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## An acyclic extension of the braid group

PETER GREENBERG and VLAD SERGIESCU

*Abstract.* We relate Artin's braid group  $B_\infty = \lim_{\rightarrow} B_n$  to a certain group  $F'$  of  $pl$ -homeomorphisms of the interval. Namely, there exists a short exact sequence  $1 \rightarrow B_\infty \rightarrow A \rightarrow F' \rightarrow 1$ , where  $H_k A = 0$ ,  $k \geq 1$ .

### 1. Introduction

Recent years have seen a growth of interest in the dynamical, combinatorial and homological aspects of groups of  $pl$ -homeomorphisms of the real line ([BS], [Br], [BrG], [Gh], [GS], [G1]).

In this paper, we link certain of these groups with the Artin braid groups in an algebraic construction which exploits the geometrical bases of the two.

Let  $F$  be the group of  $pl$ -homeomorphisms of  $[0, 1]$  whose derivative, which may be undefined at a finite subset of  $\mathbf{Z}[\frac{1}{2}]$ , is otherwise an integral power of 2, and let  $F'$  be the subgroup of elements of  $F$  which agree with the identity near 0 and 1. Let  $B_n$  be the braid group on  $n$  strings and  $B_\infty = \lim_{\rightarrow} B_n$  the usual infinite braid group. Recall that a group is acyclic provided its homology with trivial coefficients vanishes in positive dimensions.

Our main result is the following:

**THEOREM.** *There exists an exact sequence  $1 \rightarrow B_\infty \rightarrow A \rightarrow F' \rightarrow 1$ , with  $A$  an acyclic group.*

Our task in this paper is the construction of the group  $A$  and the proof of its acyclicity. We also indicate the homology of related groups, replacing either  $F'$  with a group acting on the circle or  $B_\infty$  with the group  $\Sigma_\infty$  of finitely supported permutations. The latter group is connected with the Fredholm permutations as studied by J. Wagoner and S. Priddy ([W], [P]).

The construction of  $A$  is quite geometrical. It exploits the idea of an action at infinity on a tree. The critical point is to force  $H_1(A; \mathbf{Z}) = 0$ . The role played here by the second derivative recalls the discretized Thurston cocycle introduced in [GS].

The initial evidence for the theorem is homological.

**PROPOSITION** ([G1], [GS], [S1]). *There are maps  $BF' \rightarrow \Omega S^3$ ,  $BB_\infty \rightarrow \Omega^2 S^3$  inducing isomorphisms in homology with integer coefficients.*

Thus, the path fibration  $\Omega^2 S^3 \rightarrow P\Omega S^3 \rightarrow \Omega S^3$  suggests the existence of the group  $A$  claimed by the theorem. However, as we will show in Section 6, given a fibration  $F \rightarrow E \rightarrow B$  and groups  $H$  and  $K$  with the homology of  $F$  and  $B$ , it is not generally possible to build an extension of groups  $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$  with the homology of that fibration.

It seems that  $A$  is a new kind of acyclic group. Well-known examples of such groups include Higman's finitely presented group (see [BDH] and Section 6) as well as various "large" groups: the group of compactly supported homeomorphisms of  $\mathbf{R}^n$  [M2], the group of all permutations of an infinite set or the group of continuous automorphisms of an infinite dimensional Hilbert space. The proof of acyclicity of the Higman group uses a Mayer-Vietoris argument, while for the large groups it requires an infinite repetition device due to Mather and Wagoner.

In contrast, in order to prove that  $A$  is acyclic we use a different approach involving a fairly delicate delooping argument.

We note that while a basic theorem of Kan and Thurston embeds any group in an acyclic one, our construction embeds the braid group in  $A$  as a *normal* subgroup.

This paper is organized as follows. In Section 2 we introduce a technique to build automorphisms of braid groups starting with the action of a group at infinity on a tree. Section 3 contains the definition of  $A$  and of some auxiliary monoids. In Section 4 we use the homological properties of  $B_\infty$  and  $F'$  and a delooping technique to prove the Theorem. Section 5 contains related results for other groups and Section 6 a relevant example.

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## 2. Trees and Braids

This section starts with some motivating remarks and an overall idea of our approach. We hope, thus, to make the material in the rest of the paper easier to follow.

As already stated in the Introduction, our main goal is to build an acyclic extension  $1 \rightarrow B_\infty \rightarrow A \rightarrow F' \rightarrow 1$ . The group  $B_\infty$  being centerless, it is well-known

that this comes down to a certain morphism from  $F'$  to the group of outer automorphisms  $\text{Out } B_\infty$ .

Unfortunately, there is no natural way to obtain this morphism using the standard description of  $B_\infty$ .

Instead, we note that  $F'$  acts naturally on the group  $\Sigma_\infty$  viewed as finitely supported permutations of the *dyadic* numbers in  $]0, 1[$ . Indeed,  $F'$  is a group of bijections of the dyadics. This suggests to look for some sort of braiding of the above action.

To avoid braiding a dense set, we first place the dyadic numbers as vertices of the binary tree.

A system of generators for the related braid group can be constructed from the edges of the tree. Note that  $F'$  does not act on the whole tree.

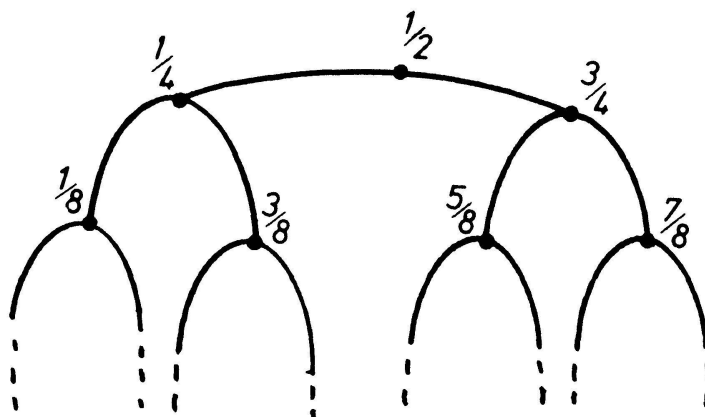


Fig. 2.0

Our basic observation is that nevertheless, each element of  $F'$  does act simplicially *outside* a finite subtree. Moreover, this action will *extend* to an automorphism of  $B_x$ , well defined up to inner automorphisms. Thus one gets a morphism from  $F'$  to  $\text{Out } B_x$ .

We mention that in fact, a slightly more involved construction is needed in order to get the extension  $A$  acyclic.

We now proceed to put these remarks in a proper context.

Recalling their relationship with configuration spaces, we define braid groups relative to a discrete set of points in the plane. When this is the set of vertices of a planar tree, the edges provide generators for the corresponding braid group.

We then introduce braid groups associated to cyclically oriented trees. Our setup is appropriate to show that an isomorphism “at infinity” of a tree becomes an outer automorphism of the associated braid group. This fact will be essential for our constructions in the next section.

2.1. DEFINITION. Let  $S$  be a discrete, closed subset of  $\mathbf{R}^2$ . For any open relatively compact, contractible set  $D \subset \mathbf{R}^2$ , let  $C_D$  be the space of injections of the finite set  $S \cap D$  into  $D$ , modulo the action of the group of permutations of  $S \cap D$ . Let  $B_D = \pi_1 C_D$ , and  $B_S = \lim_{\rightarrow} B_D$ , the limit taken over the directed system of open sets and inclusions. We call  $B_S$  the braid group of  $S$ .

2.2. Remarks. a) The isomorphism class of  $B_S$  depends only on the cardinality of  $S$ .

b) Consider an embedded arc  $c$  in  $\mathbf{R}^2$  which intersects  $S$  precisely at its endpoints.

Pick a contractible neighbourhood  $D$  such that  $S \cap D$  is the end points of  $c$ . Then  $B_D \cong \mathbf{Z}$ ; we let  $\underline{c}$  be a generator of  $B_D$  given by a counterclockwise exchange of the endpoints of  $c$ .

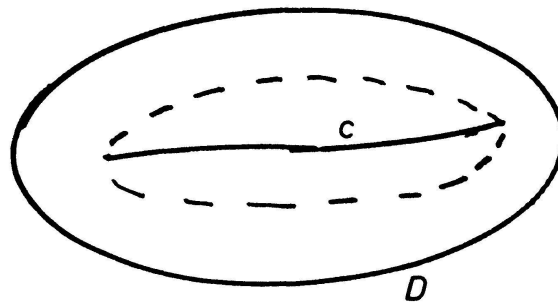


Fig. 2.2

The  $\underline{c}$  constitute a “coordinate-free” set of generators of  $B_S$ , and satisfy simple relations given by the following propositions, whose proof is left to the reader:

2.3. PROPOSITION (Triangle Rule). Let  $S$  be a discrete closed set of  $\mathbf{R}^2$ . Let  $a, b, c$  be the edges, in clockwise order, of a triangle embedded in  $\mathbf{R}^2$ . Suppose that  $a, b, c$  intersect  $S$  precisely at their end points, and the interior of the triangle contains no point of  $S$ . Then  $\underline{c} = \underline{a}^{-1}\underline{b}\underline{a} = \underline{b}\underline{a}\underline{b}^{-1}$

We recall some homological facts.

2.4. PROPOSITION. Let  $S$  be a discrete, closed subset of  $\mathbf{R}^2$ . Then  $H_1(B_S) \simeq \mathbf{Z}$ , and the map from  $B_S$  to  $H_1(B_S)$  takes all of the  $\underline{c}$  to the same generator.

It is classical ([CLM] III, App.) that  $H_1 B_S \simeq \mathbf{Z}$ , generated by some  $\underline{c}$ . The triangle rule shows that all  $\underline{c}$  lie in the same conjugacy class.  $\square$

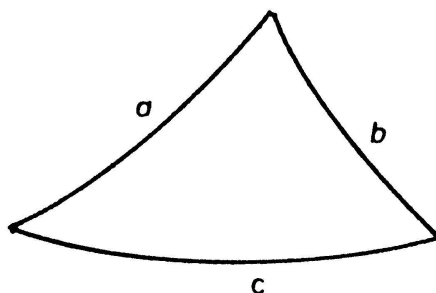


Fig. 2.3

2.5. PROPOSITION. For any integer  $m \geq 0$ , there exists a  $k > 0$  such that if  $S \subseteq S'$  are discrete, closed subsets of  $\mathbf{R}^2$  and  $\text{card } S > k$ , then the inclusion  $B_S \rightarrow B_{S'}$  induces an isomorphism in homology in degrees  $\leq m$ .

For  $S, S'$  finite, this is ([CLM], III, App]. The general result comes on taking limits. □

In our notation, the usual presentation for the braid group is as  $B_N$ , where  $N = \{(n, 0) \mid n \in \mathbf{N}\}$ ;  $B_N$  has generators  $\underline{e}_i, e_i = [i, i + 1]$  and relations  $\underline{e}_i \underline{e}_{i+1} \underline{e}_i = \underline{e}_{i+1} \underline{e}_i \underline{e}_{i+1}, [\underline{e}_i, \underline{e}_j] = 1$  if  $|i - j| \geq 2$ . One can show that these relations follow from the triangle rule. In general, the edges of any tree in the plane with the set  $S$  as vertices provide generators for  $B_S$ ; the  $\underline{c}$ , for arcs  $c$  which are not edges in the tree, may be easily calculated in terms of the edges using the triangle rule. We proceed to develop this idea.

Let  $T = (V, E)$  be a countable, locally finite tree with vertex set  $V$  and edges  $E$ . We always identify  $T$  with its geometric realization.

2.6. DEFINITION. An orientation of  $T$  is an equivalence class of collections  $\Phi = \{\varphi_v\}_{v \in V}$  of bijections  $\partial_v : E(v) \rightarrow \{1, 2, \dots, \text{card } E(v)\}$ , where we set  $\Phi \sim \Phi'$  if for all  $v \in V$ ,  $\varphi_v$  and  $\varphi'_v$  differ by a cyclic permutation of  $\{1, 2, \dots, \text{card } E(v)\}$ .

An embedding  $f : T \rightarrow \mathbf{R}^2$  is a homeomorphism onto its image, such that  $f(V)$  is a discrete, closed subset of  $\mathbf{R}^2$ . An embedding of a tree determines an orientation of the tree, the clockwise ordering of the edges  $E(v), v \in V$ . The converse is also true:

2.7. PROPOSITION. If  $T = (V, E)$  is a tree, there is a one-to-one correspondence between orientations and isotopy classes of embeddings to  $\mathbf{R}^2$ .

Namely, given an orientation, pick  $v \in V$  and define  $f(v) = (0, 0) \in \mathbf{R}^2$ . The orientation describes how to embed  $E(v)$ , and there after by induction on subtrees ( $T$  being connected) the entire tree, up to an ambient isotopy. □

Let  $T$  be an oriented tree. Let  $f : T \rightarrow \mathbf{R}^2$  be an orientation preserving embedding. The group  $B_{f(V)}$  has  $\{f(\underline{e}), e \in E\}$  as a set of generators. The relations satisfied by

the  $f(e)$  in  $B_{f(V)}$  depend only on the isotopy class of  $f$ , that is, by 2.7., only on the orientation of  $T$ . That is, we can make the following definition.

**2.8. DEFINITION.** Let  $T$  be an oriented tree. Let  $X$  be the free group on the set  $E$  of edges of  $T$ . An orientation preserving embedding  $f: T \rightarrow \mathbf{R}^2$  induces a surjection  $X \rightarrow B_{f(V)}$ . Define  $B_T$ , the braid group of  $T$ , to be the quotient of  $X$  by the kernel of  $X \rightarrow B_{f(V)}$ , for any orientation preserving  $f$ . We write  $\underline{f}: B_T \rightarrow B_{f(V)}$  for the induced isomorphism.

In [Ser] the relations amongst the generators  $e \in E$  which define the group  $B_T$  are determined. They are:  $[\underline{e}, \underline{e}'] = 1$ , if  $e$  and  $e'$  share no vertex,  $\underline{e}\underline{e}'\underline{e} = \underline{e}'\underline{e}\underline{e}'$  if  $e$  and  $e'$  share a vertex, and  $\underline{e}\underline{e}'\underline{e}''\underline{e} = \underline{e}'\underline{e}''\underline{e}\underline{e}' = \underline{e}''\underline{e}\underline{e}'\underline{e}''$  if  $e, e', e''$  share a vertex, and are cyclically oriented.

If  $f: T \rightarrow \mathbf{R}^2, f': T' \rightarrow \mathbf{R}^2$  are orientation preserving embeddings of oriented trees, such that  $f(V) = f'(V')$ , then  $\underline{f}^{-1} \circ \underline{f}'$  defines an isomorphism from  $B_{T'}$  to  $B_T$ . Our next goal is to define *outer* automorphisms of  $B_T$  via embeddings of  $T$  in  $T$  defined up to a finite subtree. The approach is suggested by a similar construction for permutations due to Wagoner ([W], [P]).

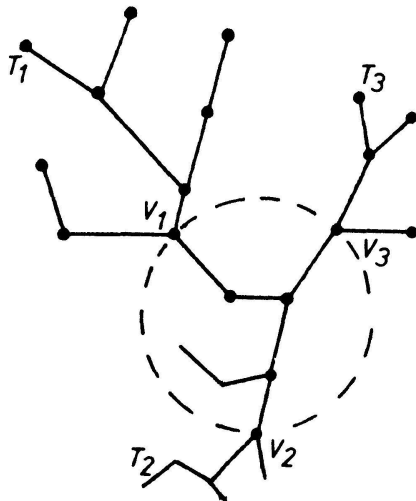
Recall that a forest is a disjoint union of trees. We keep the notation  $T = (V, E)$  for a forest. We discuss some aspects of the geometry of forests.

**2.9. DEFINITION.** Let  $T' = (V', E')$  be a subforest of a forest  $T = (V, E)$ . The complement  $T'^c$  of  $T'$  is the subforest of  $T$  with edges  $\{e \in E \mid e \notin E'\}$ . Whenever  $T'^c$  is finite,  $T'$  is called a cofinite subforest. If  $T'$  is connected and  $T' \cap T'^c$  is a single vertex  $v$ , then  $T'$  is a rooted subtree whose root is  $v$ .

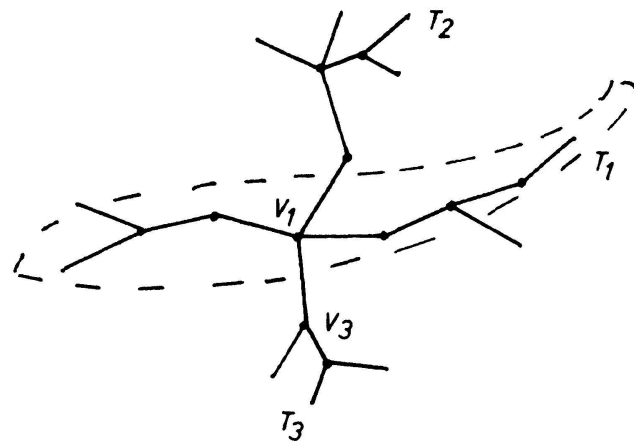
We shall need one more technical notion. Let  $T_1, \dots, T_n$  be rooted infinite disjoint subtrees of an oriented tree  $T$ . Let  $v_i$  be the root of  $T_i, i = 1, \dots, n$ .

**2.10. DEFINITION.** We say that  $T_1 \cup \dots \cup T_n$  has uncut complement if for some embedding  $f: T \rightarrow \mathbf{R}^2$  which preserves orientation, there exists an embedding of the closed unit disc  $g: D^2 \rightarrow \mathbf{R}^2$  such that  $g(D^2) \cap T_i = \{v_i\}$ . The cyclic counter clockwise order in which the roots  $v_i$  occur on  $g(S^1) = g(\partial D^2)$  is called the cyclic order of the  $T_i$  (we always assume this is  $T_1, \dots, T_n$ ).

Note that a forest can have a connected complement without having an uncut complement.



$T_1, T_2, T_3$  in cyclic order



$T_1, T_2, T_3$  do not have uncut complement

We state a technical lemma:

2.11. LEMMA. (a) Any finite subforest  $T'$  of an oriented infinite tree  $T$  may be enlarged to a finite subtree  $T'' \supseteq T'$  so that  $T''^c$  has uncut complement.

(b) Let  $T'$  be a finite subtree of an oriented tree  $T$  so that  $T'^c = T_1 \cup \dots \cup T_n$  has uncut complement. Let  $v_i$  be the root of  $T_i, i = 1, \dots, n$ . Suppose that  $f = \amalg f_i : T'^c \rightarrow \mathbf{R}^2$  is an embedding, so that each  $f_i : T_i \rightarrow \mathbf{R}^2$  is orientation preserving. Further, suppose that there is an embedding  $g : D^2 \rightarrow \mathbf{R}^2$  of the disk, with  $g(D^2) \cap f_i(T_i) = \{f_i(v_i)\}$  and such that the  $f_i(v_i)$  occur in  $g(S^1)$  in the order given by the cyclic ordering of the  $T_i$ .

Then  $f$  extends to an orientation preserving embedding  $f : T \rightarrow \mathbf{R}^2$ .

We leave part (a) to the reader, and pass to part (b). Begin with an embedding  $h : T \rightarrow \mathbf{R}^2$ , which we can assume takes  $T'$  to  $g(D^2)$ , and such that  $h(v_i) \in g(S^1)$  for each  $i$ . By definition of cyclic order, the  $h(v_i)$  occurs on  $g(S^1)$  in the same cyclic order as the  $f_i(v_i)$ . Thus we can isotope  $h$ , keeping  $h(T') \subseteq g(D^2)$ , such that  $h(v_i) = f_i(v_i)$ . Lastly, we isotope  $h$  outside  $g(D^2)$  until  $h = f_i$  on each  $T_i$ .  $\square$

One has the following key notion.

2.12. DEFINITION. Let  $T, T'$  be oriented trees. An oriented Fredholm map  $\varphi$  from  $T$  to  $T'$  is an isomorphism  $\varphi = \{\varphi_i : T_i \rightarrow T'_i, i = 1, \dots, n\}$  from a cofinite subforest  $T_1 \cup \dots \cup T_n$  of  $T$  with uncut complement to a cofinite subforest  $T'_1 \cup \dots \cup T'_n$  of  $T'$  with uncut complement. Each  $\varphi_i$  must be orientation preserving, and the cyclic orders of the  $T_i$  and  $T'_i$  must agree. The index of  $\varphi$  is defined as  $\text{ind}(\varphi) = \text{card}\{v \in V, v \notin \cup V_i\} - \text{card}\{v \in V', v \notin \cup V'_i\}$ .

The set of oriented Fredholm maps from  $T$  to  $T'$  is denoted  $\text{Fred}^+(T, T')$ . We put an equivalence relation on  $\text{Fred}^+(T, T')$  by setting  $\varphi \sim \varphi'$  if  $\varphi$  and  $\varphi'$  agree on

some cofinite subforest. The equivalence classes are called *germs* and form a set  $\text{Germ}^+(T, T')$ . For a single connected oriented tree  $T$ ,  $\text{Germ}^+(T, T)$  is a group and  $\text{ind} : \text{Germ}^+(T, T) \rightarrow \mathbf{Z}$  is a homomorphism.

Finally, we show how Fredholm maps produce automorphisms of braid groups.

**2.13. PROPOSITION.** *Let  $T = (V, E)$  and  $T' = (V', E')$  be infinite oriented trees, and let  $\varphi \in \text{Germ}^+(T, T')$  with index 0. Then there exists an isomorphism  $\Phi : B_T \rightarrow B_{T'}$ , so that  $\Phi(\underline{e}) = \underline{\varphi(e)}$  for all but a finite number of edges  $e \in E$ . Further,  $\Phi$  is well defined up to composition with an inner automorphism of  $B_T$ .*

First we show that  $\Phi$  exists. Pick  $\Psi \in \text{Fred}^+(T, T')$ ,  $\Psi = \{\Psi_i : T_i \rightarrow T'_i, i = 1, \dots, n\}$  in the equivalence class  $\varphi$ . Let  $f : T \rightarrow \mathbf{R}^2$  be an orientation preserving embedding. Define  $f'_i : T'_i \rightarrow \mathbf{R}^2$  by  $f'_i = f \circ \Phi_i^{-1}$ . Because  $\Psi$  preserves the cyclic order of the  $T_i$ , by 2.11 (b) we can extend  $f'_i$  to an embedding  $f' : T' \rightarrow \mathbf{R}^2$ . Since  $\text{ind } \varphi = 0$ , we can choose  $f'$  so that  $f'(V') = f(V)$ . Then define  $\Phi = \underline{f'}^{-1} \underline{f}$ . Clearly  $\Phi(\underline{e}) = \underline{\varphi(e)}$  for any  $e \in \cup E_i$ .

Suppose that  $\Psi : B_T \rightarrow B_{T'}$ , is an isomorphism and that  $\Psi(\underline{e}) = \underline{\varphi(e)}$  for all but a finite number of edges  $e \in E$ . Then  $\Psi^{-1}\Phi : B_T \rightarrow B_T$  is the identity on  $\underline{e}$  for all but a finite number of edges  $e \in E$ . Thus there exists some finite subtree  $T'' = (V'', E'')$  of  $T$  such that  $\Psi^{-1}\Phi$  induces an automorphism  $B_{T''}$  and  $\Psi^{-1}\Phi^{-1}(\underline{e}) = \underline{e}$ ,  $e \notin E''$ . By this latter condition and 2.4.,  $\Psi^{-1}\Phi$  induces the identity on  $H_1$  of  $B_{T''}$ . The theorem of Dyer and Grossman ([DG], 4) thus implies that  $\Psi^{-1}\Phi$ , restricted to  $B_{T''}$ , is an inner automorphism conjugation of some  $g \in B_{T''}$ . Enlarging  $T''$  if necessary, it follows that after possibly multiplying with an element in the center of  $B_{T''}$ ,  $g$  commutes with any  $\underline{e}$ ,  $e \notin E''$ . Thus  $\Psi^{-1}\Phi : B_T \rightarrow B_T$  is conjugation by  $g$ , q.e.d.  $\square$

A mild extension of 2.13 is the following.

**2.14. PROPOSITION.** *Let  $T, T'$  be infinite, oriented trees, and  $\varphi \in \text{Germ}^+(T, T')$  with index 0. Let  $\tau, \tau'$  be finite subforests of  $T, T'$  respectively, and  $\theta : \tau \rightarrow \tau'$  an orientation preserving isomorphism of trees. Then there exists an isomorphism  $\Phi : B_T \rightarrow B_{T'}$  such that  $\Phi(\underline{e}) = \underline{\varphi(e)}$  for all but a finite number of  $e \in T$ , and such that  $\Phi(\underline{e}) = \underline{\theta(e)}$  for all  $e \in \tau$ .*

First, pick  $\Psi$  as in 2.13, but so that  $\tau \cap T_i = \emptyset$ . One can then pick  $f'$ , as in the proof of 2.13, so that  $f' |_{\tau'} = f \circ \theta^{-1}$ .  $\square$

**2.15. COROLLARY.** *Let  $T$  be an infinite, oriented tree, and  $\theta : \tau \rightarrow \tau'$  an orientation preserving isomorphism between finite subforests of  $T$ . Then there exists an inner automorphism  $\Phi$  of  $B_T$  so that  $\Phi(\underline{e}) = \underline{\theta(e)}$  for all  $e \in \tau$ .*

Take  $\varphi$  to be the identity in 2.14.  $\square$

2.16. COROLLARY. *Let  $T$  be an infinite, oriented tree,  $\tau$  a finite subforest, and  $\varphi \in \text{Germ}^+(T, T)$  with index 0. Then there exists an automorphism  $\Phi$  of  $B_T$  so that  $\Phi(\underline{e}) = \underline{e}$ ,  $e \in \tau$  and  $\Phi(\underline{e}) = \underline{\varphi(e)}$  for all but a finite number of edges  $e$  of  $T$ .*

Take  $\theta$  to be the identity in 2.14. □

### 3. Constructions

In this section we perform the main construction of the paper. We will define the group  $A$ , to be shown acyclic in Section 4, and a related group  $A_G$ . Also, monoids used in the proof of acyclicity of  $A$  are constructed. We begin by defining some oriented trees.

3.1. DEFINITION. The trees  $T_N$ . For each integer  $N \geq 1$ , we construct an oriented tree  $T_N = (V_N, E_N)$ . The set of vertices of  $T_N$  is

$$V_N = \left\{ v_n^d; n \geq 0, d \in \mathbf{Z} \left[ \frac{1}{2} \right] \cap (0, N) \right\}$$

and the set of edges of  $T_N$  is  $E_N = C_N \cup \bigcup_d F_N^d$ , the latter union over  $d \in \mathbf{Z}[\frac{1}{2}] \cap (0, N)$ , where

$$F_N^d = \{e_n^d; n \geq 0\} \quad \text{and} \quad \partial e_n^d = \{v_n^d, v_{n+1}^d\}$$

$$C_N = \left\{ e_l^d, e_r^d, d \in \mathbf{Z} \left[ \frac{1}{2} \right] \cap (0, N) \right\}$$

and

$$\partial e_l^d = \{v_0^d, v_0^{l(d)}\}, \quad \partial e_r^d = \{v_0^d, v_0^{r(d)}\}$$

where if  $d = k/2^m$ ,  $k$  odd and  $m \geq 1$ , or  $m = 0$  then  $l(d) = (2k - 1)/2^{m+1}$ ,  $r(d) = (2k + 1)/2^{m+1}$ .

One should think of  $F_N^d$  as a fiber over  $d$ , and the elements of  $C_N$  as connecting the fibers. The orientation of  $T_N$  is that induced by the following embedding into  $\mathbf{R}_+^2 = \{(x, y), y > 0\}$ :

For example, the cyclic ordering of the edges  $E(v_0^{3/2})$  is as follows:  $e_r^1, e_l^{3/2}, e_0^{3/2}, e_r^{3/2}, e_l^2$ .

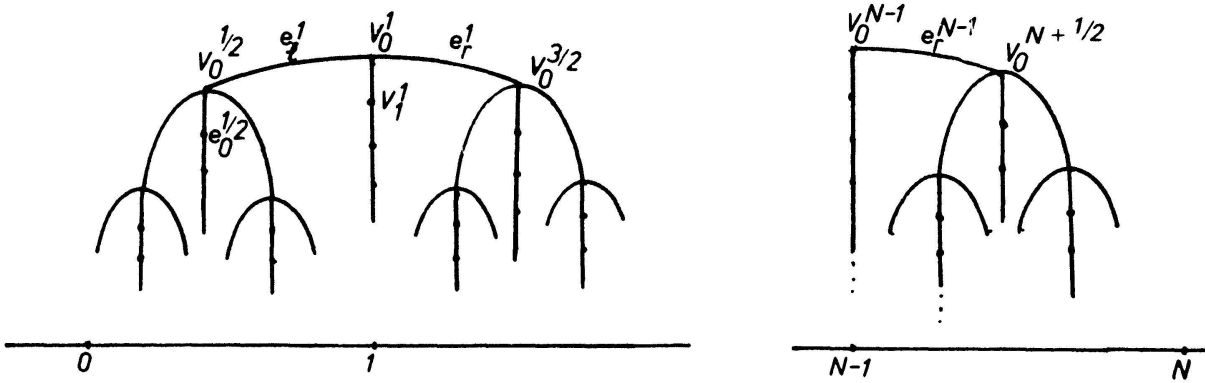


Fig. 3.2

3.3. DEFINITION.  $T_G$  is the oriented subtree of  $T_2$  with edges

$$E_G = \bigcup_{d \leq 1} F_2^d \cup \{e_l^d; d \geq 1\} \cup \{e_r^d; d > 1\}.$$

We write  $i : T_N \rightarrow T_{N+k}$  for the obvious embedding of  $T_N$  as a subtree of  $T_{N+k}$ . Other useful embeddings are the translation maps  $\tau : T_N \rightarrow T_{N+k}$  defined by  $\tau(v_n^d) = v_n^{d+k}$ ,  $\tau(e_*^d) = e_*^{d+k}$ ,  $(* = r, l, n)$ .

3.4. DEFINITION. We write  $B_{(N)}$  for the braid group  $B_{T_N}$  of the tree  $T_N$ , and  $B_G$  for the braid group of  $T_G$ .

3.5. LEMMA. The map  $H_*(i) : H_*(B_{(N)}) \rightarrow H_*(B_{(N+k)})$  is an isomorphism for all  $N, k$  and  $H_*(i) = H_*(\tau)$ .

The first assertion is a consequence of 2.5. We show that  $H_*(i) = H_*(\tau)$ . Let  $x \in H_*(B_{(N)})$ . Then, by definition of  $B_{(N)}$ ,  $x$  comes from  $B_T$ , for some finite subtree  $T \subseteq T_N$ . The trees  $i(T)$  and  $\tau(T)$  are isomorphic subtrees of  $T_{N+k}$ . Applying 2.15, there is an inner automorphism of  $B_{(N+k)}$  taking  $H_*(i)(x)$  to  $H_*(\tau)(x)$ .  $\square$

To define the groups  $A$  and  $A_G$ , and certain monoids, we make some remarks on the geometry of the trees  $T_N$  and  $T_G$ .

3.6. DEFINITION. A dyadic interval is an open interval of the form  $I = (k/2^n, (k+1)/2^n)$ ,  $n \geq k \in \mathbf{Z}$ . If  $I$  and  $J$  are dyadic intervals,  $\gamma_{IJ}$  denotes the unique element of the dyadic affine group such that  $\gamma_{IJ}(J) = I$ . If  $I$  is a dyadic interval and  $N \geq 1$  so that  $I \subseteq (0, N)$ , let  $T_I$  be the subtree of  $T_N$  with vertices  $V_I = \{v_n^d; d \in I\}$ ,  $E_I = \{e \in E_N, \partial e \subset V_I\}$ .

3.7. LEMMA. *Let  $I$  and  $J$  be dyadic intervals. Then  $G_{IJ}(v_n^d) = v_n^{\gamma_{IJ}(d)}$ ,  $G_{IJ}(e_*^d) = e_*^{\gamma_{IJ}(d)}$  ( $* = n, r, l$ ) defines an isomorphism  $G_{IJ} : T_J \rightarrow T_I$  of trees.*

Routine verification □

### 3.8. Construction of $A$ .

We now define, for any piecewise affine dyadic homeomorphism  $g : [0, n] \rightarrow [0, k]$ , an element  $\varphi_g$  of  $\text{Germ}^+(T_n, T_k)$ . Let  $M$  be an integer large enough so that for every dyadic interval of the form  $J = (k/2^M, (k+1)/2^M)$ ,  $g|_J$  is affine, and  $g(J)$  is a dyadic interval.

3.9. DEFINITION. Let  $d \in (0, n)$ . Define  $g''(d) = \log_2 g'(d) - \log_2 g^l(d)$ , where  $g^r(d) = \lim_{\varepsilon \searrow 0} g'(d + \varepsilon)$ ,  $g^l(d) = \lim_{\varepsilon \searrow 0} g'(d - \varepsilon)$ .

Let  $s_g = \max_{d \in (0, n)} |g''(d)|$ . Define a cofinite subtree of  $T_n$  to be the union of the  $T_J$ ,  $J = (k/2^M, (k+1)/2^M)$ , with subtrees  $F^d$  of the fibers  $F^d$ ,  $d = k/2^M$ ;  $F^d$  has edges  $\{e_n^d, n \geq s_g + 1\}$ . A glance at 3.2 shows that this cofinite subtree has uncut complement.

We define a representative  $\tilde{\varphi}_g \in \text{Fred}^+(T_n, T_k)$  of  $\varphi_g \in \text{Germ}^+(T_n, T_k)$  as follows:  $\tilde{\varphi}_g|_{T_J} = G_{g(J)J}$ , for any  $J = (k/2^M, (k+1)/2^M)$ . For each  $d = k/2^M$ , define  $\tilde{\varphi}_g(e_n^d) = e_{n+s_g}^{g^r(d)}$ , so that  $\tilde{\varphi}_g(F^d)$  is a subtree of  $F^{g^r(d)}$ . It is clear that the image of  $\tilde{\varphi}_g$  is a cofinite subtree with uncut complement, and that  $\tilde{\varphi}_g$  preserves the cyclic order of the components of the subtree.

3.10. PROPOSITION. *If  $g : [0, n] \rightarrow [0, k]$ ,  $h : [0, k] \rightarrow [0, m]$  are piecewise affine dyadic homeomorphisms, then  $\varphi_{hg} = \varphi_h \varphi_g$ .*

This follows in a straightforward way from the following two facts. First, if  $I, J, K$  are dyadic intervals, then  $G_{IJ}G_{JK} = G_{IK}$ . Second, the derivative defined in 3.9 satisfies the chain rule:  $h''(g(d)) + g''(d) = (hg)''(d)$ . □

3.11. PROPOSITION. *Let  $g : [0, n] \rightarrow [0, k]$  be a piecewise affine dyadic homeomorphism, such that  $g^r(0) = g^l(n) = 1$ . Then  $\text{ind } \varphi_g = 0$ .*

Since  $\tilde{\varphi}_g$  takes fibers to fibers,  $\text{ind } \varphi_g = \sum_{d \in (0, n)} g''(d)$ . Since  $g^r(0) = g^l(n) = 1$ , by the ‘‘fundamental theorem of calculus’’  $\sum_{d \in (0, n)} g''(d) = 0$ , so  $\text{ind } \varphi_g = 0$ . □

3.12. DEFINITION.  $A$  is the group of automorphisms  $h$  of  $B_{(1)}$ , such that there exists a  $g \in F'$  such that  $h(\underline{e}) = \underline{\varphi_g(e)}$  for all but a finite number of edges  $e \in E_1$ . We say that  $h$  lies over  $g$ , and define  $\rho : A \rightarrow F'$  by  $\rho(h) = g$ .

3.13. THEOREM. *There is an exact sequence  $1 \rightarrow B_{(1)} \rightarrow A \xrightarrow{\rho} F' \rightarrow 1$ .*

Definition 3.12 gives  $\rho : A \rightarrow F'$ , which is a homomorphism by 3.10, and surjective by 3.11. The kernel of  $\rho$  contains all inner automorphisms of  $B_{(1)}$ ; by definition,  $\ker(\rho)$  is the set of all automorphisms of  $B_{(1)}$  which fix  $\underline{e}$  for almost all  $e \in E_1$ . By the argument of 2.13,  $\ker(\rho)$  consists of exactly the inner automorphisms of  $B_{(1)}$ . But  $B_{(1)}$  has trivial center, and can be identified with its group of inner automorphisms. Hence  $\ker(\rho) = B_{(1)}$ .  $\square$

3.14 PROPOSITION. *The action of  $F'$  on  $H_* B_{(1)}$ , coming from the exact sequence  $1 \rightarrow B_{(1)} \rightarrow A \xrightarrow{\rho} F' \rightarrow 1$  is trivial.*

Let  $g \in F'$ ,  $x \in H_* B_{(1)}$ . Then  $x$  is the image in  $H_* B_{(1)}$  of an  $\bar{x} \in H_* B_T$ , for  $T$  a finite subtree of  $T_1$ . By corollary 2.16, we can pick an  $h \in A$  over  $g$  such that  $h$  fixes  $T$ . Thus  $g$  fixes  $x$ .  $\square$

3.15. *Construction of  $A_G$*

We briefly describe the construction of the extension  $B_G \rightarrow A_G \rightarrow G$ . Here,  $G$  is the group of orientation preserving piecewise affine dyadic homeomorphisms of  $S^1$ , thought of as  $[0, 1]$  with 0 and 1 identified, see [GS]. As in 3.8, to every  $g \in G$  we associate a  $\varphi_g \in \text{Germ}^+(T_G, T_G)$  with  $\text{ind } \varphi_g = 0$ . Then  $A_G$  is defined as the group of automorphisms  $h$  of the group  $B_G$ , such that for some  $g \in G$ ,  $h(\underline{e}) = \varphi_g(\underline{e})$  for all but a finite number of edges  $e \in T_G$ . As in 3.13 and 3.14, we obtain an exact sequence  $B_G \rightarrow A_G \rightarrow G$ , and  $G$  acts trivially on the homology of  $B_G$ . Note that we have a commutative diagram, whose vertical arrows are inclusions:

$$\begin{array}{ccccc}
 B_{(1)} & \rightarrow & A & \rightarrow & F' \\
 \downarrow & & \downarrow & & \downarrow \\
 B_G & \rightarrow & A_G & \rightarrow & G
 \end{array} \tag{3.16}$$

The proof of acyclicity of  $A$  requires the construction of strictly associative topological monoids and continuous homomorphisms  $M_B \rightarrow M_A \xrightarrow{\rho} M_F$ . The construction of  $M_F$  is due, in essence, to Quillen ([Q], §.8).

3.17 *Construction of  $M_F$*

$M_F$  is the (thin) geometrical realization  $B\mathcal{C}_F$  of a category  $\mathcal{C}_F$ . The objects of  $\mathcal{C}_F$  are  $\underline{0}, \underline{1}, \underline{2}, \dots$ . The set of morphisms  $\mathcal{C}_F(\underline{n}, \underline{k})$  is null if  $n = 0$  unless  $k = 0$ , and

$\mathcal{C}_F(\underline{0}, \underline{0}) = \{id\}$ . For  $n, k \geq 1$ ,  $\mathcal{C}_F(n, k)$  is the set of piecewise dyadic affine homeomorphisms  $g : [0, n] \rightarrow [0, k]$  such that  $g'(0) = g'(n) = 1$ . Composition in  $\mathcal{C}_F$  is via composition of homeomorphisms. Note that  $M_F = B\mathcal{C}_F$  is the union of a point  $\underline{0}$  with a  $K(F', 1)$ , and the inclusion  $F' \xrightarrow{=} \mathcal{C}_F(\underline{1}, \underline{1})$  defines a homotopy equivalence  $* \amalg BF' \rightarrow M_F$ .

The product  $\mu_F = M_F \times M_F \rightarrow M_F$  is defined via a functor, also denoted  $\mu_F : \mathcal{C}_F \times \mathcal{C}_F \rightarrow \mathcal{C}_F$ . On objects,  $\mu_F(\underline{k}, \underline{n}) = \underline{k + n}$ . If  $g \in \mathcal{C}_F(\underline{k}, \underline{n})$  and  $g' \in \mathcal{C}_F(\underline{k}', \underline{n}')$ , define  $\mu_F(g, g') \in \mathcal{C}_F(\underline{k + k'}, \underline{n + n'})$  by

$$\mu_F(g, g')(x) = \begin{cases} g(x) & x \leq k \\ g'(x - k') + n & x \geq k \end{cases}$$

Composing with the canonical homeomorphism  $B\mathcal{C}_F \times B\mathcal{C}_F \rightarrow B(\mathcal{C}_F \times \mathcal{C}_F)$ , we obtain  $\mu_F : M_F \times M_F \rightarrow M_F$ , one checks that  $\mu_F$  is strictly associative with unit  $* = \underline{0}$ .

### 3.18. Construction of $M_B$

$M_B$  is the geometrical realization  $B\mathcal{C}_B$  of a category  $\mathcal{C}_B$ . The objects of  $\mathcal{C}_B$  are  $\underline{0}, \underline{1}, \dots$ . The set of morphisms  $\mathcal{C}_B(\underline{n}, \underline{k})$  is empty unless  $n = k$ , and  $\mathcal{C}_B(\underline{0}, \underline{0}) = \{id\}$ . For  $n \geq 1$ ,  $\mathcal{C}_B(\underline{n}, \underline{n})$  is the set of elements of  $B_{(n)}$ , and composition in  $\mathcal{C}_B$  is a composition in the  $B_{(n)}$ . Thus  $M_B = B\mathcal{C}_B = \underline{0} \amalg_{k \geq 1} BB_{(k)}$ .

The product  $\mu_B : M_B \times M_B \rightarrow M_B$  is defined via a function  $\mu_B : \mathcal{C}_B \times \mathcal{C}_B \rightarrow \mathcal{C}_B$ . On objects,  $\mu_B(\underline{n}, \underline{k}) = \underline{n + k}$ . If  $g \in \mathcal{C}_B(\underline{n}, \underline{n})$  and  $h \in \mathcal{C}_B(\underline{k}, \underline{k})$  then  $\mu_B(g, h) \in \mathcal{C}_B(\underline{n + k}, \underline{n + k})$  is defined by

$$\mu_B(g, h) = i(g)\tau(h)$$

where  $i : T_n \rightarrow T_{n+k}$ ,  $\tau : T_k \rightarrow T_{n+k}$  are as defined below 3.3. Again, one checks that  $\mu_B$  is strictly associated, with unit  $\underline{0}$ .

### 3.19. Construction of $M_A$

$M_A$  is the geometrical realization  $B\mathcal{C}_A$  of a discrete category  $\mathcal{C}_A$ . The objects of  $\mathcal{C}_A$  are  $\underline{0}, \underline{1}, \dots$ . The set of morphisms  $\mathcal{C}_A(\underline{k}, \underline{n})$  is the set of isomorphisms  $h : B_{(k)} \rightarrow B_{(n)}$ , such that there exists some  $g \in \mathcal{C}_F(\underline{k}, \underline{n})$ , and  $h(\underline{e}) = \overline{\varphi_g(e)}$  for all but a finite number of edges  $e \in E_k$  (recall 3.8).  $\mathcal{C}_A(\underline{0}, \underline{k})$  is void if  $k \neq 0$ , and  $\mathcal{C}_A(\underline{0}, \underline{0}) = \{id\}$ . Composition in  $\mathcal{C}_A$  is via composition of isomorphisms. The

isomorphism  $A \xrightarrow{=} \mathcal{C}_A(\underline{1}, \underline{1})$  induces a homotopy equivalence  $* \amalg BA \rightarrow M_A$ ; we send  $*$  to  $\underline{0}$ .

As before,  $\mu_A : M_A \times M_A \rightarrow M_A$  is defined via a functor  $\mu_A : \mathcal{C}_A \times \mathcal{C}_A \rightarrow \mathcal{C}_A$ . On objects,  $\mu_A(\underline{k}, \underline{n}) = \underline{k+n}$ . If  $h \in \mathcal{C}_A(\underline{n}, \underline{k}), h' \in \mathcal{C}_A(\underline{n}', \underline{k}')$  lie over  $g \in \mathcal{C}_F(\underline{n}, \underline{k}), g' \in \mathcal{C}_F(\underline{n}', \underline{k}')$  respectively, then  $\mu_A(h, h') \in \mathcal{C}_A(\underline{n+n'}, \underline{k+k'})$ , lying over  $\mu_F(g, g')$  is defined on the generators  $\underline{e}, e \in E_{n+n'}$  by

$$\mu_A(h, h')(\underline{e}_m^n) = \underline{e}_m^k \quad n \geq 0$$

$$\mu_A(h, h')(\underline{e}) = h(\underline{e}) \quad e \in T_n$$

$$\mu_A(h, h')(\underline{\tau e}) = \tau h'(e) \quad e \in T_k$$

where  $\tau : T_k \rightarrow T_{n+k}$  is the translation map defined after 3.3. The above formulae determine  $\mu_A(h, h')(\underline{e}_i^n)$  and  $\mu_A(h, h')(\underline{e}_r^n)$ . One checks that  $\mu_A : M_A \times M_A \rightarrow M_A$  is strictly associative, with unit  $\underline{0}$ .

### 3.20. Homomorphisms

The homomorphisms  $M_B \rightarrow M_A \xrightarrow{p} M_F$  are defined via functors  $\mathcal{C}_B \rightarrow \mathcal{C}_A \xrightarrow{p} \mathcal{C}_F$ . On objects, the functors send  $\underline{n}$  to  $\underline{n}$ . The map  $\mathcal{C}_B(\underline{n}, \underline{n}) \rightarrow \mathcal{C}_A(\underline{n}, \underline{n})$  is the isomorphism of  $B_{(n)}$  with its group of inner automorphisms. The map  $\mathcal{C}_A(\underline{n}, \underline{k}) \rightarrow \mathcal{C}_F(\underline{n}, \underline{k})$  assigns to an  $h : B_{(n)} \rightarrow B_{(k)}$  the element of  $\mathcal{C}_F(\underline{n}, \underline{k})$  over which it lies. It is straightforward that these functors define homomorphisms of monoids  $M_B \rightarrow M_A \xrightarrow{p} M_F$ .

## 4. Proof of Acyclicity

We will now prove that  $A$  is acyclic, quoting propositions 4.1.–4.3. which will be proven later in this section.

Essentially, 4.1. and 4.2. show us how to “deloop” the sequence  $BB_{(1)} \rightarrow BA \rightarrow BF'$ . Acyclicity of  $A$  then reduces to proving (4.3.) that  $H_1(A; \mathbf{Z}) = 0$ . It is here that all details of our construction come into play.

To begin, recall that if  $M$  is a (strictly) associative topological monoid, we can construct  $BM$ , namely as the realization of the simplicial space

$$\begin{array}{c}
 \longleftarrow \quad \longleftarrow \\
 * \quad M \longleftarrow M \times M \cdots \\
 \longleftarrow \quad \longleftarrow
 \end{array}$$

and that this construction is functorial for monoids and continuous homomorphisms. The evident map  $\Sigma M \rightarrow BM$  has as adjoint a map  $M \rightarrow \Omega BM$  which is not, in general, a homotopy equivalence. We will use the group completion theorem [McS] (see also [D] for a very detailed treatment) to compare the homology of  $M$  and  $\Omega BM$ .

In the previous section we constructed monoids and homomorphisms  $M_B \rightarrow M_A \xrightarrow{p} M_F$ . Therefore we have spaces and maps  $BM_B \rightarrow BM_A \xrightarrow{Bp} BM_F$ , whence  $\widetilde{BM}_B \rightarrow BM_A \rightarrow BM_F$  ( $\sim$  denotes universal cover).

4.1. PROPOSITION. (a) *There is a map  $BA \rightarrow \Omega BM_A$  inducing isomorphism in homology.*

(b)  *$\widetilde{BM}_B \rightarrow BM_A \xrightarrow{Bp} BM_F$  is homotopically a fibration. There is a weak homotopy equivalence from  $\widetilde{BM}_B$  to the homotopy fiber of  $Bp$ .*

4.2. PROPOSITION. *There are weak homotopy equivalences  $\Omega S^3 \sim \widetilde{BM}_B$ ,  $BM_F \sim S^3$ .*

4.3. PROPOSITION .  $H_1 A = 0$ .

#### 4.4. Proof of Acyclicity

By 4.1. (a) and 4.2, we have, up to homotopy, a fibration  $\Omega S^3 \rightarrow BM_A \rightarrow S^3$ . By 4.3. and 4.1. (a)  $H_2 BM_A = 0$ . This, and an easy Serre cohomology spectral sequence argument, show that  $BM_A$  is contractible. Thus  $\Omega BM_A$  is contractible, and by 4.1. (a),  $A$  is acyclic.  $\square$

The proof of 4.1. begins with lemmas 4.6. and 4.7. below. Recall from Section 3 the maps  $BA \rightarrow M_A, BF' \rightarrow M_F$  induced by the identifications  $A = \mathcal{C}_A(\underline{1}, \underline{1})$ , and consider the diagram:

$$\begin{array}{ccccc}
 BA & \longrightarrow & M_A & \longrightarrow & \Omega BM_A \\
 \downarrow Bp & & \downarrow & & \downarrow \Omega Bp \\
 BF' & \longrightarrow & M_F & \longrightarrow & \Omega BM_F
 \end{array} \tag{4.5}$$

4.6. LEMMA. *The horizontal compositions in (4.5.) induce isomorphisms in homology:*

$$\begin{aligned}
 H_* BA &\xrightarrow{\cong} H_*(\Omega BM_A) \\
 H_* BF' &\xrightarrow{\cong} H_*(\Omega BM_F)
 \end{aligned}$$

We shall use the group completion theorem of [McS]. Recall that  $BA \rightarrow M_A$ ,  $BF' \rightarrow M_F$  induce homotopy equivalences  $*\mathbb{I}BA \rightarrow M_A$ ,  $*\mathbb{I}BF' \rightarrow M_F$ . Thus, for  $M = M_A$  or  $M_F$ ,  $\pi_0 M$  is a multiplicatively closed subset of  $H_* M$  with two elements (the multiplication in  $H_* M$  is induced by the product  $M \times M \rightarrow M$ ). We consider the ring  $H_*(M)[\pi_0 M^{-1}]$  obtained by inverting  $\pi_0 M$  in  $H_* M$ ; one checks easily that  $H_* BA = H_*(M_A)[\pi_0 M_A^{-1}]$ ,  $H_* BF' \simeq H_*(M_F)[\pi_0 M_F^{-1}]$ .  $\square$

The map  $H_* M \rightarrow H_* \Omega BM$  factors through  $H_*(M)[\pi_0 M^{-1}]$ , because  $\pi_0 \Omega BM$  is a group. The lemma follows if we can show that  $H_*(M)[\pi_0 M^{-1}] \rightarrow H_*(\Omega BM)$  is an isomorphism for  $M = M_A$  and  $M_F$ . But this is exactly what the group completion theorem does for us, provided that we show that  $\pi_0 M$  is in the center of  $H_* M$ . Following lines of Quillen ([Q], §8) we provide this for  $M_A$ . For  $M_F$ , the proof is parallel.

Recall that  $A$  may be identified with  $\mathcal{C}_A(\underline{1}, \underline{1})$ ; let  $A_2$  denote the set  $\mathcal{C}_A(\underline{2}, \underline{2})$  with the obvious group structure whose composition is composition of automorphisms. Proving that  $\pi_0 M_A$  is in the center of  $H_* M_A$  comes down to proving that the homomorphisms  $L, R : A \rightarrow A_2$  defined by

$$L(g) = \mu_A(g, id_1), \quad R(g) = \mu_A(id_1, g)$$

induce the same map in homology.

Let  $A_\varepsilon$  be the subgroup of  $A$  consisting of elements  $g \in A$  lying over elements of  $F'$  whose support is contained in  $(\varepsilon, 1 - \varepsilon)$  and such that  $g(e_*^d) = e_*^d$  ( $* = l, r, m$ ) if  $d < \varepsilon/2$  or  $d > 1 - \varepsilon/2$ . Clearly,  $A$  is the direct limit of the  $A_\varepsilon$ , so it suffices to show that  $L$  and  $R$ , restricted to an  $A_\varepsilon$ , induce the same map on homology. Let  $g_\varepsilon \in \mathcal{C}_F(\underline{2}, \underline{2})$  such that  $g_\varepsilon$  restricted to  $(1 + \varepsilon/4, 2 - \varepsilon/4)$  is translation by  $-1$ , and pick  $h_\varepsilon \in A_2$ ,  $h_\varepsilon$  lying over  $g_\varepsilon$  such that  $h_\varepsilon(e_*^d) = e_*^{d-1}$  if  $d \in (1 + \varepsilon/4, 2 - \varepsilon/4)$ ,  $* = l, r, m$ . Then for any  $g \in A_\varepsilon$ ,  $h_\varepsilon R(g) h_\varepsilon^{-1} = L(g)$ . Hence  $R$  and  $L$  induce the same map on  $H_* A_\varepsilon$ .  $\square$

**4.7. LEMMA.** *There is a map  $BB_{(1)} \rightarrow \Omega \widetilde{B}M_B$  inducing an isomorphism in homology, so that the square*

$$\begin{array}{ccc} BB_{(1)} & \rightarrow & \Omega \widetilde{B}M_B \\ \downarrow & & \downarrow \\ BA & \rightarrow & \Omega BM_A \end{array}$$

*commutes up to homotopy.*

We first apply the group completion theorem of [McS] to prove that  $H_*(M_B)[\pi_0 M_B^{-1}] \rightarrow H_*(\Omega BM_B)$  is an isomorphism. Thus we need that  $\pi_0 M_B \simeq \mathbb{N}$  is in the center of  $H_* M_B$ . But this is a consequence of 3.5.  $\square$

Now  $H_*(M_B)[\pi_0 M_B^{-1}] = H_*(\mathbb{Z} \times \lim_{\rightarrow} BB_{(n)})$ , where  $i : B_{(n)} \rightarrow B_{(n+1)}$  is the usual inclusion. Since by 3.5, these inclusions induce isomorphism in homology, and since  $BB_{(1)} \rightarrow \Omega BM_B$  takes  $BB_{(1)}$  to the component  $(\Omega BM_B)_1$  of  $1 \in \pi_1 BM_B \simeq \mathbb{Z}$ , we have an isomorphism in homology  $BB_{(1)} \rightarrow (\Omega BM_B)_1$ . Composing with a representative of  $-1 \in \pi_1 BM_B$  gives a homotopy equivalence  $(\Omega BM_B)_1 \rightarrow (\Omega BM_B)_0$ . Composing with the homeomorphism  $(\Omega BM_B)_0 \rightarrow \Omega \widetilde{BM}_B$ , we obtain

$$\begin{array}{ccccc} BB_{(1)} & \rightarrow & (\Omega BM_B)_1 & \xrightarrow{\circ-1} & (\Omega BM_B)_0 & \longrightarrow & \Omega \widetilde{BM}_B \\ \downarrow & & \downarrow & & & & \\ BA & \rightarrow & \Omega BM_A & & & & \end{array}$$

which commutes up to homotopy, because  $1 = 0$  in  $\pi_1 \Omega BM_A$ .

#### 4.8. Proof of 4.1.

4.1. (a) is part of lemma 4.6. We pass to the proof of 4.1. (b).

Let  $* \in BM_F$  be the canonical basepoint arising from the definition of  $BM_F$  as the geometrical realization of the simplicial space  $* \leftarrow M_F \leftarrow \dots$ . Let  $\text{Fib}_*(Bp)$  be the homotopy fiber [Sp] of  $Bp : BM_A \rightarrow BM_F$  over  $*$ . The natural numbers  $\mathbb{N}$  are a submonoid of  $M_F$ , coming from the objects  $\underline{0}, \underline{1}, \dots$  of  $\mathcal{C}_F$ , and the image of  $\widetilde{BM}_B$  in  $BM_F$  is  $B\mathbb{N}$ . Now  $B\mathbb{N}$  is contractible in  $BM_F$ . Picking a contraction defines a map  $\beta : \widetilde{BM}_B \rightarrow \text{Fib}_*(Bp)$ . We aim to show that  $\beta$  is a weak equivalence. It suffices to show that  $\Omega\beta : (\Omega BM_B)_0 \rightarrow \text{Fib}_*(\Omega Bp)$  is a homotopy equivalence; here we identify  $\Omega \widetilde{BM}_B = (\Omega BM_B)_0$ , and  $\Omega \text{Fib}_*(Bp)$  with  $\text{Fib}_*(\Omega Bp)$ , the homotopy fiber of  $\Omega Bp : \Omega BM_A \rightarrow \Omega BM_F$ , over the constant loop at  $* \in BM_F$ .

Add a whisker to  $BF'$ , so that  $BF' \rightarrow \Omega BM_F$  takes the new basepoint to the constant loop at  $*$ . As before, denote by  $B\rho : BA \rightarrow BF'$  the map from  $BA$  to (the new)  $BF'$ . The whisker gives an obvious homotopy equivalence  $BB_{(1)} \rightarrow \text{Fib}_*(B\rho)$ , and naturality of homotopy fibers gives a map  $\text{Fib}_*(B\rho) \rightarrow \text{Fib}_*(\Omega B\rho)$ . Now  $\Omega BM_A \xrightarrow{\Omega B\rho} \Omega BM_F$  is the  $+$  construction [Be] of  $B\rho : BA \rightarrow BF'$ , by 4.6. Further,  $F'$  acts trivially on the homology of  $B_{(1)}$ , by 3.14. Thus by ([Be], 6.4),  $\text{Fib}_*(B\rho) \rightarrow \text{Fib}_*(\Omega B\rho)$  is an isomorphism in homology.

We have a square

$$\begin{array}{ccc}
 BB(1) & \xrightarrow{\circ^{-1}} & (\Omega BM_B)_0 \\
 \downarrow & & \downarrow \Omega\beta \\
 \text{Fib}_*(B\rho) & \rightarrow & \text{Fib}_*(\Omega B\rho)
 \end{array} \tag{4.9}$$

which, one checks by hand, is homotopy commutative. Further, every arrow, except possibly  $\Omega\beta$ , induces isomorphism in homology, and so  $\Omega\beta$  must as well.

So  $\Omega\beta$  is a map between loop spaces, which induces isomorphism in homology. We can apply Whitehead’s theorem to see that  $\Omega\beta$ , and hence  $\beta$ , is a weak equivalence.  $\square$

The proof of proposition 4.2 is divided between lemmas 4.10 and 4.12.

4.10. LEMMA. *There is a weak equivalence  $S^3 \rightarrow BM_F$ .*

Let  $\Gamma$  be the pseudogroup of orientation preserving, piecewise affine dyadic homeomorphisms between open subsets of  $\mathbf{R}$ . In [GS], using techniques of [G1], it is shown tht  $B\Gamma \simeq S^3$ . Results of [G2] extending a theorem of Mather show that there is a homology equivalence  $BF' \rightarrow \Omega B\Gamma$ , hence  $BF' \rightarrow \Omega S^3$ . But by 4.6, there is a homology isomorphism  $BF' \rightarrow \Omega BM_F$ . Further,  $\pi_1 \Omega BM_F = 0$ . Thus (see e.g. [Be])  $\Omega BM_F$  and  $\Omega S^3$  are both the plus construction of  $BF'$  with respect to  $\pi_1 BF' = F'$ , and 5.1. of [Be] implies that  $\Omega BM_F$  and  $\Omega S^3$  are weakly equivalent.  $\square$

4.11. LEMMA. *Let  $X$  be a space such that  $\Omega X$  is weakly equivalent to  $\Omega S^3$ . Then  $X$  is weakly equivalent to  $S^3$ .*

By the Hurewicz theorem, it suffices to show that  $X$  has the homology of  $S^3$ . Consider the Serre cohomology spectral sequence of  $\Omega X \rightarrow PX \rightarrow X$ , where  $PX$  is contractible. Let  $a \in H^2 \Omega X = E_2^{0,2}$  be a generator. Let  $b \in H^3 X = \mathbf{Z}$  a generator so that  $d_3^{3,0}(b) = a$ . A little work with the multiplicative structure shows that  $d_3^{3,2n} : E_3^{3,2n} \rightarrow E_3^{0,2n+2}$  is an isomorphism for all  $n$ . Suppose that for some  $k > 3$ ,  $H^k X \neq 0$ , and let  $y \in H^k(X)$ ,  $y \neq 0$ , for smallest such  $k$ . Then  $y$  must survive to  $E_\infty$ , a contradiction.  $\square$

4.12. LEMMA.  *$\widetilde{BM}_B$  is weakly homotopy equivalent to  $\Omega S^3$ .*

It suffices to show that  $BM_B \sim \Omega S^2$ , since  $\Omega S^2 = S^1 \times \Omega S^3$ . Let  $M = * \amalg_{k \geq 1} BB_k$  be the disjoint union of the classifying spaces of the finite braid groups, considered as a monoid as in [S1]. We will define a homomorphism  $a : M \rightarrow M_B$ , and prove that  $Ba$  is a weak equivalence. Since by [S1] (see also [CLM], III, 3, for an alternative approach),  $BM \sim \Omega S^2$ , the lemma follows.  $\square$

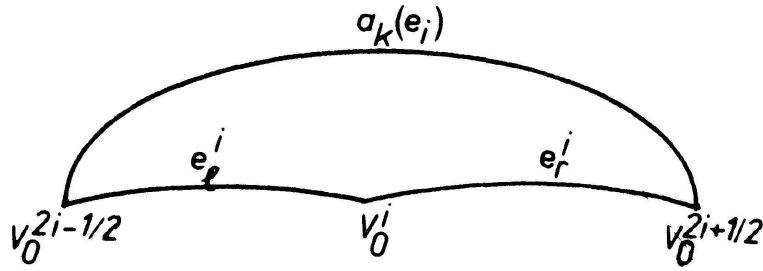


Fig. 4.13

Let  $B_k = \langle e_1, \dots, e_{k-1} \mid [e_i, e_j] = 1, |i - j| \geq 2, e_i e_{i+1} e_i = e_{i+1} e_i e_{i+1} \rangle$  as usual. We define  $a : M \rightarrow M_B$  by  $a(*) = \mathbb{Q}$ , and with homomorphisms  $a_k : B_k \rightarrow B_{(k)}$  defined by “braiding the  $v_0^{1/2}, \dots, v_0^{2k-1/2}$ .” That is (see Figure 4.13), using the triangle rule we define  $a_k(e_i) = (\underline{e}_r^i)^{-1} \underline{e}_l^i \underline{e}_r^i$ . One checks that  $M \rightarrow M_B$  thus defined is a homomorphism of monoids. Applying group completion to both monoids, we obtain a diagram:

$$\begin{array}{ccccc} H_*(\mathbf{Z} \times \lim_{\rightarrow} BB_k) & \simeq & H_*(M) [\pi_0 M^{-1}] & \rightarrow & H_* \Omega BM \\ \downarrow & & \downarrow & & \downarrow \\ H_*(\mathbf{Z} \times \lim_{\rightarrow} BB_k) & \simeq & H_*(M_B) [\pi_0 M_B^{-1}] & \rightarrow & H_* \Omega BM_B \end{array}$$

By 2.5,  $\lim_{\rightarrow} BB_k \rightarrow \lim_{\rightarrow} BB_{(k)}$  induces an isomorphism in homology. Therefore  $\Omega BM \rightarrow \Omega BM_B$  is a homology isomorphism and a loop map, and hence a weak equivalence, whence  $BM \rightarrow BM_B$  is a weak equivalence.  $\square$

4.14. Proof of 4.3.

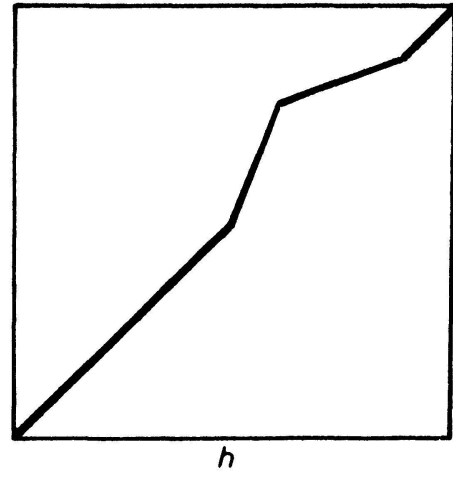
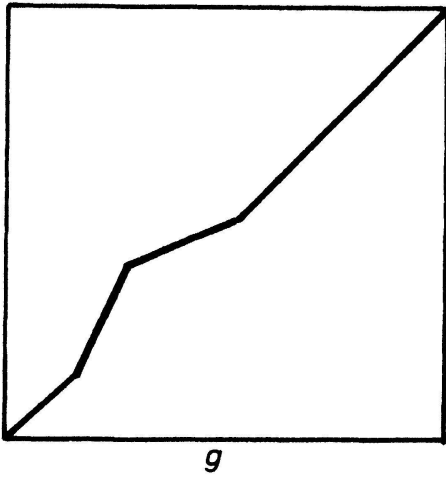
We have an exact sequence  $B_{(1)} \rightarrow A \rightarrow F'$ , and  $F'$  acts trivially on the homology of  $B_{(1)}$ . Further, since  $BF'$  has the homology of  $\Omega S^3$ , we know that  $H_1 B_{(1)} \simeq H_2 F' \simeq \mathbf{Z}$ ,  $H_1 F' = 0$ . Thus, to prove that  $H_1 A = 0$  it suffices to show that the differential  $d_1 : H_2 F' \rightarrow H_1 B_{(1)}$  in the Leray-Serre spectral sequence is an isomorphism. We will explicitly calculate the image in  $H_1 B_{(1)}$  of a generator of  $H_2 F'$  as follows.

$H_2 F'$  is generated (via Hopf’s formula) by the relation  $[g, h] = 1$ , for  $g, h \in F'$  described below. To calculate the image  $k \in H_1 B_{(1)}$ , we lift  $g, h$  to  $\tilde{g}, \tilde{h} \in A$ , and compute the commutator  $[\tilde{g}, \tilde{h}]$  which will lie in  $B_{(1)}$ . The image of the commutator in  $H_1 B_{(1)}$  is  $k$ . Indeed, we shall see that  $[\tilde{g}, \tilde{h}]$ , thought of as an element of  $A$ , is an inner automorphism of  $B_{(1)}$  by an element  $\underline{e}$ , for a certain edge in  $T_{(1)}$ . By 2.4, the homology class of  $\underline{e}$  generates  $H_1 B_{(1)}$ .

It is found in in [GS] that the commutator of the following  $g, h \in F'$  generates  $H_2F'$ :

$$g(x) = \begin{cases} x & x \leq 1/8 \\ 2x - 1/8 & 1/8 \leq x \leq 1/4 \\ 1/2x + 1/4 & 1/4 \leq x \leq 1/2 \\ x & x \geq 1/2 \end{cases}$$

$$h(x) = \begin{cases} x & x \leq 1/2 \\ 2x - 1/2 & 1/2 \leq x \leq 5/8 \\ 1/2x + 7/16 & 5/8 \leq x \leq 7/8 \\ x & 7/8 \leq x \end{cases}$$



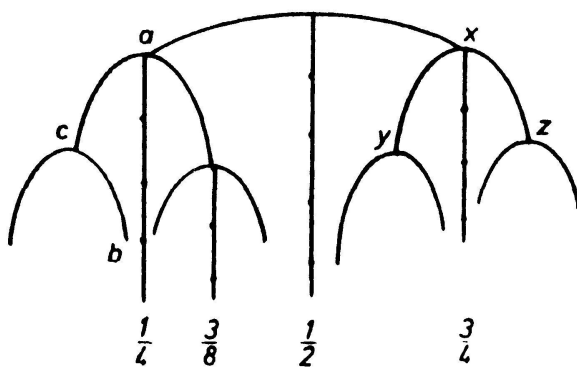
Recall (Figure 4.15(a)) the standard embedding  $\sigma : T_1 \rightarrow \mathbf{R}^2$ . As in the proof of 2.13,  $\tilde{g}$  and  $\tilde{h}$  are defined via embeddings  $G, H : T_1 \rightarrow \mathbf{R}^2$  (Figure 4.15 (b), (c)) which agrees with  $\sigma \circ \varphi_g, \sigma \circ \varphi_h$  near infinity; the  $\varphi_g, \varphi_h$  being defined in 3.8. Namely,  $\tilde{h} = H^{-1}\sigma, \tilde{g} = G^{-1}\sigma$ . Note that, restricted to  $T_{(1/2,1)}, G \equiv \sigma$ . Similarly,  $H \equiv \sigma$  restricted to  $T_{(0,1/2)}$ . Also,  $G(e_r^{1/2}) = \sigma(e_r^{1/2}), H(e_l^{1/2}) = \sigma(e_l^{1/2})$ . It follows that  $\tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}(\underline{e}) = \underline{e}$ , for  $e = e_l^{1/2}, e_r^{1/2}, e \in E_{(0,1/2)} \cup E_{(1/2,1)}$ . Further for  $n \geq 2, G(e_n^{1/2}) = H(e_n^{1/2}) = e_{n-1}^{1/2}$ . Hence, for  $n \geq 3, \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}(\underline{e}_n^{1/2}) = \underline{e}_n^{1/2}$ . It remains to calculate  $\tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}$  on  $\underline{e}_n^{1/2}, n = 0, 1, 2$ .

Figure 4.16 is useful for the application of the triangle rule to the calculation of  $\tilde{g}^{-1}(\underline{e}_0^{1/2}), \tilde{g}^{-1}(\underline{e}_1^{1/2}), \tilde{h}^{-1}(\underline{e}_0^{1/2}), \tilde{h}^{-1}(\underline{e}_1^{1/2})$ . From 4.16 (a), writing  $\alpha = (e_r^{1/4})^{-1}e_l^{1/2}e_r^{1/4}$ , we have

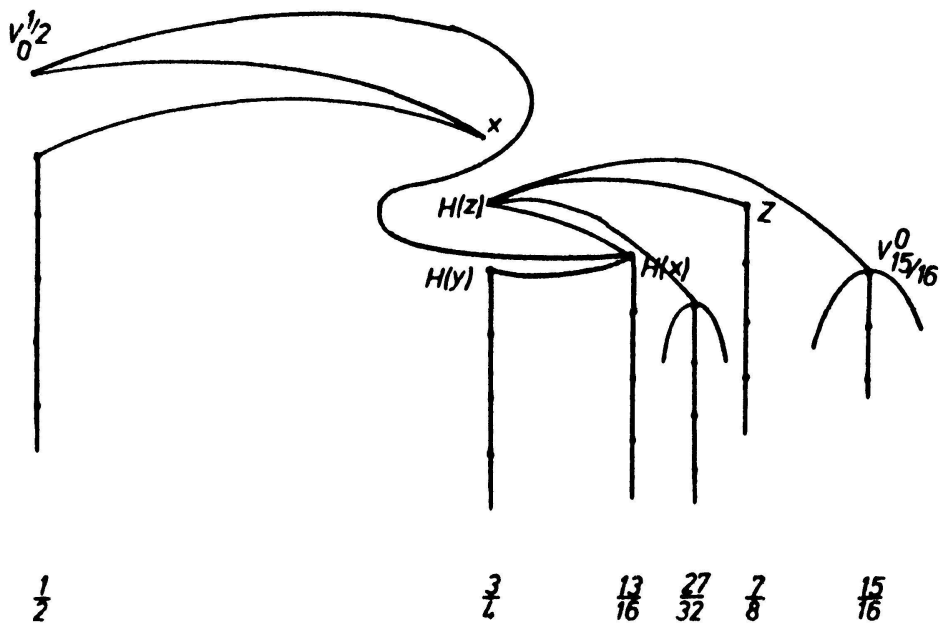
$$\tilde{g}^{-1}(\underline{e}_0^{1/2}) = (\underline{e}_0^{3/8})^{-1}\alpha\underline{e}_0^{3/8} \tag{4.17}$$

and it follows that

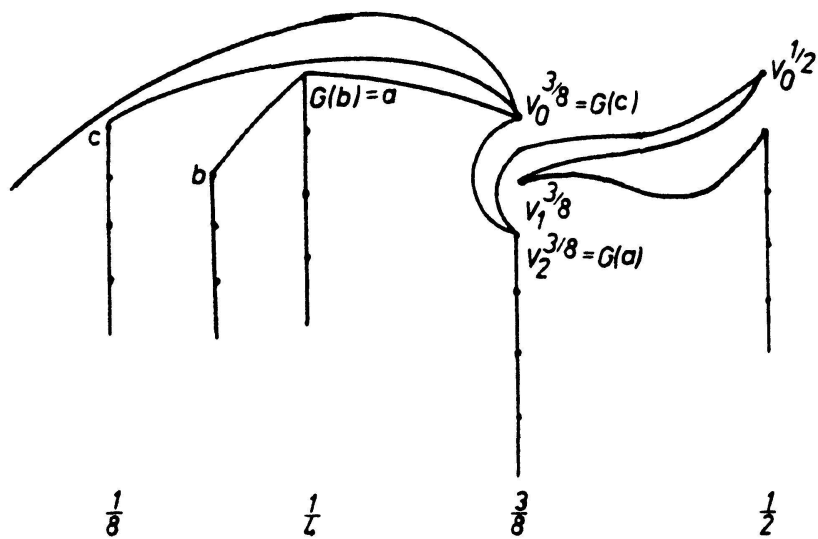
$$\tilde{h}^{-1}(\tilde{g}^{-1}(\underline{e}_0^{1/2})) = \tilde{g}^{-1}(\underline{e}_0^{1/2}). \tag{4.18}$$



(a)  $\sigma : T_1 \rightarrow \mathbb{R}^2$  standard embedding.



(b)  $G : T_1 \rightarrow \mathbb{R}^2$  (only  $G$  restricted to  $T_{(0, \frac{1}{2})}$  pictured)



(c)  $H : T_1 \rightarrow \mathbb{R}^2$  (only  $H$  restricted to  $T_{(\frac{1}{2}, 1)}$  pictured)

Fig. 4.15

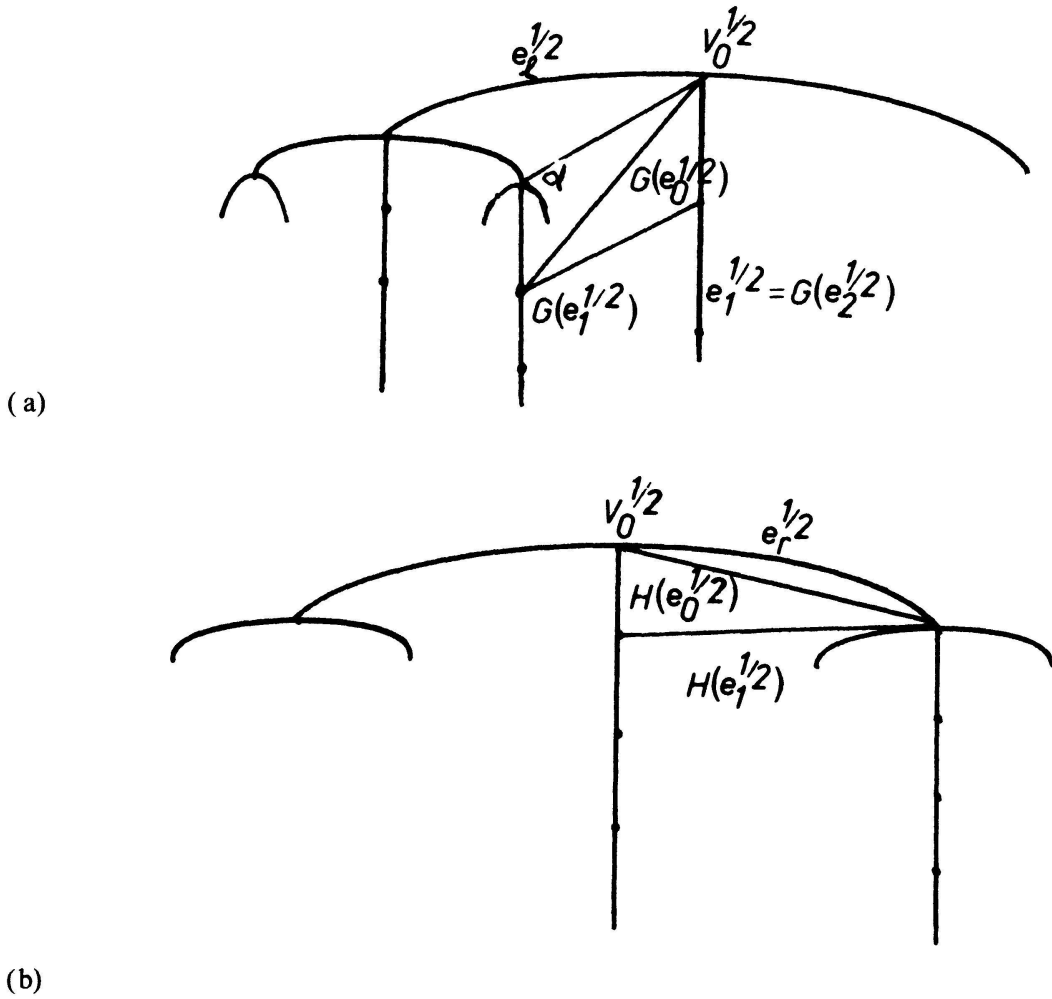


Fig. 4.16

We also see from 4.16 (a) that

$$\tilde{g}^{-1}(\underline{e}_1^{1/2}) = [\tilde{g}^{-1}(\underline{e}_0^{1/2})]^{-1} \underline{e}_0^{1/2} \tilde{g}^{-1}(\underline{e}_0^{1/2}) \tag{4.19}$$

and consequently

$$\tilde{g}(\underline{e}_1^{1/2}) = (\underline{e}_1^{1/2})^{-1} \underline{e}_0^{1/2} \underline{e}_1^{1/2} \tag{4.20}$$

From 4.16 (b) we see  $\tilde{h}^{-1}(\underline{e}_0^{1/2}) = \underline{e}_r^{1/2}$ , and

$$\tilde{h}^{-1}(\underline{e}_1^{1/2}) = (\underline{e}_0^{1/2})^{-1} \underline{e}_r^{1/2} \underline{e}_0^{1/2} \tag{4.21}$$

Further, from 4.16 (b),  $\underline{e}_0^{1/2} = \tilde{h}^{-1}((\underline{e}_0^{1/2})^{-1}) \tilde{h}^{-1}(\underline{e}_1^{1/2}) \tilde{h}^{-1}(\underline{e}_0^{1/2})$ , and so

$$\tilde{h}(\underline{e}_0^{1/2}) = (\underline{e}_0^{1/2})^{-1} \underline{e}_1^{1/2} \underline{e}_0^{1/2} \tag{4.22}$$

Now, let us calculate. Let  $c = \tilde{g}\tilde{h}\tilde{g}^{-1}\tilde{h}^{-1}$ . Then

$$c(\underline{e}_0^{1/2}) = \tilde{g}\tilde{h}(\underline{e}_r^{1/2}) = \tilde{g}(\underline{e}_0^{1/2}) = (\underline{e}_1^{1/2})^{-1}\underline{e}_0^{1/2}\underline{e}_1^{1/2} \quad (4.23)$$

$$\begin{aligned} c(\underline{e}_2^{1/2}) &= \tilde{g}(\tilde{g}^{-1}(\underline{e}_1^{1/2})^{-1}\underline{e}_1^{1/2}\tilde{g}^{-1}(\underline{e}_1^{1/2})) \\ &= (\underline{e}_1^{1/2})^{-1}\underline{e}_2^{1/2}\underline{e}_1^{1/2} \end{aligned} \quad (4.24)$$

Lastly,  $c(\underline{e}_1^{1/2}) = \tilde{g}\tilde{h}\tilde{g}^{-1}((\underline{e}_0^{1/2})^{-1}\underline{e}_r^{1/2}\underline{e}_0^{1/2})$ . Using 4.18 and  $\tilde{h}(\underline{e}_r^{1/2}) = \underline{e}_0^{1/2}$ ,

$$\tilde{h}\tilde{g}^{-1}((\underline{e}_0^{1/2})^{-1}\underline{e}_r^{1/2}\underline{e}_0^{1/2}) = \tilde{g}^{-1}(\underline{e}_1^{1/2}),$$

and thus

$$c(\underline{e}_1^{1/2}) = \underline{e}_1^{1/2} \quad (4.25).$$

Now, (4.23) – (4.25) affirm that  $c$  is conjugation by  $\underline{e}_1^{1/2}$ , hence, by 2.4 a generator of  $H_1 B_{(1)}$ .

## 5. Related Groups

In this section we describe the homology of two groups closely related to  $A$ .

It will be convenient to make use of the plus construction of Quillen ([Q], [Be]). Recall that if  $X$  is a space, and  $N \subseteq \pi_1 X$  is the maximal perfect subgroup, there exists a space  $X_+$  and a map  $X \rightarrow X_+$ , well defined up to homotopy, such that  $\pi_1 X_+ = \pi_1 X/N$ , and  $X \rightarrow X_+$  is an equivalence in homology. We will often invoke the fact ([Be], 6.4) that if  $1 \rightarrow H \rightarrow G \rightarrow K \rightarrow 1$  is an exact sequence of groups, such that  $BH_+$  is a nilpotent space and such that  $\pi_1 K$  acts trivially on  $H_*(H; \mathbf{Z})$ , then  $BH_+ \rightarrow BG_+ \rightarrow BK_+$  is a quasifibration.

### 5.1. The group $A_G$

Recall (3.15) the group  $A_G$  which was constructed as an extension  $B_G \rightarrow A_G \rightarrow G$ . We will prove:

**5.2. PROPOSITION.** *The cohomology ring  $H^*(A_G; \mathbf{Z})$  is the free graded  $\mathbf{Z}$ -algebra with generators in dimensions 2 and 3.*

We do not know whether  $BA_{G_+}$  is homotopy equivalent to  $S^3 \times \mathbf{C}P^\infty$ .

The proof of 5.2 will involve an auxiliary group  $A_{\tilde{G}}$ . Let  $\tilde{G}$  be the group of homeomorphisms of  $\mathbf{R} = \tilde{S}^1$  which are lifts of elements of  $G$ . Let  $B_G \rightarrow A_{\tilde{G}} \rightarrow \tilde{G}$  and  $B_G \rightarrow A^{\text{aug}} \rightarrow F'$  be the extensions obtained by pullback over the natural maps  $F' \rightarrow \tilde{G} \rightarrow G$ . We have the following diagram:

$$\begin{array}{cccc}
 B_{T_1} & \rightarrow & B_G & \rightarrow & B_G & \rightarrow & B_G \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 A & \rightarrow & A^{\text{aug}} & \rightarrow & A_{\tilde{G}} & \rightarrow & A_G \\
 \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 F' & \rightarrow & F' & \rightarrow & \tilde{G} & \rightarrow & G
 \end{array} \tag{5.3}$$

The inclusion  $B_{T_1} \rightarrow B_G$  induces isomorphisms in homology, and thus  $A^{\text{aug}}$  is an acyclic group.

Let  $LS^3$  denote the space of unbased maps of a circle to  $S^3$ , and let  $\mathcal{L}S^3 = ES^1 \times_{S^1} LS^3$  denote the homotopy quotient of  $LS^3$  by  $S^1$ , acting by reparametrization of loops. One can apply the plus construction to (5.3), obtaining the following diagram commuting up to homotopy, whose vertical arrows are fibrations.

$$\begin{array}{cccc}
 \Omega^2 S^3 & \xrightarrow{=} & \Omega^2 S^3 & \xrightarrow{=} & \Omega^2 S^3 \\
 \downarrow & & \downarrow & & \downarrow \\
 * & \rightarrow & BA_{\tilde{G}_+} & \rightarrow & BA_{G_+} \\
 \downarrow & & \downarrow & & \downarrow \\
 \Omega S^3 & \rightarrow & LS^3 & \rightarrow & \mathcal{L}S^3
 \end{array} \tag{5.4}$$

The map  $\Omega S^3 \rightarrow LS^3$  has homotopy fiber  $\Omega S^3$ , because it is simply the inclusion  $\Omega S^3 \rightarrow S^3 \times \Omega S^3 = LS^3$ . Further, it is not hard to see that the plus construction commutes with pullbacks of surjective homomorphisms. We thus obtain a fibration  $\Omega S^3 \rightarrow * \rightarrow BA_{\tilde{G}_+}$ .

**5.5. PROPOSITION.** *There is a homotopy equivalence  $S^3 \rightarrow BA_{\tilde{G}_+}$ .*

Since  $\pi_1 BA_{\tilde{G}_+} = 0$ , it suffices to show that  $BA_{\tilde{G}_+}$  has the integral cohomology of  $S^3$ . This follows from an easy argument on the cohomology spectral sequence of the fibration  $\Omega S^3 \rightarrow * \rightarrow BA_{\tilde{G}_+}$ .  $\square$

**5.6. LEMMA.** *The homomorphism  $A_{\tilde{G}} \rightarrow \tilde{G}$  induces an isomorphism  $\mathbf{Z} \simeq H^3(\tilde{G}) \rightarrow H^3(A_{\tilde{G}}) \simeq \mathbf{Z}$ .*

Consider the fibration  $\Omega^2 S^3 \rightarrow S^3 \rightarrow LS^3$ , arising, as an application of 5.5, from the middle column of 5.4. The identification of  $LS^3$  with  $\Omega S^3 \times S^3$  gives an

element of  $\pi_3 \Omega S^3 \times \pi_3 S^3 = \mathbf{Z}/2 \times \mathbf{Z}$  which is either  $0 \times 1$  or  $1 \times 1$ . In either case, the map  $H^3(LS^3) \rightarrow H^3(S^3)$  is an isomorphism.  $\square$

The short exact sequence  $\mathbf{Z} \rightarrow \tilde{G} \rightarrow G$  lifts to a short exact sequence  $\mathbf{Z} \rightarrow A_{\tilde{G}} \rightarrow A_G$ . Let  $e \in H^2(A_G; \mathbf{Z})$  be the Euler class of this extension, and consider the associated Gysin sequence:

$$\begin{aligned} 0 \rightarrow H^3(A_G; \mathbf{Z}) \rightarrow H^3(A_{\tilde{G}}; \mathbf{Z}) \xrightarrow{f} H^2(A_G; \mathbf{Z}) \xrightarrow{\cup e} H^4(A_G; \mathbf{Z}) \rightarrow \cdots \rightarrow H^n(A_{\tilde{G}}; \mathbf{Z}) \\ \rightarrow H^{n-1}(A_G; \mathbf{Z}) \xrightarrow{\cup e} H^{n+1}(A_G; \mathbf{Z}) \rightarrow H^{n+1}(A_{\tilde{G}}; \mathbf{Z}) \rightarrow \cdots \end{aligned} \quad (5.7)$$

Proposition 5.2 follows from the following:

**5.8. PROPOSITION.**  $H^3(A_{\tilde{G}}; \mathbf{Z}) \rightarrow H^2(A_G; \mathbf{Z})$  is the zero map.

*Proof of 5.2.* Since  $BA_{\tilde{G}^+} \simeq S^3$ , we have isomorphisms  $H^3(A_G; \mathbf{Z}) \simeq H^3(A_{\tilde{G}}; \mathbf{Z})$ , and  $\cup e : H^n(A_G; \mathbf{Z}) \rightarrow H^{n+2}(A_G; \mathbf{Z})$ ,  $n \geq 2$ . This implies the proposition; indeed, if  $y \in H^3(A_G; \mathbf{Z})$  is a generator,  $H^*(A_G; \mathbf{Z}) = \mathbf{Z}[e, y]/(y^2 = 0)$ .

*Proof of 5.8.* Considering the Gysin sequences arising from the extensions  $0 \rightarrow \mathbf{Z} \rightarrow A_{\tilde{G}} \rightarrow A_G \rightarrow 1$  and  $0 \rightarrow \mathbf{Z} \rightarrow \tilde{G} \rightarrow G \rightarrow 1$ , we obtain a commuting square:

$$\begin{array}{ccc} H^3(A_{\tilde{G}}; \mathbf{Z}) & \xrightarrow{f} & H^2(A_G; \mathbf{Z}) \\ \uparrow & & \uparrow \\ H^3(\tilde{G}; \mathbf{Z}) & \xrightarrow{f} & H^2(G; \mathbf{Z}) \end{array} \quad (5.9)$$

Let  $\beta \in H^3(\tilde{G}; \mathbf{Z})$  be a generator, and  $\alpha = f(\beta)$ . By lemma 5.6, it suffices to show:

**5.10. ASSERTION.** The image of  $\alpha$  in  $H^2(A_G; \mathbf{Z})$  is 0.

Consider the differential  $d_2 : H_2(G; \mathbf{Z}) \rightarrow H_1(B_G; \mathbf{Z}) = \mathbf{Z}$  in the homology spectral sequence of the extension  $B_G \rightarrow A_G \rightarrow G$ . It is not hard to see that the kernel of  $d_2$  is  $H_2(A_G; \mathbf{Z})$ . Let  $\gamma \in H^2(G; \mathbf{Z})$  be the cohomology class defined by  $d_2$ . We will show that  $\gamma = \alpha$ , proving 5.10.

Write  $\gamma = m\alpha + ne$ , where  $e \in H^2(G; \mathbf{Z})$  is the Euler class of the extension. Evaluating both sides of this equality on the image of a generator of  $H_2(F'; \mathbf{Z}) \simeq \mathbf{Z}$ , and using the computation in the proof of 4.3, we see that  $m = 1$ . The proof of 5.10 is complete when we show that  $n = 0$ .

Now  $G$  is a group of homeomorphisms of the circle, and contains the cyclic subgroups  $\mathbf{Z}/2^r$ ,  $r \geq 1$  generated by rotations  $R_r(x) = x + 2^{-r}$ ,  $x \in \mathbf{R}/\mathbf{Z}$ . Suppose that the inclusion  $\mathbf{Z}/2^r \rightarrow G$  lifts to  $A_G$ . Since  $H^2(\mathbf{Z}/2^r; \mathbf{Z}) \simeq \mathbf{Z}/2^r$ , generated by the

pullbacks of the Euler class  $e \in H^2(G; \mathbf{Z})$ , one would obtain that  $n = 0 \pmod{2^r}$ ,  $r \geq 1$ , and thus that  $n = 0$ .

It thus remains to show that the subgroups  $\mathbf{Z}/2^r$  lift to  $A_G$ . We do this explicitly for  $r = 1$ ; the general case follows similarly, but is more intricate.

The rotation  $R_1$  acts naturally on  $T_G$  away from the edges  $e_i^{1/2}, e_r^{1/2}$  (see Figure 5.11). We define a lift  $\tilde{R}_1 \in A_G$  of  $R_1$  by sending  $\underline{e}_i^{1/2}$  to  $\underline{\delta}$ , and  $\underline{e}_r^{1/2}$  to  $\underline{\varepsilon}$ , where  $\delta$  and  $\varepsilon$  are as shown in Figure 5.11.

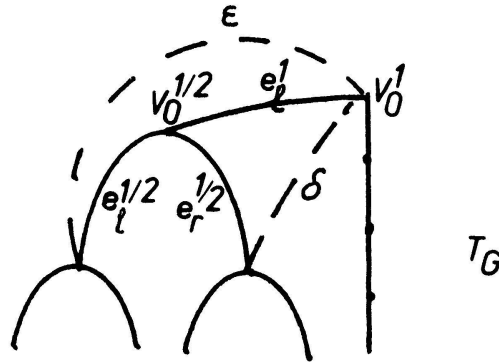


Fig. 5.11

Let us verify that  $\tilde{R}_1^2$  is the identity.

This is clear away from  $\underline{e}_r^{1/2}$  and  $\underline{e}_i^{1/2}$ . Using the triangle rule, we find:

$$\tilde{R}_1^2(\underline{e}_r^{1/2}) = \tilde{R}_1(\underline{e}_i^{1/2} \underline{e}_i^1 (\underline{e}^{1/2})_i^{-1}) = \underline{\delta} \underline{e}_i^1 \underline{\delta}^{-1} = \underline{e}_r^{1/2}$$

$$\tilde{R}_1^2(\underline{e}_i^{1/2}) = \tilde{R}_1((\underline{e}^{1/2})_r^{-1} \underline{e}_i^1 \underline{e}_r^{1/2}) = \underline{\varepsilon}^{-1} \underline{e}_i^1 \underline{\varepsilon} = \underline{e}_i^{1/2}$$

So  $\tilde{R}_1^2$  is the identity, as claimed.

### 5.11. The group $A_\Sigma$

We now introduce a second group related to  $A$ . Recall the tree  $T_1$  fundamental in the construction of  $A$ , and let  $V(T_1)$  be the set of vertices of  $T_1$ . Let  $A_\Sigma$  be the set of bijections  $\varphi : V(T_1) \rightarrow V(T_1)$  such that there is some  $g \in F'$ , such that  $\varphi(v_n^d) = v_{n+g''(d)}^{g(d)}$ ,  $v_n^d \in V(T_1)$  except for a finite number of points.

Clearly,  $A_\Sigma$  surjects to  $F'$  with kernel  $\Sigma_\infty$ , the group of finitely supported permutations. If we consider  $A_\Sigma$  as embedded in the natural way in  $\text{Aut}(\Sigma_\infty)$ , we have a map of exact sequences

$$\begin{array}{ccccccc} 1 & \rightarrow & B_\infty & \rightarrow & A & \rightarrow & F' \rightarrow 1 \\ & & \downarrow & & \downarrow & & \parallel \\ 1 & \rightarrow & \Sigma_\infty & \rightarrow & A_\Sigma & \rightarrow & F' \rightarrow 1 \end{array}$$

Thus the group  $A_\Sigma$  is an analogue of  $A$ . In the rest of §5, we shall identify the space  $BA_{\Sigma+}$ . Two auxiliary groups introduced by Wagoner [W] will be useful.

5.12. DEFINITION. Let  $P_\infty$  be the group of bijections  $\varphi$  of  $V(T_1)$  such that for some  $\varepsilon > 0$ ,  $\varphi(v_n^d) = v_n^d$  for  $d < \varepsilon$ ,  $1 - \varepsilon < d$ . Set  $F'_\infty = P_\infty/\Sigma_\infty$ .

5.13. *Identifying the space  $BA_{\Sigma+}$*

It is clear that  $A_\Sigma \subset P_\infty$  and that this inclusion induces an inclusion  $F' \subset F'_\infty$ , so that we have a pullback

$$\begin{array}{ccccccc} 1 & \rightarrow & \Sigma_\infty & \rightarrow & A_\Sigma & \rightarrow & F' \rightarrow 1 \\ & & \parallel & & \downarrow & & \downarrow \\ 1 & \rightarrow & \Sigma_\infty & \rightarrow & P_\infty & \rightarrow & F'_\infty \rightarrow 1 \end{array}$$

Passing to the plus construction, we obtain a pullback of fibrations:

$$\begin{array}{ccccc} B\Sigma_{\infty+} & \rightarrow & BA_{\Sigma+} & \rightarrow & BF'_+ \\ \parallel & & \downarrow & & \downarrow \\ B\Sigma_{\infty+} & \rightarrow & BP_{\infty+} & \rightarrow & BF'_{\infty+} \end{array}$$

As  $BP_{\infty+}$  is contractible [W], we see that  $BA_{\Sigma+}$  is the homotopy fibre of the map  $BF'_+ \rightarrow BF'_{\infty+}$ . Now, we have already used the fact that  $BF'_+ \simeq \widetilde{\Omega}S^3 = \Omega S^2$ , and a theorem of Priddy [P] identifies  $BF'_{\infty+}$  as  $\Omega^{\infty-1}S^\infty$ . We conclude this section with a sketch of the following:

5.14. *Assertion*

$BA_{\Sigma+}$  is the homotopy fibre of the inclusions  $\Omega S^2 \rightarrow \Omega^{\infty-1}S^\infty$ .

- i) Let  $M_B$  be the monoid, associated to the braid groups, that we considered in §3, and let  $M_\Sigma$  be its analogue for permutations. The results of Cohen ([C] p. 106–108) imply the existence of a homotopy commutative diagram

$$\begin{array}{ccc} \widetilde{B}M_B & \rightarrow & \widetilde{B}M_\Sigma \\ \downarrow \simeq & & \downarrow \simeq \\ \widetilde{\Omega}S^2 & \rightarrow & \Omega^{\infty-1}S^\infty \end{array} \tag{5.15}$$

whose vertical arrows are homotopy equivalences.

- ii) Recall (3.19.) the monoid  $M_A$  associated to the group  $A$ , and let  $M_{P_\infty}$  the monoid constructed in the same way for  $P_\infty$ . Further, let  $M_{F'_\infty}$  be the monoid constructed analogously to  $M_F$ . As we have seen in (4.1. (b)), we have a quasi-fibration  $\widetilde{BM}_B \rightarrow BM_A \rightarrow BM_F$ ; the same is true for  $\widetilde{BM}_\Sigma \rightarrow BM_{P_\infty} \rightarrow BM_{F'_\infty}$ . The total spaces of these quasifibrations are contractible (cf. (4.4) for  $BM_A$ ). We therefore have a homotopy commutative square.

$$\begin{array}{ccc}
 \Omega BM_F & \rightarrow & \Omega BM_{F_\infty} \\
 \downarrow \simeq & & \downarrow \simeq \\
 \widetilde{BM}_B & \rightarrow & \widetilde{BM}_\Sigma
 \end{array} \tag{5.16}$$

whose vertical arrows are homotopy equivalences.

- iii) As we have seen (4.6), there is a homology equivalence  $BF' \rightarrow \Omega BM_F$  and thus a homotopy equivalence  $BF'_+ \rightarrow \Omega BM_F$ . Similarly, we have a homotopy equivalence  $BF'_{\infty+} \rightarrow \Omega BM_{F_\infty}$ . Further, the square

$$\begin{array}{ccc}
 BF'_+ & \rightarrow & BF'_{\infty+} \\
 \downarrow \simeq & & \downarrow \simeq \\
 \Omega BM_F & \rightarrow & \Omega BM_{F_\infty}
 \end{array} \tag{5.17}$$

is homotopy commutative.

- iv) Assembling the diagrams 5.15, 5.16 and 5.17, we can identify the map  $BF'_+ \rightarrow BF'_{\infty+}$  with the inclusions  $\widetilde{\Omega}S^2 \rightarrow \Omega^{\infty-1}S^\infty$ , thus establishing the assertion. □

## 6. An example

In this section we provide the example referred to in the introduction. We construct a fibration  $F \rightarrow E \rightarrow B$ , and groups  $L$  and  $K$  with the homology of  $F$  and  $B$ , such that there is no exact sequence  $1 \rightarrow L \rightarrow P \rightarrow K \rightarrow 1$  so that  $BL_+ \rightarrow BP_+ \rightarrow BK_+$  is equivalent to the original fibration.

### 6.1. The idea of the construction

Start with the fibrations  $S^1 \times S^1 \rightarrow E \rightarrow S^1$  whose monodromy is the involution  $(x, y) \rightarrow (y, x)$ . The exact sequence of fundamental groups  $\mathbf{Z} \times \mathbf{Z} \rightarrow \pi_1 E \rightarrow \mathbf{Z}$  has plus construction the initial fibration. We will enlarge  $\mathbf{Z} \times \mathbf{Z}$  to a group  $L$  with the same homology, such that the involution does not extend, and this leads to our example.

## 6.2. Main construction

Let  $\mathcal{H}$  be Higman's acyclic group [BDH] :  $\mathcal{H} = \langle a, b, c, d \mid aba^{-1} = b^2, bcb^{-1} = c^2, cdc^{-1} = d^2, dad^{-1} = a^2 \rangle$ . Let  $\alpha$  be the automorphism of  $\mathcal{H}$  which cyclically permutes  $a, b, c, d$ . This automorphism determines an extension  $1 \rightarrow \mathcal{H} \rightarrow H \xrightarrow{\pi} \mathbf{Z} \rightarrow 0$  where  $\pi$  is a homology equivalence since  $\mathcal{H}$  acyclic, and  $[H, H] = \mathcal{H}$ .

Let  $L = \mathbf{Z} \times H$ . Obviously  $BL_+ \simeq S^1 \times S^1$ . We shall show that there is no automorphism  $\varphi$  of  $L$  such that  $B\varphi_+ : S^1 \times S^1 \rightarrow S^1 \times S^1$  is homotopic to the involution  $(x, y) \rightarrow (y, x)$ . This proves that no exact sequence  $1 \rightarrow L \rightarrow P \rightarrow \mathbf{Z} \rightarrow 0$  induces a sequence  $BL_+ \rightarrow BP_+ \rightarrow S^1$  equivalent to the fibration  $S^1 \times S^1 \rightarrow E \rightarrow S^1$ .

The nonexistence of such a  $\varphi$  is established by the following three claims.

6.3. CLAIM. The automorphism  $\alpha$  is not inner. Clearly,  $\alpha$  is of order 4. If  $\alpha(x) = \omega^{-1}x\omega$ , then  $\omega^4$  is an element of the center of  $\mathcal{H}$ . But  $\mathcal{H}$  is an iterated amalgamated free product, starting with a centerless torsion free group. The center theorem and torsion theorem for amalgamated products ([MKS], [LS]) show that  $\omega = e$ , a contradiction.

6.4. CLAIM. No element  $y \in H$  such that  $\pi(y) = 1$  commutes with  $\mathcal{H}$ . Indeed, such an element allows us to identify  $H$  with  $\mathcal{H} \times \mathbf{Z}$ , contradicting the fact that  $\alpha$  is not inner.

6.5. CLAIM. No automorphism  $\varphi$  of  $L$  induces on  $H_1(L) = \mathbf{Z} \oplus \mathbf{Z}$  the involution  $(n, m) \rightarrow (m, n)$ .

If such a  $\varphi$  exists, then  $\varphi(1, e) = (0, y)$ , for some  $y \in H$  such that  $\pi(y) = 1$ . Moreover,  $\varphi(0, x) = (\pi(x), \chi(x))$  where  $\chi : H \rightarrow H$  is a morphism such that  $y\chi(x) = \chi(x)y$ . But  $\chi(\mathcal{H}) = \mathcal{H}$  as  $\varphi([Z \times H, Z \times H]) = \varphi(0 \times [H, H]) = 0 \times [H, H]$ . Thus  $y$  and  $\mathcal{H}$  commute, contradicting claim 6.4.

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