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On small eigenvalues of the Laplacian for $\Gamma_0(q)\backslash\mathscr{H}$

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Let $\mathscr{H}$ be the complex upper half plane, and $\Gamma_0(q)$ be the subgroup of matrices

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

in $SL(2, \mathbb{Z})$ with $c \equiv 0 \pmod q$. Suppose f is a Maass cusp form with eigenvalue λ; i.e., a non-constant function $f: \mathscr{H} \to \mathbb{C}$ satisfying

$$f(\gamma z) = f(z) \qquad \text{for } \gamma \text{ in } \Gamma_0(q), z \text{ in } \mathscr{H}$$

$$y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)f + \lambda f = 0$$

$$\int_{\mathscr{F}} |f(z)|^2 \frac{dx\,dx}{y^2} < \infty$$

where $\mathscr{F}$ is a fundamental domain for $\Gamma_0(q)$. Selberg conjectured [8] that $\lambda \geqq 1/4$, and showed that $\lambda \geqq 3/16$. Iwaniec has a statistical result [5] that shows the rarity of small eigenvalues, similar to density theorems about real zeros of Dirichlet's L-series.

For an odd prime q, whether q is ramified, split, or inert in a real quadratic field $\mathbb{Q}(\sqrt{\Delta})$ depends only on the Legendre symbol (Δ/q), and so is periodic in $\Delta(\text{mod } q)$. If we consider instead the set of all norm 1 units $\epsilon > 1$ in all real quadratic fields, ordered by the size of ϵ, we expect the behavior of q in $\mathbb{Q}(\epsilon)$ to be more or less random. We use the trace formula to show that if there is an eigenvalue λ less than 1/4 then that expectation is wrong; instead q has a bias towards a certain behavior.

Specifically let $t \geqq 3$ in $\mathbb{Z}$, and write $t^2 - 4 = u^2\Delta$ with Δ a discriminant of a real quadratic field. Then $\mathbb{Q}(\sqrt{\Delta}) = \mathbb{Q}(\epsilon)$ for ϵ the larger root of $x^2 - tx + 1 = 0$. Let $h(\Delta)$ be the narrow class number and ϵ_1 the fundamental norm 1 unit. Define

$$\Theta(t) = \begin{cases} q & \text{if } t \equiv \pm 2 (\text{mod } q^2) \\ \left(\dfrac{\Delta}{q}\right) & \text{otherwise} \end{cases}$$

If λ is the smallest eigenvalue; $\lambda = 1/4 + r^2$ with $r = i\rho$ purely imaginary, $0 < \rho < 1/2$ then for $T \to \infty$ we have

THEOREM.

$$\frac{1}{\sqrt{\pi T}} \sum_{t \geqq 3} \frac{2h(\Delta) \log (\epsilon_1)}{\sqrt{\Delta}} \sum_{m \mid u} \left(\frac{\Delta}{m}\right) \sigma(u/m) \mu(m) u^{-1} \Theta(t) \exp(-\log^2(t)/T) \sim \exp(\rho^2 T)$$

Here $\sigma(n) = \Sigma_{d \mid n} d$ and $\mu(n)$ is the Möbius function. In the course of proving this formula we will see that

$$\sum_{m \mid u} \left(\frac{\Delta}{m}\right) \sigma(u/m) \mu(m) > 0;$$

one can show that it is less than $\sigma(u)$. From [2], Theorem 322 we know that

$$\frac{\sigma(u)}{u} = O(u^{\kappa}) = O(t^{\kappa})$$

for any $\kappa > 0$. By the Brauer–Siegel theorem and the fact that $\Delta \leqq t^2 - 4$,

$$t^{-\kappa} \leqq \frac{2h(\Delta) \log(\epsilon_1)}{\sqrt{\Delta}} \leqq t^{\kappa}$$

for any $\kappa > 0$, for Δ sufficiently large. Thus if one expects that (Δ/q) is random for $\Delta = \Delta(t)$ as t increases, then there should be cancellation in the sum and it will not grow like $\exp(\rho^2 T)$, so $\lambda < 1/4$ will not occur.

The proof of the theorem depends, of course, on the trace formula. Let f be a Maass cusp form corresponding to the eigenvalue $\lambda < 1/4$. Then for γ in $\Gamma = SL(2, \mathbb{Z}), f(\gamma z)$ is also a Maass cusp form for the same eigenvalue, so Γ acts in this finite dimensional space. The principal congruence subgroup $\Gamma(q)$ acts trivially giving a representation π of the factor group $G = \Gamma/\Gamma(q)$, (isomorphic to $SL(2, \mathbb{F}_q)$). Let $B = \Gamma_0(q)/\Gamma(q)$, then since f is fixed by $\Gamma_0(q)$, the multiplicity of the trivial representation in the restriction of π to B is $\geqq 1$. By Frobenius Reciprocity, the multiplicity of π in the induced representation $\mathrm{Ind}_B(1)$ is also $\geqq 1$. This is a $q + 1$ dimensional representation isomorphic to the space of functions

$$\{f : B\backslash G \to \mathbb{C}\}$$

where the group G acts by right translation

$$\mathrm{Ind}_B(1)(g)f(Bx) = f(Bxg).$$

By Mackey's Irreducibility Criterion (see e.g. [9] p. 59) the induced representation has two irreducible components,

$$\mathrm{Ind}_B(1) = 1 \oplus \theta.$$

Here 1 is the trivial representation of G and θ is realized in the q dimensional subspace orthogonal to the constant functions; i.e. in the space of functions

$$\{f : B\backslash G \to \mathbb{C} \mid \Sigma f(Bg) = 0\}.$$

Since f is not fixed by $\Gamma = SL(2, \mathbb{Z})$ ($\lambda > 1/4$ is known), the projection of f on the space isomorphic to that of θ is not 0. Thus there exist cusp forms which transform according to θ. Then by [4] ((16) on page 182), λ is an eigenvalue for the Laplacian acting in the space of vector valued Maass cusp forms for Γ with multiplier θ:

$$F : \mathscr{H} \to \mathbb{C}^q \quad \text{such that} \quad F(\gamma z) = \theta(\gamma)F(z).$$

We briefly recall the trace formula for such forms, as in Theorem 4.2 in [3], page 315. Let

$$g(u) = \exp(-u^2/4T)/\sqrt{4\pi T}, \quad \text{and} \quad h(r) = \exp(-r^2 T)$$

its Fourier transform be our test functions. The eigenvalues of the Laplacian

$\lambda_n = 1/4 + r_n^2$ are related to the primitive hyperbolic conjugacy classes $\{P\}$ by

$$\begin{aligned}\sum_n h(r_n) = \sum_{\{P\}} \sum_{k=1}^{\infty} \frac{\text{Trace}\,\theta(P^k)\log(NP)}{NP^{k/2} - NP^{-k/2}} g(\log(NP^k))\\ + \int_{-\infty}^{\infty} h(r)\{\text{contribution from central class}\}\,dr\\ + \int_{-\infty}^{\infty} h(r)\{\text{contribution from elliptic classes}\}\,dr\\ + \int_{-\infty}^{\infty} h(r)\{\text{parabolic contribution to continuous spectrum}\}\,dr\\ + g(0)\{\text{parabolic contribution to discrete spectrum}\}\\ + h(0)\{\text{parabolic contribution to discrete spectrum}\}.\end{aligned}$$

All sums and integrals are absolutely convergent. Recall that $NP = \epsilon^2$ where ϵ is the larger root of the characteristic polynomial of P. By the Dominated Convergence Theorem, we have, as $T \to \infty$,

$$\int_{-\infty}^{\infty} h(r)\{*\}\,dr \to 0.$$

The terms with $h(0)$ and $g(0)$ are $O(1)$ as $T \to \infty$. We next consider the terms from the spectral side. For all but finitely many n, say $n > N$ we have $\lambda_n > 1/4$ so r_n is in $\mathbb{R}$. Thus

$$\sum_{n>N} h(r_n) \to 0$$

again by the Dominated Convergence Theorem. The finitely many eigenvalues less than $1/4$ have r_n purely imaginary. Note that the contribution of the smallest such eigenvalue dominates the others, and 0 does not occur as an eigenvalue as θ is a nontrivial representation. Thus as $T \to \infty$ we have

$$\frac{1}{\sqrt{4\pi T}} \sum_{\{P\}} \sum_{k=1}^{\infty} \frac{\text{Trace}\,\theta(P^k)\log(NP)}{NP^{k/2} - NP^{-k/2}} \exp(-\log^2(NP^k)/4T) \sim \exp(\rho^2 T). \qquad (1)$$

We define the usual map ϕ from matrices to binary quadratic forms

$$\phi : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \to \left[\frac{b}{v}, \frac{d-a}{v}, \frac{-c}{v}\right]$$

where $v = gcd(b, d-a, c)$. A form $[\alpha, \beta, \gamma]$ with discriminant D is the image of

$$\begin{bmatrix} \frac{t-u\beta}{2} & u\alpha \\ -u\gamma & \frac{t+u\beta}{2} \end{bmatrix}.$$

where $t^2 - u^2 D = 4$ is the fundamental solution to Pell's equation for D. Sarnak shows in Proposition 1.4 of [7] that ϕ is a $2-1$ map and commutes with the action of the modular group giving a $2-1$ correspondence between primitive hyperbolic conjugacy classes and equivalence classes of indefinite binary quadratic forms. For a $\{P\}$ corresponding to a class $\{\phi\}$ of discriminant D, write $D = d^2 \Delta$, with Δ the discriminant of a real quadratic field. We will often suppress Δ from the notation. Note that NP is the square of the larger root of the characteristic polynomial and so only depends on the discriminant D. Write

$$NP = \epsilon_d^2 \quad \text{for} \quad \epsilon_d = \frac{t_d + u_d\sqrt{d^2\Delta}}{2}.$$

We next analyze Trace θ. We will show that Trace $\theta(P^k)$ depends only on the characteristic polynomial of P^k which is the minimum polynomial $x^2 - tx + 1$ of ϵ_d^k. First suppose

$$t \equiv \pm(\text{mod } q^2).$$

Sarnak ([7], Proposition 3.3) shows that

$$P^k \equiv \pm I(\text{mod } q) \Leftrightarrow q \mid u \Leftrightarrow q^2 \mid t^2 - 4 \Leftrightarrow t \equiv \pm 2(\text{mod } q^2).$$

Tables ([1] vol. IV, p. 1829) show this is the case when Trace θ is equal to q. Now suppose

$$t \not\equiv \pm 2(\text{mod } q).$$

From the tables Trace $\theta = 1$ if and only if the matrix element $P^k(\text{mod } q)$ is diagonalizable over $\mathbb{F}_q$ but not central. This occurs if and only if the discriminant $t^2 - 4 = u^2\Delta$ is a square in $\mathbb{F}_q^*$; i.e., $(\Delta/q) = 1$. Similarly Trace $\theta = -1$ if and only if the matrix element $P^k(\text{mod } q)$ is not diagonalizable over $\mathbb{F}_q$ but is diagonalizable over $\mathbb{F}_{q^2}$. This occurs when the discriminant $t^2 - 4 = u^2\Delta$ is not a square in $\mathbb{F}_q^*$; i.e.,

$(\Delta/q) = -1$. Finally if

$$t \equiv \pm(\mathrm{mod}\, q) \quad \text{but not} \quad (\mathrm{mod}\, q^2),$$

then

$$t^2 - 4 \equiv 0(\mathrm{mod}\, q) \quad \text{but not} \quad (\mathrm{mod}\, q^2).$$

By the above, $q \nmid u$, so $q \mid \Delta$. Also, $x^2 - tx + 1$ has repeated roots (mod q), but P^k(mod q) is not central. Thus P^k is unipotent (mod q) and the table shows that Trace $\theta(t) = 0 = (\Delta/q)$. This shows that the Trace θ depends only on the characteristic polynomial and is equal to the function Θ defined above. (Since the determinant is always 1 only the x coefficient, yet another trace, matters.) It will be convenient to write $\Theta(\epsilon_d^k)$.

This gives the formula

$$\frac{1}{\sqrt{\pi T}} \sum_{D=d^2\Delta} \sum_{k=1}^{\infty} 2h(d^2\Delta) \log(\epsilon_d) \frac{\Theta(\epsilon_d^k)}{\epsilon_d^k - \epsilon_d^{-k}} \exp(-\log^2(\epsilon_d^k)/T) \sim \exp(\rho^2 T). \quad (2)$$

The class number $h(d^2\Delta)$ of forms is related to the ideal class number $h(\Delta)$ of $\mathbb{Q}(\sqrt{\Delta})$ by formula (see e.g. [6] p. 95)

$$h(d^2\Delta) = \frac{h(\Delta)d}{[O_\Delta^* : \mathbb{Z}[\epsilon_d]^*]} \prod_{l \mid d} \left(1 - \frac{\left(\frac{\Delta}{l}\right)}{l}\right).$$

And for ϵ_1 the fundamental norm 1 unit in $\mathbb{Q}(\sqrt{\Delta})$ we have

$$\epsilon_d = \epsilon_1^{[O_\Delta^* : \mathbb{Z}[\epsilon_d]^*]}.$$

In fact this follows the definition of ϵ_1 and $\mathbb{Z}[\epsilon_d]$. Substituting this in (2) gives

$$\frac{1}{\sqrt{\pi T}} \sum_{D=d^2\Delta} 2h(\Delta) \log(\epsilon_1)\, d \prod_{l \mid d} \left(1 - \frac{\left(\frac{\Delta}{l}\right)}{l}\right) \sum_{k=1}^{\infty} \frac{\Theta(\epsilon_d^k)}{\epsilon_d^k - \epsilon_d^{-k}} \exp(-\log^2(\epsilon_d^k)/T)$$
$$\sim \exp(\rho^2 T) \quad (3)$$

Still viewing Δ as fixed we want to group all terms of the form $\epsilon_d^k = \epsilon_1^n$.

We have

$$\epsilon_d = \frac{t_d + u_d\sqrt{d^2\Delta}}{2}$$

and suppose

$$\epsilon_1^n = \frac{t(n) + u(n)\sqrt{\Delta}}{2}$$

then

$$\epsilon_d^k - \epsilon_d^{-k} = \epsilon_1^n - \epsilon_1^{-n} = u(n)\sqrt{\Delta}.$$

We combine the infinite sum on d and k to sum on n and $d \mid u(n)$ to get

$$\frac{1}{\sqrt{\pi T}} \sum_{\Delta} \frac{2h(\Delta)\log(\epsilon_1)}{\sqrt{\Delta}} \sum_{n=1}^{\infty} \sum_{d \mid u(n)} \frac{d}{u(n)} \prod_{l \mid d} \left(1 - \frac{\left(\frac{\Delta}{l}\right)}{l}\right) \Theta(\epsilon_1^n) \exp(-\log^2(\epsilon_1^n)/T) \sim \exp(\rho^2 T) \quad (4)$$

Now

$$\begin{aligned}
\sum_{d \mid u(n)} \frac{d}{u(n)} \prod_{l \mid d} \left(1 - \frac{\left(\frac{\Delta}{l}\right)}{l}\right) &= \sum_{d \mid u(n)} \frac{d}{u(n)} \sum_{m \mid d} \frac{\mu(m)}{m} \left(\frac{\Delta}{m}\right) \\
&= \frac{1}{u(n)} \sum_{dd' = u(n)} \sum_{mm' = d} m'\mu(m) \left(\frac{\Delta}{m}\right) \\
&= \frac{1}{u(n)} \sum_{m \mid u(n)} \mu(m) \left(\frac{\Delta}{m}\right) \sum_{m' \mid u(n)/m} m' \\
&= \frac{1}{u(n)} \sum_{m \mid u(n)} \mu(m) \left(\frac{\Delta}{m}\right) \sigma\left(\frac{u(n)}{m}\right).
\end{aligned}$$

From this we get that (4) is equal

$$\frac{1}{\sqrt{\pi T}} \sum_{\Delta} \frac{2h(\Delta)\log(\epsilon_1)}{\sqrt{\Delta}} \sum_{n=1}^{\infty} \sum_{m \mid u(n)} \mu(m) \left(\frac{\Delta}{m}\right) \sigma\left(\frac{u(n)}{m}\right) \frac{\Theta(\epsilon_1^n)}{u(n)} \exp(-\log^2(\epsilon_1^n)/T) \sim \exp(\rho^2 T). \quad (5)$$

To get the theorem we now need to reorder the terms in the sum. We have a sum over all real quadratic fields, and over all positive powers of the fundamental norm 1 unit of that field. These units are in $1-1$ correspondence with their minimum polynomial $x^2 - tx + 1$ ordered by their trace t. Note that Θ depends only on t above. As the units $\epsilon_1^n \to \infty$; we have $\epsilon_1^{-n} \to 0$, and since $t = \epsilon_1^n + \epsilon_1^{-n}$ we can replace ϵ_1^n with t in the statement of the theorem.

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