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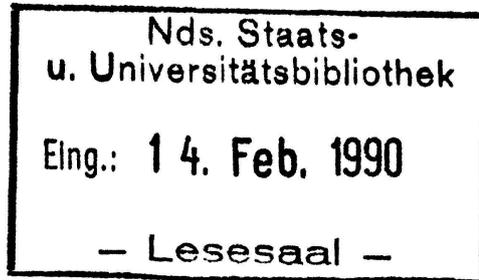
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On the periodic spectrum of the 1-dimensional Schrödinger operator

TH. KAPPELER



1. Introduction

Let us consider Hill's equation

$$-y''(x) + q(x)y(x) = \lambda y(x) \quad (x \text{ in } \mathbf{R}) \tag{1}$$

$$y(x + 2) = y(x) \quad (x \text{ in } \mathbf{R}) \tag{2}$$

where q is a potential in $L^2[0, 1]$, periodically extended to all of \mathbf{R} . It is well known that those λ 's for which (1)–(2) admits a non zero solution, form a non decreasing sequence of real numbers $\lambda_k = \lambda_k(q)$ ($k \geq 0$), written with multiplicities. $(\lambda_k)_{k \geq 0}$ is called the periodic spectrum of q . Observe that, for convenience, the period in (2) has been chosen equal to 2 rather than 1 in order to include the so called antiperiodic eigenvalues as well. For q in $L^2[0, 1]$, the isospectral set $\text{Iso}(q)$ is defined to be the set of all potentials p in $L^2[0, 1]$ such that $\lambda_k(p) = \lambda_k(q)$ ($k \geq 0$) and $G(q)$ denotes the set of all potentials p in $L^2[0, 1]$ with the same gaps as q , i.e. $\lambda_0(p) = \lambda_0(q)$ and $\lambda_{2k}(p) - \lambda_{2k-1}(p) = \lambda_{2k}(q) - \lambda_{2k-1}(q)$ ($k \geq 1$). Then $\text{Iso}(q) \subseteq G(q)$.

This paper presents an elementary proof of a result due to J. Garnett and E. Trubowitz [GT1] which says that the converse inclusion $G(q) \subseteq \text{Iso}(q)$ also holds:

THEOREM (Garnett, Trubowitz). *For all q in $L^2[0, 1]$, $\text{Iso}(q) = G(q)$.*

In [GT1], this theorem is proved by applying harmonic measure arguments to the identification, due to Marcenko and Ostrovskii [MO], of band configurations with certain slit quarter planes. In this paper it is shown that the theorem is a direct consequence of the spectral theory for even potentials q in $L^2[0, 1]$ (i.e. $q(x) = q(1 - x)$), as it is presented in the beautiful paper [GT2], using analysis in Hilbert space.

2. Proof of theorem

First observe that due to the fact that $\lambda_k(q+c) = \lambda_k(q) + c$ ($k \geq 0$; c real) it suffices to prove the theorem for potentials q in V where V is given by $V := \{q \in L^2[0, 1] : \lambda_0(q) = 0\}$.

Let q be a fixed element in V . Clearly, for p in $G(q)$, $\text{Iso}(p) \subset G(q)$ and thus $G(q) = \bigcup \text{Iso}(p)$ where the union extends over all p in $G(q)$.

Denote by $\mu_n = \mu_n(p)$ ($n \geq 1$) and $\nu_n = \nu_n(p)$ ($n \geq 0$) the Dirichlet and Neuman spectrum of p in $L^2[0, 1]$, that is the spectrum of (1) for the boundary conditions $y(0)=0, y(1)=0$ and $y'(0)=0, y'(1)=0$ respectively. It is well known (cf. e.g. [MW]) that $\nu_0 \leq \lambda_0$ and $\lambda_{2n-1} \leq \mu_n, \nu_n \leq \lambda_{2n}$ ($n \geq 1$). By the lemma below one can find for a given p in V an element p_{\max} in $\text{Iso}(p) \cap E$ with the properties that $\mu_n(p_{\max}) = \lambda_{2n}(p)$ ($n \geq 1$), $\nu_0(p_{\max}) = \lambda_0(p)$ and $\nu_n(p_{\max}) = \lambda_{2n-1}(p)$ ($n \geq 1$) where E denotes the subspace of $L^2[0, 1]$ of all even potentials p (i.e. $p(x) = p(1-x)$). Thus for all $n \geq 1$, $\mu_n(p_{\max}) - \nu_n(p_{\max}) = \lambda_{2n}(q) - \lambda_{2n-1}(q)$. For p in $L^2[0, 1]$, define $\sigma(p)$ to be the sequence $(\mu_n(p) - \nu_n(p))_{n \geq 1}$. From the asymptotics (cf. e.g. [PT]) $\mu_n(p), \nu_n(p) = n^2\pi^2 + \int_0^1 p(x) dx + a_n, b_n$, where $\sum (a_n^2 + b_n^2) < \infty$, one concludes that $\sigma(p)$ is an element in l^2 . In [GT2] it is proved that the restriction of σ to $V \cap E$ is 1-1. Therefore $p_{\max} = q_{\max}$ for all p in $G(q)$ and thus $\text{Iso}(p) = \text{Iso}(q)$. This implies that $G(q) = \text{Iso}(q)$.

LEMMA. *Let p be in $L^2[0, 1]$. Then there exists a potential p_{\max} in $\text{Iso}(p)$, such that*

- (1) $\mu_n(p_{\max}) = \lambda_{2n}(p)$ ($n \geq 1$)
- (2) $\nu_0(p_{\max}) = \lambda_0(p)$ and $\nu_n(p_{\max}) = \lambda_{2n-1}(p)$ ($n \geq 1$)
- (3) p_{\max} is even, i.e. an element in E .

Proof. For p in $L^2[0, 1]$ with only a finite number of simple periodic eigenvalues, the existence of p_{\max} with property (1) together with $\|p_{\max}\|_{L^2} = \|p\|_{L^2}$ is a direct consequence of results presented in [M, M]. By standard arguments one proves (2) and (3). To be more precise, denote by $y_1(x, \lambda)$ and $y_2(x, \lambda)$ the fundamental solutions of (1), i.e. the solutions $y(x, \lambda)$ of (1) with the initial conditions $y(0, \lambda) = 1, y'(0, \lambda) = 0$ and $y(0, \lambda) = 0, y'(0, \lambda) = 1$ respectively. For $\lambda_{2n}(p_{\max}) = \mu_n(p_{\max})$, $y_2(x, \mu_n)$ is the corresponding eigenfunction and thus $y_2'(2, \mu_n) = 1$. By investigating the Floquet matrix

$$F(\lambda) = \begin{pmatrix} y_1(1, \lambda) & y_2(1, \lambda) \\ y_1'(1, \lambda) & y_2'(1, \lambda) \end{pmatrix}$$

one concludes that $|y_2'(1, \mu_n)| = 1$. Combining Corollary 2.2 and Lemma 3.4 in [PT], it follows that p_{\max} is even. Using this fact together with reflection one now

verifies that $\nu_n(p_{\max})$ is a periodic eigenvalue of p_{\max} ($n \geq 0$). From $\nu_0 \leq \lambda_0$ and $\lambda_{2n-1} \leq \mu_n$, $\nu_n \leq \lambda_{2n}$ ($n \geq 1$) it then follows that $\nu_0(p_{\max}) = \lambda_0(p_{\max})$ and $\nu_n(p_{\max}) = \lambda_{2n-1}(p_{\max})$ ($n \geq 1$).

Towards the general case, choose a sequence $(p_n)_{n \geq 1}$ of potentials in $L^2[0, 1]$ such that $p = \lim_{n \rightarrow \infty} p_n$ in the norm topology of $L^2[0, 1]$ and such that, for $n \geq 1$, p_n has only a finite number of periodic eigenvalues. (Cf. [CK] for an elementary proof concerning the existence of such a sequence). This implies that $\lim_{n \rightarrow \infty} \lambda_k(p_n) = \lambda_k(p)$ ($k \geq 0$) as the eigenvalues depend continuously on the potential. Denote by q_n the potential $(p_n)_{\max}$ in $\text{Iso}(p_n)$. Then $\lambda_{2k}(p_n) = \mu_k(q_n)$ ($k \geq 1$), $\lambda_0(p_n) = \nu_0(q_n)$ and $\lambda_{2k-1}(p_n) = \nu_{2k-1}(q_n)$ ($k \geq 1$). Moreover $\|p_n\|_{L^2[0,1]} = \|q_n\|_{L^2[0,1]}$. Thus $(q_n)_{n \geq 1}$ is a sequence, bounded in $L^2[0, 1]$; without loss of any generality we may assume that $(q_n)_{n \geq 1}$ converges weakly to a potential q in $L^2[0, 1]$. As E is a closed subspace of L^2 , q must be an element of E . Now use that all eigenvalues λ_k ($k \geq 0$), ν_k ($k \geq 0$) and μ_k ($k \geq 1$) depend continuously on the potential with respect to the weak topology of $L^2[0, 1]$ to conclude that $\lambda_k(q) = \lim_{n \rightarrow \infty} \lambda_k(q_n) = \lim_{n \rightarrow \infty} \lambda_k(p_n) = \lambda_k(p)$ ($k \geq 0$) as well as $\mu_k(q) = \lambda_{2k}(p)$ ($k \geq 1$), $\nu_0(q) = \lambda_0(p)$ and $\nu_k(q) = \lambda_{2k-1}(p)$ ($k \geq 1$). Thus $p_{\max} := q$ has the desired properties.

REMARK. The proof of the lemma shows that for a given q in $L^2[0, 1]$ one has $\|p\|_{L^2} = \|q\|_{L^2}$ for all potentials p in $\text{Iso}(q)$. For q in $C^\infty(\mathbf{R}/\mathbf{Z})$ this is a consequence of results in [MT].

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