

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 65 (1990)

Artikel: Rational category of the space of sections of a nilpotent bundle.
Autor: Félix, Y.
DOI: <https://doi.org/10.5169/seals-49746>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 13.01.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Rational category of the space of sections of a nilpotent bundle

Y. FÉLIX*

Abstract. Denote by $\zeta : F \rightarrow E \xrightarrow{p} B$ a nilpotent fibration where F is a 1-connected space of finite category and B a finite c.w. complex with non trivial rational cohomology. In this note we compute the rational category of the space Γ_* of continuous pointed sections of ζ .

§1. Introduction

In 1956, R. Thom studied the homotopy type of the space F_f^x of continuous maps of X into F homotopic to a given map f . He computed explicitly the cohomology of F_f^x when F is a product of Eilenberg–Mac Lane spaces [12].

Later on, following ideas of Sullivan, A. Haefliger gave the rational minimal model of the space of sections of a nilpotent bundle [7]. This model has been extensively studied by K. Shibata and M. Vigué-Poirrier [14]. In particular, M. Vigué-Poirrier noted that, if the dimension of X is less than the connectivity of Y , then the rational homotopy Lie algebra of Y^X is isomorphic as a Lie algebra to $H^*(X; \mathbb{Q}) \otimes (\pi_*(\Omega Y) \otimes \mathbb{Q})$.

The aim of this paper is to show that the category of the space of continuous maps from X into Y , and more generally of pointed sections of a fibration, is often infinite.

To be more precise, we prove in fact the following two theorems.

THEOREM 1. *Let Γ_* be the space of continuous pointed sections of a nilpotent fibration $F \rightarrow E \rightarrow B$. Suppose that*

- (1) $\Gamma_* \neq \emptyset$
- (2) F is a nilpotent space of finite category
- (3) $H^+(B; \mathbb{Q}) \neq 0$ and $\dim H^*(B; \mathbb{Q}) < \infty$
- (4) $\dim (\pi_*(F) \otimes \mathbb{Q})$ is infinite.

Then, $\text{cat}(\Gamma_) = \infty$.*

* Chercheur qualifié FNRS.

In the case where the fibration is trivial, the result is more precise:

THEOREM 2. *Let X be a finite nilpotent c.w. complex and Y be a nilpotent space. We suppose that the rational cohomologies of X and Y are not trivial and one of the two following conditions is satisfied:*

- (1) $\dim \pi_*(Y) \otimes \mathbb{Q} = \infty$
- (2) $\dim \pi_*(Y) \otimes \mathbb{Q} < \infty$ and there are odd integers p and q such that $H^p(X; \mathbb{Q}) \neq 0$, $\pi_q(Y) \otimes \mathbb{Q} \neq 0$ and $q - p \geq 2$.

Then the functional space Y^X has infinite category.

This result clearly yields the following corollary previously proved by E. Fadell and S. Husseini.

COROLLARY [3]. *If Y is a 1-connected space of finite category, such that $\tilde{H}^*(Y; \mathbb{Q}) \neq 0$; then the free loop space Y^{S^1} has infinite category.*

The organization of the paper is as follows. We first recall the construction of the Sullivan–Haefliger model for the space of continuous (resp. pointed) sections Γ (resp. Γ_*). We then show how to deduce the two theorems from the model. We also deduce a way to compute explicitly the rational homotopy groups of Γ .

In fact, if X is a 1-connected space and X_0 its rationalization, we have the inequality $\text{cat } X_0 \leq \text{cat } X$ [13]. The integer $\text{cat}(X_0)$ is called the rational category of X and is denoted $\text{cat}_0 X$. Its relevance comes from the fact that $\text{cat}_0(X)$ can be obtained from the minimal model of the space [5].

§2. The Sullivan–Haefliger model

Let $\zeta : F \rightarrow E \xrightarrow{p} B$ be a fibration. We suppose that B is a finite nilpotent c.w. complex and F is a nilpotent space with finite Betti numbers. We suppose that $\Gamma_* \neq \phi$.

Let $(A, d_A) \rightarrow (A \otimes \Lambda V, D) \rightarrow (\Lambda V, d)$ be a minimal K.S. model of ζ [9]. As $\Gamma_* \neq \phi$, we can also suppose that the differential D satisfies:

$$D(V) \subset A \otimes \Lambda^+ V.$$

B is a finite nilpotent c.w. complex. Therefore we can average A is a finite dimensional graded \mathbb{Q} -vector space. Denote by S a graded supplementary subspace of the graded vector space formed by the cocycles in A . This gives a direct sum decomposition of $A : A = S \oplus d(S) \oplus T$. We then choose a homogeneous basis

$(a_i)_{i \in I}$ of A by taking the union of homogeneous bases of S , $d(S)$ and T . The graded dual vector space of A will be denoted by A^\vee :

$$(A^\vee)_n = \text{Hom}(A^n, \mathbb{Q}).$$

A^\vee is naturally equipped with the dual basis a_i^* :

$$\langle a_i^*; a_j \rangle = \delta_{ij}.$$

We now look at the map of algebras defined by:

$$\varepsilon : A \otimes \Lambda V \rightarrow A \otimes \Lambda(A^\vee \otimes V) : \varepsilon(v) = \sum_{i \in I} a_i \otimes (a_i^* \otimes v); \quad \varepsilon(a) = a, \quad a \in A.$$

In [7], A. Haefliger shows how to put a uniquely defined differential $d_A \otimes \delta$ on $A \otimes \Lambda(A^\vee \otimes V)$ in such a way that ε becomes a morphism of commutative differential graded algebras. Let W be the quotient of $A^\vee \otimes V$ by the subspace of elements of degree < 0 , and by the subspace formed by the δ -cocycles in degree 0.

A short computation shows that $\delta(1 \otimes v) = 1 \otimes dv$, so that the injection $V \cong \mathbb{Q} \otimes V \hookrightarrow A^\vee \otimes V$ induces a K.S. extension:

$$\theta : (\Lambda V, d) \rightarrow (\Lambda W, \delta) \rightarrow (\Lambda(W/V), \delta).$$

THEOREM A (Haefliger, [7]). *θ is a model for the canonical fibration $\Gamma_* \rightarrow \Gamma \xrightarrow{p} F$ where p denotes the evaluation on the basis point of B .*

With this model, we can for instance give a rational analogue of the Cohen–Taylor theorem [2].

PROPOSITION. *Let X be a finite wedge of spheres of dimension less than m ($X = \vee_{i=1}^r S^{n_i}$) and Y be a $(m+2)$ -connected space, then we have a rational homotopy equivalence*

$$(Y^X)_* \cong \prod_{i=1}^r (Y^{S^{n_i}})_*.$$

Proof. The Haefliger model for $(Y^X)_*$ is $(\Lambda(H_*^+(X; \mathbb{Q}) \otimes (\pi_*(Y))^\vee), 0)$. \square

§3. The rational homotopy Lie algebra of a space

If S is a nilpotent space with finite Betti numbers, the minimal model of S is a free commutative differential graded algebra $(\Lambda Z, d)$. The graded vector spaces Z^* and $\text{Hom}(\pi_*(S), \mathbb{Q})$ are then isomorphic [11].

The differential d always decomposes in the form $d = d_2 + d_3 + \dots$, where $d_i(Z) \subset \Lambda^i Z$. This gives on $s^{-1} \text{Hom}(Z, \mathbb{Q})$ a structure of Lie algebra by putting:

$$\langle d_2 z; f, g \rangle = (-1)^{\deg(f) + 1} \langle sz; [s^{-1}f, s^{-1}g] \rangle$$

$z \in Z; f, g \in \text{Hom}(Z, \mathbb{Q})$.

It is a result of Andrews and Arkowitz [1] that this Lie algebra is isomorphic to the Lie algebra $L_S = \pi_*(\Omega S) \otimes \mathbb{Q}$ obtained on the rational homotopy groups by means of the Whitehead product. An extensive study of L_S has been made these last years with for instance the following result:

THEOREM B ([6], [4]). *If S is a space of finite category, then every nilpotent ideal I of L_S is finite dimensional.*

We now want to compute L_Γ for a given fibration. With the notations of §2, we decompose the differentials D and δ in the form

$$\begin{aligned} D &= D_1 + D_2 + \dots \quad D_i(V) \subset A \otimes \Lambda^i(V) \\ \delta &= \delta_1 + \delta_2 + \dots \quad \delta_i(A^\vee \otimes V) \subset \Lambda^i(A^\vee \otimes V) \end{aligned}$$

δ_1 is completely defined by d_A and D_1 . In fact, put

$$D_1 v_r = \sum_s \alpha_{rs} v_s, \quad \alpha_{rs} \in A^+.$$

We then have:

$$\begin{aligned} (*) \quad \sum_i (-1)^{\deg(a_i)} a_i \otimes \delta_1(a_i^* \otimes v_r) &= - \sum_i d_A(a_i) \otimes (a_i^* \otimes v_r) \\ &\quad + \sum_i \sum_s \alpha_{rs} \cdot a_i \otimes (a_i^* \otimes v_s) \end{aligned}$$

The homology of $(A^\vee \otimes V, \delta_1)$ and $(A_+^\vee \otimes V, \delta_1)$ are respectively isomorphic to the vector spaces of indecomposable elements of the minimal models of Γ and Γ_* :

We thus have isomorphisms:

$$(1) \ H^*(A^\vee \otimes V, \delta_1) \cong (\pi_*(\Gamma) \otimes \mathbb{Q})^\vee$$

and

$$(2) \ H^*(A_+^\vee \otimes V, \delta_1) \cong (\pi_*(\Gamma_*) \otimes \mathbb{Q})^\vee.$$

Moreover, the short exact sequence of complexes

$$0 \rightarrow (V, 0) \rightarrow (A^\vee \otimes V, \delta_1) \rightarrow (A_+^\vee \otimes V, \delta_1) \rightarrow 0$$

induces in homology a long exact sequence isomorphic to the dual of the homotopy long exact sequence of the fibration $\Gamma_* \rightarrow \Gamma \rightarrow F$ [9]:

$$\begin{array}{ccccccc} \cdots \rightarrow H^q(A_+^\vee \otimes V, \delta_1) & \xrightarrow{\Delta} & V^{q+1} & \rightarrow & H^{q+1}(A^\vee \otimes V, \delta_1) & \longrightarrow & H^{q+1}(A_+^\vee \otimes V, \delta_1) \rightarrow \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \rightarrow (\pi_q(\Gamma_*))^\vee & \longrightarrow & (\pi_{q+1}(F))^\vee & \longrightarrow & (\pi_{q+1}(\Gamma))^\vee & \longrightarrow & (\pi_{q+1}(\Gamma_*))^\vee \rightarrow \cdots \end{array}$$

REMARK. “ D_1 is differential” can be expressed by the fact that the matrix \mathbf{a} consisting of the α_{rs} satisfies $\mathbf{a}^2 + d\mathbf{a} = 0$.

§4. Proof of Theorem 1

We use the notations of §§2 and 3, $(\Lambda V, d) \rightarrow (\Lambda W, \delta) \rightarrow (\Lambda(W/V, \delta))$ is a model of the fibration $\Gamma_* \rightarrow \Gamma \xrightarrow{p} F$. We consider the linear map

$$D_1 : V \rightarrow A \otimes V.$$

There are two cases: Either there exists an infinite sequence of homogeneous linearly independent elements v_1, v_2, \dots belonging to V such that $D_1(v_i)$ doesn't belong to $D_1(A^+ \otimes V)$, or we can suppose that there exists an integer N such that for every v in V of degree larger than N , $D(v)$ belongs to $A \otimes \Lambda^{\geq 2}V$.

We take a K.S. basis $(v_i)_{i \geq 1}$ of V : $D(v_i) \in A \otimes (v_j)_{j < i}$. Write

$$D_1(v_n) = \sum_{r=1}^s \alpha_r \cdot v_r.$$

We obtain $d_A(\alpha_s) = 0$. If $[\alpha_s] = 0$, then $\alpha_s = d_A(b)$ and $D_1(v_n - b \cdot v_s) \in A \otimes (v_j)_{j < s}$.

We then replace v_n by $v'_n = v_n - b \cdot v_s$. If $D_1(v)$ does not belong to $D_1(A^+ \otimes V)$, we can suppose $[a_s] \neq 0$. In this case, formula $(*)$ gives the equality

$$\delta_1(a_s^* \otimes v_s) - (-1)^{\deg(a_s)} 1 \otimes v_n.$$

This means that the element $1 \otimes v_n$ belongs to the image of Δ in the dual homotopy long exact sequence. Recall that the elements in the image of Δ are called the Gottlieb elements of the fibration and let us come back to our dichotomy:

In the first case, the v_i are Gottlieb elements of Γ_* [5]. By [5], we know that the category of a space is greater or equal to the number of its linearly independent Gottlieb elements, so that $\text{cat}(\Gamma_*) = \infty$.

In the second case, put $n = \max \{p \text{ such that } A^p \neq 0\}$. For $q > n + N$, formula $(*)$ yields the isomorphisms

$$H^q((A^\vee \otimes V), \delta_1) = (H(A, d_A)^\vee \otimes V)^q.$$

The injections $\delta(A_p^\vee \otimes V) \subset \Lambda(A_{<p}^\vee \otimes V)$, valid for $p > 0$, imply that the Lie algebra $L = (H(A, d_A)^\vee \otimes V)^{>n+N}$ is a nilpotent Lie algebra of infinite dimension, which is impossible by Theorem B. \square

§5. Proof of Theorem 2

In this case, the fibration $(Y^X)_* \xrightarrow{i} (Y^X) \rightarrow Y$ admits a section and so $\pi_*(i) \otimes \mathbb{Q}$ is injective. It then results from [5] that $\text{cat}_0((Y^X)_*) \leq \text{cat}_0(Y^X)$. If (1) is satisfied, we deduce from Theorem 1 that $\text{cat}_0(Y^X)$ is infinite.

If (2) is satisfied, choose a non homologically trivial cycle α in A^p and a nonzero element v in V^q . We now have $D_1 = 0$. The formula $(*)$ shows that $\alpha^* \otimes v$ is a δ_1 -cycle which is not a δ_1 -boundary. $\alpha^* \otimes v$ defines thus a nonzero indecomposable element in the minimal model of Y^X . The definition of δ , as given in §2 implies the following formulas $(**)$ and $(***)$:

$$(**) \sum_i (-)^{\deg(a_i)} a_i \otimes \delta(a_i^* \otimes t) = - \sum_i d_A(a_i) \otimes (a_i^* \otimes t) + \varepsilon(D(t)).$$

$$(***) \varepsilon(v_1 \cdot v_2 \cdots v_r) = \sum_{i_1} \sum_{i_2} \cdots \sum_{i_r} a_{i_1} \cdot a_{i_2} \cdots a_{i_r} \otimes (a_{i_1}^* \otimes v_1)(a_{i_2}^* \otimes v_2) \cdots (a_{i_r}^* \otimes v_r).$$

As $\alpha^2 = 0$ formula $(***)$ shows that $(\alpha^* \otimes v)^n$ can never appear in the decomposition of $\varepsilon(v_1 \cdot v_2 \cdots v_r)$, and so by $(**)$, in the differential of an element $\beta^* \otimes t$. It then results from ([8] Proposition 1) that $\text{cat}_0(Y^X) = \infty$. \square

If the dimension of X is less than the connectivity of Y , the result we obtain is better.

THEOREM 3. *Let X be a nilpotent space such that there exists an integer $k \geq 1$ with $H^p(X; \mathbb{Q}) = 0$, $p > k$ and $H^k(X; \mathbb{Q}) \neq 0$ and let Y be a $(m - 1)$ -connected space, non contractible over \mathbb{Q} , with $m \geq k + 2$, then the functional space Y^X has finite rational category iff the three following conditions are satisfied:*

- (1) $\pi_*(Y) \otimes \mathbb{Q} = \pi_{\text{odd}}(Y) \otimes \mathbb{Q}$.
- (2) $H^*(X; \mathbb{Q}) = H^{\text{even}}(X; \mathbb{Q})$.
- (3) $\dim \pi_*(Y) \otimes \mathbb{Q} < \infty$.

Proof. By Theorem 2, Condition 3 is necessary. In this case, we have $\dim \pi_{\text{even}}(Y) \otimes \mathbb{Q} \leq \dim \pi_{\text{odd}}(Y) \otimes \mathbb{Q}$ ([8], Proposition 1), so that $\pi_{\text{odd}}(Y) \otimes \mathbb{Q} \neq 0$. By Theorem 2, the second condition is thus also necessary.

Suppose thus that $H^*(X; \mathbb{Q}) = H^{\text{even}}(X; \mathbb{Q})$. We can suppose that $A^{>k} = 0$. If $\pi_{\text{even}}(Y) \otimes \mathbb{Q} \neq 0$, let's choose a cycle α of A^k defining a nonzero element of $H^k(X; \mathbb{Q})$ and let's choose a nonzero element v in V^{even} . Then, formula (****) shows that no power of $a^* \otimes v$ can appear in the decomposition of $\varepsilon(v_1 \cdot v_2 \cdots v_r)$ for any choice of v_1, v_2, \dots, v_r . Now by (**) no power of $a^* \otimes v$ appear in the expression of a boundary, so that, by ([8]) the category of Y^X has to be infinite.

On the other hand, if the three conditions are satisfied,

$$\pi_*(Y^X) \otimes \mathbb{Q} \cong \pi_{\text{odd}}(Y^S) \otimes \mathbb{Q}$$

is finite dimensional and concentrated in odd degrees. The minimal model of Y^X is thus finite dimensional. This implies that Y^X has the rational homotopy type of a finite c.w. complex. \square

REFERENCES

- [1] P. ANDREWS and M. ARKOWITZ, *Sullivan's minimal models and higher order Whitehead products.* Can. J. of Math. 30 (1978), 961–982.
- [2] F. R. COHEN and L. R. TAYLOR, *Homology of function spaces.* Math. Z. 198 (1988), 299–316.
- [3] E. FADELL and S. HUSSEINI, *A note on the category of the Free loop space.* (Preprint) 1988.
- [4] Y. FÉLIX, *La dichotomie Elliptique-Hyperbolique en homotopie rationnelle.* Asterisque n° 179, 1989, Société Mathématique de France.
- [5] Y. FÉLIX and S. HALPERIN, *Rational L.S. category and its applications.* Trans. Amer. Math. Soc. 273 (1982), 1–37.
- [6] Y. FÉLIX, S. HALPERIN, C. JACOBSON, C. LÖFWALL and J. C. THOMAS, *The radical of the homotopy Lie algebra.* Amer. J. of Math. (1988), 301–322.
- [7] A. HAEFLIGER, *Rational homotopy of the space of sections of a nilpotent bundle.* Trans. Amer. Math. Soc. 273 (1982), 609–620.
- [8] S. HALPERIN, *Finiteness in the minimal models of Sullivan.* Trans. Amer. Math. Soc. 230 (1977), 173–199.

- [9] S. HALPERIN, *Lectures on minimal models*. Mémoire Soc. Math. France n° 9/10 (1983).
- [10] K. SHIBATA, *On Haefliger's model for the Gelfand–Fuchs cohomology*. Japan J. Math. 7 (1981), 379–415.
- [11] D. SULLIVAN, *Infinitesimal computations in topology*. Publ. IHES 47 (1977), 269–331.
- [12] R. THOM, *L'homologie des espaces fonctionnels*. Colloque Top. Alg. Louvain (1956), 29–39.
- [13] G. TOOMER, *Two applications of homology decompositions*. Can. J. Math. vol. XXVII (1975), 323–329.
- [14] M. VIGUÉ-POIRIER, *Sur l'homotopie rationnelle des espaces fonctionnels*. Manuscripta Math. 56 (1986), 177–191.

Institut Mathématique

2, chemin du Cyclotron

1348 Louvain-la-Neuve

Belgique

Received December 4, 1989