it is conjugate in $M(\bar{R}^n)$ to a map as in (A1) where $\beta = \mathrm{id}$. If $g \in M(\bar{R}^n)$ is not loxodromic, then it is either elliptic or parabolic. If g is *elliptic*, then it is conjugate in $M(\bar{R}^n)$, or in $M(\bar{R}^{n+1})$, to a map as in (A1) where $\lambda = 1$, and g is *parabolic* if it is conjugate in $M(\bar{R}^n)$ to a map of the form

$$x \mapsto \beta(x) + a \tag{A4}$$

where $a \in \mathbb{R}^n$, $a \neq 0$, and β is an orthogonal linear map such that $\beta(a) = a$ (cf. [T3, p. 560]).

A Möbius group G is a subgroup of $M(\bar{R}^n)$ and such a group is discrete if it is discrete in the compact-open topology of \bar{R}^n . A set A is G-invariant if gA = A for every $g \in G$.

The *limit set* L(G) of G is

$$L(G) = \operatorname{cl} Gz \cap \bar{R}^n \tag{A5}$$

where $z \in H^{n+1}$ (and where cl is the closure). This does not depend on the choice of $z \in H^{n+1}$ and is a reasonable definition of L(G) also for non-discrete G.

We define that a Möbius group G is non-elementary if it contains two loxodromic elements with disjoint fixed point sets. If G is discrete, then it is well-known that G is non-elementary if and only if L(G) contains more than two points (see e.g. [T3, Theorem B2].)

We usually work in \overline{R}^n but we find it more natural to formulate Theorem D for Möbius groups of the *n*-sphere $S^n = \{x \in R^{n+1} : |x| = 1\}$. We also use above definitions with appropriate modifications for Möbius groups of S^n .

B. Representation of Möbius transformations by matrices

The proof of our main theorem depends on matrix representations of Möbius transformations. Let O(1, n + 1) be the group of $(n + 2) \times (n + 2)$ -matrices which preserves the quadratic form $x_1^2 - x_2^2 - \cdots - x_{n+2}^2$ and let $O_+(1, n + 1)$ be the subgroup of O(1, n + 1) which preserves

$$\{(x_1,\ldots,x_{n+2})\in R^{n+2}: x_1^2-x_2^2-\cdots-x_{n+2}^2=1 \text{ and } x_1>0\}.$$

Then, as is well-known [W], every $g \in M(\bar{R}^n)$ can be represented by a unique matrix $A \in O_+(1, n+1)$.

If n=2, then we identify R^2 and the complex plane C. If $g \in M(\overline{R}^2)$ is orientation preserving, then it can be represented by a matrix of SL(2, C), that is, by a complex 2×2 -matrix with determinant 1.

We will now give two simple formulas that relate the multiplier of $g \in M(\bar{R}^n)$ and the trace tr A of the matrix $A \in O_+(1, n+1)$ or $A \in SL(2, C)$ representing g. If $g \in M(\bar{R}^n)$ is represented by a matrix $A \in O_+(1, n+1)$, then

$$\operatorname{mul} g = \operatorname{tr} A + M(A) \tag{B1}$$

where $|M(A)| \le n+2$. This follows from explicit matrix representations for loxodromic, elliptic or parabolic Möbius transformations, see Wielenberg [W, Section 5] and the classification of a Möbius transformation as loxodromic, elliptic or parabolic mentioned above. Recall that nnul g = 1 for non-loxodromic g. Note that if g is elliptic, then we possibly need to extend g to a Möbius transformation of \bar{R}^{n+1} in order to obtain that g is conjugate to an orthogonal linear map.

If $g \in M(\bar{R}^2)$ is orientation preserving, then g can be represented by $B \in SL(2, C)$, and

$$\operatorname{mul} g = |\operatorname{tr} B|^2 + M'(B)$$
 (B2)

where $|M'(B)| \le 3$ as a simple calculation shows. The next lemma is based on these estimates.

LEMMA B. Let $g, h \in M(\bar{R}^n)$ be loxodromic and let $\gamma = (\text{mul } g)^{1/2}$ and $\chi = (\text{mul } h)^{1/2}$. Then, for $m, k \in \mathbb{Z}$,

where a, b are complex numbers such that a + b = 1 and depending only on the quadruple (P_g, N_g, P_h, N_h) of the fixed points. The constants c_{ik} , d_{im} , and e_{mk} are bounded and, furthermore,

- (a) $a \neq 0 \neq b$ if and only if g and h do not have common fixed points,
- (b) if g and h are hyperbolic, then $c_{mk} = d_{mk} = 0$ and $|e_{mk}| \le 3$ for all m, k; in the general loxodromic case there is a sequence $r_1 < r_2 < \cdots$ such that as $j \to \infty$,

$$c_{i,\pm r_i} \to 0$$
 and $d_{i,\pm r_i} \to 0$ $(i = 1, 2),$

(c) if the fixed fixed points of g and h are in $\bar{R}^2 = \bar{C}$ and $N_g = 0$, $P_g = \infty$, $N_h = 1$, and $P_h = p$, then there are the following relations between the numbers p, a and b:

$$p = -\frac{a}{b}$$
, $a = \frac{p}{p-1}$ and $b = \frac{1}{1-p}$.

Proof. The fixed points of g and h lie in a 2-dimensional sphere and hence we may assume that their fixed points lie in \bar{C} and that

$$P_g = \infty$$
 and $N_g = 0$.

Let \bar{g} and \bar{h} be the corresponding hyperbolic Möbius transformations, i.e. they have the same multiplier and the same repelling and attractive fixed points. Then \bar{g} and \bar{h} preserve \bar{C} and can be represented by matrices \tilde{A} , $\tilde{B} \in SL(2, C)$, respectively. We can conjugate in SL(2, C) to obtain

$$\tilde{A} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix}$$
 and $\tilde{B} = \begin{pmatrix} s & t \\ u & v \end{pmatrix} \begin{pmatrix} \chi & 0 \\ 0 & \chi^{-1} \end{pmatrix} \begin{pmatrix} v & -t \\ -u & s \end{pmatrix}$

where s, t, u and v are complex numbers such that sv - tu = 1. A simple calculation shows that if

$$a = sv$$
 and $b = -tu$,

then

$$\operatorname{tr} \tilde{A}^{m} \tilde{B}^{k} = a(\gamma^{m} \chi^{k} + \gamma^{-m} \chi^{-k}) + b(\gamma^{m} \gamma^{-k} + \gamma^{-m} \chi^{k}).$$
(B4)

Obviously, a and b depend only on the fixed points and a + b = 1. Since $P_h = s/u$ and $N_h = t/v$, we have that $a \neq 0 \neq b$ if and only if g and h do not have common fixed points. Remembering that a + b = 1, the formulas in (c) also follow.

We then represent g, h, \bar{g} , \bar{h} by matrices A, B, \bar{A} , \bar{B} of $O_+(1, n+1)$, respectively, and perform similar calculations. The matrix A has eigenvalues $\alpha_1, \ldots, \alpha_{n+2}$ which we can enumerate so that $\alpha_1 = \gamma^2 = \text{mul } g$ and $\alpha_2 = \gamma^{-2}$ and that α_i , i > 2 are complex numbers of modulus 1 as follows from the canonical forms for matrices of $O_+(1, n+1)$ representing loxodromic Möbius transformation [W, Section 5]. Similarly, B has eigenvalues $\beta_1 = \chi^2$, $\beta_2 = \chi^{-2}$, ..., β_{n+2} . We can assume that A is diagonal (so that the diagonal entries are the eigenvalues) and that

$$B = EDE^{-1} \tag{B5}$$

for some matrices E, D where D is diagonal (they need not be matrices of O(1, n + 1).) A calculation shows that there are constants a_{ij} , $i, j \le n + 2$ such that

$$\operatorname{tr} A^m B^k = \sum_{i,j} a_{ij} \alpha_i^m \beta_j^k.$$
 (B6)

Thus if we set

$$c'_{ik} = \sum_{j>2} a_{ij} \beta_j^k, \qquad (i = 1, 2, k \in \mathbb{Z}),$$

$$d'_{im} = \sum_{i>2} a_{ji} \alpha_j^m, \qquad (i = 1, 2, m \in Z),$$

$$e'_{mk} = \sum_{i>2, i>2} a_{ij} \alpha_i^m \beta_j^k, \qquad (m, k \in \mathbb{Z}),$$

we obtain bounded numbers (since $|\alpha_j| = |\beta_j| = 1$ if j > 2) such that

$$\operatorname{tr} A^{m}B^{k} = a_{11}\gamma^{2m}\chi^{2k} + a_{12}\gamma^{2m}\chi^{-2k} + a_{21}\gamma^{-2m}\chi^{2k} + a_{22}\gamma^{-2m}\chi^{-2k} + c'_{1k}\gamma^{2m} + c'_{2k}\gamma^{-2m} + d'_{1m}\chi^{2k} + d'_{2m}\chi^{-2k} + e'_{mk}.$$
(B7)

We obtain the matrix \bar{A} from that of A by substituting 1 for α_i if i > 2 (and leaving α_1 and α_2 unchanged). Similarly, substituting 1 for β_j in D if j > 2, we obtain \bar{B} from the right hand side of (B5). With these substitutions (B6) gives tr $\bar{A}^m \bar{B}^k$ with a_{ij} unchanged but with the new α_i and β_j , and (B7) is valid if we substitute in it for c'_{im} , d'_{im} and e'_{mk} the numbers

$$\bar{c}_i = \sum_{j>2} a_{ij},$$

$$\overline{d}_i = \sum_{j>2} a_{ji},$$

$$\bar{e} = \sum_{i > 2, j > 2} a_{ij}.$$

(They do not depend on m and k and so we have not marked them.) It follows by Kronecker's theorem ([A, Theorem 7.10] or [C, p. 53]) that there is a sequence $r_1 < r_2 < \cdots$ such that, as $i \to \infty$,

$$c'_{i,\pm r_i} \to \bar{c}_i \quad \text{and} \quad d'_{i,\pm r_i} \to \bar{d}_i$$
 (B8)

for i = 1, 2.

Applying (B1) and (B2) to mul $\bar{g}^m \bar{h}^k$ we obtain that

$$|\operatorname{tr} \bar{A}^m \bar{B}^k - |\operatorname{tr} \tilde{A}^m \tilde{B}^k|^2| \le n + 5.$$
(B9)

Set

$$c_{im} = c'_{im} - \bar{c}_i,$$
 $d_{ik} = d'_{ik} - \bar{d}_i,$
 $e''_{mk} = e'_{mk} - \bar{e}.$
(B10)

These numbers are bounded and satisfy (b) with respect to the r_i in (B8). Write

$$\operatorname{mul} g^{m}h^{k} = (\operatorname{mul} g^{m}h^{k} - \operatorname{tr} A^{m}B^{k}) + (\operatorname{tr} A^{m}B^{k} - \operatorname{tr} \bar{A}^{m}\bar{B}^{k}) + (\operatorname{tr} \bar{A}^{m}\bar{B}^{k} - |\operatorname{tr} \tilde{A}^{m}\tilde{B}^{k}|^{2}) + |\operatorname{tr} \tilde{A}^{m}\tilde{B}^{k}|^{2}.$$

On the right-hand sum the first parenthesis is bounded by (B1) and the third parenthesis by (B9). The second parenthesis can be estimated by (B7) when it is applied to tr $\bar{A}^m B^k$ and to tr $\bar{A}^m \bar{B}^k$, and the last term is given by (B4). Combining all this, we have

$$\operatorname{mul} g^{m} h^{k} = \left| a(\gamma^{m} \chi^{k} + \gamma^{-m} \chi^{-k}) + b(\gamma^{m} \chi^{-k} + \gamma^{-m} \chi^{k}) \right|^{2}$$

$$+ c_{1m} \gamma^{2m} + c_{2m} \gamma^{-2m} + d_{1k} \chi^{2k} + d_{2k} \chi^{-2k} + e_{mk}.$$

Here e_{mk} is the sum of e''_{mk} in (B10) and of the first and third parenthesis. They are bounded since e''_{mk} are bounded, and by what has been said above, and so are the numbers c_{im} and d_{ik} .

If g and h are hyperbolic, then we can use (B4) and (B2) to conclude that (B3) is true with $c_{im} = d_{ik} = 0$ and $|e_{mk}| \le 3$.

Finally, (b) follows from (B8) and (B10).

Remark. If we have two (or, in fact, any number of) pairs g, h and \tilde{g} , \tilde{h} of loxodromic Möbius transformations and if \tilde{c}_{ij} and \tilde{d}_{ij} are the numbers in the expression for mul $\tilde{g}^m \tilde{h}^k$, and if \tilde{r}_i is the corresponding sequence in (b), then exactly as in (b), by Kronecker's theorem, one can choose these sequences so that $r_i = \tilde{r}_i$.

C. The main theorem

We can now prove our main

THEOREM C. Let $\varphi: G \to H$ be a surjective homomorphism of two Möbius groups of \overline{R}^n such that one of the groups G and H is non-elementary. Then φ is multiplier preserving if it is of constant stretch d > 0.

Furthermore, let S be the k-sphere of smallest dimension k such that $S = g\bar{R}^k$ for some $g \in M(\bar{R}^n)$ and that $S \supset L(G)$, where L(G) is the limit set of G (see (A5)). Then S is G-invariant and there is $h \in M(\bar{R}^n)$ such that

$$hg(x) = \varphi(g)h(x) \tag{C0}$$

for $x \in S$ and $g \in G$.

In particular, if $S = \overline{R}^n$, then φ is a conjugation by a Möbius transformation.

Remark. Actually, it would suffice to assume that (A2) is true for all $g \in G$ such that g is loxodromic (if G is non-elementary) or such that $\varphi(g)$ is loxodromic (if H is non-elementary).

The sphere S in the theorem is well-defined if L(G) contains at least two points. Since either G or H is non-elementary, and φ is of constant stretch, G has loxodromics and hence S is well-defined (the proof shows that both groups are non-elementary).

Proof. We first assume that G is non-elementary and that φ is an isomorphism. We first prove that d=1. Since G is non-elementary, there are two loxodromic elements $g, h \in G$ without common fixed points. Then also $\varphi(g)$ and $\varphi(h)$ are loxodromic by (A2).

Let $\bar{g} = \varphi(g)$ and $\bar{h} = \varphi(h)$. Then $\text{mul } \bar{g} = (\text{mul } g)^d = \gamma^{2d}$ and $\text{mul } \bar{h} = (\text{mul } h)^d = \chi^{2d}$ and hence if \bar{a} , \bar{b} , \bar{c}_{im} , \bar{d}_{ik} and \bar{e}_{mk} are numbers as in Lemma B, we have that

We now use the equality

$$\operatorname{mul} \bar{g}^m \bar{h}^k = (\operatorname{mul} g^m h^k)^d \tag{C2}$$

together with (B3) and (C1) and let m, k tend to $+\infty$ or to $-\infty$. Since g and h do not have common fixed points, $a \neq 0 \neq b$ by Lemma B (a) and it follows that

$$|\bar{a}| = |a|^d$$
 and $|\bar{b}| = |b|^d$. (C3)

In particular, it follows that $\bar{a} \neq 0 \neq \bar{b}$ and hence \bar{g} and \bar{h} do not have common fixed points by Lemma B (a). Thus H is also non-elementary and, if necessary, we can replace φ by φ^{-1} and thereby assume that

$$d \ge 1$$
. (C4)

Next, substitute again (B3) and (C1) into (C2) and divide both sides of the resulting equation by $|a|^{d\gamma^{2dm}}\chi^{2dk} = |\bar{a}|\gamma^{2dm}\chi^{2dk}$. Keep k fixed and let m assume the values r_i of Lemma B (b) (see also the Remark following Lemma B) and let $i \to \infty$. We obtain

$$|1 + \bar{b}\chi^{-2kd}/\bar{a}|^2 = |1 + b\chi^{-2k}/a|^{2d}$$
(C5)

which is valid for every $k \in \mathbb{Z}$.

Let

$$r = |b|/|a|,$$
 $\alpha = \arg b/a,$ $\bar{\alpha} = \arg \bar{b}/\bar{a},$

so that $r^d = |\bar{b}|/|\bar{a}|$. Substituting this into (C5) and using elementary trigonometry, we obtain for all $k \in \mathbb{Z}$

$$1 + 2r^{d}\chi^{-2kd}\cos\bar{\alpha} + r^{2d}\chi^{-4kd} = (1 + 2r\chi^{-2k}\cos\alpha + r^{2}\chi^{-4k})^{d}.$$

We develop the right hand side into a power series for $t = \chi^{-2k}$ and compare it to the left side. When $\cos \alpha \neq 0$, we obtain immediately a contradiction if d > 1.

We look at the geometric situation when $\cos \alpha = 0$, that is, $\alpha = \pm \pi/2$. Suppose that g fixes 0 and ∞ and that h fixes 1 so that 0 and 1 are the repelling fixed points. By Lemma B (c), h fixes also the point -a/b and we know that $\arg -a/b = -\alpha = \pm \pi/2$ and hence this point lies on the imaginary axis.

We state this in terms independent of normalization. Let S_1 and S_2 be the two circles through the fixed points of g and through one fixed point of h. Then S_1 and S_2 intersect orthogonally.

But this is absurd since g and h can be any two loxodromic elements of G without common fixed points. If we have chosen $g, h \in G$ and S_1 and S_2 are orthogonal for these g and h, we can replace h by $h^k g h^{-k}$, k big, in such a way that they are no more orthogonal. This contradiction concludes the proof that d = 1.

We now assume that d = 1 and prove the remaining part of the theorem. As we have already done this in [T1, pp. 338-339] in more detail, we present only the main points.

We continue from the preceding situation with $g, h \in G$ loxodromic without common fixed points. Since d = 1, (C3) becomes $|\bar{a}| = |a|$, $|\bar{b}| = |b|$ and in addition we know that a + b = 1 and $\bar{a} + \bar{b} = 1$. Hence the two triangles with vertices 0, 1, a and 0, 1, \bar{a} , respectively, have the same sidelengths and consequently either

$$\bar{a}=a$$
 and $\bar{b}=b$, or $\bar{a}=a^*$ and $\bar{b}=b^*$

where * is the complex conjugation. Conjugating by a Möbius transformation we obtain that the fixed points are

$$P_{g} = P_{\bar{g}} = \infty$$
, $N_{g} = N_{\bar{g}} = 0$, $N_{h} = N_{\bar{h}} = 1$ $P_{h} = -a/b$, $P_{\bar{h}} = -\bar{a}/\bar{b}$.

Here P and N denote the attractive and repelling fixed points (see Section A) and we have also used (c) of Lemma B for P_h and $P_{\bar{h}}$. Hence at least we can conjugate the fixed points of g and h to the fixed points of \bar{g} and \bar{h} . In particular, if

 $N_h = N_{\bar{h}} = \infty$, then the two triangles with vertices P_g , N_g , P_h and $P_{\bar{g}}$, $N_{\bar{g}}$, $P_{\bar{h}}$, respectively, are similar.

Now, g and h can be any two loxodromic elements in G without common fixed points. Using this and the fact that every distinct point-pair of $L(G) \times L(G)$ can be approximated arbitrarily closely by the fixed points of a loxodromic map in G (this follows from [T3, Theorem B1]), we can show that the map defined by

$$P_g \mapsto P_{\varphi(g)}$$
 (C6)

 $(g \in G \text{ loxodromic})$ is the restriction of a Möbius transformation f. It follows that $fg|L(G) = \varphi(g)f|L(G)$ for $g \in G$ from which fact the rest of Theorem C follows.

Finally, we remove the assumptions that φ was an isomorphism and G non-elementary. If φ is not an isomorphism but G is non-elementary, we pick as above loxodromic $g, h \in G$ without common fixed points. Then for big enough k, the group G' generated by g^k and h^k is a Schottky group which is a free group such that every element of $G'\setminus\{id\}$ is loxodromic (e.g. [T, p. 333] contains the simple argument). Then every $\varphi(g'), g' \in G'\setminus\{id\}$ is loxodromic by (A2) and hence $\varphi(G')$ is an isomorphism onto $\varphi(G')$ and we can apply above reasoning with G replaced by G' and G' and G' by G' and G' and

If G is elementary, then H is non-elementary. Thus there are loxodromic $g, h \in H$ without common fixed points. As above, for big enough k, the group H' generated by g^k and h^k is a Schottky group. Find $g_0, h_0 \in G$ such that $\varphi(g_0) = g^k$ and $\varphi(h_0) = h^k$ and let G' be the group generated by them. Since H' is free, $\varphi(G')$ is an isomorphism onto H' and we can apply the above reasoning to H', G' and $\varphi(g_0) = g^k$ and show that g_0 and g_0 are loxodromic and without common fixed points and hence G was in fact non-elementary, contrary to the assumption.

Remark. It is clear that there are non-trivial situations in which Theorem C is not true. For instance, let G be generated by $g: x \mapsto 2x$ and H by $h: x \mapsto 4x$ which are Möbius groups of \overline{R}^n . Then the isomorphism mapping g onto h is of constant stretch 2.

Another example is given by the group G whose elements are of the form $x \mapsto \lambda x + a$ where $\lambda > 0$ and $a \in \mathbb{R}^n$. Let α be an affine homeomorphism of \mathbb{R}^n and let $\varphi(g) = \alpha g \alpha^{-1}$. Then φ is an isomorphism $G \to G$ which preserves multipliers but is not a conjugation by a Möbius transformation if α is not a similarity.

Thus it is necessary to assume something on the groups G and H although it might be, as is suggested by the last example, that if the groups contain two loxodromic elements with different fixed point sets (but which may have a common fixed point), then, if φ preserves multipliers, (A3) might be true for some affine h

(i.e. h is a map such that $h_0hh_1|R^n$ is affine for some Möbius transformations h_0 and h_1).

D. Bilipschitz maps and rigidity

In this last section we note a consequence of Theorem C. Roughly, it says that if φ is induced by f, i.e. (A3) is true, and f is a bilipschitz map, then φ preserves multipliers and hence is, or almost is, a conjugation by a Möbius transformation. Since we use the euclidean metric which is not a metric of whole \overline{R}^n , we transfer the situation to the n-sphere $S^n \subset R^{n+1}$. It turns out that the bilipschitz condition need not be satisfied everywhere, and taking account that in Theorem C we actually considered homomorphisms of constant stretch, we can generalize this as

THEOREM D. Let $\varphi: G \to H$ be a homomorphism of two Möbius groups of S^n such that G is non-elementary. Let $A \subset S^n$ be a non-empty G-invariant set and let $f: A \to S^n$ be a map inducing φ . Suppose that there are an open set $U \subset S^n$ and numbers $L \geq 1$ and d > 0 such that $U \cap L(G) \neq \varphi$ and that

$$|x - y|^d / L \le |f(x) - f(y)| \le L|x - y|^d$$
 (D1)

for $x, y \in U \cap A$. Then d = 1, φ preserves multipliers and, if in addition, $L(G) \subset h(S^k)$ for no Möbius transformation h and no k < n, φ is a conjugation by a Möbius transformation.

Proof. Pick $z \in L(G) \cap U$. Thus there are $g_i \in G$ and $w \in S^n$ such that

$$g_i|S^n\setminus\{w\}\to z$$

locally uniformly, as follows easily from the definition of the limit set (cf. (A5)) and the convergence property of Möbius groups (see [GM, Theorem 3.2]). This fact has two consequences. The first is that if acc A denotes the accumulation points of A, then

$$\operatorname{acc} A \supset L(G),$$
 (D2)

(for (D2) we remark that A is in any case actually infinite by non-elementariness) and the second is that

$$\{g_i^{-1}(U)\}\$$
 is a cover of $L(G)\setminus\{w\}$. (D3)

It follows that if $g \in G$ is loxodromic, then there is $h \in G$ which is conjugate to g in G such that at least one of the fixed points of h is in U. Consequently, if we can prove that

$$\operatorname{mul} \varphi(g) = (\operatorname{mul} g)^d \tag{D4}$$

for all loxodromic $g \in G$ with one fixed point in U, then this is actually valid for all loxodromic $g \in G$.

So suppose that $g \in G$ is loxodromic and fixes $u \in U$. We can assume that u is the attractive fixed point of g. Then $u \in L(G)$ and hence, by (D2), there are distinct $x, y \in U \cap A$ not fixed by g. Under these circumstances we have, as can be seen from (A1),

$$\operatorname{mul} g = \lim_{k \to \infty} |g^{k}(x) - g^{k}(y)|^{-1/k}.$$
 (D5)

We observe that (D5) gives mul g for any Möbius transformation g as follows from the representations (A1) and (A4), provided that x and y are not fixed by g. By (D1) and G-compatibility, f(x) and f(y) are not fixed by $\varphi(g)$. Hence (D5) and (D1) imply that

$$(\operatorname{mul} g)^{d} = \lim_{k \to \infty} |fg^{k}(x) - fg^{k}(y)|^{-1/k}$$

$$= \lim_{k \to \infty} |\varphi(g)^{k} f(x) - \varphi(g)^{k} f(y)|^{-1/k}$$

$$= \operatorname{mul} \varphi(g)$$

and (D4) is valid for all loxodromic $g \in G$.

This is all that is needed for the validity of Theorem C if G is non-elementary (see the Remark after it). Theorem C implies the rest of the present theorem, for instance that φ preserves multipliers for all $g \in G$.

Remarks. 1. We needed the assumption that G is non-elementary in order to apply Theorem C but to obtain (D4) for loxodromic g, this assumption was not used (though we must assume that A contains at least three points if G is elementary). In fact, if g is parabolic such that g is conjugate to some h with a fixed point in U or if g is elliptic, then basically as above one obtains that (D4) is valid for g; in the non-elementary case a parabolic g is always conjugate to such g. It is valid even if g is parabolic and not conjugate to such a map g but then a more complicated reasoning, given below, is necessary. Thus even if g is elementary g is still of constant stretch g, provided that g contains at least 3 points.

Suppose that g is parabolic with the fixed point v and $h(v) \in U$ for no $h \in G$. It follows that v must be the point w in (D3) and that v is fixed by every $g \in G$. We cannot have that $\{v\} = L(G)$ (since then $v \in U$) and hence there are at least two points in L(G). Consequently, there is loxodromic $h \in G$ [T3, proof of Theorem E]. Then h fixes v and we assume that v is the repelling fixed point. It follows from (A1) and (A4) (transform the situation to \overline{R}^n so that $v = \infty$) that there are $k_i > 0$ and $n_i > 0$ such that if

$$g_i = h^{k_i} g^{n_i} h^{-k_i},$$

then $g_i(x) \to x$ for all $x \in S^n$.

It follows that also $\varphi(g_i)(x) \to x$ for all $x \in f(U \cap A)$. Since $f(U \cap A)$ is infinite, it follows by the convergence property [GM, 3.2] that we can pass to a subsequence in such a way that $\varphi(g_i) \to \bar{g}$ where g is a Möbius transformation such that $\bar{g} \mid f(U \cap A) = \text{id}$. hence \bar{g} is elliptic and $\text{mul } \bar{g} = 1 = \lim_{i \to \infty} \text{mul } \varphi(g_i)$. However, $\varphi(g_i)$ is conjugate to $\varphi(g)^{n_i}$. Consequently, $\text{mul } \varphi(g) = (\text{mul } \varphi(g_i))^{1/n_i} = 1$.

2. Actually, we need not assume that (D1) is true for all $x, y \in U \cap A$, only that for each loxodromic $g \in G$ there are distinct points $u, v \in A$, not both of them fixed by g, such that (D1) is valid for $x = g^k(u)$ and $y = g^k(v)$ when k > 0 with some $k \geq 1$ which may depend on $k \geq 0$ which does not.

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