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**Autor:** Tukia, Pekka  
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## Homomorphisms of constant stretch between Möbius groups

PEKKA TUKIA

### A. Introduction

A Möbius transformation  $g$  of  $\bar{R}^n$  is *loxodromic* if it can be conjugated by another Möbius transformation to the form

$$x \mapsto \lambda\beta(x) \quad (x \in R^n) \quad (\text{A1})$$

where  $\lambda > 1$  and  $\beta$  is an orthogonal linear map. The number  $\lambda > 1$  does not depend on how the conjugacy is chosen and it is the *multiplier*  $\text{mul } g$  of  $g$ ; for non-loxodromic  $g$  we set  $\text{mul } g = 1$ .

If  $G$  and  $H$  are two groups of Möbius transformations of  $\bar{R}^n$  and  $\varphi : G \rightarrow H$  is a homomorphism between them,  $\varphi$  is said to be of *constant stretch* if there is  $d > 0$  such that

$$\text{mul } \varphi(g) = (\text{mul } g)^d \quad (\text{A2})$$

for all  $g \in G$ ; more precisely we can say that  $\varphi$  is of constant stretch  $d$ . Note that  $d$  is well-defined if there are loxodromic elements in  $G$  and that  $g$  is loxodromic if and only if  $\varphi(g)$  is. If  $d = 1$ , then we say that  $\varphi$  is *multiplier preserving*.

Our main Theorem C says that a homomorphism  $\varphi$  of non-elementary groups is of constant stretch if and only if it is multiplier preserving. Furthermore, such a  $\varphi$  comes very near to being a conjugation by a Möbius transformation. If the limit set  $L(G)$  of  $G$  “fills”  $\bar{R}^n$ , that is,  $h(L(G)) \not\subset \bar{R}^k$  for no  $k < n$  and no Möbius transformation  $h$ , then we can actually show that  $\varphi$  is a conjugation by a Möbius transformation.

A consequence of Theorem C is that if a map  $f : A \rightarrow \bar{R}^n$  is compatible with a homomorphism  $\varphi : G \rightarrow H$  of non-elementary Möbius groups, that is for every  $g \in G$ ,  $gA = A$  and

$$fg(x) = \varphi(g)f(x) \quad (\text{A3})$$

when  $x \in A$ , then  $\varphi$  is a conjugation by a Möbius transformation as soon as  $f$  satisfies a bilipschitz property and the above mentioned condition for the limit set is satisfied (Theorem D).

Originally, we needed Theorem C in [T2] but after we found a simpler method for [T2], we separated these results into the present paper. In [T1] we have already treated the case of a multiplier preserving  $\varphi$ . The present arrangement of the proof seems to be slightly simpler also for the multiplier preserving case. In our proof of Theorem C we will first show that a homomorphism of constant stretch is multiplier preserving and then sketch the remaining part for completeness although we could here refer to [T1].

**DEFINITIONS AND NOTATIONS.** We denote the group of all Möbius transformations of  $\bar{R}^n$  by  $M(\bar{R}^n)$ . Each  $g \in M(\bar{R}^n)$  has a unique extension to a Möbius transformation of  $\bar{R}^{n+1}$  such that  $g(H^{n+1}) = H^{n+1}$  when  $H^{n+1}$  is the  $(n+1)$ -dimensional hyperbolic space

$$H^{n+1} = \{x \in R^{n+1} : x = (x_1, \dots, x_{n+1}) \text{ where } x_{n+1} > 0\}.$$

We identify  $g$  and this extension of  $g$  to  $\bar{R}^{n+1}$ ; thus  $M(\bar{R}^n) \subset M(\bar{R}^{n+1})$ .

A loxodromic  $g \in M(\bar{R}^n)$  has two fixed points denoted by  $P_g = P(g)$  and  $N_g = N(g)$  so that  $P_g$  is the attracting fixed point and  $N_g$  the repelling fixed point; these names are self-explanatory. A loxodromic map  $g$  is *hyperbolic* if it is conjugate in  $M(\bar{R}^n)$  to a map as in (A1) where  $\beta = \text{id}$ . If  $g \in M(\bar{R}^n)$  is not loxodromic, then it is either elliptic or parabolic. If  $g$  is *elliptic*, then it is conjugate in  $M(\bar{R}^n)$ , or in  $M(\bar{R}^{n+1})$ , to a map as in (A1) where  $\lambda = 1$ , and  $g$  is *parabolic* if it is conjugate in  $M(\bar{R}^n)$  to a map of the form

$$x \mapsto \beta(x) + a \tag{A4}$$

where  $a \in R^n$ ,  $a \neq 0$ , and  $\beta$  is an orthogonal linear map such that  $\beta(a) = a$  (cf. [T3, p. 560]).

A *Möbius group*  $G$  is a subgroup of  $M(\bar{R}^n)$  and such a group is discrete if it is discrete in the compact-open topology of  $\bar{R}^n$ . A set  $A$  is *G-invariant* if  $gA = A$  for every  $g \in G$ .

The *limit set*  $L(G)$  of  $G$  is

$$L(G) = \text{cl } Gz \cap \bar{R}^n \tag{A5}$$

where  $z \in H^{n+1}$  (and where  $\text{cl}$  is the closure). This does not depend on the choice of  $z \in H^{n+1}$  and is a reasonable definition of  $L(G)$  also for non-discrete  $G$ .

We define that a Möbius group  $G$  is *non-elementary* if it contains two loxodromic elements with disjoint fixed point sets. If  $G$  is discrete, then it is well-known that  $G$  is non-elementary if and only if  $L(G)$  contains more than two points (see e.g. [T3, Theorem B2].)

We usually work in  $\bar{R}^n$  but we find it more natural to formulate Theorem D for Möbius groups of the  $n$ -sphere  $S^n = \{x \in R^{n+1} : |x| = 1\}$ . We also use above definitions with appropriate modifications for Möbius groups of  $S^n$ .

## B. Representation of Möbius transformations by matrices

The proof of our main theorem depends on matrix representations of Möbius transformations. Let  $O(1, n+1)$  be the group of  $(n+2) \times (n+2)$ -matrices which preserves the quadratic form  $x_1^2 - x_2^2 - \cdots - x_{n+2}^2$  and let  $O_+(1, n+1)$  be the subgroup of  $O(1, n+1)$  which preserves

$$\{(x_1, \dots, x_{n+2}) \in R^{n+2} : x_1^2 - x_2^2 - \cdots - x_{n+2}^2 = 1 \text{ and } x_1 > 0\}.$$

Then, as is well-known [W], every  $g \in M(\bar{R}^n)$  can be represented by a unique matrix  $A \in O_+(1, n+1)$ .

If  $n = 2$ , then we identify  $R^2$  and the complex plane  $C$ . If  $g \in M(\bar{R}^2)$  is orientation preserving, then it can be represented by a matrix of  $SL(2, C)$ , that is, by a complex  $2 \times 2$ -matrix with determinant 1.

We will now give two simple formulas that relate the multiplier of  $g \in M(\bar{R}^n)$  and the trace  $\text{tr } A$  of the matrix  $A \in O_+(1, n+1)$  or  $A \in SL(2, C)$  representing  $g$ . If  $g \in M(\bar{R}^n)$  is represented by a matrix  $A \in O_+(1, n+1)$ , then

$$\text{mul } g = \text{tr } A + M(A) \tag{B1}$$

where  $|M(A)| \leq n+2$ . This follows from explicit matrix representations for loxodromic, elliptic or parabolic Möbius transformations, see Wielenberg [W, Section 5] and the classification of a Möbius transformation as loxodromic, elliptic or parabolic mentioned above. Recall that  $\text{mul } g = 1$  for non-loxodromic  $g$ . Note that if  $g$  is elliptic, then we possibly need to extend  $g$  to a Möbius transformation of  $\bar{R}^{n+1}$  in order to obtain that  $g$  is conjugate to an orthogonal linear map.

If  $g \in M(\bar{R}^2)$  is orientation preserving, then  $g$  can be represented by  $B \in SL(2, C)$ , and

$$\text{mul } g = |\text{tr } B|^2 + M'(B) \tag{B2}$$



where  $|M'(B)| \leq 3$  as a simple calculation shows. The next lemma is based on these estimates.

**LEMMA B.** *Let  $g, h \in M(\bar{R}^n)$  be loxodromic and let  $\gamma = (\text{mul } g)^{1/2}$  and  $\chi = (\text{mul } h)^{1/2}$ . Then, for  $m, k \in \mathbb{Z}$ ,*

$$\begin{aligned} \text{mul } g^m h^k = & |a(\gamma^m \chi^k + \gamma^{-m} \chi^{-k}) + b(\gamma^m \chi^{-k} + \gamma^{-m} \chi^k)|^2 \\ & + c_{1m} \gamma^{2m} + c_{2m} \gamma^{-2m} + d_{1k} \chi^{2k} + d_{2k} \chi^{-2k} + e_{mk} \end{aligned} \quad (\text{B3})$$

where  $a, b$  are complex numbers such that  $a + b = 1$  and depending only on the quadruple  $(P_g, N_g, P_h, N_h)$  of the fixed points. The constants  $c_{ik}$ ,  $d_{im}$ , and  $e_{mk}$  are bounded and, furthermore,

- (a)  $a \neq 0 \neq b$  if and only if  $g$  and  $h$  do not have common fixed points,
- (b) if  $g$  and  $h$  are hyperbolic, then  $c_{mk} = d_{mk} = 0$  and  $|e_{mk}| \leq 3$  for all  $m, k$ ; in the general loxodromic case there is a sequence  $r_1 < r_2 < \dots$  such that as  $j \rightarrow \infty$ ,

$$c_{i, \pm r_j} \rightarrow 0 \quad \text{and} \quad d_{i, \pm r_j} \rightarrow 0 \quad (i = 1, 2),$$

- (c) if the fixed fixed points of  $g$  and  $h$  are in  $\bar{R}^2 = \bar{C}$  and  $N_g = 0$ ,  $P_g = \infty$ ,  $N_h = 1$ , and  $P_h = p$ , then there are the following relations between the numbers  $p$ ,  $a$  and  $b$ :

$$p = -\frac{a}{b}, \quad a = \frac{p}{p-1} \quad \text{and} \quad b = \frac{1}{1-p}.$$

*Proof.* The fixed points of  $g$  and  $h$  lie in a 2-dimensional sphere and hence we may assume that their fixed points lie in  $\bar{C}$  and that

$$P_g = \infty \quad \text{and} \quad N_g = 0.$$

Let  $\bar{g}$  and  $\bar{h}$  be the corresponding hyperbolic Möbius transformations, i.e. they have the same multiplier and the same repelling and attractive fixed points. Then  $\bar{g}$  and  $\bar{h}$  preserve  $\bar{C}$  and can be represented by matrices  $\tilde{A}, \tilde{B} \in SL(2, C)$ , respectively. We can conjugate in  $SL(2, C)$  to obtain

$$\tilde{A} = \begin{pmatrix} \gamma & 0 \\ 0 & \gamma^{-1} \end{pmatrix} \quad \text{and} \quad \tilde{B} = \begin{pmatrix} s & t \\ u & v \end{pmatrix} \begin{pmatrix} \chi & 0 \\ 0 & \chi^{-1} \end{pmatrix} \begin{pmatrix} v & -t \\ -u & s \end{pmatrix}$$

where  $s, t, u$  and  $v$  are complex numbers such that  $sv - tu = 1$ . A simple calculation shows that if

$$a = sv \quad \text{and} \quad b = -tu,$$

then

$$\text{tr } \tilde{A}^m \tilde{B}^k = a(\gamma^m \chi^k + \gamma^{-m} \chi^{-k}) + b(\gamma^m \gamma^{-k} + \gamma^{-m} \chi^k). \quad (\text{B4})$$

Obviously,  $a$  and  $b$  depend only on the fixed points and  $a + b = 1$ . Since  $P_h = s/u$  and  $N_h = t/v$ , we have that  $a \neq 0 \neq b$  if and only if  $g$  and  $h$  do not have common fixed points. Remembering that  $a + b = 1$ , the formulas in (c) also follow.

We then represent  $g, h, \bar{g}, \bar{h}$  by matrices  $A, B, \bar{A}, \bar{B}$  of  $O_+(1, n+1)$ , respectively, and perform similar calculations. The matrix  $A$  has eigenvalues  $\alpha_1, \dots, \alpha_{n+2}$  which we can enumerate so that  $\alpha_1 = \gamma^2 = \text{mul } g$  and  $\alpha_2 = \gamma^{-2}$  and that  $\alpha_i, i > 2$  are complex numbers of modulus 1 as follows from the canonical forms for matrices of  $O_+(1, n+1)$  representing loxodromic Möbius transformation [W, Section 5]. Similarly,  $B$  has eigenvalues  $\beta_1 = \chi^2, \beta_2 = \chi^{-2}, \dots, \beta_{n+2}$ . We can assume that  $A$  is diagonal (so that the diagonal entries are the eigenvalues) and that

$$B = EDE^{-1} \quad (\text{B5})$$

for some matrices  $E, D$  where  $D$  is diagonal (they need not be matrices of  $O(1, n+1)$ .) A calculation shows that there are constants  $a_{ij}, i, j \leq n+2$  such that

$$\text{tr } A^m B^k = \sum_{i,j} a_{ij} \alpha_i^m \beta_j^k. \quad (\text{B6})$$

Thus if we set

$$c'_{ik} = \sum_{j>2} a_{ij} \beta_j^k, \quad (i = 1, 2, k \in \mathbb{Z}),$$

$$d'_{im} = \sum_{j>2} a_{ji} \alpha_j^m, \quad (i = 1, 2, m \in \mathbb{Z}),$$

$$e'_{mk} = \sum_{i>2, j>2} a_{ij} \alpha_i^m \beta_j^k, \quad (m, k \in \mathbb{Z}),$$

we obtain bounded numbers (since  $|\alpha_j| = |\beta_j| = 1$  if  $j > 2$ ) such that

$$\begin{aligned} \operatorname{tr} A^m B^k &= a_{11} \gamma^{2m} \chi^{2k} + a_{12} \gamma^{2m} \chi^{-2k} + a_{21} \gamma^{-2m} \chi^{2k} + a_{22} \gamma^{-2m} \chi^{-2k} \\ &\quad + c'_{1k} \gamma^{2m} + c'_{2k} \gamma^{-2m} + d'_{1m} \chi^{2k} + d'_{2m} \chi^{-2k} + e'_{mk}. \end{aligned} \quad (\text{B7})$$

We obtain the matrix  $\bar{A}$  from that of  $A$  by substituting 1 for  $\alpha_i$  if  $i > 2$  (and leaving  $\alpha_1$  and  $\alpha_2$  unchanged). Similarly, substituting 1 for  $\beta_j$  in  $D$  if  $j > 2$ , we obtain  $\bar{B}$  from the right hand side of (B5). With these substitutions (B6) gives  $\operatorname{tr} \bar{A}^m \bar{B}^k$  with  $a_{ij}$  unchanged but with the new  $\alpha_i$  and  $\beta_j$ , and (B7) is valid if we substitute in it for  $c'_{im}$ ,  $d'_{im}$  and  $e'_{mk}$  the numbers

$$\bar{c}_i = \sum_{j>2} a_{ij},$$

$$\bar{d}_i = \sum_{j>2} a_{ji},$$

$$\bar{e} = \sum_{i>2, j>2} a_{ij}.$$

(They do not depend on  $m$  and  $k$  and so we have not marked them.) It follows by Kronecker's theorem ([A, Theorem 7.10] or [C, p. 53]) that there is a sequence  $r_1 < r_2 < \dots$  such that, as  $i \rightarrow \infty$ ,

$$c'_{i, \pm r_i} \rightarrow \bar{c}_i \quad \text{and} \quad d'_{i, \pm r_i} \rightarrow \bar{d}_i \quad (\text{B8})$$

for  $i = 1, 2$ .

Applying (B1) and (B2) to  $\operatorname{mul} \bar{g}^m \bar{h}^k$  we obtain that

$$|\operatorname{tr} \bar{A}^m \bar{B}^k - |\operatorname{tr} \tilde{A}^m \tilde{B}^k|^2| \leq n + 5. \quad (\text{B9})$$

Set

$$\begin{aligned} c_{im} &= c'_{im} - \bar{c}_i, \\ d_{ik} &= d'_{ik} - \bar{d}_i, \\ e''_{mk} &= e'_{mk} - \bar{e}. \end{aligned} \quad (\text{B10})$$

These numbers are bounded and satisfy (b) with respect to the  $r_i$  in (B8). Write

$$\begin{aligned} \operatorname{mul} g^m h^k &= (\operatorname{mul} g^m h^k - \operatorname{tr} A^m B^k) + (\operatorname{tr} A^m B^k - \operatorname{tr} \bar{A}^m \bar{B}^k) \\ &\quad + (\operatorname{tr} \bar{A}^m \bar{B}^k - |\operatorname{tr} \tilde{A}^m \tilde{B}^k|^2) + |\operatorname{tr} \tilde{A}^m \tilde{B}^k|^2. \end{aligned}$$

On the right-hand sum the first parenthesis is bounded by (B1) and the third parenthesis by (B9). The second parenthesis can be estimated by (B7) when it is applied to  $\text{tr } A^m B^k$  and to  $\text{tr } \bar{A}^m \bar{B}^k$ , and the last term is given by (B4). Combining all this, we have

$$\begin{aligned} \text{mul } g^m h^k &= |a(\gamma^m \chi^k + \gamma^{-m} \chi^{-k}) + b(\gamma^m \chi^{-k} + \gamma^{-m} \chi^k)|^2 \\ &\quad + c_{1m} \gamma^{2m} + c_{2m} \gamma^{-2m} + d_{1k} \chi^{2k} + d_{2k} \chi^{-2k} + e_{mk}. \end{aligned}$$

Here  $e_{mk}$  is the sum of  $e''_{mk}$  in (B10) and of the first and third parenthesis. They are bounded since  $e''_{mk}$  are bounded, and by what has been said above, and so are the numbers  $c_{im}$  and  $d_{ik}$ .

If  $g$  and  $h$  are hyperbolic, then we can use (B4) and (B2) to conclude that (B3) is true with  $c_{im} = d_{jk} = 0$  and  $|e_{mk}| \leq 3$ .

Finally, (b) follows from (B8) and (B10).

*Remark.* If we have two (or, in fact, any number of) pairs  $g, h$  and  $\tilde{g}, \tilde{h}$  of loxodromic Möbius transformations and if  $\tilde{c}_{ij}$  and  $\tilde{d}_{ij}$  are the numbers in the expression for  $\text{mul } \tilde{g}^m \tilde{h}^k$ , and if  $\tilde{r}_i$  is the corresponding sequence in (b), then exactly as in (b), by Kronecker's theorem, one can choose these sequences so that  $r_i = \tilde{r}_i$ .

### C. The main theorem

We can now prove our main

**THEOREM C.** *Let  $\varphi : G \rightarrow H$  be a surjective homomorphism of two Möbius groups of  $\bar{R}^n$  such that one of the groups  $G$  and  $H$  is non-elementary. Then  $\varphi$  is multiplier preserving if it is of constant stretch  $d > 0$ .*

*Furthermore, let  $S$  be the  $k$ -sphere of smallest dimension  $k$  such that  $S = g\bar{R}^k$  for some  $g \in M(\bar{R}^n)$  and that  $S \supset L(G)$ , where  $L(G)$  is the limit set of  $G$  (see (A5)). Then  $S$  is  $G$ -invariant and there is  $h \in M(\bar{R}^n)$  such that*

$$hg(x) = \varphi(g)h(x) \tag{C0}$$

*for  $x \in S$  and  $g \in G$ .*

*In particular, if  $S = \bar{R}^n$ , then  $\varphi$  is a conjugation by a Möbius transformation.*

*Remark.* Actually, it would suffice to assume that (A2) is true for all  $g \in G$  such that  $g$  is loxodromic (if  $G$  is non-elementary) or such that  $\varphi(g)$  is loxodromic (if  $H$  is non-elementary).

The sphere  $S$  in the theorem is well-defined if  $L(G)$  contains at least two points. Since either  $G$  or  $H$  is non-elementary, and  $\varphi$  is of constant stretch,  $G$  has loxodromics and hence  $S$  is well-defined (the proof shows that both groups are non-elementary).

*Proof.* We first assume that  $G$  is non-elementary and that  $\varphi$  is an isomorphism.

We first prove that  $d = 1$ . Since  $G$  is non-elementary, there are two loxodromic elements  $g, h \in G$  without common fixed points. Then also  $\varphi(g)$  and  $\varphi(h)$  are loxodromic by (A2).

Let  $\bar{g} = \varphi(g)$  and  $\bar{h} = \varphi(h)$ . Then  $\text{mul } \bar{g} = (\text{mul } g)^d = \gamma^{2d}$  and  $\text{mul } \bar{h} = (\text{mul } h)^d = \chi^{2d}$  and hence if  $\bar{a}$ ,  $\bar{b}$ ,  $\bar{c}_{im}$ ,  $\bar{d}_{ik}$  and  $\bar{e}_{mk}$  are numbers as in Lemma B, we have that

$$\begin{aligned} \text{mul } \bar{g}^m \bar{h}^k &= |\bar{a}(\gamma^{dm} \chi^{dk} + \gamma^{-dm} \chi^{-dk}) + \bar{b}(\gamma^{dm} \chi^{-dk} + \gamma^{-dm} \chi^{dk})|^2 \\ &\quad + \bar{c}_{1m} \gamma^{2dm} + \bar{c}_{2m} \gamma^{-2dm} + \bar{d}_{1k} \chi^{2dk} + \bar{d}_{2k} \chi^{-2dk} + \bar{e}_{mk}. \end{aligned} \quad (\text{C1})$$

We now use the equality

$$\text{mul } \bar{g}^m \bar{h}^k = (\text{mul } g^m h^k)^d \quad (\text{C2})$$

together with (B3) and (C1) and let  $m, k$  tend to  $+\infty$  or to  $-\infty$ . Since  $g$  and  $h$  do not have common fixed points,  $a \neq 0 \neq b$  by Lemma B (a) and it follows that

$$|\bar{a}| = |a|^d \quad \text{and} \quad |\bar{b}| = |b|^d. \quad (\text{C3})$$

In particular, it follows that  $\bar{a} \neq 0 \neq \bar{b}$  and hence  $\bar{g}$  and  $\bar{h}$  do not have common fixed points by Lemma B (a). Thus  $H$  is also non-elementary and, if necessary, we can replace  $\varphi$  by  $\varphi^{-1}$  and thereby assume that

$$d \geq 1. \quad (\text{C4})$$

Next, substitute again (B3) and (C1) into (C2) and divide both sides of the resulting equation by  $|a|^d \gamma^{2dm} \chi^{2dk} = |\bar{a}| \gamma^{2dm} \chi^{2dk}$ . Keep  $k$  fixed and let  $m$  assume the values  $r_i$  of Lemma B (b) (see also the Remark following Lemma B) and let  $i \rightarrow \infty$ . We obtain

$$|1 + \bar{b} \chi^{-2kd} / \bar{a}|^2 = |1 + b \chi^{-2k} / a|^{2d} \quad (\text{C5})$$

which is valid for every  $k \in \mathbb{Z}$ .

Let

$$r = |b|/|a|, \quad \alpha = \arg b/a, \quad \bar{\alpha} = \arg \bar{b}/\bar{a},$$

so that  $r^d = |\bar{b}|/|\bar{a}|$ . Substituting this into (C5) and using elementary trigonometry, we obtain for all  $k \in \mathbb{Z}$

$$1 + 2r^d \chi^{-2kd} \cos \bar{\alpha} + r^{2d} \chi^{-4kd} = (1 + 2r \chi^{-2k} \cos \alpha + r^2 \chi^{-4k})^d.$$

We develop the right hand side into a power series for  $t = \chi^{-2k}$  and compare it to the left side. When  $\cos \alpha \neq 0$ , we obtain immediately a contradiction if  $d > 1$ .

We look at the geometric situation when  $\cos \alpha = 0$ , that is,  $\alpha = \pm \pi/2$ . Suppose that  $g$  fixes 0 and  $\infty$  and that  $h$  fixes 1 so that 0 and 1 are the repelling fixed points. By Lemma B (c),  $h$  fixes also the point  $-a/b$  and we know that  $\arg -a/b = -\alpha = \pm \pi/2$  and hence this point lies on the imaginary axis.

We state this in terms independent of normalization. Let  $S_1$  and  $S_2$  be the two circles through the fixed points of  $g$  and through one fixed point of  $h$ . Then  $S_1$  and  $S_2$  intersect orthogonally.

But this is absurd since  $g$  and  $h$  can be any two loxodromic elements of  $G$  without common fixed points. If we have chosen  $g, h \in G$  and  $S_1$  and  $S_2$  are orthogonal for these  $g$  and  $h$ , we can replace  $h$  by  $h^k g h^{-k}$ ,  $k$  big, in such a way that they are no more orthogonal. This contradiction concludes the proof that  $d = 1$ .

We now assume that  $d = 1$  and prove the remaining part of the theorem. As we have already done this in [T1, pp. 338–339] in more detail, we present only the main points.

We continue from the preceding situation with  $g, h \in G$  loxodromic without common fixed points. Since  $d = 1$ , (C3) becomes  $|\bar{a}| = |a|$ ,  $|\bar{b}| = |b|$  and in addition we know that  $a + b = 1$  and  $\bar{a} + \bar{b} = 1$ . Hence the two triangles with vertices 0, 1,  $a$  and 0, 1,  $\bar{a}$ , respectively, have the same sidelengths and consequently either

$$\bar{a} = a \quad \text{and} \quad \bar{b} = b, \quad \text{or} \quad \bar{a} = a^* \quad \text{and} \quad \bar{b} = b^*$$

where  $*$  is the complex conjugation. Conjugating by a Möbius transformation we obtain that the fixed points are

$$P_g = P_{\bar{g}} = \infty, \quad N_g = N_{\bar{g}} = 0, \quad N_h = N_{\bar{h}} = 1 \quad P_h = -a/b, \quad P_{\bar{h}} = -\bar{a}/\bar{b}.$$

Here  $P$  and  $N$  denote the attractive and repelling fixed points (see Section A) and we have also used (c) of Lemma B for  $P_h$  and  $P_{\bar{h}}$ . Hence at least we can conjugate the fixed points of  $g$  and  $h$  to the fixed points of  $\bar{g}$  and  $\bar{h}$ . In particular, if

$N_h = N_{\bar{h}} = \infty$ , then the two triangles with vertices  $P_g, N_g, P_h$  and  $P_{\bar{g}}, N_{\bar{g}}, P_{\bar{h}}$ , respectively, are similar.

Now,  $g$  and  $h$  can be any two loxodromic elements in  $G$  without common fixed points. Using this and the fact that every distinct point-pair of  $L(G) \times L(G)$  can be approximated arbitrarily closely by the fixed points of a loxodromic map in  $G$  (this follows from [T3, Theorem B1]), we can show that the map defined by

$$P_g \mapsto P_{\varphi(g)} \tag{C6}$$

( $g \in G$  loxodromic) is the restriction of a Möbius transformation  $f$ . It follows that  $fg|L(G) = \varphi(g)f|L(G)$  for  $g \in G$  from which fact the rest of Theorem C follows.

Finally, we remove the assumptions that  $\varphi$  was an isomorphism and  $G$  non-elementary. If  $\varphi$  is not an isomorphism but  $G$  is non-elementary, we pick as above loxodromic  $g, h \in G$  without common fixed points. Then for big enough  $k$ , the group  $G'$  generated by  $g^k$  and  $h^k$  is a Schottky group which is a free group such that every element of  $G' \setminus \{\text{id}\}$  is loxodromic (e.g. [T, p. 333] contains the simple argument). Then every  $\varphi(g'), g' \in G' \setminus \{\text{id}\}$  is loxodromic by (A2) and hence  $\varphi|G'$  is an isomorphism onto  $\varphi(G')$  and we can apply above reasoning with  $G$  replaced by  $G'$  and  $H$  by  $\varphi(G')$ . Since replacing  $g$  by  $g^k$  and  $h$  by  $h^k$  does not affect the attractive and repelling fixed points, the reasoning leading to (C6) is still valid.

If  $G$  is elementary, then  $H$  is non-elementary. Thus there are loxodromic  $g, h \in H$  without common fixed points. As above, for big enough  $k$ , the group  $H'$  generated by  $g^k$  and  $h^k$  is a Schottky group. Find  $g_0, h_0 \in G$  such that  $\varphi(g_0) = g^k$  and  $\varphi(h_0) = h^k$  and let  $G'$  be the group generated by them. Since  $H'$  is free,  $\varphi|G'$  is an isomorphism onto  $H'$  and we can apply the above reasoning to  $H', G'$  and  $\varphi^{-1}|H'$  and show that  $g_0$  and  $h_0$  are loxodromic and without common fixed points and hence  $G$  was in fact non-elementary, contrary to the assumption.

*Remark.* It is clear that there are non-trivial situations in which Theorem C is not true. For instance, let  $G$  be generated by  $g : x \mapsto 2x$  and  $H$  by  $h : x \mapsto 4x$  which are Möbius groups of  $\bar{R}^n$ . Then the isomorphism mapping  $g$  onto  $h$  is of constant stretch 2.

Another example is given by the group  $G$  whose elements are of the form  $x \mapsto \lambda x + a$  where  $\lambda > 0$  and  $a \in R^n$ . Let  $\alpha$  be an affine homeomorphism of  $R^n$  and let  $\varphi(g) = \alpha g \alpha^{-1}$ . Then  $\varphi$  is an isomorphism  $G \rightarrow G$  which preserves multipliers but is not a conjugation by a Möbius transformation if  $\alpha$  is not a similarity.

Thus it is necessary to assume something on the groups  $G$  and  $H$  although it might be, as is suggested by the last example, that if the groups contain two loxodromic elements with different fixed point sets (but which may have a common fixed point), then, if  $\varphi$  preserves multipliers, (A3) might be true for some affine  $h$

(i.e.  $h$  is a map such that  $h_0 h h_1|_{R^n}$  is affine for some Möbius transformations  $h_0$  and  $h_1$ ).

#### D. Bilipschitz maps and rigidity

In this last section we note a consequence of Theorem C. Roughly, it says that if  $\varphi$  is induced by  $f$ , i.e. (A3) is true, and  $f$  is a bilipschitz map, then  $\varphi$  preserves multipliers and hence is, or almost is, a conjugation by a Möbius transformation. Since we use the euclidean metric which is not a metric of whole  $\bar{R}^n$ , we transfer the situation to the  $n$ -sphere  $S^n \subset R^{n+1}$ . It turns out that the bilipschitz condition need not be satisfied everywhere, and taking account that in Theorem C we actually considered homomorphisms of constant stretch, we can generalize this as

**THEOREM D.** *Let  $\varphi : G \rightarrow H$  be a homomorphism of two Möbius groups of  $S^n$  such that  $G$  is non-elementary. Let  $A \subset S^n$  be a non-empty  $G$ -invariant set and let  $f : A \rightarrow S^n$  be a map inducing  $\varphi$ . Suppose that there are an open set  $U \subset S^n$  and numbers  $L \geq 1$  and  $d > 0$  such that  $U \cap L(G) \neq \emptyset$  and that*

$$|x - y|^d / L \leq |f(x) - f(y)| \leq L|x - y|^d \quad (\text{D1})$$

*for  $x, y \in U \cap A$ . Then  $d = 1$ ,  $\varphi$  preserves multipliers and, if in addition,  $L(G) \subset h(S^k)$  for no Möbius transformation  $h$  and no  $k < n$ ,  $\varphi$  is a conjugation by a Möbius transformation.*

*Proof.* Pick  $z \in L(G) \cap U$ . Thus there are  $g_i \in G$  and  $w \in S^n$  such that

$$g_i|_{S^n \setminus \{w\}} \rightarrow z$$

locally uniformly, as follows easily from the definition of the limit set (cf. (A5)) and the convergence property of Möbius groups (see [GM, Theorem 3.2]). This fact has two consequences. The first is that if  $\text{acc } A$  denotes the accumulation points of  $A$ , then

$$\text{acc } A \supset L(G), \quad (\text{D2})$$

(for (D2) we remark that  $A$  is in any case actually infinite by non-elementariness) and the second is that

$$\{g_i^{-1}(U)\} \text{ is a cover of } L(G) \setminus \{w\}. \quad (\text{D3})$$



It follows that if  $g \in G$  is loxodromic, then there is  $h \in G$  which is conjugate to  $g$  in  $G$  such that at least one of the fixed points of  $h$  is in  $U$ . Consequently, if we can prove that

$$\text{mul } \varphi(g) = (\text{mul } g)^d \quad (\text{D4})$$

for all loxodromic  $g \in G$  with one fixed point in  $U$ , then this is actually valid for all loxodromic  $g \in G$ .

So suppose that  $g \in G$  is loxodromic and fixes  $u \in U$ . We can assume that  $u$  is the attractive fixed point of  $g$ . Then  $u \in L(G)$  and hence, by (D2), there are distinct  $x, y \in U \cap A$  not fixed by  $g$ . Under these circumstances we have, as can be seen from (A1),

$$\text{mul } g = \lim_{k \rightarrow \infty} |g^k(x) - g^k(y)|^{-1/k}. \quad (\text{D5})$$

We observe that (D5) gives  $\text{mul } g$  for any Möbius transformation  $g$  as follows from the representations (A1) and (A4), provided that  $x$  and  $y$  are not fixed by  $g$ . By (D1) and  $G$ -compatibility,  $f(x)$  and  $f(y)$  are not fixed by  $\varphi(g)$ . Hence (D5) and (D1) imply that

$$\begin{aligned} (\text{mul } g)^d &= \lim_{k \rightarrow \infty} |fg^k(x) - fg^k(y)|^{-1/k} \\ &= \lim_{k \rightarrow \infty} |\varphi(g)^k f(x) - \varphi(g)^k f(y)|^{-1/k} \\ &= \text{mul } \varphi(g) \end{aligned}$$

and (D4) is valid for all loxodromic  $g \in G$ .

This is all that is needed for the validity of Theorem C if  $G$  is non-elementary (see the Remark after it). Theorem C implies the rest of the present theorem, for instance that  $\varphi$  preserves multipliers for all  $g \in G$ .

*Remarks.* 1. We needed the assumption that  $G$  is non-elementary in order to apply Theorem C but to obtain (D4) for loxodromic  $g$ , this assumption was not used (though we must assume that  $A$  contains at least three points if  $G$  is elementary). In fact, if  $g$  is parabolic such that  $g$  is conjugate to some  $h$  with a fixed point in  $U$  or if  $g$  is elliptic, then basically as above one obtains that (D4) is valid for  $g$ ; in the non-elementary case a parabolic  $g$  is always conjugate to such  $h$ . It is valid even if  $g$  is parabolic and not conjugate to such a map  $h$  but then a more complicated reasoning, given below, is necessary. Thus even if  $G$  is elementary  $\varphi$  is still of constant stretch  $d$ , provided that  $A$  contains at least 3 points.

Suppose that  $g$  is parabolic with the fixed point  $v$  and  $h(v) \in U$  for no  $h \in G$ . It follows that  $v$  must be the point  $w$  in (D3) and that  $v$  is fixed by every  $g \in G$ . We cannot have that  $\{v\} = L(G)$  (since then  $v \in U$ ) and hence there are at least two points in  $L(G)$ . Consequently, there is loxodromic  $h \in G$  [T3, proof of Theorem E]. Then  $h$  fixes  $v$  and we assume that  $v$  is the repelling fixed point. It follows from (A1) and (A4) (transform the situation to  $\bar{R}^n$  so that  $v = \infty$ ) that there are  $k_i > 0$  and  $n_i > 0$  such that if

$$g_i = h^{k_i} g^{n_i} h^{-k_i},$$

then  $g_i(x) \rightarrow x$  for all  $x \in S^n$ .

It follows that also  $\varphi(g_i)(x) \rightarrow x$  for all  $x \in f(U \cap A)$ . Since  $f(U \cap A)$  is infinite, it follows by the convergence property [GM, 3.2] that we can pass to a subsequence in such a way that  $\varphi(g_i) \rightarrow \bar{g}$  where  $g$  is a Möbius transformation such that  $\bar{g}|f(U \cap A) = \text{id}$ . hence  $\bar{g}$  is elliptic and  $\text{mul } \bar{g} = 1 = \lim_{i \rightarrow \infty} \text{mul } \varphi(g_i)$ . However,  $\varphi(g_i)$  is conjugate to  $\varphi(g)^{n_i}$ . Consequently,  $\text{mul } \varphi(g) = (\text{mul } \varphi(g_i))^{1/n_i} = 1$ .

2. Actually, we need not assume that (D1) is true for all  $x, y \in U \cap A$ , only that for each loxodromic  $g \in G$  there are distinct points  $u, v \in A$ , not both of them fixed by  $g$ , such that (D1) is valid for  $x = g^k(u)$  and  $y = g^k(v)$  when  $k > 0$  with some  $L \geq 1$  which may depend on  $g$  and some  $d > 0$  which does not.

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*Department of Mathematics*  
*University of Helsinki*  
*Hallituskatu 15*  
*SF-00100 Helsinki*  
*Finland*

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## Buchanzeigen

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MICHAŁ KAROŃSKI, JERZY JAWORSKI, ANDRZEJ RUCIŃSKI, **Random Graphs**, John Wiley & Sons, 1990, 368 pages, £40.00.

Preface – List of Participants – Bollobás B., Sharp concentration of measure phenomena in the theory of random graphs – Buckley F. and Z. Palka, Property preserving spanning trees in random graphs – Cooper C. and A. M. Frieze, Pancyclic random graphs – Frieze A. M. and L. Kučera, Parallel colouring of random graphs – Frieze A. M. and T. Łuczak, Hamiltonian cycles in a class of random graphs: one step further – Godehardt E., Connectivity of random graphs of small order and statistical testing – Janson S., T. Łuczak and A. Ruciński, An exponential bound for the probability of nonexistence of a specified subgraph of a random graph – Jaworski J., Random mapping with independent choices of images – Kemp R., Further results on leftist trees – Kordecki W., normal approximation and isolated vertices in random graphs – Kratochvíl M., J. Malý and J. Matoušek, on the existence of perfect codes in a random graph – Łuczak T., On the equivalence of two basic models of random graphs – Maehara H., On the intersection graph of random arcs on a circle – Matula D. and L. Kučera, An expose-and-merge algorithm and the chromatic number of a random graph – Mutařchiev L., Large components and cycles in a random mapping pattern – Nowicki K., Asymptotic distributions of graph statistics for colored graphs – Pittel B., W. A. Wořczynski and J. A. Mann, Random tree-type partitions as a model for acyclic polymerization: Gaussian behavior of the subcritical sol phase – Prömel H. J., Almost bipartite-making graphs – Ruciński A., Small subgraphs of random graphs – a survey – Spencer J., Undecidable probabilities – Szymański J., On the maximum degree and the height of a random recursive tree – Tomescu I., Almost all digraphs have a kernel – Vahidi-Asl M. Q. and J. C. Wierman, First-passage percolation on the Voronoi tessellation and Delaunay triangulation – Weber K., Random spread of information, random graph processes and random walks – Random graphs '87: Open problems.

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