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Rational Co-H-Spaces

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Section 1—Introduction

H-spaces, or spaces having a continuous multiplication with homotopy unit, have been intensively studied in topology. In the last twenty five years or so, much work has also been done on the dual notion of a co-H-space. This is a space X together with a map $X \to X \vee X$, called the comultiplication, whose composition with each projection $X \vee X \to X$ is homotopic to the identity map. The primary example of a co-H-space is the suspension of a space with the natural pinching map. A number of authors have investigated basic properties of comultiplications such as homotopy-associativity [B–C], [Be_2], [Ga], homotopy-commutativity [B–G] and existence of homotopy-inverses [H–M–R_2]. Others have considered when a co-H-space is equivalent to a suspension [B–Ha], [B–Hi], [Sc] and extensions of one co-H-space by another [C–N].

In this paper we use the technique of rationalization or $\mathbb{Q}$-localization to study co-H-spaces. This leads to a consideration of rational co-H-spaces, i.e., co-H-spaces which are also rational spaces. We study the totality of homotopy classes of comultiplications on a rational co-H-space. In particular, we are interested in whether or not there are infinitely many homotopy classes of homotopy-associative comultiplications, homotopy-commutative comultiplications, etc. on a given rational co-H-space.

It is well-known that a rational co-H-space X has the homotopy type of a wedge of rational spheres. The latter space admits a standard comultiplication arising from the pinching map. However, X with its given comultiplication and the wedge with the standard comultiplication need not be co-H-equivalent. Thus a rational co-H-space may admit many comultiplications with different properties. For instance, a rational co-H-space that has the homotopy type of a wedge of three rational spheres of dimensions 2, 3 and 5 has infinitely many homotopy classes of homotopy-associative comultiplications, non-homotopy-associative comultiplications, homotopy-commutative comultiplications and non-homotopy-commutative comultiplications.

The principal tool of the first part of this paper is the Quillen minimal model. This is a functor which assigns a differential graded Lie algebra L_X to a space X and a homotopy class of differential graded Lie algebra homomorphisms to a homotopy class of maps. If X is a co-H-space, then we see that L_X and $L_{X \vee X}$ are free graded Lie algebras with zero differential. From this it follows that a homotopy class of comultiplications on X corresponds to a unique homomorphism $L_X \to L_{X \vee X}$. The study of homotopy classes of comultiplications on a rational co-H-space X is therefore replaced by a study of certain homomorphisms $L_X \to L_{X \vee X}$ of free graded Lie algebras. We have found this to be a very effective setting to work in. It would be possible to phrase the early material of this paper in terms of the rational homotopy Lie algebra $\pi_*(\Omega X) \otimes \mathbb{Q}$ of a co-H-space X, instead of the Quillen minimal model. In order to do this it would be necessary to relate homotopy classes of maps of a rational co-H-space into a rational space to the corresponding homomorphisms of rational homotopy Lie algebras (see [Sc, Prop. 2]). Since the Quillen minimal model of a rational co-H-space is the rational homotopy Lie algebra, these two viewpoints are equivalent.

A main objective of this paper is a study of homotopy-associativity of a rational co-H-space. Homotopy-associativity is perhaps the most basic property of a comultiplication and has been widely investigated. Homotopy-associative co-H-spaces are dual to homotopy-associative H-spaces. The latter occupy a central position in topology, and are the homotopy analogue of topological groups. A one-connected homotopy-associative co-H-space X also has special properties; for instance, the homotopy set $[X, Y]$ has a natural group structure, for any space Y.

Although we consider rational co-H-spaces for most of the paper, we have been able to carry over many results to wedges of ordinary spheres. In a sense these are the simplest finite CW-co-H-spaces; nonetheless they turn out to have a surprisingly rich set of comultiplications—cf. Ganea's discussion of comultiplications on $S^3 \vee S^{15}$ in [Ga, Sec. 4]. In this context we replace the free graded Lie algebras of the Quillen minimal model by the homotopy groups of a wedge of spheres with Whitehead product. Hilton's theorem is fundamental in this approach for it allows arguments similar to those given earlier to be made. However, the presence of torsion in the homotopy groups gives rise to additional complexities.

The paper is organised as follows: Section 2 contains definitions and establishes the basic framework in which we work. We begin by showing the equivalence of comultiplications with certain Lie algebra homomorphisms, and we then show that the latter are equivalent to certain vector space homomorphisms called perturbations. We next give necessary and sufficient conditions for a rational co-H-space to admit infinitely many homotopy classes of comultiplications in Proposition 2.7. In Section 3 we begin our study of associativity of rational co-H-spaces. The main result is Theorem 3.11, which provides a characterization of associativity for a broad class of comultiplications. Theorems 3.14 and 3.15 give necessary and sufficient conditions for a rational co-H-space to admit infinitely many homotopy classes of homotopy-associative comultiplications or infinitely many homotopy classes of non-homotopy-associative comultiplications, respectively. Also, we give a dual version of a result of Leray-Samelson in Theorem 3.18, which implies that any two homotopy-associative comultiplications on certain rational co-H-spaces are equivalent.

Sections 4 and 5 contain our study of two further topics concerning comultiplications on a rational co-H-space: homotopy-commutativity and homotopy-inverses. Proposition 4.2 provides necessary and sufficient conditions for a rational co-H-space to admit infinitely many homotopy classes of homotopy-commutative comultiplications. Much of Sections 4 and 5, though, is concerned with examples which demonstrate many of the possibilities that can occur. Examples 4.3 illustrate the independence of the notions of homotopy-associativity and homotopy-commutativity. Corollary 5.6 shows that left and right homotopy-inverses with respect to any comultiplication on a wedge of two rational spheres always agree. Together with Proposition 3.16 and Examples 4.3, this gives a rather complete picture of the comultiplications on a rational co-H-space of the homotopy type of a wedge of two rational spheres. In Example 5.7, we give an example of a rational co-H-space whose left and right homotopy inverses do not agree.

Section 6 contains results on wedges of ordinary spheres. Most of the previous results carry over to wedges of spheres, but others require modification. We also give two examples to illustrate how certain phenomena, when considered over the integers, are more complicated than when considered over the rationals. In Example 6.9 we present a wedge of spheres that admits infinitely many homotopy classes of non-homotopy-associative comultiplications, but whose rationalization does not. Example 6.13 provides a comultiplication on a wedge of two spheres for which the left and right homotopy-inverses do not agree. In Section 7 we work in the universal enveloping algebra of a Lie algebra and prove the result on which Theorem 3.11, the main result on homotopy-associativity, depends. The fact that the universal enveloping algebra is free and associative make it more suitable than the Lie algebra for establishing our main result on associative comultiplications.

We close this section by giving some notation and terminology used in the rest of the paper. All topological spaces which we consider are 1-connected, based spaces of the based homotopy type of a based CW-complex. All maps and homotopies preserve base point. We do not distinguish notationally between a map and its homotopy class. All vector spaces are vector spaces over $\mathbb{Q}$, the field of rationals. We frequently work with a positively graded vector space V, that is, a sequence of vector spaces $V_1, V_2, \ldots$. By $v \in V$ we mean $v \in V_i$ for some i. We write $V = \langle x_1, x_2, \ldots \rangle$ to indicate that $x_1, x_2, \ldots$ is a graded basis of V, and write $|x_i|$ for the degree of x_i, i.e., $|x_i| = m$ if $x_i \in V_m$. A space Y is called a rational space if the total homotopy group $\pi_*(Y)$ is a graded vector space. The technique of rationalization or $\mathbb{Q}$-localization assigns to a space X, a rational space $X_{\mathbb{Q}}$, called the rationalization of X. It also assigns to a map of spaces β, a map of rationalized spaces $\beta_{\mathbb{Q}}$ [H-M-R$_1$, Ch. 2].

We next summarize briefly some of the salient features of the Quillen minimal model—for details see [B–L], [Ne], [Ta, Ch. 2, 3]. For a graded vector space $V = \langle x_1, x_2, \ldots \rangle$, we denote the free graded Lie algebra generated by V by $\mathbb{L}(V)$ or $\mathbb{L}(x_1, x_2, \ldots)$. For free graded Lie algebras $\mathbb{L}(V)$ and $\mathbb{L}(W)$, we denote the coproduct by $\mathbb{L}(V) \sqcup \mathbb{L}(W)$, and observe that $\mathbb{L}(V) \sqcup \mathbb{L}(W) = \mathbb{L}(V \oplus W)$. For elements x, y in a graded Lie algebra, we sometimes denote the Lie bracket $[x, y]$ by $\mathrm{ad}(x)(y)$, and similarly $[x, [x, \ldots [x, y] \ldots]]$, with x occuring r times, by $\mathrm{ad}^r(x)(y)$. A minimal differential graded Lie algebra is a free graded Lie algebra with decomposable differential. The Quillen minimal model functor assigns to a space X a minimal differential graded Lie algebra (L_X, d_X), and to a map $f : X \to Y$ a differential graded Lie algebra homomorphism $\hat{f} : (L_X, d_X) \to (L_Y, d_Y)$. If $s^{-1}\tilde{H}_*(X; \mathbb{Q})$ denotes the desuspension of the reduced, rational homology of X, i.e., $(s^{-1}\tilde{H}_*(X; \mathbb{Q}))_n = (\tilde{H}_*(X; \mathbb{Q}))_{n+1}$, then as a Lie algebra $L_X = \mathbb{L}(s^{-1}\tilde{H}_*(X; \mathbb{Q}))$. There is the notion of homotopy for homomorphisms of minimal differential graded Lie algebras, and the Quillen minimal model functor provides a bijection of the homotopy set $[L_X, L_Y]$ with the homotopy set $[X, Y]$, for rational spaces X and Y. Further properties of the Quillen minimal model will be recalled as needed.

Section 2—Comultiplications and Lie Algebra Comultiplications

A *comultiplication* on a space X is a map $\alpha : X \to X \vee X$ such that $p \circ \alpha \simeq \mathrm{id}$ and $p' \circ \alpha \simeq \mathrm{id}$, where $p, p' : X \vee X \to X$ are the projections on the first and second summands, respectively. A *co-H-space* is a pair (X, α), where α is a comultiplication on X. A comultiplication α is called *homotopy-associative* if the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\alpha"'] & X \vee X \arrow[d, "\alpha \vee 1"] \\ X \vee X \arrow[r, "1 \vee \alpha"] & X \vee X \vee X \end{tikzcd}$$

is commutative up to homotopy. A comultiplication is called *homotopy-commutative* if the diagram

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[dr, "\alpha"'] & X \vee X \arrow[d, "T"] \\ & X \vee X \end{tikzcd}$$

is commutative up to homotopy, where T is the twisting map given by $T(x, *) = (*, x)$ and $T(*, x) = (x, *)$. A comultiplication α on X induces a binary operation on the set $[X, Z]$, for any Z. The operation is written additively and defined as follows: If $f, g \in [X, Z]$, then $f +_\alpha g = (f \mid g) \circ \alpha$, where $(f \mid g): X \vee X \to Z$ is the map determined by f and g. It is known that this binary operation gives $[X, Y]$ the structure of an algebraic loop [H-M-R$_2$, Th. 2.3]. In particular, every element f of $[X, Z]$ has a unique left inverse $L(f)$ and a unique right inverse $R(f)$; i.e., $L(f) +_\alpha f = 0$ and $f +_\alpha R(f) = 0$, where 0 is the homotopy class of the constant map. If α is a homotopy-associative comultiplication, the left and right inverses are identical.

Now suppose that Y is a 1-connected rational space with $H_*(Y)$ finite dimensional. It is known that if Y admits a comultiplication, then Y has the homotopy type of a finite wedge of rational spheres, i.e., $Y \simeq S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_k+1}$, for some integers $n_1, n_2, \ldots, n_k$ [Be$_1$, Th. 2.2], [He]. Thus, to investigate comultiplications on a rational space Y, it is sufficient to study wedges of rational spheres. Consider the following example:

2.1 EXAMPLE. The pinching map $S^p \to S^p \vee S^p$ is a comultiplication on the p-sphere, for any p [Sp, p. 41]. Therefore, any wedge of spheres $S^{n_1+1} \vee \cdots \vee S^{n_k+1}$ is a co-H-space, with comultiplication constructed from the pinching map on each factor. If we denote this map by σ, then $(S^{n_1+1} \vee \cdots \vee S^{n_k+1}, \sigma)$ is a homotopy-associative, homotopy-commutative co-H-space.

Thus any wedge of rational spheres $S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_k+1}$ admits a standard comultiplication $\sigma_{\mathbb{Q}}$ obtained by rationalizing σ. It follows that $(S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_k+1}, \sigma_{\mathbb{Q}})$ is homotopy-associative and homotopy-commutative. If (Y, α) is a rational co-H-space, then $Y \simeq S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_k+1}$ as spaces, but the homotopy equivalence from Y to $S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_k+1}$ need not be a co-H-map from (Y, α) to $(S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_k+1}, \sigma_{\mathbb{Q}})$.

The preceding discussion translates easily into *DG* Lie algebra terms, via the Quillen minimal model functor. If Y is a rational co-H-space, then Y is a wedge of rational spheres, up to homotopy. Consequently, the Quillen model $\mathbb{L}(s^{-1}\tilde{H}_*(Y))$ of Y has trivial differential. Furthermore, the Quillen minimal model preserves coproducts [Ne, Lem. 8.6], and so the Quillen model of the wedge $Y \vee Y$ is the coproduct $\mathbb{L}(s^{-1}\tilde{H}_*(Y)) \sqcup \mathbb{L}(s^{-1}\tilde{H}_*(Y))$, with trivial differential. Thus a comultiplication α on a rational co-H-space Y induces a map of free graded Lie algebras $\hat{\alpha} : \mathbb{L}(s^{-1}\tilde{H}_*(Y)) \to \mathbb{L}(s^{-1}\tilde{H}_*(Y)) \sqcup \mathbb{L}(s^{-1}\tilde{H}_*(Y))$. Denote the Quillen model of Y by L_Y, and the Quillen model of $Y \vee Y$ by $L_Y \sqcup L'_Y$, where L'_Y is just a copy of L_Y. Then the projection $p : Y \vee Y \to Y$ induces the canonical projection $\pi : L_Y \sqcup L'_Y \to L_Y$ of Quillen models, where $\pi(x) = x$ and $\pi(x') = 0$; similarly the projection $p' : Y \vee Y \to Y$ induces $\pi' : L_Y \sqcup L'_Y \to L_Y$, $\pi'(x) = 0$ and $\pi'(x') = x$.

2.2 LEMMA. *Let Y be a rational co-H-space and let L_Y be the Quillen model of Y. There exists a bijection of sets*:

$$\left\{\begin{matrix}\textit{Homotopy classes of}\\ \textit{comultiplications on } Y\end{matrix}\right\} \longleftrightarrow \left\{\begin{matrix}\textit{Homomorphisms } \phi : L_Y \to L_Y \sqcup L'_Y\\ \textit{with } \pi \circ \phi = id = \pi' \circ \phi\end{matrix}\right\},$$

where π and π' are the projections.

Proof. The bijection between the homotopy set $[Y, Y \vee Y]$ and homotopy classes of differential graded Lie algebra maps $[L_Y, L_{Y \vee Y}]$ restricts to a bijection between homotopy classes of comultiplications on Y and homotopy classes of differential graded Lie algebra maps $\phi : L_Y \to L_Y \sqcup L'_Y$ with $\pi \circ \phi \simeq \mathrm{id}$ and $\pi' \circ \phi \simeq \mathrm{id}$. However, L_Y and $L_{Y \vee Y}$ have zero differential. For differential graded Lie algebra maps between Lie algebras with zero differential, homotopy reduces to equality, and the result now follows. □

2.3 DEFINITION. (cf. [Sc, p. 67]) A map of Lie algebras $\phi : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ is called a *Lie algebra comultiplication*, or simply a *comultiplication*, if $\pi \circ \phi = \mathrm{id}$ and $\pi' \circ \phi = \mathrm{id}$, where V' is a copy of V.

2.4 EXAMPLE. The standard comultiplication map $\sigma_{\mathbb{Q}}$ on a wedge of rational spheres $S^{n_1+1}_{\mathbb{Q}} \vee \cdots \vee S^{n_k+1}_{\mathbb{Q}}$ induces the comultiplication $\phi_0 : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ given by $\phi_0(v) = v + v'$, for all $v \in V$. We call ϕ_0 the *standard comultiplication*.

By Lemma 2.2, the set of homotopy classes of comultiplications on a rational co-H-space Y is equivalent to the set of comultiplications on $\mathbb{L}(V)$, where $V = s^{-1}\tilde{H}_*(Y)$. Thus the focus of most of this paper is the study of those Lie algebra homomorphisms $\phi : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ such that $\phi(v) = v + v' + \xi_v$, for some $\xi_v \in \mathbb{L}(V) \sqcup \mathbb{L}(V')$ with $\pi(\xi_v) = 0 = \pi'(\xi_v)$.

The following terminology is adapted from [N-M]:

2.5 DEFINITION. A graded linear transformation $P : V \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ is called a *perturbation* if $\pi \circ P$ and $\pi' \circ P$ are both zero. A perturbation P is called a *one-stage perturbation* if there is an integer n, such that $P(V_i) = 0$, for all $i \neq n$.

Now let $V = \langle x_1, \ldots, x_k \rangle$ and $V' = \langle x'_1, \ldots, x'_k \rangle$, where $|x_i| = |x'_i| = n_i$. Note that for a perturbation P, $P(x_i)$ is a sum of brackets of the elements $x_1, x_2, \ldots, x_k, x'_1, x'_2, \ldots, x'_k$, with no bracket containing only elements from $x_1, x_2, \ldots, x_k$ or only elements from $x'_1, x'_2, \ldots, x'_k$. Also observe that any Lie algebra comultiplication ϕ restricted to V is of the form $\phi_0|_V + P$, for P a perturbation. We write this as $\phi = \phi_0 + P$.

2.6 COROLLARY. *If Y is a rational co-H-space, then the set of homotopy classes of comultiplications on Y is in* 1-1 *correspondence with the set of all perturbations $P : s^{-1}\tilde{H}_*(Y) \to \mathbb{L}(s^{-1}\tilde{H}_*(Y)) \sqcup \mathbb{L}(s^{-1}\tilde{H}_*(Y)')$.*

Proof. Suppose $\phi : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ is a Lie algebra comultiplication. Then $\phi = \phi_0 + P$, where P is a perturbation. The result follows from Lemma 2.2. □

Notice Corollary 2.6 implies that a rational co-H-space admits either a single comultiplication or infinitely many, since the set of perturbations is a vector space over $\mathbb{Q}$. The dimension of this vector space could be used to measure, in some sense, the size of the set of homotopy classes of comultiplications on a rational co-H-space. Indeed, to calculate this dimension, it would be sufficient to know $\dim(V_i)$ and $\dim((\mathbb{L}(V) \sqcup \mathbb{L}(V'))_i)$, for each i. A Witt formula for graded Lie algebras (cf. [Hi_2, p. 155]) provides a formula for the latter in terms of the dimensions of the V_i's. However, we content ourselves with giving necessary and sufficient conditions for a rational co-H-space to admit infinitely many comultiplications.

2.7 PROPOSITION. *Let Y be a rational co-H-space with $Y \simeq S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_k+1}$, for some integers $n_1, n_2, \ldots, n_k$ with $n_i \geq 1$. Then Y admits infinitely many comultiplications if and only if there is a j such that $n_j = \Sigma_{i=1}^r a_i n_i$, for some integers $a_i \geq 0$ with $\Sigma_{i=1}^r a_i \geq 2$.*

Proof. Y admits infinitely many comultiplications if and only if there exists a non-zero perturbation $P : V \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$, where $V = s^{-1}\tilde{H}_*(Y) = \langle x_1, \ldots, x_k \rangle$. If $n_j = \Sigma_{i=1}^r a_i n_i$, with $a_r \geq 1$ and $\Sigma_{i=1}^r a_i \geq 2$, then define the non-zero perturbation $P(x_j) = \mathrm{ad}^{a_1}(x_1)\mathrm{ad}^{a_2}(x_2) \cdots \mathrm{ad}^{a_r-1}(x_r)(x_r')$. On the other hand, if $P(x_j) \neq 0$ for some j, then $n_j = \Sigma_{i=1}^r a_i n_i$ with $\Sigma_{i=1}^r a_i \geq 2$. □

Section 3—Associativity

In this section we give a complete determination of associative Lie algebra comultiplications with one-stage perturbation. This leads to necessary and sufficient conditions for a rational co-H-space to admit infinitely many homotopy classes of homotopy-associative comultiplications and to admit infinitely many homotopy classes of non-homotopy-associative comultiplications. We assume for the remainder of the paper that Y is a rational co-H-space with $Y \simeq S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_k+1}$ and that $1 \leq n_1 \leq n_2 \leq \cdots \leq n_k$. We set $V = s^{-1}\tilde{H}_*(Y)$, and write $x_1, x_2, \ldots, x_k$ for a fixed basis of V, where $|x_i| = n_i$. The following notation will be used freely in this section:

3.1 NOTATION. For the graded vector space V, let V' and V'' be copies of V. The free Lie algebra $\mathbb{L}(V) \sqcup \mathbb{L}(V') = \mathbb{L}(V \oplus V')$ will often be denoted $\mathbb{L}(V, V')$, and the free Lie algebra $\mathbb{L}(V) \sqcup \mathbb{L}(V') \sqcup \mathbb{L}(V'') = \mathbb{L}(V \oplus V' \oplus V'')$ will similarly be denoted $\mathbb{L}(V, V', V'')$. We define homomorphisms $\beta, \gamma, \delta : \mathbb{L}(V, V') \to \mathbb{L}(V, V', V'')$ by first defining them on vector space generators and then extending to the free Lie algebras. For $v \in V$ set $\beta(v) = v + v'$, $\beta(v') = v''$; $\gamma(v) = v$, $\gamma(v') = v' + v''$; and $\delta(v) = v'$, $\delta(v') = v''$. We regard $\mathbb{L}(V, V')$ and $\mathbb{L}(V', V'')$ as contained in $\mathbb{L}(V, V', V'')$.

Recall from Section 2 the condition for a comultiplication to be homotopy-associative. It follows, as in the discussion there, that a comultiplication $\alpha : Y \to Y \vee Y$ on a rational co-H-space Y is homotopy-associative if and only if the corresponding induced map $\hat{\alpha} : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ satisfies $(1 \sqcup \hat{\alpha})\hat{\alpha} = (\hat{\alpha} \sqcup 1)\hat{\alpha}$:

$$\begin{tikzcd}
\mathbb{L}(V) \arrow[r, "\hat{\alpha}"] \arrow[d, "\hat{\alpha}"'] & \mathbb{L}(V) \sqcup \mathbb{L}(V') \arrow[d, "1 \sqcup \hat{\alpha}"] \\
\mathbb{L}(V) \sqcup \mathbb{L}(V') \arrow[r, "\hat{\alpha} \sqcup 1"] & \mathbb{L}(V) \sqcup \mathbb{L}(V') \sqcup \mathbb{L}(V'').
\end{tikzcd}$$

3.2 DEFINITION. A Lie algebra comultiplication $\phi : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ for which $(1 \sqcup \phi)\phi = (\phi \sqcup 1)\phi$ is called *associative.*

3.3 LEMMA. *Suppose* $\phi = \phi_0 + P : \mathbb{L}(V) \to \mathbb{L}(V, V')$ *is a comultiplication, where* $\phi_0(v) = v + v'$ *for all* v *and* $P = 0$ *on* V_i, $i < n$.

(i) *If* ϕ *is associative, then* $P(v) + \beta(P(v)) = \delta(P(v)) + \gamma(P(v))$ *for all* $v \in V_n$.

(ii) *Suppose further that* P *is a one-stage perturbation with* $P = 0$ *on* V_i, $i > n$. *If* $P(v) + \beta(P(v)) = \delta(P(v)) + \gamma(P(v))$ *for all* $v \in V_n$, *then* ϕ *is associative.*

Proof. We have $(\phi \sqcup 1)\phi(v) = (\phi \sqcup 1)(v + v' + P(v)) = v + v' + P(v) + v'' + (\phi \sqcup 1)P(v)$, and similarly $(1 \sqcup \phi)\phi(v) = v + v' + v'' + P(v') + (1 \sqcup \phi)P(v)$. Thus ϕ is associative if and only if $P(v) + (\phi \sqcup 1)P(v) = P(v') + (1 \sqcup \phi)P(v)$ for all v. If $v \in V_n$, then $P(v)$ is a linear combination of brackets of elements of degree less than n, and thus $(\phi \sqcup 1)P(v) = \beta(P(v))$ and $(1 \sqcup \phi)P(v) = \gamma(P(v))$. Furthermore $P(v') = \delta(P(v))$. Hence for $v \in V_n$, $(\phi \sqcup 1)\phi(v) = (1 \sqcup \phi)\phi(v)$ if and only if $P(v) + \beta(P(v)) = \delta(P(v)) + \gamma(P(v))$. This implies (i). For (ii), note that $\phi = \phi_0$ on elements not of degree n and that ϕ_0 is associative. $\square$

Subsequent results in this section require additional notation for perturbations.

3.4 NOTATION. Let $P : V \to \mathbb{L}(V, V')$ be a perturbation. We denote by $P_r(x_i)$ the linear combination in the expression for $P(x_i)$ involving only Lie brackets in $x_1, \ldots, x_k, x'_1, \ldots, x'_k$ of length r. If $P(x_i) = P_2(x_i)$ for all i, we say that the perturbation P is *quadratic.*

3.5 PROPOSITION. *Suppose $\phi : \mathbb{L}(V) \to \mathbb{L}(V, V')$ is a comultiplication of the form $\phi = \phi_0 + P$, with P a one-stage perturbation. If P is quadratic, then ϕ is associative.*

Proof. Let P be non-zero on V_n. Suppose $x_r \in V_n$, and write $P(x_r) = \Sigma\, \lambda_{i,j}[x_i, x_j']$. Then $P(x_r) + \beta(P(x_r)) = \Sigma\, \lambda_{i,j}[x_i, x_j'] + \Sigma\, \lambda_{i,j}[x_i + x_i', x_j'']$, and $\delta(P(x_r)) + \gamma(P(x_r)) = \Sigma\, \lambda_{i,j}[x_i', x_j''] + \Sigma\, \lambda_{i,j}[x_i, x_j' + x_j'']$. Since these are equal, it follows from Lemma 3.3 that ϕ is associative. □

3.6 REMARK. A similar argument to that of 3.5 holds for certain quadratic perturbations P that are not necessarily one-stage. One simply requires that, if $P(x_r) = \Sigma\, \lambda_{i,j}^r[x_i, x_j']$ for all r, then $P(x_i) = 0$ and $P(x_j') = 0$ for each r, i, j such that $\lambda_{i,j}^r \neq 0$.

We need to consider the universal enveloping algebra of a free graded Lie algebra. For a graded vector space V, the universal enveloping algebra of $\mathbb{L}(V)$ is the graded tensor algebra $T(V)$ on V [M-M], [Ta, p. 19]. Multiplication of elements v and w in $T(V)$ is denoted by $vw = v \otimes w$. An element of $T(V)$ is said to be of *homogeneous length* $r \geq 1$ if it can be written as a linear combination, with non-zero coefficients, of products of r elements of V.

3.7 EXAMPLE. Let $V = \langle x, y \rangle$, where $|x| = p$ and $|y| = 3p$. For degree reasons, any comultiplication is of the form $\phi(x) = \phi_0(x) = x + x'$, and $\phi(y) = \phi_0(y) + P(y)$, where $P(y) = \lambda[x, [x, x']] + \mu[x', [x, x']]$ with $\lambda, \mu \in \mathbb{Q}$. Consider those cases with $\mu = 0$, i.e., $P(y) = \lambda[x, [x, x']]$. A simple calculation shows that $P(y) + \beta(P(y)) - \delta(P(y)) - \gamma(P(y)) = \lambda\{[x + x', [x + x', x'']] - [x, [x, x'']] - [x', [x', x'']]\}$, and we claim that this is non-zero unless $\lambda = 0$. Inspection alone suffices in this simple case, but consider the following argument which is applicable to other situations. Denote the element in the braces by χ, and consider the image of χ under the standard map $i : \mathbb{L}(V \oplus V' \oplus V'') \to T(V \oplus V' \oplus V'')$ into the universal enveloping algebra. Since i is injective, it is sufficient to show $i(\chi) \neq 0$. Now $i(\chi) = (x + x') \otimes (x + x') \otimes x'' - x \otimes x \otimes x'' - x' \otimes x' \otimes x'' + \eta$, where η is a sum of terms of homogeneous length 3, each having x'' in the first or second place. By expanding the parentheses, it is clear that for $i(\chi)$ to be zero, $(x \otimes x' + x' \otimes x) \otimes x''$ must be zero. This is not the case, and so $\chi \neq 0$. Hence ϕ is a non-associative comultiplication if $\lambda \neq 0$, by Lemma 3.3.

The basic idea behind this last example can be generalized to give the following:

3.8 PROPOSITION. *Let $V = \langle x_1, \ldots, x_k \rangle$ and let $\phi = \phi_0 + P : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ be a comultiplication with $P = 0$ on V_i for $i \leq n - 1$. Suppose $P(x_r) = \lambda[x_{i_1}, [\ldots, [x_{i_{s-1}}, x_{i_s}'] \ldots]]$ for some $x_r \in V_n$ and some $i_1, \ldots, i_s$, where $s \geq 3$ and $\lambda \neq 0$. Then ϕ is non-associative.*

Proof. We check that $P(x_r) + \beta(P(x_r)) \neq \delta(P(x_r)) + \gamma(P(x_r))$. As in Example 3.7, a simple calculation shows that

$$\begin{aligned} &P(x_r) + \beta(P(x_r)) - \delta(P(x_r)) - \gamma(P(x_r)) \\ &\quad = \lambda\{[x_{i_1} + x'_{i_1}, [\ldots, [x_{i_{s-1}} + x'_{i_{s-1}}, x''_{i_s}]\ldots]] \\ &\qquad - [x_{i_1}, [\ldots, [x_{i_{s-1}}, x''_{i_s}]\ldots]] - [x'_{i_1}, [\ldots, [x'_{i_{s-1}}, x''_{i_s}]\ldots]]\}. \end{aligned}$$

Denote the term in braces by χ. The argument of Example 3.7 using the universal enveloping algebra can readily be generalized to show this latter term is non-zero. Inductively, one shows that in the universal enveloping algebra,

$$\begin{aligned} i(\chi) &= ((x_{i_1} + x'_{i_1}) \otimes \cdots \otimes (x_{i_{s-1}} + x'_{i_{s-1}}) \otimes x''_{i_s}) \\ &\quad - (x_{i_1} \otimes \cdots \otimes x_{i_{s-1}} \otimes x''_{i_s}) \\ &\quad - (x'_{i_1} \otimes \cdots \otimes x'_{i_{s-1}} \otimes x''_{i_s}) + \eta, \end{aligned}$$

where η is a sum of homogeneous length s terms, none of which has x''_{i_s} in the last place. Thus $P(x_r) + \beta(P(x_r)) - \delta(P(x_r)) - \gamma(P(x_r)) \neq 0$, and so ϕ is non-associative unless $\lambda = 0$, by Lemma 3.3. □

3.9 NOTATION. Let $T(V, V') = T(V \oplus V')$ and $T(V, V', V'') = T(V \oplus V' \oplus V'')$. In analogy to 3.1, define maps $\beta, \gamma, \delta : T(V, V') \to T(V, V', V'')$ as follows. On generators $v \in V$ and $v' \in V'$, set $\beta(v) = v + v'$, $\beta(v') = v''$; $\gamma(v) = v$, $\gamma(v') = v' + v''$; and $\delta(v) = v'$, $\delta(v') = v''$; and extend to algebra maps. If $V = \langle x_1, x_2, \ldots, x_k \rangle$, define the binomial $B_i \in T(V, V')$ by $B_i = x_i + x'_i$.

The central theorem of this section requires the following proposition:

3.10 PROPOSITION. *Let $\xi \in T(V, V')$ be of homogeneous length r, with $r \geq 3$. Then the equality $\xi + \beta(\xi) = \delta(\xi) + \gamma(\xi)$ holds if and only if*

$$\begin{aligned} \xi = \sum_J \lambda_J\{&B_{j_1} \otimes \cdots \otimes B_{j_{r-1}} \otimes B_{j_r} - x_{j_1} \otimes \cdots \otimes x_{j_{r-1}} \otimes x_{j_r} \\ &- x'_{j_1} \otimes \cdots \otimes x'_{j_{r-1}} \otimes x'_{j_r}\}, \end{aligned}$$

where each J is a sequence $j_1, \ldots, j_r$ with $j_i \in \{1, \ldots, k\}$, $|\xi| = n_{j_1} + \cdots + n_{j_r}$ and $\lambda_J \in \mathbb{Q}$.

The proof of 3.10 is postponed until the last section.

3.11 THEOREM. *Let* $V=\langle x_1,\ldots,x_k\rangle$ *and let* $\phi=\phi_0+P$ *be a comultiplication on* $\mathbb{L}(V)$ *with perturbation* P, *where* $P=0$ *on* V_i *for* $i<n$. *Write* $P(x_j)=\Sigma_{r\geq 2}P_r(x_j)$ *for each* $x_j\in V_n$.

(i) *If* ϕ *is associative, then for* $r\geq 3$, $P_r(x_j)$ *can be written*

$$\begin{aligned}P_r(x_j)=\sum_J \lambda^j_J\{&[x_{j_1}+x'_{j_1},[\ldots,[x_{j_{r-1}}+x'_{j_{r-1}},x_{j_r}+x'_{j_r}]\ldots]]\\&-[x_{j_1},[\ldots,[x_{j_{r-1}},x_{j_r}]\ldots]]-[x'_{j_1},[\ldots,[x'_{j_{r-1}},x'_{j_r}]\ldots]]\}\end{aligned}\qquad(*)$$

for each $x_j\in V_n$, *where* $\lambda^j_J\in\mathbb{Q}$ *and each* J *is a finite sequence* $j_1,j_2,\ldots,j_r$ with $j_i\in\{1,\ldots,k\}$ *and* $n=n_{j_1}+\cdots+n_{j_r}$.

(ii) *If* P *is a one-stage perturbation with* $P\neq 0$ *on* V_n *and if, for* $r\geq 3$, $P_r(x_j)$ *can be written as in* $(*)$ *for each* $x_j\in V_n$, *then* ϕ *is associative.*

Proof. We work in the universal enveloping algebra of $\mathbb{L}(V,V')$, then use a result of Quillen's to return to the Lie algebra. Since ϕ is associative, Lemma 3.3 implies

$$P(x_j)+\beta(P(x_j))=\delta(P(x_j))+\gamma(P(x_j))$$

for each $x_j\in V_n$. The maps β, γ and δ all preserve bracket length, so this equation splits into homogeneous bracket length components. Thus ϕ is associative implies that

$$P_r(x_j)+\beta(P_r(x_j))=\delta(P_r(x_j))+\gamma(P_r(x_j))\qquad(3.12)$$

for all $r\geq 2$. In order to work with this equality, we pass to the universal enveloping algebra $T(V,V',V'')$. Since the standard map $i:\mathbb{L}(V,V',V'')\to T(V,V',V'')$ is injective, equation (3.12) holds if and only if

$$iP_r(x_j)+i\beta(P_r(x_j))=i\delta(P_r(x_j))+i\gamma(P_r(x_j)).$$

Using the maps β,γ,δ of 3.1 and 3.9, we see that $i\beta=\beta i$, $i\gamma=\gamma i$ and $i\delta=\delta i$:

$$\begin{tikzcd}\mathbb{L}(V,V') \arrow[r, "\beta,\gamma,\delta"] \arrow[d, "i"'] & \mathbb{L}(V,V',V'') \arrow[d, "i"] \\ T(V,V') \arrow[r, "\beta,\gamma,\delta"] & T(V,V',V'').\end{tikzcd}$$

Therefore (3.12) holds if and only if $\xi_r + \beta(\xi_r) = \delta(\xi_r) + \gamma(\xi_r)$, where $\xi_r = i(P_r(x_j))$. By Proposition 3.10, this implies that, for each $r \geq 3$,

$$\xi_r = i(P_r(x_j)) = \sum_J \mu^j_J \{B_{j_1} \otimes \cdots \otimes B_{j_{r-1}} \otimes B_{j_r}$$
$$- x_{j_1} \otimes \cdots \otimes x_{j_{r-1}} \otimes x_{j_r} - x'_{j_1} \otimes \cdots \otimes x'_{j_{r-1}} \otimes x'_{j_r}\}.$$

But Lemma 2.2 of [Qu, Ap. B] asserts that, for any vector space W, the map $\rho : T(W) \to \mathbb{L}(W)$ given by

$$\rho(w_1 \otimes \cdots \otimes w_{m-1} \otimes w_m) = \frac{1}{m}[w_1, [\ldots, [w_{m-1}, w_m] \ldots]] \quad m > 0$$

is a left inverse for $i : \mathbb{L}(W) \to T(W)$. Hence $P_r(x_j) = \rho i(P_r(x_j))$, for each $r \geq 3$. This proves (i).

(ii) By Lemma 3.3, it is sufficient to show that $P(x_j) + \beta(P(x_j)) = \delta(P(x_j)) + \gamma(P(x_j))$ for each $x_j \in V_n$. Since the maps β, γ and δ all preserve bracket length, it suffices to check that $P_r(x_j) + \beta(P_r(x_j)) = \delta(P_r(x_j)) + \gamma(P_r(x_j))$ for each $r \geq 2$. For $r = 2$, the proof of Proposition 3.5 shows this equality holds. It also holds for $r \geq 3$, as is easily shown, since $P_r(x_j)$ has the form described by $(*)$. Thus ϕ is associative. $\square$

3.13 EXAMPLES. (i) Let $V = \langle x_p, y_{2p}, z_{5p} \rangle$ with subscripts denoting degree. Define a one-stage perturbation by

$$P(z) = [B_y, [B_x, B_y]] - [y, [x, y]] - [y', [x', y']]$$
$$+ [B_x, [B_x, [B_x, B_y]]] - [x, [x, [x, y]]] - [x', [x', [x', y']]],$$

where $B_x = x + x'$ and $B_y = y + y'$. Then Theorem 3.11 implies $\phi = \phi_0 + P$ is an associative comultiplication. On the other hand, consider the one-stage perturbation defined by

$$Q(z) = [y, [x, y']] + [y, [x', y]] + [y, [x', y']] + [y', [x, y]]$$
$$+ [y', [x, y']] + [y', [x', y]].$$

A direct calculation shows the comultiplication $\phi_0 + Q$ to be associative, and so $Q(z)$ can be written as in Theorem 3.11. In fact $Q(z) = [B_y, [B_x, B_y]] - [y, [x, y]] - [y', [x', y']]$. Thus it may be difficult to recognize that a given one-stage perturbation has the form of Theorem 3.11. Nonetheless, 3.11 provides a useful criterion for associativity, as will be seen.

(ii) Let $V = \langle x_p, y_{2p}, z_{3p} \rangle$ with subscripts denoting degree. Consider the comultiplication ϕ on $\mathbb{L}(x, y, z)$ defined by $\phi = \phi_0 + P$, with $P(x) = 0$, $P(y) = [x, x']$ and $P(z) = [x, y'] + [x', y] + [x, [x, x']] + [x', [x, x']]$. A straightforward computation shows that ϕ is an associative comultiplication, although P is not a one-stage perturbation. This example can be generalised to produce many associative comultiplications which do not arise from one-stage perturbations.

Theorem 3.11 leads to necessary and sufficient conditions for a rational co-H-space to admit infinitely many homotopy-associative or infinitely many non-homotopy-associative comultiplications. Let $Y \simeq S^{n_1+1}_{\mathbb{Q}} \vee \cdots \vee S^{n_k+1}_{\mathbb{Q}}$, and consider the following three conditions on the integers $n_1, n_2, \ldots, n_k$.

Condition (i): For some j, $n_j = a_1 n_{i_1} + a_2 n_{i_2}$ with a_1 and $a_2 \geq 0$ and $a_1 + a_2 = 2$.
Condition (ii): For some j, $n_j = a_1 n_1 + \cdots + a_r n_r$ with each $a_i \geq 0$, at least two of the a_i's non-zero and $\Sigma_{i=1}^r a_i \geq 3$.
Condition (iii): For some j, $n_j = a n_i$ for some i and $a \geq 3$.

These conditions are not mutually exclusive, but are exhaustive of the condition on $n_1, n_2, \ldots, n_k$ given in Proposition 2.7. Furthermore, when combined with earlier results, they yield the following theorems.

3.14 THEOREM. *If Y is a rational co-H-space with $Y \simeq S^{n_1+1}_{\mathbb{Q}} \vee \cdots \vee S^{n_k+1}_{\mathbb{Q}}$, then Y admits infinitely many homotopy classes of homotopy-associative comultiplications if and only if the integers $n_1, n_2, \ldots, n_k$ satisfy condition (i) or condition (ii).*

Proof. Suppose (i) or (ii) is satisfied. If (ii) holds, write $n_j = a_1 n_1 + \cdots + a_r n_r$, with $a_r \geq 1$. The one-stage perturbation

$$\begin{aligned} P(x_j) &= \mathrm{ad}^{a_r-1}(B_r)\mathrm{ad}^{a_1}(B_1) \cdots \mathrm{ad}^{a_{r-1}}(B_{r-1})(B_r) \\ &\quad - \mathrm{ad}^{a_r-1}(x_r)\mathrm{ad}^{a_1}(x_1) \cdots \mathrm{ad}^{a_{r-1}}(x_{r-1})(x_r) \\ &\quad - \mathrm{ad}^{a_r-1}(x'_r)\mathrm{ad}^{a_1}(x'_1) \cdots \mathrm{ad}^{a_{r-1}}(x'_{r-1})(x'_r) \end{aligned}$$

has length at least three and is non-zero. This gives an associative comultiplication $\phi_0 + P$ on $\mathbb{L}(V)$ by Theorem 3.11. Suppose (i) is satisfied. If $n_j = 2n_{i_t}$ for $t = 1$ or 2, set $Q(x_j) = [x_{i_t}, x'_{i_t}]$; otherwise, set $Q(x_j) = [x_{i_1}, x'_{i_2}]$. Then the one-stage perturbation Q gives an associative comultiplication $\phi_0 + Q$ on $\mathbb{L}(V)$ by Proposition 3.5. As was remarked in Section 2, the set of perturbations is a vector space over $\mathbb{Q}$. Thus one non-zero perturbation yields infinitely many perturbations by taking scalar multiples. It follows that Y admits infinitely many homotopy classes of homotopy-associative comultiplications.

On the other hand, assume there exist infinitely many homotopy classes of homotopy-associative comultiplications on Y, and that $\phi = \phi_0 + R$ is an associative comultiplication on $\mathbb{L}(V)$ with $R \neq 0$. If $R_2 \neq 0$, condition (i) must hold, so assume $R_2 = 0$. Consider the first n for which R is non-zero on V_n. Theorem 3.11 implies that

$$R_r(x_j) = \sum \lambda_J^j \{[B_{j_1}, [\ldots, [B_{j_{r-1}}, B_{j_r}] \ldots]] - [x_{j_1}, [\ldots, [x_{j_{r-1}}, x_{j_r}] \ldots]]$$
$$- [x'_{j_1}, [\ldots, [x'_{j_{r-1}}, x'_{j_r}] \ldots]]\},$$

for each $x_j \in V_n$ and for each $r \geq 3$. If, in each summand, $j_{r-2} = j_{r-1} = j_r$, then $R_r(x_j)$ would be zero, since $[\eta, [\eta, \eta]] = 0$ for all η. By assumption, this is not true for some r and j. Therefore some n_j can be written as a linear combination of $n_1, n_2, \ldots, n_{j-1}$ as in condition (ii). □

3.15 THEOREM. *If Y is a rational co-H-space with $Y \simeq S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_k+1}$, then Y admits infinitely many homotopy classes of non-homotopy-associative comultiplications if and only if the integers $n_1, n_2, \ldots, n_k$ satisfy condition (ii) or condition (iii).*

Proof. Suppose (ii) is satisfied and $n_j = a_1 n_1 + \cdots + a_r n_r$, with $a_r \geq 1$. Then the one-stage perturbation defined by $P(x_i) = 0$, for $i \neq j$, and

$$P(x_j) = \mathrm{ad}^{a_1}(x_1)\mathrm{ad}^{a_2}(x_2) \cdots \mathrm{ad}^{a_r - 1}(x_r)(x'_r),$$

gives a comultiplication $\phi = \phi_0 + P$ which is non-associative by Proposition 3.8. Similarly, suppose (iii) is satisfied. The one-stage perturbation defined by $P(x_i) = 0$, for $i \neq j$, and $P(x_j) = \mathrm{ad}^{a-1}(x_i)(x'_i)$ gives a non-associative comultiplication, again by Proposition 3.8. Thus if either (ii) or (iii) holds, Y admits infinitely many homotopy classes of non-homotopy-associative comultiplications.

Conversely, suppose $\phi_0 + P$ is a non-associative comultiplication. If $P_r \neq 0$ for some $r \geq 3$, then either (ii) or (iii) must hold for degree reasons. So suppose P is a quadratic perturbation, with $P(x_j) = \Sigma_{q,r} \lambda_{q,r}^j [x_q, x'_r]$, for each j. If $P(x_j)$ is a sum of brackets of elements on which P is zero for every j, then by Remark 3.6, $\phi_0 + P$ is associative. Thus for $\phi_0 + P$ to be non-associative, there must exist a j such that $P(x_j) = \Sigma_{q,r} \lambda_{q,r}^j [x_q, x'_r]$, and $P(x_q) \neq 0$ or $P(x_r) \neq 0$, for some non-zero $\lambda_{q,r}^j$. If $P(x_q) \neq 0$, then $P(x_q) = \Sigma_{s,t} \lambda_{s,t}^q [x_s, x'_t]$, and $n_q = n_s + n_t$ for each s and t in the sum. Therefore there exists some n_q, n_r, n_s, n_t with $n_j = n_q + n_r = n_s + n_t + n_r$, and so condition (ii) is satisfied. A similar argument holds if $P(x_r) \neq 0$. Hence if Y admits infinitely many homotopy classes of non-homotopy-associative comultiplications, condition (ii) or (iii) must hold. □

As an application of Theorems 3.14 and 3.15, we consider the case where Y is a wedge of two rational spheres.

3.16 PROPOSITION. *Let $Y \simeq S_{\mathbb{Q}}^{p+1} \vee S_{\mathbb{Q}}^{q+1}$. If $q = 2p$, then Y admits infinitely many homotopy classes of homotopy-associative comultiplications and no homotopy classes of non-homotopy-associative comultiplications. If $q = np$ with $n \geq 3$, then Y admits infinitely many homotopy classes of non-homotopy-associative comultiplications and a unique homotopy class of homotopy-associative comultiplications. Otherwise, i.e., if p does not divide q, Y admits a unique homotopy class of comultiplications, which is necessarily homotopy-associative.*

Proof. If p does not divide q, there is a unique homotopy class of comultiplications on Y by Proposition 2.7. Let $\{x_p, y_q\}$ be a basis for $s^{-1}\tilde{H}_*(Y)$, with subscripts denoting degrees. If $q = 2p$, the only possible perturbations are of the form $P(y) = \lambda[x, x']$, for $\lambda \in \mathbb{Q}$. These all give associative comultiplications by Proposition 3.5. This proves the first assertion. Now suppose $q = np$, with $n \geq 3$. By Theorem 3.15, Y admits infinitely many homotopy classes of non-homotopy-associative comultiplications. By Theorem 3.14, Y admits a unique homotopy class of homotopy-associative comultiplications. □

As a further application of Theorem 3.11, we give a dual version of a theorem of Leray-Samelson. To state the theorem, we need the concept of equivalence of comultiplications.

3.17 DEFINITION. If α and α' are two comultiplications on the space X, then α is *equivalent* to α' if there is a homotopy equivalence $f: X \to X$ such that $\alpha' \circ f = (f \vee f) \circ \alpha$. If ϕ, ϕ': $\mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ are Lie algebra comultiplications, then ϕ is *equivalent* to ϕ' if there is an isomorphism of Lie algebras $\theta : \mathbb{L}(V) \to \mathbb{L}(V)$ such that $\phi' \circ \theta = (\theta \sqcup \theta) \circ \phi$.

The notion of equivalence of Lie algebra comultiplications corresponds to the notion of equivalence of comultiplications on a co-H-space. The latter in turn is dual to equivalence of multiplications on an H-space. We show next that if V is oddly graded, then any two associative comultiplications on $\mathbb{L}(V)$ are equivalent. This is an appropriate dual of the Leray-Samelson theorem which asserts that any two associative diagonals on an oddly generated, free, commutative Hopf algebra over $\mathbb{Q}$ are equivalent [M–M, p. 258], [Cu, p. 8].

3.18 THEOREM. *Suppose $\phi : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ is an associative comultiplication. If $V_i = 0$ for even i, then ϕ is equivalent to the standard comultiplication ϕ_0.*

Proof. Let $V_{(n)}$ be the graded vector sub-space of V consisting of all elements of degree $\leq n$ and let $\{x_1, x_2, \ldots, x_k\}$ be a basis of V. We show inductively that, for all positive integers n, there are comultiplications ϕ^n on $\mathbb{L}(V)$ such that ϕ is equivalent to ϕ^n and $\phi^n = \phi_0$ on $\mathbb{L}(V_{(n)})$. This is true for $n = 1$ since $\phi = \phi_0$ on $\mathbb{L}(V_{(1)})$. Assume ϕ^{n-1} exists with $\phi^{n-1} = \phi_0$ on $\mathbb{L}(V_{(n-1)})$. Write $\phi^{n-1} = \phi_0 + P$. By Theorem 3.11, for each $x_j \in V_n$ and each $r \geq 3$,

$$\begin{aligned} P_r(x_j) = \sum_J \lambda^j_J\{&[x_{j_1} + x'_{j_1}, [\ldots, [x_{j_{r-1}} + x'_{j_{r-1}}, x_{j_r} + x'_{j_r}] \ldots]] \\ &- [x_{j_1}, [\ldots, [x_{j_{r-1}}, x_{j_r}] \ldots]] - [x'_{j_1}, [\ldots, [x'_{j_{r-1}}, x'_{j_r}] \ldots]]\}, \end{aligned}$$

where each $J = j_1, j_2, \ldots, j_r$. Furthermore, P has zero quadratic part, since $V_i = 0$ for even i. Thus $\phi^{n-1}(x_j) = \phi_0(x_j) + \Sigma_{r \geq 3} P_r(x_j)$. Now define a map $\theta : V \to \mathbb{L}(V)$ by putting $\theta(x_j) = x_j$ if $|x_j| \neq n$ and for each $x_j \in V_n$, $\theta(x_j) = x_j + \Sigma_{r \geq 3} \theta_r(x_j)$, where $\theta_r(x_j) = \Sigma_J \lambda^j_J [x_{j_1}, [\ldots, [x_{j_{r-1}}, x_{j_r}] \ldots]]$. The summation is taken over the same $J = j_1, \ldots, j_r$ which appear in the decomposition of $P_r(x_j)$ above. Extend θ to a map of Lie algebras $\theta : \mathbb{L}(V) \to \mathbb{L}(V)$. Then θ is an isomorphism of Lie algebras; indeed for $x_j \in V_n$, $\theta^{-1}(x_j) = x_j - \Sigma_{r \geq 3} \theta_r(x_j)$ while for $x_j \notin V_n$, $\theta^{-1}(x_j) = x_j$. Define a comultiplication $\phi^n : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ by $\phi^n = (\theta \sqcup \theta)\phi^{n-1}\theta^{-1}$. By construction, ϕ^n is equivalent to ϕ^{n-1}. On elements $v \in V_{(n-1)}$, $\theta(v) = v$, and so $\phi^n(v) = \phi^{n-1}(v)$. Thus

$$\begin{aligned} \phi^n(x_j) &= (\theta \sqcup \theta)\phi^{n-1}\left(x_j - \sum_{r,J} \lambda^j_J [x_{j_1}, [\ldots, [x_{j_{r-1}}, x_{j_r}] \ldots]]\right) \\ &= (\theta \sqcup \theta)\left(x_j + x'_j - \sum_{r,J} \lambda^j_J [x_{j_1}, [\ldots, [x_{j_{r-1}}, x_{j_r}] \ldots]]\right. \\ &\qquad \left. - \sum_{r,J} \lambda^j_J [x'_{j_1}, [\ldots, [x'_{j_{r-1}}, x'_{j_r}] \ldots]]\right), \end{aligned}$$

since $\phi^{n-1}(x_{j_s}) = \phi_0(x_{j_s}) = x_{j_s} + x'_{j_s}$, for each x_{j_s} appearing in $\theta_r(x_j)$. Now $\theta \sqcup \theta = 1$ on the entries appearing in the sums, so $\phi^n(x_j) = x_j + x'_j$. Hence ϕ^{n-1} is equivalent to ϕ^n and $\phi^n = \phi_0$ on $V_{(n)}$. This completes the induction. □

3.19 COROLLARY. *If Y is a rational co-H-space with $Y \simeq S^{n_1+1}_{\mathbb{Q}} \vee \cdots \vee S^{n_k+1}_{\mathbb{Q}}$ with all n_i odd, then any two homotopy-associative comultiplications on Y are equivalent.*

Section 4—Homotopy-Commutativity

The definition of homotopy-commutativity for co-H-space given in Section 2 can be translated into the following Lie algebra definition:

4.1 DEFINITION. A comultiplication $\phi : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ is called *commutative* if $\tau\phi = \phi$, where $\tau : \mathbb{L}(V) \sqcup \mathbb{L}(V') \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ is the twisting isomorphism given by $\tau(v) = v'$, $\tau(v') = v$ for $v \in V$ and $v' \in V'$.

Now let (Y, α) be a rational co-H-space and let $\hat{\alpha} : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ be the map induced by α, where $V = s^{-1}\tilde{H}_*(Y)$. It is straightforward to see that α is homotopy-commutative if and only if $\hat{\alpha}$ is commutative. Write $\hat{\alpha} = \phi_0 + P$, where ϕ_0 is the standard comultiplication and P is a perturbation. Since $\tau\phi_0 = \phi_0$, $\hat{\alpha}$ is commutative if and only if $\tau P = P$. Clearly, for $v \in V$, $\tau P(v) = P(v)$ precisely when $P(v) = w + \tau(w)$, for some $w \in \mathbb{L}(V)$. This fact provides the following analogue of Theorem 3.14.

4.2 PROPOSITION. *Let Y be a rational co-H-space with $Y \simeq S^{n_1+1}_{\mathbb{Q}} \vee \cdots \vee S^{n_k+1}_{\mathbb{Q}}$. Then Y admits infinitely many homotopy classes of homotopy-commutative comultiplications if and only if one of the following conditions holds: (i) For some $j, n_j = n_{i_1} + n_{i_2}$ with $i_1 \neq i_2$ or $n_j = 2n_i$ with n_i odd. (ii) For some $j, n_j = a_1n_1 + a_2n_2 + \cdots + a_rn_r$, with each $a_i \geq 0$ and $\Sigma_{i=1}^r a_i \geq 3$.*

Proof. Suppose (ii) is satisfied, with $a_r \geq 1$. Define a one-stage perturbation P as follows: $P(x_j) = w + \tau(w)$, where $w = \mathrm{ad}^{a_1}(x_1)\mathrm{ad}^{a_2}(x_2) \cdots \mathrm{ad}^{a_r-1}(x_r)(x_r')$ and $P(x_i) = 0$ for $i \neq j$. It is easy to see that $\lambda P(x_j) \neq 0$, for $\lambda \neq 0 \in \mathbb{Q}$. By construction, this provides infinitely many commutative comultiplications. Suppose that (i) is satisfied. Then set $w = [x_{i_1}, x_{i_2}']$ if $n_j = n_{i_1} + n_{i_2}$, and set $w = [x_{i_1}, x_{i_1}']$ if $n_j = 2n_{i_1}$. As before, define $P(x_j) = w + \tau(w)$ and $P(x_i) = 0$ for $i \neq j$. In either case, $\lambda P(x_j) \neq 0$, for $\lambda \neq 0 \in \mathbb{Q}$, and so we have infinitely many commutative comultiplications.

Conversely, suppose neither (i) nor (ii) holds. Then, for any comultiplication $\phi = \phi_0 + P$, P must have the form $P(x_j) = \lambda[x_i, x_i']$, with x_i and x_i' of even degree and $\lambda \in \mathbb{Q}$. If ϕ is commutative, then $\lambda[x_i, x_i'] = P(x_j) = \tau P(x_j) = \lambda[x_i', x_i] = -\lambda[x_i, x_i']$, so $\lambda = 0$. Hence, $\phi = \phi_0$ is the unique commutative comultiplication. $\square$

4.3 EXAMPLES. Let $Y \simeq S^{p+1}_{\mathbb{Q}} \vee S^{rp+1}_{\mathbb{Q}}$ and let $V = \langle x_p, y_{rp} \rangle$ with subscripts denoting degrees. Consider the comultiplications $\phi_\lambda = \phi_0 + \lambda P : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$, where $\lambda \in \mathbb{Q}$ and P is a one-stage perturbation with $P(x) = 0$.

(i) Let $r = 2$ with p even and put $P(y) = [x, x']$. The comultiplications ϕ_λ are associative by Proposition 3.5, but are clearly non-commutative if $\lambda \neq 0$. Thus Y admits infinitely many homotopy classes of homotopy-associative, non-homotopy-commutative comultiplications.

(ii) Let $r = 2$ with p odd and put $P(y) = [x, x']$. The comultiplications ϕ_λ are associative by Proposition 3.5 and are clearly commutative. Thus Y admits infinitely many homotopy classes of homotopy-associative, homotopy-commutative comultiplications.

(iii) Let $r \geq 3$ and put $P(y) = \mathrm{ad}^{r-1}(x)(x') + \mathrm{ad}^{r-1}(x')(x)$. The comultiplications ϕ_λ are non-associative if $\lambda \neq 0$ by Proposition 3.16, but are clearly commutative. Thus Y admits infinitely many homotopy classes of homotopy-commutative, non-homotopy-associative comultiplications.

4.4 EXAMPLE. Every comultiplication whose perturbations P_r, for $r \geq 3$, have the form $(*)$ of Theorem 3.11 and is such that $\tau P_2 = P_2$ is also commutative. For example, let $V = \langle x_p, y_p, z_{3p} \rangle$. Then the one-stage perturbation given by $P(z) = [x + x', [x + x', y + y']] + [y + y', [x + x', y + y']] - [x, [x, y]] - [x', [x', y']] - [y, [x, y]] - [y', [x', y']]$, defines a commutative comultiplication.

4.5 REMARK. Let $Y \simeq S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_k+1}$, with the n_i's all odd. Then every homotopy-associative comultiplication on Y is also homotopy-commutative. This follows, since Corollary 3.19 asserts that any homotopy-associative comultiplication is equivalent to the standard one, which is homotopy-commutative.

Section 5—Homotopy-Inverses

In analogy to the discussion of left and right inverses for a co-H-space in Section 2, we consider left and right inverses for Lie algebra comultiplications. Let $\phi : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ be a comultiplication and let M be a free differential graded Lie algebra. Then ϕ induces a binary operation on the set of homotopy classes of differential graded Lie algebra maps $[\mathbb{L}(V), M]$. This is denoted additively and defined as follows: If $f, g \in [\mathbb{L}(V), M]$, then $f +_\phi g = (f \mid g) \circ \phi$, where $(f \mid g) : \mathbb{L}(V) \sqcup \mathbb{L}(V') \to M$ is the homotopy class given by f and g on each summand. The class of the zero homomorphism is a two-sided identity element for $+_\phi$. As in the topological case, this operation admits unique left and right inverses.

5.1 LEMMA. *If $\mathbb{L}(V)$ is a free Lie algebra with comultiplication ϕ and M is a free DG Lie algebra, then every element $f \in [\mathbb{L}(V), M]$ has a unique left inverse and a unique right inverse with respect to the binary operation $+_\phi$. In particular, the identity element $1 \in [\mathbb{L}(V), \mathbb{L}(V)]$ has unique left inverse λ and unique right inverse ρ.*

Proof. It suffices to prove the existence of a unique λ and ρ in $[\mathbb{L}(V), \mathbb{L}(V)]$; for then the left inverse of f is just $f \circ \lambda$, and the right inverse is $f \circ \rho$. Note that $\lambda +_\phi 1 = 0$ is equivalent to $(\lambda \mid 1)\phi(v) = 0$, for all $v \in V$. Writing $\phi(v) = v + v' + P(v)$, we obtain

$$\lambda(v) = -v - (\lambda \mid 1)P(v). \tag{5.2}$$

This gives an inductive method for constructing λ: Let $V_{(i)}$ denote the graded vector subspace of V of all elements of degree $\leq i$ and suppose λ is defined on $\mathbb{L}(V_{(n-1)})$. If $v \in V_n$, then for degree reasons, $P(v) \in \mathbb{L}(V_{(n-1)}) \sqcup \mathbb{L}(V'_{(n-1)})$. Now $(\lambda \mid 1)$ is defined on $\mathbb{L}(V_{(n-1)}) \sqcup \mathbb{L}(V'_{(n-1)})$ by the inductive hypothesis, so $\lambda(v)$ is defined by (5.2). The induction starts by setting $\lambda(v) = -v$ for $v \in V_1$. This proves the existence and uniqueness of λ. A similar argument for ρ holds by setting

$$\rho(v) = -v - (1 \mid \rho)P(v). \qquad \square \tag{5.3}$$

We now investigate whether or not the left inverse λ agrees with the right inverse ρ, for a free Lie algebra $\mathbb{L}(V)$.

5.4 DEFINITION. Let $P : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ be a perturbation, where $V = \langle x_1, \ldots, x_k \rangle$. P is called an *even* perturbation if, for each i, $P_r(x_i) = 0$ for all odd r. Note that if P is even, then $P(x_i)$ can be written as a linear combination of brackets in the elements $x_1, x_2, \ldots, x_k, x'_1, x'_2, \ldots, x'_k$, such that in each bracket the number of primed entries is congruent modulo two to the number of un-primed entries.

5.5 PROPOSITION. *Let $\phi : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ be a Lie algebra comultiplication with $V = \langle x_1, \ldots, x_k \rangle$. If $\phi = \phi_0 + P$, with P a one-stage, even perturbation, then the left inverse λ equals the right inverse ρ.*

Proof. Suppose $P \neq 0$ on V_n. If $x_j \in V_m$ for $m \neq n$, then $\lambda(x_j) = -x_j = \rho(x_j)$ by (5.2) and (5.3). Note that $(\lambda \mid 1)(x_j) = -x_j$, $(\lambda \mid 1)(x'_j) = x_j$, $(1 \mid \rho)(x_j) = x_j$ and $(1 \mid \rho)(x'_j) = -x_j$. Now let $x_i \in V_n$ and let χ be a bracket in the elements $x_1, x_2, \ldots, x_k, x'_1, x'_2, \ldots, x'_k$ appearing in $P(x_i)$ with non-zero coefficient. Suppose χ is made up of r of $x_1, x_2, \ldots, x_k$ and s of $x'_1, x'_2, \ldots, x'_k$. Then $(\lambda \mid 1)(\chi) = (-1)^r \tilde{\chi}$ and $(1 \mid \rho)(\chi) = (-1)^s \tilde{\chi}$, where $\tilde{\chi} \in \mathbb{L}(V)$ is obtained from χ by removing all primes from the $x'_1, \ldots, x'_k$ which appear in χ. Since P is even, $(\lambda \mid 1)(\chi) = (1 \mid \rho)(\chi)$. Thus $(\lambda \mid 1)P(x_i) = (1 \mid \rho)P(x_i)$. By (5.2) and (5.3), $\lambda(x_i) = \rho(x_i)$, and so $\lambda = \rho$. $\square$

5.6 COROLLARY. *Every comultiplication on a rational co-H-space Y which has the homotopy type of a wedge of two rational spheres has left homotopy-inverse equal to right homotopy-inverse.*

Proof. Let $V = s^{-1}\tilde{H}_*(Y) = \langle x_p, y_q \rangle$. If $q = 2p$, then $P(y) = \mu[x, x']$ for some $\mu \in \mathbb{Q}$. This is a one-stage, even perturbation, so $\lambda = \rho$ by 5.5. Now let $q = rp$ with $r \geq 3$. Then $P(y)$ is a linear combination of r-fold brackets, each bracket involving only x and x'. Hence $(\lambda \mid 1)P(y) = 0$, since $[x, [x, x]] = 0$. Similarly $(1 \mid \rho)P(y) = 0$. Thus by (5.2) and (5.3), $\lambda = \rho$. If $q \neq rp$ for any integer r, then $P = 0$, and $\lambda = \rho$ by (5.2) and (5.3). Consequently, for any comultiplication on Y with left inverse l and right inverse r, $l = r$. $\square$

Notice that by combining 5.6 with 3.16, we have examples of comultiplications on $S_{\mathbb{Q}}^{p+1} \vee S_{\mathbb{Q}}^{rp+1}$, for $r \geq 3$, which are not homotopy-associative and yet for which $l = r$. We conclude this section with an example of a wedge of three rational spheres that admits a comultiplication for which $l \neq r$.

5.7 EXAMPLE. Let $Y \simeq S_{\mathbb{Q}}^{p+1} \vee S_{\mathbb{Q}}^{q+1} \vee S_{\mathbb{Q}}^{2p+q+1}$ and let $V = s^{-1}\tilde{H}_*(Y) = \langle x_p, y_q, z_{2p+q} \rangle$. Define a comultiplication $\phi : \mathbb{L}(V) \to \mathbb{L}(V) \sqcup \mathbb{L}(V')$ by $\phi = \phi_0 + P$, where $P(x) = 0 = P(y)$ and $P(z) = [x, [x, y']]$. Using (5.2) and (5.3), we see that $\lambda(z) = -z - [x, [x, y]]$, and $\rho(z) = -z + [x, [x, y]]$. Thus, if α is the comultiplication on Y corresponding to ϕ, then $l \neq r$.

Section 6—Wedges of Spheres

We begin with some terminology and notation. Denote by X the wedge of spheres $S^{n_1+1} \vee \cdots \vee S^{n_k+1}$ with $1 \leq n_1 \leq \cdots \leq n_k$. Let $\iota_j \in \pi_*(S^{n_1+1} \vee \cdots \vee S^{n_k+1})$ be the inclusion into the jth summand. Since X is a wedge of spheres, $\alpha : X \to Z$ is completely determined by the k elements $\alpha_*(\iota_1), \ldots, \alpha_*(\iota_k) \in \pi_*(Z)$. Let $\iota_1, \ldots, \iota_k, \iota'_1, \ldots, \iota'_k$ be the elements of $\pi_*(X \vee X)$ given by the inclusions into the summands. From the definition in Section 2, if $\alpha : X \to X \vee X$ is a comultiplication then

$$\alpha_*(\iota_j) = \iota_j + \iota'_j + P(\iota_j)$$

for some $P(\iota_j) \in \pi_{n_j+1}(X \vee X)$ such that $p_* P(\iota_j) = 0 = p'_* P(\iota_j)$ for the two projections $p, p' : X \vee X \to X$. We call P the *homotopy perturbation* of the comultiplication α (cf. Definition 2.5). A comultiplication on X is thus completely specified by the elements $P(\iota_1), \ldots, P(\iota_k) \in \pi_*(X \vee X)$. We use homotopy perturbations in order to construct and analyse comultiplications on wedges of spheres following the methods of the previous sections. In particular, we say that a homotopy perturbation P is *one-stage* if there exists some $m \in \{n_1, \ldots, n_k\}$ such that $P(\iota_j) = 0$ except possibly for those $\iota_j \in \pi_{m+1}(X)$.

6.1 PROPOSITION. *If $X = S^{n_1+1} \vee \cdots \vee S^{n_k+1}$, then X admits infinitely many homotopy classes of comultiplications if and only if for some j, $n_j = \Sigma_{i=1}^{r} a_i n_i$ for integers $a_i \geq 0$ with $\Sigma_{i=1}^{r} a_i \geq 2$.*

Proof. Let $\mathscr{C}(X)$ denote the set of homotopy classes of comultiplications on X and let $Y = X_{\mathbb{Q}} = S_{\mathbb{Q}}^{n_1+1} \vee \cdots \vee S_{\mathbb{Q}}^{n_k+1}$. Then the rationalization functor θ defined by $\theta(\alpha) = \alpha_{\mathbb{Q}}$ induces a mapping $\theta : \mathscr{C}(X) \to \mathscr{C}(Y)$. It is known that this mapping is finite-to-one [H-M-R$_1$, Cor. 5.4]. Thus if $\mathscr{C}(X)$ is infinite, then some $n_j = \Sigma_{i=1}^{r} a_i n_i$ as in the hypothesis, by Proposition 2.7. Conversely, if some $n_j = \Sigma_{i=1}^{r} a_i n_i$, then an infinite family of comultiplications can be defined on X, in analogy to the proof of Proposition 2.7. □

6.2 REMARK. Proposition 6.1 was known to Arkowitz and Curjel (unpublished) and is also a consequence of a result of Naylor [Na].

For the remainder of this section, the bracket operation in homotopy groups will be the Whitehead product. For properties of the Whitehead product refer to [Wh, Ch. 10].

6.3 PROPOSITION. *Let α be a comultiplication on $X = S^{n_1+1} \vee \cdots \vee S^{n_k+1}$ with homotopy perturbation P.*

(i) *Suppose P is a one-stage homotopy perturbation with $P \neq 0$ on $\pi_{m+1}(X)$. If, for each $\iota_j \in \pi_{m+1}(X)$, $P(\iota_j) = \Sigma_{r,s} a^j_{r,s}[\iota_r, \iota'_s]$ for integers $a^j_{r,s}$, then α is homotopy-associative.*

(ii) *Suppose that $P(\iota_j)$ is a torsion element for $\iota_j \in \pi_{r+1}(X)$ if $r < m$, and that there exists some $\iota_j \in \pi_{m+1}(X)$ with $P(\iota_j) = a[\iota_{i_1}, [\iota_{i_2}, \ldots, [\iota_{i_{s-1}}, \iota'_{i_s}] \ldots]]$ for some non-zero integer a, where $s \geq 3$ and $i_r \in \{1, \ldots, k\}$ for each r. Then α is non-homotopy-associative.*

Proof. (i) The proof given in Proposition 3.5 holds over the integers.

(ii) The rationalization $\alpha_{\mathbb{Q}}$ is a comultiplication on $X_{\mathbb{Q}}$. The associated Lie algebra comultiplication ϕ has perturbation as in Proposition 3.8. Since ϕ is non-homotopy-associative, α is non-homotopy-associative. □

6.4 NOTATION. Let $J = j_1, \ldots, j_r$ be a sequence of integers with each $j_i \in \{1, \ldots, k\}$. Define the elements

$$B_J = [(\iota_{j_1} + \iota'_{j_1}), [\ldots, [(\iota_{j_{r-1}} + \iota'_{j_{r-1}}), (\iota_{j_r} + \iota'_{j_r})] \ldots]],$$

$$\iota_J = [\iota_{j_1}, [\ldots, [\iota_{j_{r-1}}, \iota_{j_r}] \ldots]] \quad \text{and} \quad \iota'_J = [\iota'_{j_1}, [\ldots, [\iota'_{j_{r-1}}, \iota'_{j_r}] \ldots]]$$

in $\pi_*(X \vee X)$. An element $\chi \in \pi_*(X \vee X)$ is said to be of *standard associative form* if

$$\chi = \sum_J a_J(B_J - \iota_J - \iota'_J),$$

with $a_J \in \mathbb{Z}$, where the sum is over sequences J of length at least three (cf. Theorem 3.11). In particular, zero is an element of standard associative form.

6.5 THEOREM. *Let $X = S^{n_1+1} \vee \cdots \vee S^{n_k+1}$ and let α be a comultiplication on X with homotopy perturbation P.*

(i) *Suppose that P is a one-stage homotopy perturbation with $P \neq 0$ on $\pi_{m+1}(X)$. If, for each $\iota_j \in \pi_{m+1}(X)$, $P(\iota_j)$ can be written in standard associative form, then α is homotopy-associative.*

(ii) *Suppose that α is homotopy-associative and that $P(\iota_j)$ is a torsion element for all $\iota_j \in \pi_{r+1}(X)$ such that $r < m$. Then there exists an integer N such that, for each $\iota_j \in \pi_{m+1}(X)$,*

$$N \cdot P(\iota_j) = \sum_{r,s} a^j_{r,s}[\iota_r, \iota'_s] \ + \ \chi_j,$$

where χ_j is in standard associative form.

Proof. (i) This is checked by a straightforward calculation.

(ii) Since α is homotopy-associative, so is $\alpha_{\mathbb{Q}}$. Furthermore, $\alpha_{\mathbb{Q}}$ has homotopy perturbation $P_{\mathbb{Q}}$ which is zero on $\pi_{r+1}(X_{\mathbb{Q}})$ for $r < m$. Thus Theorem 3.11 applies, and so for each $\iota_j \in \pi_{m+1}(X_{\mathbb{Q}})$,

$$P_{\mathbb{Q}}(\iota_j) = \sum_{r,s} \lambda^j_{r,s}[\iota_r, \iota'_s] + \sum_J \lambda^j_J(B_J - \iota_J - \iota'_J),$$

where the $\lambda^j_{r,s}$ and λ^j_J are rationals, and the ι_i and ι'_i now denote the two inclusions of $S^{n_i+1}_{\mathbb{Q}}$ into $X_{\mathbb{Q}} \vee X_{\mathbb{Q}}$. Choose a finite collection of non-zero integers as follows: for each non-zero $\lambda^j_{r,s}$, let $N^j_{r,s}$ be such that $N^j_{r,s}\lambda^j_{r,s}$ is an integer; for each non-zero λ^j_J, let N^j_J be such that $N^j_J\lambda^j_J$ is an integer; for each $\iota_j \in \pi_{r+1}(X_{\mathbb{Q}})$ with $r > m$, let N_j be such that $N_jP_{\mathbb{Q}}(\iota_j)$ is a sum of brackets in the ι_i and the ι'_i with integer coefficients. Let M be the product of all the $N^j_{r,s}$, the N^j_J and the N_j. Write $\mu^j_{r,s} = M\lambda^j_{r,s}$ and $\mu^j_J = M\lambda^j_J$. Define a homotopy perturbation $\hat{P}$ on X by $\hat{P}(\iota_j) = 0$ for $\iota_j \in \pi_{r+1}(X)$ such that $r < m$,

$$\hat{P}(\iota_j) = \sum_{r,s} \mu^j_{r,s}[\iota_r, \iota'_s] + \sum_J \mu^j_J(B_J - \iota_J - \iota'_J)$$

for $\iota_j \in \pi_{m+1}(X)$, and $\hat{P}(\iota_j) = MP_{\mathbb{Q}}(\iota_j)$ for $\iota_j \in \pi_{r+1}(X)$ with $r > m$. Then $\hat{P}(\iota_j)$ and $MP(\iota_j)$ rationalize to the same element in $\pi_{n_j+1}(X_{\mathbb{Q}} \vee X_{\mathbb{Q}})$, for each j. Therefore, $MP(\iota_j) = \hat{P}(\iota_j) + \tau_j$ for some torsion element $\tau_j \in \pi_{n_j+1}(X_{\mathbb{Q}} \vee X_{\mathbb{Q}})$. Suppose that each τ_j has order M_j and set $N = MM_1 \cdots M_k$. Then for $\iota_j \in \pi_{m+1}(X)$, $NP(\iota_j) = M_1 \cdots M_k \hat{P}(\iota_j)$, which is easily seen to have the desired form. □

The following theorems refer to conditions (i), (ii) and (iii) preceding Theorem 3.14.

6.6 THEOREM. *A wedge of spheres $S^{n_1+1} \vee \cdots \vee S^{n_k+1}$ admits infinitely many homotopy classes of homotopy-associative comultiplications if and only if condition (i) or (ii) on the integers $n_1, \ldots, n_k$ holds.*

Proof. If (i) or (ii) holds, then proceed as in the proof of Theorem 3.14. Conversely, assume that neither (i) nor (ii) holds. Let α be a homotopy-associative comultiplication with homotopy perturbation P. We prove inductively that each $P(\iota_j)$ is a torsion element in $\pi_{n_j+1}(X \vee X)$. Assume this is true for all $\iota_j \in \pi_{r+1}(X)$ such that $r < m$. Then, by Theorem 6.5, there is some integer N such that for each $\iota_j \in \pi_{m+1}(X)$, $NP(\iota_j) = \Sigma_{r,s} a^j_{r,s}[\iota_r, \iota'_s] + \chi_j$, where χ_j can be written in standard associative form. Since condition (i) does not hold, then $N \cdot P(\iota_j) = \chi_j$. Since condition (ii) does not hold, then $\chi_j = a^j_J(B_J - \iota_J - \iota'_J)$, where $J = j_1, j_1, \ldots, j_1$. However, J is a sequence of length at least three and $[\eta, [\eta, \eta]]$ is a torsion element for any η, so each $N \cdot P(\iota_j)$ is a torsion element. Thus $P(\iota_j)$ is a torsion element for each $\iota_j \in \pi_{m+1}(X)$. This completes the induction. Since there are only finitely many torsion elements in the group $\Sigma^k_{j=1} \pi_{n_j+1}(X \vee X)$, there are only finitely many homotopy perturbations that give homotopy-associative comultiplications. This completes the proof. □

6.7 PROPOSITION. *A wedge of spheres $S^{n_1+1} \vee \cdots \vee S^{n_k+1}$ admits infinitely many homotopy classes of non-homotopy-associative comultiplications if condition (ii) or (iii) on the integers $n_1, \ldots, n_k$* holds.

Proof. The proof given for this implication in Theorem 3.15 holds over the integers. □

Unlike Theorem 3.15, the converse of Proposition 6.7 is not true, as Example 6.9 below shows.

In what follows, we let $b_r \in \pi_*(X \vee X)$ denote a *basic product* in the $\iota_1, \ldots, \iota_k, \iota'_1, \ldots, \iota'_k$ [Hi$_2$, p. 154], such that b_r contains at least one entry from the $\iota_1, \ldots, \iota_k$ and at least one entry from the $\iota'_1, \ldots, \iota'_k$. If P is a homotopy perturbation, it follows from Hilton's theorem [Hi$_2$, Th. A] that $P(\iota_j) = \Sigma_r b_r \circ \theta_r$ for each $\iota_j \in \pi_{n_j+1}(X)$, where b_r is a basic product, $|b_r|$ denotes the degree of b_r and $\theta_r \in \pi_{n_j+1}(S^{|b_r|})$. This sum can be decomposed into a torsion part and a free part:

$$P(\iota_j) = \sum_{s,\, \theta_s \text{ torsion}} b_s \circ \theta_s + \sum_{t,\, \theta_t \text{ free}} b_t \circ \theta_t. \tag{6.8}$$

Note that if θ_s is a suspension, then left-additivity holds, i.e., $(a_1 + a_2) \circ \theta_s = a_1 \circ \theta_s + a_2 \circ \theta_s$ for any homotopy elements a_1, a_2. In particular, $(na_1) \circ \theta_s = n(a_1 \circ \theta_s) = a_1 \circ (n\theta_s)$.

6.9 EXAMPLE. Let $X = S^7 \vee S^{11} \vee S^{21}$ and observe that neither condition (ii) nor condition (iii) holds for 6, 10, 20. Define a collection of comultiplications α^n by $P^n(\iota_1) = 0$, $P^n(\iota_2) = 0$ and $P^n(\iota_3) = [\iota_1, [\iota_1, \iota_1']] \circ \theta + n[\iota_2, \iota_2']$, where n is an integer and $\theta \neq 0 \in \pi_{21}(S^{19}) = \mathbb{Z}_2$ [To, Ch. 14]. We check that $(\alpha^n \vee 1)\alpha^n(\iota_3) - (1 \vee \alpha^n)\alpha^n(\iota_3) \neq 0$:

$$\begin{aligned}(\alpha^n \vee 1)\alpha^n(\iota_3) &= (\alpha^n \vee 1)(\iota_3 + \iota_3' + [\iota_1, [\iota_1, \iota_1']] \circ \theta + n[\iota_2, \iota_2']) \\ &= \iota_3 + \iota_3' + [\iota_1, [\iota_1, \iota_1']] \circ \theta + n[\iota_2, \iota_2'] \\ &\quad + \iota_3'' + [\iota_1 + \iota_1', [\iota_1 + \iota_1', \iota_1'']] \circ \theta + n[\iota_2 + \iota_2', \iota_2''].\end{aligned}$$

Similarly,

$$\begin{aligned}(1 \vee \alpha^n)\alpha^n(\iota_3) &= \iota_3 + \iota_3' + \iota_3'' + [\iota_1', [\iota_1', \iota_1'']] \circ \theta + n[\iota_2', \iota_2''] \\ &\quad + [\iota_1, [\iota_1, \iota_1' + \iota_1'']] \circ \theta + n[\iota_2, \iota_2' + \iota_2''].\end{aligned}$$

Since θ is a suspension, we obtain

$$(\alpha^n \vee 1)\alpha^n(\iota_3) - (1 \vee \alpha^n)\alpha^n(\iota_3) = [\iota_1, [\iota_1', \iota_1'']] \circ \theta + [\iota_1', [\iota_1, \iota_1'']] \circ \theta.$$

Commutativity and the Jacobi identity for the Whitehead product and the fact that θ has order two yield

$$(\alpha^n \vee 1)\alpha^n(\iota_3) - (1 \vee \alpha^n)\alpha^n(\iota_3) = [\iota_1'', [\iota_1, \iota_1']] \circ \theta,$$

which is non-zero by [Hi$_2$, Th. A]. Thus α^n is a non-homotopy-associative comultiplication, for each n. Notice that $X_{\mathbb{Q}} = S^7_{\mathbb{Q}} \vee S^{11}_{\mathbb{Q}} \vee S^{21}_{\mathbb{Q}}$ only admits homotopy-associative comultiplications, whereas X admits infinitely many non-homotopy-associative comultiplications.

We now consider homotopy-commutativity for a wedge of spheres. Proposition 4.2 and Examples 4.3 carry over verbatim into this context. Of more interest, perhaps, is a consideration of how torsion elements can affect homotopy-commutativity. Let $\tau : \pi_*(X \vee X) \to \pi_*(X \vee X)$ denote the homomorphism induced by the twisting map $T : X \vee X \to X \vee X$. Let α be a comultiplication on $X = S^{n_1+1} \vee \cdots \vee S^{n_k+1}$, with homotopy perturbation P. Notice that α is homotopy-commutative if and only if $\tau P = P$.

6.10 EXAMPLES. Let $X = S^{p+1} \vee S^{q+1}$ and let $\iota_1 \in \pi_{p+1}(X)$ and $\iota_2 \in \pi_{q+1}(X)$ be the inclusions. We shall consider two different pairs (p, q) to illustrate the rôle of torsion. In each case, we define a one-stage homotopy perturbation P with $P(\iota_1) = 0$.

(i) Let $X = S^3 \vee S^7$ and define $P(\iota_2) = [\iota_1, \iota_1'] \circ \theta$, where $\theta \neq 0 \in \pi_7(S^5) = \mathbb{Z}_2$, which is a suspension [To, Ch. 14]. Then $\tau([\iota_1, \iota_1'] \circ \theta) = (-[\iota_1, \iota_1']) \circ \theta = [\iota_1, \iota_1'] \circ (-\theta)$. But θ is of order two, so $\tau([\iota_1, \iota_1'] \circ \theta) = [\iota_1, \iota_1'] \circ \theta$. Also, the one-stage homotopy perturbations $Q^n(\iota_2) = n([\iota_1, [\iota_1, \iota_1']] + [\iota_1', [\iota_1', \iota_1]])$, for n an integer, give homotopy-commutative comultiplications. Thus there are two infinite families of homotopy-commutative comultiplications on X, one given by the Q^n and one by the $P + Q^n$.

(ii) Let $X = S^3 \vee S^8$ and define $P(\iota_2) = [\iota_1, \iota_1'] \circ \theta$, where θ is an element of order three in $\pi_8(S^5) = \mathbb{Z}_{24}$ [To, Ch. 14]. Then $\tau P(\iota_2) = (-[\iota_1, \iota_1']) \circ \theta$, so α is homotopy-commutative if and only if $(-[\iota_1, \iota_1']) \circ \theta = [\iota_1, \iota_1'] \circ \theta$. Since θ is a suspension, α is homotopy-commutative if and only if $[\iota_1, \iota_1'] \circ 2\theta = 0$. The latter holds if and only if $2\theta = 0$. However, $2\theta \neq 0$, since θ has order three. Therefore α is non-homotopy-commutative. The rationalization of α, however, must be homotopic to the standard comultiplication on $S^3_{\mathbb{Q}} \vee S^8_{\mathbb{Q}}$ since $P(\iota_2)$ is a torsion element, and hence $\alpha_{\mathbb{Q}}$ must be homotopy-commutative. This gives an example of a non-homotopy-commutative comultiplication whose rationalization is homotopy-commutative.

We next present a brief treatment of homotopy-inverses for comultiplications on a wedge of spheres. We do not give general results, but instead emphasize examples. It is straightforward to construct examples of comultiplications whose left and right homotopy-inverses are equal, as in the proof of Proposition 5.5. However, the presence of torsion allows more possibilities as the following example shows.

6.11 EXAMPLE. Let $X = S^2 \vee S^2 \vee S^5$ and let P be the one-stage homotopy perturbation given by $P(\iota_3) = [\iota_1, [\iota_1, \iota_2']] \circ \theta + [\iota_1, [\iota_1, [\iota_1', \iota_2']]]$, where $\theta \neq 0 \in \pi_5(S^4) = \mathbb{Z}_2$[To, Ch. 14]. Then by (5.2) and (5.3) and the fact that θ has order two, we have that $l = r$ for the corresponding comultiplication on X.

We next show how torsion can lead to different left and right homotopy-inverses.

6.12 EXAMPLE. Let $X = S^6 \vee S^6 \vee S^{29}$ and let P be the one-stage homotopy perturbation given by $P(\iota_3) = [\iota_1, [\iota_1, \iota_2']] \circ \theta$, where θ is an element of order three in $\pi_{29}(S^{16}) = \mathbb{Z}_3$, which is a suspension [To, Ch. 14]. If l and r are the left and right homotopy inverses for the corresponding comultiplication α, then it is easily shown that $l \neq r$. Notice that $P(\iota_3)$ is a torsion element and so $\alpha_{\mathbb{Q}}$ is the standard comultiplication on $S^6_{\mathbb{Q}} \vee S^6_{\mathbb{Q}} \vee S^{29}_{\mathbb{Q}}$. Thus $\alpha_{\mathbb{Q}}$ has identical left and right homotopy inverses.

We conclude this section with an example which shows Corollary 5.6 is not true for a wedge of two ordinary spheres. This example again illustrates the variety of behaviour which can be displayed by comultiplications on a wedge of two spheres.

6.13 EXAMPLE. Let $X = S^4 \vee S^{10}$ and define a one-stage homotopy perturbation by $P(\iota_2) = [\iota_1, [\iota_1, \iota_1']]$. Let l and r be the left and right homotopy inverses for the comultiplication α corresponding to P. Then $(l, 1)_* P(\iota_2) = [\iota_1, [\iota_1, \iota_1]] = \iota_{1*}[\iota, [\iota, \iota]]$, where $\iota \in \pi_4(S^4)$ is the identity class. Similarly $(1, r)_* P(\iota_2) = -\iota_{1*}[\iota, [\iota, \iota]]$. If $l = r$, then $(l, 1)_* P(\iota_2) = (1, r)_* P(\iota_2)$, by (5.2) and (5.3), and so $[\iota, [\iota, \iota]] = -[\iota, [\iota, \iota]]$. This is impossible, since $[\iota, [\iota, \iota]] \in \pi_{10}(S^4)$ has order three [Hi$_1$]. Thus α has distinct left and right homotopy-inverses. This example extends to $S^{2n} \vee S^{6n-2}$ for $n \geqslant 2$, since $[\iota, [\iota, \iota]] \in \pi_{6n-2}(S^{2n})$ is known to have order three for $n \geq 2$.

Section 7—Proof of Proposition 3.10

This final section is a technical one devoted to the proof of Proposition 3.10. The proof is given after several lemmas concerning the maps $\beta, \gamma, \delta : T(V, V') \to T(V, V', V'')$. For the purposes of the proof, we introduce some notation. Suppose the vector space V has basis $\{x_1, x_2, \ldots, x_k\}$. As before, denote the binomial $x_i + x_i' \in T(V, V')$ by B_i. Now extend this notation by defining, for each sequence $I = i_1, i_2, \ldots, i_r$, with $i_j \in \{1, 2, \ldots, k\}$,

$$B_I = B_{i_1} B_{i_2} \cdots B_{i_r},$$

$$x_I = x_{i_1} x_{i_2} \cdots x_{i_r} \quad \text{and} \quad x_I' = x_{i_1}' x_{i_2}' \cdots x_{i_r}'.$$

Observe that we denote multiplication in the tensor algebra by juxtaposition. Thus, for example, $B_{1,2} = B_1 B_2 = (x_1 + x_1')(x_2 + x_2') = x_1 x_2 + x_1 x_2' + x_1' x_2 + x_1' x_2'$. Notice that $\beta(x_i) = B_i$ and $\gamma(x_i') = \delta(B_i)$. Throughout the following lemmas, χ denotes an element of $T(V, V')$ of fixed homogeneous length.

7.1 LEMMA. (i) $\beta(\chi) = 0$ *if and only if* $\chi = 0$, *and* (ii) $\gamma(\chi) = 0$ *if and only if* $\chi = 0$.

Proof. (i) Clearly $\beta(0) = 0$. For the converse, β preserves length, so we use induction on the length of χ. Suppose χ is of length 1. Write $\chi = \Sigma\,(\lambda_i x_i + \mu_i x'_i)$, so that $\beta(\chi) = \Sigma\,(\lambda_i(x_i + x'_i) + \mu_i x''_i)$. Thus $\beta(\chi) = 0$ implies $\lambda_i = 0$ and $\mu_i = 0$, for all i. Now suppose χ is of length $r+1$ for some $r \geqslant 1$. Write $\chi = \Sigma\,(x_i A_i + x'_i C_i)$, where A_i and $C_i \in T(V, V')$, and both are of length r. Then $\beta(\chi) = \Sigma\,(x_i\beta(A_i) + x'_i\beta(A_i) + x''_i\beta(C_i))$, so $\beta(\chi) = 0$ implies $\beta(A_i) = 0$ and $\beta(C_i) = 0$. Thus $A_i = 0$ and $C_i = 0$ by the inductive hypothesis. Hence (i) is proved by induction. The proof of (ii) is similar. □

7.2 LEMMA. *If* χ *has homogeneous length* r, *then* $\beta(\chi) = \gamma(\chi)$ *if and only if* $\chi = \Sigma_I\,\lambda_I B_I$, *where the sum is taken over all sequences* I *of length* r.

Proof. If $\chi = \Sigma_I\,\lambda_I B_I$, then $\beta(\chi) = \gamma(\chi)$ since $\beta(B_I) = \gamma(B_I)$ for each I. For the converse we again argue by induction. If χ has length 1, then $\chi = \Sigma\,(\lambda_i x_i + \mu_i x'_i)$. So $\beta(\chi) = \Sigma(\lambda_i x_i + \lambda_i x'_i + \mu_i x''_i)$ and $\gamma(\chi) = \Sigma(\lambda_i x_i + \mu_i x'_i + \mu_i x''_i)$. Thus $\beta(\chi) = \gamma(\chi)$ implies $\Sigma\,\lambda_i x'_i = \Sigma\,\mu_i x'_i$ and so $\lambda_i = \mu_i$ for all i. Hence $\chi = \Sigma\,\lambda_i B_i$. Now suppose χ has length $r+1$, for some $r \geqslant 1$, and assume the result holds for length r terms. Write $\chi = \Sigma\,(x_i A_i + x'_i C_i)$, where $A_i, C_i \in T(V, V')$ both have length r. Then $\beta(\chi) = \Sigma\,(x_i\beta(A_i) + x'_i\beta(A_i) + x''_i\beta(C_i))$ and $\gamma(\chi) = \Sigma\,(x_i\gamma(A_i) + x'_i\gamma(C_i) + x''_i\gamma(C_i))$. Therefore $\beta(\chi) = \gamma(\chi)$ implies $\beta(A_i) = \gamma(A_i)$, $\beta(A_i) = \gamma(C_i)$ and $\beta(C_i) = \gamma(C_i)$. The first and third of these latter equalities give $A_i = \Sigma_I\,\lambda^i_I B_I$ and $C_i = \Sigma_I\,\mu^i_I B_I$ for some λ^i_I, μ^i_I, by the inductive hypothesis. From the second and third equalities, we obtain $\beta(A_i) = \beta(C_i)$. This implies $A_i = C_i$ by Lemma 7.1. Thus $\chi = \Sigma_{i,I}\,\lambda^i_I(x_i + x'_i)B_I = \Sigma_{i,I}\,\lambda^i_I B_{i,I}$. □

The proofs of the next four lemmas are omitted. They follow an identical pattern to the proofs of the previous two by induction over the length of χ.

7.3 LEMMA. (i) $\gamma(\chi) = \chi$ *if and only if* $\chi \in T(V)$, *and* (ii) $\beta(\chi) = \chi$ *if and only if* $\chi = 0$.

7.4 LEMMA. (i) $\gamma(\chi) = \delta(\chi)$ *if and only if* $\chi = 0$, *and* (ii) $\beta(\chi) = \delta(\chi)$ *if and only if* $\chi \in T(V')$.

7.5 LEMMA. $\chi + \beta(\chi) = \gamma(\chi)$ *if and only if* $\chi = \Sigma\,\lambda_I\{B_I - x_I\}$.

7.6 LEMMA. $\beta(\chi) = \delta(\chi) + \gamma(\chi)$ *if and only if* $\chi = \Sigma\,\lambda_I\{B_I - x'_I\}$.

7.7 LEMMA. *Let A and C be elements of the form $A = \Sigma\, \lambda_I(B_I - x_I)$* and $C = \Sigma\, \mu_I(B_I - x'_I)$, where each I in the sum is a sequence of length r with $r \geqslant 2$. Then $C + \beta(A) = \delta(A) + \gamma(C)$ if and only if $\lambda_I = \mu_I$ for all I.

Proof. If $\lambda_I = \mu_I$ for all I, then clearly $C + \beta(A) = \delta(A) + \gamma(C)$. We prove the converse by induction on r. Suppose $r = 2$, so that $B_I = B_{i_1, i_2}$ for each I. Expand B_I as $x_{i_1} B_{i_2} + x'_{i_1} B_{i_2}$, and write $A = \Sigma\, \lambda_I \{x_{i_1}(B_{i_2} - x_{i_2}) + x'_{i_1} B_{i_2}\}$ and $C = \Sigma\, \mu_I \{x_{i_1} B_{i_2} + x'_{i_1}(B_{i_2} - x'_{i_2})\}$. Then

$$C + \beta(A) = \sum x_{i_1}(\mu_I B_{i_2} + \lambda_I \beta(B_{i_2} - x_{i_2})) + \text{terms with leading entry } x'_{i_1} \text{ or } x''_{i_1};$$

and

$$\delta(A) + \gamma(C) = \sum x_{i_1} \mu_I \gamma(B_{i_2}) + \text{terms with leading entry } x'_{i_1} \text{ or } x''_{i_1}.$$

Thus $C + \beta(A) = \delta(A) + \gamma(C)$ implies $\Sigma_{i_2}\, \{\mu_I B_{i_2} + \lambda_I \beta(x'_{i_2})\} = \Sigma_{i_2}\, \mu_I \gamma(B_{i_2})$, for each fixed i_1. The latter equality implies $\lambda_I = \mu_I$ for each I, since $\beta(x'_{i_2}) = x''_{i_2}$ and $\gamma(B_{i_2}) = x_{i_2} + x'_{i_2} + x''_{i_2}$. This starts the induction.

Assume the result holds for length r elements and let A and C be of length $r+1$, for some $r \geqslant 2$. Expand A and C as $A = \Sigma\, \lambda_I \{x_{i_1}(B_{i_2, \ldots, i_{r+1}} - x_{i_2, \ldots, i_{r+1}}) + x'_{i_1} B_{i_2, \ldots, i_{r+1}}\}$ and $C = \Sigma\, \mu_I \{x_{i_1} B_{i_2, \ldots, i_{r+1}} + x'_{i_1}(B_{i_2, \ldots, i_{r+1}} - x'_{i_2, \ldots, i_{r+1}})\}$, where $I = i_1, i_2, \ldots, i_{r+1}$. Then

$$\begin{aligned} C + \beta(A) = {} & \sum \mu_I x'_{i_1}(B_{i_2, \ldots, i_{r+1}} - x'_{i_2, \ldots, i_{r+1}}) \\ & + \sum \lambda_I x'_{i_1} \beta(B_{i_2, \ldots, i_{r+1}} - x_{i_2, \ldots, i_{r+1}}) \\ & + \text{terms with first entry } x_{i_1} \text{ or } x''_{i_1}; \end{aligned}$$

and

$$\begin{aligned} \delta(A) + \gamma(C) = {} & \sum \lambda_I x'_{i_1} \delta(B_{i_2, \ldots, i_{r+1}} - x_{i_2, \ldots, i_{r+1}}) \\ & + \sum \mu_I x'_{i_1} \gamma(B_{i_2, \ldots, i_{r+1}} - x'_{i_2, \ldots, i_{r+1}}) \\ & + \text{terms with first entry } x_{i_1} \text{ or } x''_{i_1}. \end{aligned}$$

Hence $C + \beta(A) = \delta(A) + \gamma(C)$ implies, for each fixed i_1, that

$$\begin{aligned} & \sum \mu_I(B_{i_2, \ldots, i_{r+1}} - x'_{i_2, \ldots, i_{r+1}}) + \beta\Big(\sum \lambda_I(B_{i_2, \ldots, i_{r+1}} - x_{i_2, \ldots, i_{r+1}})\Big) \\ & \quad = \delta\Big(\sum \lambda_I(B_{i_2, \ldots, i_{r+1}} - x_{i_2, \ldots, i_{r+1}})\Big) + \gamma\Big(\sum \mu_I(B_{i_2, \ldots, i_{r+1}} - x'_{i_2, \ldots, i_{r+1}})\Big), \end{aligned}$$

where the sums are taken over all sequences $i_2, \ldots, i_{i+1}$. The induction hypothesis applied to the terms $\tilde{A} = \Sigma\, \lambda_I(B_{i_2,\ldots,i_{r+1}} - x_{i_2,\ldots,i_{r+1}})$ and $\tilde{C} = \Sigma\, \mu_I(B_{i_2,\ldots,i_{r+1}} - x'_{i_2,\ldots,i_{r+1}})$ yields $\lambda_I = \mu_I$ for all I. Thus induction is complete, and the result follows. $\square$

Proof of Proposition 3.10. It is straightforward to check that if ξ has the form described in Proposition 3.10, then $\xi + \beta(\xi) = \delta(\xi) + \gamma(\xi)$. We prove the converse. Suppose $\xi \in T(V, V')$ is of length $r + 1$, with $r \geqslant 2$. Write $\xi = \Sigma_j (x_j A_j + x'_j C_j)$, where A_j, $C_j \in T(V, V')$ are of length r. Expanding $\xi + \beta(\xi) = \delta(\xi) + \gamma(\xi)$ and equating terms with first entry x_j, x'_j, or x''_j, respectively, we have the following three equations for each j:

$$A_j + \beta(A_j) = \gamma(A_j) \tag{7.8}$$

$$C_j + \beta(A_j) = \delta(A_j) + \gamma(C_j) \tag{7.9}$$

$$\beta(C_j) = \delta(C_j) + \gamma(C_j). \tag{7.10}$$

Now, Lemma 7.5 and (7.8) imply that $A_j = \Sigma\, \lambda^j_I(B_I - x_I)$. Similarly, Lemma 7.6 and (7.10) imply $C_j = \Sigma\, \mu^j_I(B_I - x'_I)$. Thus Lemma 7.7 and (7.9) show $\lambda^j_I = \mu^j_I$ for each j and I. Hence

$$\begin{aligned}\xi &= \sum_j \sum_I \lambda^j_I(x_j(B_I - x_I) + x'_j(B_I - x'_I)) \\ &= \sum_J \lambda_J(B_J - x_J - x'_J),\end{aligned}$$

where J is the sequence j, I and $\lambda_J = \lambda^j_I$. $\square$

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