

**Zeitschrift:** Commentarii Mathematici Helvetici  
**Herausgeber:** Schweizerische Mathematische Gesellschaft  
**Band:** 65 (1990)

**Artikel:** On small eigenvalues of the Laplacian for  $\dots(q)\backslash\dots$  .  
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**DOI:** <https://doi.org/10.5169/seals-49748>

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## On small eigenvalues of the Laplacian for $\Gamma_0(q)\backslash\mathcal{H}$

JEFFREY STOPPLE

Let  $\mathcal{H}$  be the complex upper half plane, and  $\Gamma_0(q)$  be the subgroup of matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in  $SL(2, \mathbb{Z})$  with  $c \equiv 0 \pmod{q}$ . Suppose  $f$  is a Maass cusp form with eigenvalue  $\lambda$ ; i.e., a non-constant function  $f: \mathcal{H} \rightarrow \mathbb{C}$  satisfying

$$f(\gamma z) = f(z) \quad \text{for } \gamma \text{ in } \Gamma_0(q), z \text{ in } \mathcal{H}$$

$$y^2 \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) f + \lambda f = 0$$

$$\int_{\mathcal{F}} |f(z)|^2 \frac{dx \, dy}{y^2} < \infty$$

where  $\mathcal{F}$  is a fundamental domain for  $\Gamma_0(q)$ . Selberg conjectured [8] that  $\lambda \geq 1/4$ , and showed that  $\lambda \geq 3/16$ . Iwaniec has a statistical result [5] that shows the rarity of small eigenvalues, similar to density theorems about real zeros of Dirichlet's  $L$ -series.

For an odd prime  $q$ , whether  $q$  is ramified, split, or inert in a real quadratic field  $\mathbb{Q}(\sqrt{\Delta})$  depends only on the Legendre symbol  $(\Delta/q)$ , and so is periodic in  $\Delta \pmod{q}$ . If we consider instead the set of all norm 1 units  $\epsilon > 1$  in all real quadratic fields, ordered by the size of  $\epsilon$ , we expect the behavior of  $q$  in  $\mathbb{Q}(\epsilon)$  to be more or less random. We use the trace formula to show that if there is an eigenvalue  $\lambda$  less than  $1/4$  then that expectation is wrong; instead  $q$  has a bias towards a certain behavior.

Specifically let  $t \geq 3$  in  $\mathbb{Z}$ , and write  $t^2 - 4 = u^2 \Delta$  with  $\Delta$  a discriminant of a real quadratic field. Then  $\mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\epsilon)$  for  $\epsilon$  the larger root of  $x^2 - tx + 1 = 0$ . Let  $h(\Delta)$  be the narrow class number and  $\epsilon_1$  the fundamental

norm 1 unit. Define

$$\Theta(t) = \begin{cases} q & \text{if } t \equiv \pm 2 \pmod{q^2} \\ \left(\frac{\Delta}{q}\right) & \text{otherwise} \end{cases}$$

If  $\lambda$  is the smallest eigenvalue;  $\lambda = 1/4 + r^2$  with  $r = i\rho$  purely imaginary,  $0 < \rho < 1/2$  then for  $T \rightarrow \infty$  we have

**THEOREM.**

$$\frac{1}{\sqrt{\pi T}} \sum_{t \geq 3} \frac{2h(\Delta) \log(\epsilon_1)}{\sqrt{\Delta}} \sum_{m|u} \left(\frac{\Delta}{m}\right) \sigma(u/m) \mu(m) u^{-1} \Theta(t) \exp(-\log^2(t)/T) \sim \exp(\rho^2 T)$$

Here  $\sigma(n) = \sum_{d|n} d$  and  $\mu(n)$  is the Möbius function. In the course of proving this formula we will see that

$$\sum_{m|u} \left(\frac{\Delta}{m}\right) \sigma(u/m) \mu(m) > 0;$$

one can show that it is less than  $\sigma(u)$ . From [2], Theorem 322 we know that

$$\frac{\sigma(u)}{u} = O(u^\kappa) = O(t^\kappa)$$

for any  $\kappa > 0$ . By the Brauer–Siegel theorem and the fact that  $\Delta \leq t^2 - 4$ ,

$$t^{-\kappa} \leq \frac{2h(\Delta) \log(\epsilon_1)}{\sqrt{\Delta}} \leq t^\kappa$$

for any  $\kappa > 0$ , for  $\Delta$  sufficiently large. Thus if one expects that  $(\Delta/q)$  is random for  $\Delta = \Delta(t)$  as  $t$  increases, then there should be cancellation in the sum and it will not grow like  $\exp(\rho^2 T)$ , so  $\lambda < 1/4$  will not occur.

The proof of the theorem depends, of course, on the trace formula. Let  $f$  be a Maass cusp form corresponding to the eigenvalue  $\lambda < 1/4$ . Then for  $\gamma$  in  $\Gamma = SL(2, \mathbb{Z})$ ,  $f(\gamma z)$  is also a Maass cusp form for the same eigenvalue, so  $\Gamma$  acts in this finite dimensional space. The principal congruence subgroup  $\Gamma(q)$  acts trivially giving a representation  $\pi$  of the factor group  $G = \Gamma/\Gamma(q)$ , (isomorphic to  $SL(2, \mathbb{F}_q)$ ). Let  $B = \Gamma_0(q)/\Gamma(q)$ , then since  $f$  is fixed by  $\Gamma_0(q)$ , the multiplicity of the trivial representation in the restriction of  $\pi$  to  $B$  is  $\geq 1$ . By Frobenius Reciprocity, the multiplicity of  $\pi$  in the induced representation  $\text{Ind}_B(1)$  is also  $\geq 1$ . This is a  $q + 1$  dimensional representation isomorphic to the space of functions

$$\{f : B \backslash G \rightarrow \mathbb{C}\}$$

where the group  $G$  acts by right translation

$$\text{Ind}_B(1)(g)f(Bx) = f(Bxg).$$

By Mackey's Irreducibility Criterion (see e.g. [9] p. 59) the induced representation has two irreducible components,

$$\text{Ind}_B(1) = 1 \oplus \theta.$$

Here  $1$  is the trivial representation of  $G$  and  $\theta$  is realized in the  $q$  dimensional subspace orthogonal to the constant functions; i.e. in the space of functions

$$\{f : B \backslash G \rightarrow \mathbb{C} \mid \sum f(Bg) = 0\}.$$

Since  $f$  is not fixed by  $\Gamma = SL(2, \mathbb{Z})$  ( $\lambda > 1/4$  is known), the projection of  $f$  on the space isomorphic to that of  $\theta$  is not 0. Thus there exist cusp forms which transform according to  $\theta$ . Then by [4] ((16) on page 182),  $\lambda$  is an eigenvalue for the Laplacian acting in the space of vector valued Maass cusp forms for  $\Gamma$  with multiplier  $\theta$ :

$$F : \mathcal{H} \rightarrow \mathbb{C}^q \quad \text{such that} \quad F(\gamma z) = \theta(\gamma)F(z).$$

We briefly recall the trace formula for such forms, as in Theorem 4.2 in [3], page 315. Let

$$g(u) = \exp(-u^2/4T)/\sqrt{4\pi T}, \quad \text{and} \quad h(r) = \exp(-r^2T)$$

its Fourier transform be our test functions. The eigenvalues of the Laplacian

$\lambda_n = 1/4 + r_n^2$  are related to the primitive hyperbolic conjugacy classes  $\{P\}$  by

$$\begin{aligned} \sum_n h(r_n) &= \sum_{\{P\}} \sum_{k=1}^{\infty} \frac{\text{Trace } \theta(P^k) \log(NP)}{NP^{k/2} - NP^{-k/2}} g(\log(NP^k)) \\ &+ \int_{-\infty}^{\infty} h(r) \{\text{contribution from central class}\} dr \\ &+ \int_{-\infty}^{\infty} h(r) \{\text{contribution from elliptic classes}\} dr \\ &+ \int_{-\infty}^{\infty} h(r) \{\text{parabolic contribution to continuous spectrum}\} dr \\ &+ g(0) \{\text{parabolic contribution to discrete spectrum}\} \\ &+ h(0) \{\text{parabolic contribution to discrete spectrum}\}. \end{aligned}$$

All sums and integrals are absolutely convergent. Recall that  $NP = \epsilon^2$  where  $\epsilon$  is the larger root of the characteristic polynomial of  $P$ . By the Dominated Convergence Theorem, we have, as  $T \rightarrow \infty$ ,

$$\int_{-\infty}^{\infty} h(r) \{*\} dr \rightarrow 0.$$

The terms with  $h(0)$  and  $g(0)$  are  $O(1)$  as  $T \rightarrow \infty$ . We next consider the terms from the spectral side. For all but finitely many  $n$ , say  $n > N$  we have  $\lambda_n > 1/4$  so  $r_n$  is in  $\mathbb{R}$ . Thus

$$\sum_{n > N} h(r_n) \rightarrow 0$$

again by the Dominated Convergence Theorem. The finitely many eigenvalues less than  $1/4$  have  $r_n$  purely imaginary. Note that the contribution of the smallest such eigenvalue dominates the others, and 0 does not occur as an eigenvalue as  $\theta$  is a nontrivial representation. Thus as  $T \rightarrow \infty$  we have

$$\frac{1}{\sqrt{4\pi T}} \sum_{\{P\}} \sum_{k=1}^{\infty} \frac{\text{Trace } \theta(P^k) \log(NP)}{NP^{k/2} - NP^{-k/2}} \exp(-\log^2(NP^k)/4T) \sim \exp(\rho^2 T). \tag{1}$$

We define the usual map  $\phi$  from matrices to binary quadratic forms

$$\phi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \rightarrow \left[ v, \frac{d-a}{v}, \frac{-c}{v} \right]$$

where  $v = gcd(b, d - a, c)$ . A form  $[\alpha, \beta, \gamma]$  with discriminant  $D$  is the image of

$$\begin{bmatrix} \frac{t - u\beta}{2} & u\alpha \\ -u\gamma & \frac{t + u\beta}{2} \end{bmatrix}.$$

where  $t^2 - u^2D = 4$  is the fundamental solution to Pell's equation for  $D$ . Sarnak shows in Proposition 1.4 of [7] that  $\phi$  is a 2 - 1 map and commutes with the action of the modular group giving a 2 - 1 correspondence between primitive hyperbolic conjugacy classes and equivalence classes of indefinite binary quadratic forms. For a  $\{P\}$  corresponding to a class  $\{\phi\}$  of discriminant  $D$ , write  $D = d^2\Delta$ , with  $\Delta$  the discriminant of a real quadratic field. We will often suppress  $\Delta$  from the notation. Note that  $NP$  is the square of the larger root of the characteristic polynomial and so only depends on the discriminant  $D$ . Write

$$NP = \epsilon_d^2 \quad \text{for} \quad \epsilon_d = \frac{t_d + u_d \sqrt{d^2 \Delta}}{2}.$$

We next analyze Trace  $\theta$ . We will show that Trace  $\theta(P^k)$  depends only on the characteristic polynomial of  $P^k$  which is the minimum polynomial  $x^2 - tx + 1$  of  $\epsilon_d^k$ . First suppose

$$t \equiv \pm 1 \pmod{q^2}.$$

Sarnak ([7], Proposition 3.3) shows that

$$P^k \equiv \pm I \pmod{q} \Leftrightarrow q \mid u \Leftrightarrow q^2 \mid t^2 - 4 \Leftrightarrow t \equiv \pm 2 \pmod{q^2}.$$

Tables ([1] vol. IV, p. 1829) show this is the case when Trace  $\theta$  is equal to  $q$ . Now suppose

$$t \not\equiv \pm 2 \pmod{q}.$$

From the tables Trace  $\theta = 1$  if and only if the matrix element  $P^k \pmod{q}$  is diagonalizable over  $\mathbb{F}_q$  but not central. This occurs if and only if the discriminant  $t^2 - 4 = u^2\Delta$  is a square in  $\mathbb{F}_q^*$ ; i.e.,  $(\Delta/q) = 1$ . Similarly Trace  $\theta = -1$  if and only if the matrix element  $P^k \pmod{q}$  is not diagonalizable over  $\mathbb{F}_q$  but is diagonalizable over  $\mathbb{F}_{q^2}$ . This occurs when the discriminant  $t^2 - 4 = u^2\Delta$  is not a square in  $\mathbb{F}_q^*$ ; i.e.,

$(\Delta/q) = -1$ . Finally if

$$t \equiv \pm 1 \pmod{q} \quad \text{but not} \quad \pmod{q^2},$$

then

$$t^2 - 4 \equiv 0 \pmod{q} \quad \text{but not} \quad \pmod{q^2}.$$

By the above,  $q \nmid u$ , so  $q \mid \Delta$ . Also,  $x^2 - tx + 1$  has repeated roots  $\pmod{q}$ , but  $P^k \pmod{q}$  is not central. Thus  $P^k$  is unipotent  $\pmod{q}$  and the table shows that  $\text{Trace } \theta(t) = 0 = (\Delta/q)$ . This shows that the Trace  $\theta$  depends only on the characteristic polynomial and is equal to the function  $\Theta$  defined above. (Since the determinant is always 1 only the  $x$  coefficient, yet another trace, matters.) It will be convenient to write  $\Theta(\epsilon_d^k)$ .

This gives the formula

$$\frac{1}{\sqrt{\pi T}} \sum_{D=d^2\Delta} \sum_{k=1}^{\infty} 2h(d^2\Delta) \log(\epsilon_d) \frac{\Theta(\epsilon_d^k)}{\epsilon_d^k - \epsilon_d^{-k}} \exp(-\log^2(\epsilon_d^k)/T) \sim \exp(\rho^2 T). \quad (2)$$

The class number  $h(d^2\Delta)$  of forms is related to the ideal class number  $h(\Delta)$  of  $\mathbb{Q}(\sqrt{\Delta})$  by formula (see e.g. [6] p. 95)

$$h(d^2\Delta) = \frac{h(\Delta)d}{[O_{\Delta}^* : \mathbb{Z}[\epsilon_d]^*]} \prod_{l \mid d} \left( 1 - \frac{\left(\frac{\Delta}{l}\right)}{l} \right).$$

And for  $\epsilon_1$  the fundamental norm 1 unit in  $\mathbb{Q}(\sqrt{\Delta})$  we have

$$\epsilon_d = \epsilon_1^{[O_{\Delta}^* : \mathbb{Z}[\epsilon_d]^*]}.$$

In fact this follows the definition of  $\epsilon_1$  and  $\mathbb{Z}[\epsilon_d]$ . Substituting this in (2) gives

$$\frac{1}{\sqrt{\pi T}} \sum_{D=d^2\Delta} 2h(\Delta) \log(\epsilon_1) d \prod_{l \mid d} \left( 1 - \frac{\left(\frac{\Delta}{l}\right)}{l} \right) \sum_{k=1}^{\infty} \frac{\Theta(\epsilon_d^k)}{\epsilon_d^k - \epsilon_d^{-k}} \exp(-\log^2(\epsilon_d^k)/T) \sim \exp(\rho^2 T) \quad (3)$$

Still viewing  $\Delta$  as fixed we want to group all terms of the form  $\epsilon_d^k = \epsilon_1^n$ .

We have

$$\epsilon_d = \frac{t_d + u_d \sqrt{d^2 \Delta}}{2}$$

and suppose

$$\epsilon_1^n = \frac{t(n) + u(n) \sqrt{\Delta}}{2}$$

then

$$\epsilon_d^k - \epsilon_d^{-k} = \epsilon_1^n - \epsilon_1^{-n} = u(n) \sqrt{\Delta}.$$

We combine the infinite sum on  $d$  and  $k$  to sum on  $n$  and  $d | u(n)$  to get

$$\begin{aligned} \frac{1}{\sqrt{\pi T}} \sum_{\Delta} \frac{2h(\Delta) \log(\epsilon_1)}{\sqrt{\Delta}} \sum_{n=1}^{\infty} \sum_{d|u(n)} \frac{d}{u(n)} \prod_{l|d} \left( 1 - \frac{\left(\frac{\Delta}{l}\right)}{l} \right) \Theta(\epsilon_1^n) \exp(-\log^2(\epsilon_1^n)/T) \\ \sim \exp(\rho^2 T) \quad (4) \end{aligned}$$

Now

$$\begin{aligned} \sum_{d|u(n)} \frac{d}{u(n)} \prod_{l|d} \left( 1 - \frac{\left(\frac{\Delta}{l}\right)}{l} \right) &= \sum_{d|u(n)} \frac{d}{u(n)} \sum_{m|d} \frac{\mu(m)}{m} \left(\frac{\Delta}{m}\right) \\ &= \frac{1}{u(n)} \sum_{dd'=u(n)} \sum_{mm'=d} m' \mu(m) \left(\frac{\Delta}{m}\right) \\ &= \frac{1}{u(n)} \sum_{m|u(n)} \mu(m) \left(\frac{\Delta}{m}\right) \sum_{m'|u(n)/m} m' \\ &= \frac{1}{u(n)} \sum_{m|u(n)} \mu(m) \left(\frac{\Delta}{m}\right) \sigma\left(\frac{u(n)}{m}\right). \end{aligned}$$

From this we get that (4) is equal

$$\begin{aligned} \frac{1}{\sqrt{\pi T}} \sum_{\Delta} \frac{2h(\Delta) \log(\epsilon_1)}{\sqrt{\Delta}} \sum_{n=1}^{\infty} \sum_{m|u(n)} \mu(m) \left(\frac{\Delta}{m}\right) \sigma\left(\frac{u(n)}{m}\right) \frac{\Theta(\epsilon_1^n)}{u(n)} \exp(-\log^2(\epsilon_1^n)/T) \\ \sim \exp(\rho^2 T). \quad (5) \end{aligned}$$

To get the theorem we now need to reorder the terms in the sum. We have a sum over all real quadratic fields, and over all positive powers of the fundamental norm 1 unit of that field. These units are in 1–1 correspondence with their minimum polynomial  $x^2 - tx + 1$  ordered by their trace  $t$ . Note that  $\Theta$  depends only on  $t$  above. As the units  $\epsilon_1^n \rightarrow \infty$ ; we have  $\epsilon_1^{-n} \rightarrow 0$ , and since  $t = \epsilon_1^n + \epsilon_1^{-n}$  we can replace  $\epsilon_1^n$  with  $t$  in the statement of the theorem.

The authors would like to thank Morris Newman for correcting some mistakes in an earlier version of this paper.

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Received January 30, 1990