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## Rational category of the space of sections of a nilpotent bundle

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*Abstract.* Denote by  $\zeta : F \rightarrow E \xrightarrow{p} B$  a nilpotent fibration where  $F$  is a 1-connected space of finite category and  $B$  a finite c.w. complex with non trivial rational cohomology. In this note we compute the rational category of the space  $\Gamma_*$  of continuous pointed sections of  $\zeta$ .

### §1. Introduction

In 1956, R. Thom studied the homotopy type of the space  $F_f^X$  of continuous maps of  $X$  into  $F$  homotopic to a given map  $f$ . He computed explicitly the cohomology of  $F_f^X$  when  $F$  is a product of Eilenberg–Mac Lane spaces [12].

Later on, following ideas of Sullivan, A. Haefliger gave the rational minimal model of the space of sections of a nilpotent bundle [7]. This model has been extensively studied by K. Shibata and M. Vigué-Poirrier [14]. In particular, M. Vigué-Poirrier noted that, if the dimension of  $X$  is less than the connectivity of  $Y$ , then the rational homotopy Lie algebra of  $Y^X$  is isomorphic as a Lie algebra to  $H^*(X; \mathbb{Q}) \otimes (\pi_*(\Omega Y) \otimes \mathbb{Q})$ .

The aim of this paper is to show that the category of the space of continuous maps from  $X$  into  $Y$ , and more generally of pointed sections of a fibration, is often infinite.

To be more precise, we prove in fact the following two theorems.

**THEOREM 1.** *Let  $\Gamma_*$  be the space of continuous pointed sections of a nilpotent fibration  $F \rightarrow E \rightarrow B$ . Suppose that*

- (1)  $\Gamma_* \neq \phi$
- (2)  $F$  is a nilpotent space of finite category
- (3)  $H^+(B; \mathbb{Q}) \neq 0$  and  $\dim H^*(B; \mathbb{Q}) < \infty$
- (4)  $\dim (\pi_*(F) \otimes \mathbb{Q})$  is infinite.

*Then,  $\text{cat}(\Gamma_*) = \infty$ .*

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\* Chercheur qualifié FNRS.

In the case where the fibration is trivial, the result is more precise:

**THEOREM 2.** *Let  $X$  be a finite nilpotent c.w. complex and  $Y$  be a nilpotent space. We suppose that the rational cohomologies of  $X$  and  $Y$  are not trivial and one of the two following conditions is satisfied:*

- (1)  $\dim \pi_*(Y) \otimes \mathbb{Q} = \infty$
- (2)  $\dim \pi_*(Y) \otimes \mathbb{Q} < \infty$  and there are odd integers  $p$  and  $q$  such that  $H^p(X; \mathbb{Q}) \neq 0$ ,  $\pi_q(Y) \otimes \mathbb{Q} \neq 0$  and  $q - p \geq 2$ .

*Then the functional space  $Y^X$  has infinite category.*

This result clearly yields the following corollary previously proved by E. Fadell and S. Husseini.

**COROLLARY [3].** *If  $Y$  is a 1-connected space of finite category, such that  $\tilde{H}^*(Y; \mathbb{Q}) \neq 0$ ; then the free loop space  $Y^{S^1}$  has infinite category.*

The organization of the paper is as follows. We first recall the construction of the Sullivan–Haefliger model for the space of continuous (resp. pointed) sections  $\Gamma$  (resp.  $\Gamma_*$ ). We then show how to deduce the two theorems from the model. We also deduce a way to compute explicitly the rational homotopy groups of  $\Gamma$ .

In fact, if  $X$  is a 1-connected space and  $X_0$  its rationalization, we have the inequality  $\text{cat } X_0 \leq \text{cat } X$  [13]. The integer  $\text{cat}(X_0)$  is called the rational category of  $X$  and is denoted  $\text{cat}_0 X$ . Its relevance comes from the fact that  $\text{cat}_0(X)$  can be obtained from the minimal model of the space [5].

## §2. The Sullivan–Haefliger model

Let  $\zeta : F \rightarrow E \xrightarrow{p} B$  be a fibration. We suppose that  $B$  is a finite nilpotent c.w. complex and  $F$  is a nilpotent space with finite Betti numbers. We suppose that  $\Gamma_* \neq \phi$ .

Let  $(A, d_A) \rightarrow (A \otimes \Lambda V, D) \rightarrow (\Lambda V, d)$  be a minimal K.S. model of  $\zeta$  [9]. As  $\Gamma_* \neq \phi$ , we can also suppose that the differential  $D$  satisfies:

$$D(V) \subset A \otimes \Lambda^+ V.$$

$B$  is a finite nilpotent c.w. complex. Therefore we can average  $A$  is a finite dimensional graded  $\mathbb{Q}$ -vector space. Denote by  $S$  a graded supplementary subspace of the graded vector space formed by the cocycles in  $A$ . This gives a direct sum decomposition of  $A : A = S \oplus d(S) \oplus T$ . We then choose a homogeneous basis

$(a_i)_{i \in I}$  of  $A$  by taking the union of homogeneous bases of  $S$ ,  $d(S)$  and  $T$ . The graded dual vector space of  $A$  will be denoted by  $A^\vee$ :

$$(A^\vee)_n = \text{Hom}(A^n, \mathbb{Q}).$$

$A^\vee$  is naturally equipped with the dual basis  $a_i^*$ :

$$\langle a_i^*; a_j \rangle = \delta_{ij}.$$

We now look at the map of algebras defined by:

$$\varepsilon : A \otimes \Lambda V \rightarrow A \otimes \Lambda(A^\vee \otimes V) : \varepsilon(v) = \sum_{i \in I} a_i \otimes (a_i^* \otimes v); \quad \varepsilon(a) = a, \quad a \in A.$$

In [7], A. Haefliger shows how to put a uniquely defined differential  $d_A \otimes \delta$  on  $A \otimes \Lambda(A^\vee \otimes V)$  in such a way that  $\varepsilon$  becomes a morphism of commutative differential graded algebras. Let  $W$  be the quotient of  $A^\vee \otimes V$  by the subspace of elements of degree  $< 0$ , and by the subspace formed by the  $\delta$ -cocycles in degree 0.

A short computation shows that  $\delta(1 \otimes v) = 1 \otimes dv$ , so that the injection  $V \cong \mathbb{Q} \otimes V \hookrightarrow A^\vee \otimes V$  induces a K.S. extension:

$$\theta : (\Lambda V, d) \rightarrow (\Lambda W, \delta) \rightarrow (\Lambda(W/V), \delta).$$

**THEOREM A** (Haefliger, [7]).  *$\theta$  is a model for the canonical fibration  $\Gamma_* \rightarrow \Gamma \xrightarrow{p} F$  where  $p$  denotes the evaluation on the basis point of  $B$ .*

With this model, we can for instance give a rational analogue of the Cohen–Taylor theorem [2].

**PROPOSITION.** *Let  $X$  be a finite wedge of spheres of dimension less than  $m$  ( $X = \vee_{i=1}^r S^{n_i}$ ) and  $Y$  be a  $(m+2)$ -connected space, then we have a rational homotopy equivalence*

$$(Y^X)_* \cong \prod_{i=1}^r (Y^{S^{n_i}})_*.$$

*Proof.* The Haefliger model for  $(Y^X)_*$  is  $(\Lambda(H_*^+(X; \mathbb{Q}) \otimes (\pi_*(Y))^\vee), 0)$ . □

**§3. The rational homotopy Lie algebra of a space**

If  $S$  is a nilpotent space with finite Betti numbers, the minimal model of  $S$  is a free commutative differential graded algebra  $(\Lambda Z, d)$ . The graded vector spaces  $Z^*$  and  $\text{Hom}(\pi_*(S), \mathbb{Q})$  are then isomorphic [11].

The differential  $d$  always decomposes in the form  $d = d_2 + d_3 + \dots$ , where  $d_i(Z) \subset \Lambda^i Z$ . This gives on  $s^{-1} \text{Hom}(Z, \mathbb{Q})$  a structure of Lie algebra by putting:

$$\langle d_2 z; f, g \rangle = (-1)^{\text{deg}(f)+1} \langle s z; [s^{-1} f, s^{-1} g] \rangle$$

$z \in Z; f, g \in \text{Hom}(Z, \mathbb{Q})$ .

It is a result of Andrews and Arkowitz [1] that this Lie algebra is isomorphic to the Lie algebra  $L_S = \pi_*(\Omega S) \otimes \mathbb{Q}$  obtained on the rational homotopy groups by means of the Whitehead product. An extensive study of  $L_S$  has been made these last years with for instance the following result:

**THEOREM B ([6], [4]).** *If  $S$  is a space of finite category, then every nilpotent ideal  $I$  of  $L_S$  is finite dimensional.*

We now want to compute  $L_r$  for a given fibration. With the notations of §2, we decompose the differentials  $D$  and  $\delta$  in the form

$$D = D_1 + D_2 + \dots \quad D_i(V) \subset A \otimes \Lambda^i(V)$$

$$\delta = \delta_1 + \delta_2 + \dots \quad \delta_i(A^\vee \otimes V) \subset \Lambda^i(A^\vee \otimes V)$$

$\delta_1$  is completely defined by  $d_A$  and  $D_1$ . In fact, put

$$D_1 v_r = \sum_s \alpha_{rs} v_s, \quad \alpha_{rs} \in A^+.$$

We then have:

$$(*) \sum_i (-1)^{\text{deg}(a_i)} a_i \otimes \delta_1(a_i^* \otimes v_r) = - \sum_i d_A(a_i) \otimes (a_i^* \otimes v_r)$$

$$+ \sum_i \sum_s \alpha_{rs} \cdot a_i \otimes (a_i^* \otimes v_s)$$

The homology of  $(A^\vee \otimes V, \delta_1)$  and  $(A^+ \otimes V, \delta_1)$  are respectively isomorphic to the vector spaces of indecomposable elements of the minimal models of  $\Gamma$  and  $\Gamma_*$ :

We thus have isomorphisms:

$$(1) H^*(A^\vee \otimes V, \delta_1) \cong (\pi_*(\Gamma) \otimes \mathbb{Q})^\vee$$

and

$$(2) H^*(A_+^\vee \otimes V, \delta_1) \cong (\pi_*(\Gamma_*) \otimes \mathbb{Q})^\vee.$$

Moreover, the short exact sequence of complexes

$$0 \rightarrow (V, 0) \rightarrow (A^\vee \otimes V, \delta_1) \rightarrow (A_+^\vee \otimes V, \delta_1) \rightarrow 0$$

induces in homology a long exact sequence isomorphic to the dual of the homotopy long exact sequence of the fibration  $\Gamma_* \rightarrow \Gamma \rightarrow F$  [9]:

$$\begin{array}{ccccccc} \cdots \rightarrow H^q(A_+^\vee \otimes V, \delta_1) & \xrightarrow{\Delta} & V^{q+1} & \rightarrow & H^{q+1}(A^\vee \otimes V, \delta_1) & \rightarrow & H^{q+1}(A_+^\vee \otimes V, \delta_1) \rightarrow \cdots \\ & & \downarrow & & \downarrow & & \downarrow \\ \cdots \rightarrow (\pi_q(\Gamma_*))^\vee & \rightarrow & (\pi_{q+1}(F))^\vee & \rightarrow & (\pi_{q+1}(\Gamma))^\vee & \rightarrow & (\pi_{q+1}(\Gamma_*))^\vee \rightarrow \cdots \end{array}$$

REMARK. “ $D_1$  is differential” can be expressed by the fact that the matrix  $\mathbf{a}$  consisting of the  $\alpha_{r,s}$  satisfies  $\mathbf{a}^2 + d\mathbf{a} = 0$ .

#### §4. Proof of Theorem 1

We use the notations of §§2 and 3,  $(AV, d) \rightarrow (AW, \delta) \rightarrow (A(W/V), \delta)$  is a model of the fibration  $\Gamma_* \rightarrow \Gamma \xrightarrow{p} F$ . We consider the linear map

$$D_1 : V \rightarrow A \otimes V.$$

There are two cases: Either there exists an infinite sequence of homogeneous linearly independent elements  $v_1, v_2, \dots$  belonging to  $V$  such that  $D_1(v_i)$  doesn't belong to  $D_1(A^+ \otimes V)$ , or we can suppose that there exists an integer  $N$  such that for every  $v$  in  $V$  of degree larger than  $N$ ,  $D(v)$  belongs to  $A \otimes A^{\geq 2}V$ .

We take a K.S. basis  $(v_i)_{i \geq 1}$  of  $V$ :  $D(v_i) \in A \otimes (v_j)_{j < i}$ . Write

$$D_1(v_n) = \sum_{r=1}^s \alpha_r \cdot v_r.$$

We obtain  $d_A(\alpha_s) = 0$ . If  $[\alpha_s] = 0$ , then  $\alpha_s = d_A(b)$  and  $D_1(v_n - b \cdot v_s) \in A \otimes (v_j)_{j < s}$ .

We then replace  $v_n$  by  $v'_n = v_n - b \cdot v_s$ . If  $D_1(v)$  does not belong to  $D_1(A^+ \otimes V)$ , we can suppose  $[a_s] \neq 0$ . In this case, formula (\*) gives the equality

$$\delta_1(\alpha_s^* \otimes v_s) - (-1)^{\text{deg}(a_s)} 1 \otimes v_n.$$

This means that the element  $1 \otimes v_n$  belongs to the image of  $\Delta$  in the dual homotopy long exact sequence. Recall that the elements in the image of  $\Delta$  are called the Gottlieb elements of the fibration and let us come back to our dichotomy:

In the first case, the  $v_i$  are Gottlieb elements of  $\Gamma_*$  [5]. By [5], we know that the category of a space is greater or equal to the number of its linearly independent Gottlieb elements, so that  $\text{cat}(\Gamma_*) = \infty$ .

In the second case, put  $n = \max \{p \text{ such that } A^p \neq 0\}$ . For  $q > n + N$ , formula (\*) yields the isomorphisms

$$H^q((A^\vee \otimes V), \delta_1) = (H(A, d_A)^\vee \otimes V)^q.$$

The injections  $\delta(A_p^\vee \otimes V) \subset \Lambda(A_{\leq p}^\vee \otimes V)$ , valid for  $p > 0$ , imply that the Lie algebra  $L = (H(A, d_A)_+^\vee \otimes V)^{>n+N}$  is a nilpotent Lie algebra of infinite dimension, which is impossible by Theorem B. □

**§5. Proof of Theorem 2**

In this case, the fibration  $(Y^X)_* \xrightarrow{i} (Y^X) \rightarrow Y$  admits a section and so  $\pi_*(i) \otimes \mathbb{Q}$  is injective. It then results from [5] that  $\text{cat}_0((Y^X)_*) \leq \text{cat}_0(Y^X)$ . If (1) is satisfied, we deduce from Theorem 1 that  $\text{cat}_0(Y^X)$  is infinite.

If (2) is satisfied, choose a non homologically trivial cycle  $\alpha$  in  $A^p$  and a nonzero element  $v$  in  $V^q$ . We now have  $D_1 = 0$ . The formula (\*) shows that  $\alpha^* \otimes v$  is a  $\delta_1$ -cycle which is not a  $\delta_1$ -boundary.  $\alpha^* \otimes v$  defines thus a nonzero indecomposable element in the minimal model of  $Y^X$ . The definition of  $\delta$ , as given in §2 implies the following formulas (\*\*): and (\*\*\*):

$$(**) \sum_i (-1)^{\text{deg}(a_i)} a_i \otimes \delta(a_i^* \otimes t) = -\sum_i d_A(a_i) \otimes (a_i^* \otimes t) + \varepsilon(D(t)).$$

$$(***) \varepsilon(v_1 \cdot v_2 \cdots v_r) = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r} \otimes (a_{i_1}^* \otimes v_1)(a_{i_2}^* \otimes v_2) \cdots (a_{i_r}^* \otimes v_r).$$

As  $\alpha^2 = 0$  formula (\*\*\*) shows that  $(\alpha^* \otimes v)^n$  can never appear in the decomposition of  $\varepsilon(v_q \cdot v_2 \cdots v_r)$ , and so by (\*\*), in the differential of an element  $\beta^* \otimes t$ . It then results from ([8] Proposition 1) that  $\text{cat}_0(Y^X) = \infty$ . □

If the dimension of  $X$  is less than the connectivity of  $Y$ , the result we obtain is better.

**THEOREM 3.** *Let  $X$  be a nilpotent space such that there exists an integer  $k \geq 1$  with  $H^p(X; \mathbb{Q}) = 0$ ,  $p > k$  and  $H^k(X; \mathbb{Q}) \neq 0$  and let  $Y$  be a  $(m - 1)$ -connected space, non contractible over  $\mathbb{Q}$ , with  $m \geq k + 2$ , then the functional space  $Y^X$  has finite rational category iff the three following conditions are satisfied:*

- (1)  $\pi_*(Y) \otimes \mathbb{Q} = \pi_{\text{odd}}(Y) \otimes \mathbb{Q}$ .
- (2)  $H^*(X; \mathbb{Q}) = H^{\text{even}}(X; \mathbb{Q})$ .
- (3)  $\dim \pi_*(Y) \otimes \mathbb{Q} < \infty$ .

*Proof.* By Theorem 2, Condition 3 is necessary. In this case, we have  $\dim \pi_{\text{even}}(Y) \otimes \mathbb{Q} \leq \dim \pi_{\text{odd}}(Y) \otimes \mathbb{Q}$  ([8], Proposition 1), so that  $\pi_{\text{odd}}(Y) \otimes \mathbb{Q} \neq 0$ . By Theorem 2, the second condition is thus also necessary.

Suppose thus that  $H^*(X; \mathbb{Q}) = H^{\text{even}}(X; \mathbb{Q})$ . We can suppose that  $A^{>k} = 0$ . If  $\pi_{\text{even}}(Y) \otimes \mathbb{Q} \neq 0$ , let's choose a cycle  $\alpha$  of  $A^k$  defining a nonzero element of  $H^k(X; \mathbb{Q})$  and let's choose a nonzero element  $v$  in  $V^{\text{even}}$ . Then, formula (\*\*\*) shows that no power of  $a^* \otimes v$  can appear in the decomposition of  $\varepsilon(v_1 \cdot v_2 \cdots v_r)$  for any choice of  $v_1, v_2, \dots, v_r$ . Now by (\*\*) no power of  $a^* \otimes v$  appear in the expression of a boundary, so that, by ([8]) the category of  $Y^X$  has to be infinite.

On the other hand, if the three conditions are satisfied,

$$\pi_*(Y^X) \otimes \mathbb{Q} \cong \pi_{\text{odd}}(Y^S) \otimes \mathbb{Q}$$

is finite dimensional and concentrated in odd degrees. The minimal model of  $Y^X$  is thus finite dimensional. This implies that  $Y^X$  has the rational homotopy type of a finite c.w. complex.  $\square$

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