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## Small eigenvalues on Y-pieces and on Riemann surfaces

PAUL SCHMUTZ

### I. Introduction

We treat eigenvalues of the Laplacian on Riemann surfaces whose Gauss curvature is identically  $-1$ . We label the eigenvalues in ascending order:

$$0 \leq \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

Each eigenvalue is repeated according to its multiplicity.

We define as *small eigenvalues* those which are less than  $\frac{1}{4}$ . In particular, 0 is taken to be a small eigenvalue. An introduction to the subject is found, for example, in Chapters 1 and 10 of [6].

The question of how many small eigenvalues can exist on closed Riemann surfaces has been treated in two theorems of [3]:

**THEOREM 1.** *Given any  $\varepsilon > 0$  and integer  $g \geq 2$ , there exists a closed Riemann surface of genus  $g$  with  $2g - 2$  eigenvalues smaller than  $\varepsilon$ .*

**THEOREM 2.** *A closed Riemann surface of genus  $g \geq 2$  has at most  $4g - 2$  small eigenvalues.*

In this article we present an improvement of Theorem 2:

**THEOREM 3.** *A closed Riemann surface of genus  $g \geq 2$  has at most  $4g - 4$  small eigenvalues.*

These theorems are proved using the principle of monotonicity. Cut the surface  $M$  into pieces. Then:

- (a) The number of *all* small eigenvalues of all pieces with respect to Neumann boundary conditions is an upper bound for the number of small eigenvalues on  $M$ .
- (b) The number of *all* small eigenvalues of all pieces with respect to Dirichlet boundary conditions is a lower bound for the number of small eigenvalues on  $M$ .

Thus, we must determine the number of small eigenvalues of the pieces.

Considering the fact that a closed Riemann surface of genus  $g$  can be cut into  $2g - 2$   $Y$ -pieces (these are Riemann surfaces of signature  $(0, 3)$  with closed geodesics as boundary components) or also into  $4g - 2$  geodesic triangles, the propositions above follow as corollaries of the following more general theorems:

**THEOREM 1'.** *Given any  $\varepsilon > 0$ , there exists a  $Y$ -piece which has an eigenvalue smaller than  $\varepsilon$  with respect to Dirichlet boundary conditions.*

**THEOREM 2'.** *A geodesic triangle has 0 as its only small eigenvalue with respect to Neumann boundary conditions.*

**THEOREM 3'.** *A  $Y$ -piece has at most two small eigenvalues with respect to Neumann boundary conditions.*

We proceed as follows with the proof of theorem 3', our main theorem. In Section II we provide the necessary base which includes information about the small eigenvalues in the right-angled hexagon (hexagons in the hyperbolic plane  $\mathbb{H}^2$  with six right angles), the Symmetry-Lemma and the Quadrilateral-Lemma. In Section III we prove the main theorem with two different methods. We also prove that a closed Riemann surface of genus  $g$  can be cut into  $4g - 4$  geodesic triangles. In Section IV we classify the  $Y$ -pieces into four types. Finally, in Section V we add some remarks concerning the number of small eigenvalues which can exist on Riemann surfaces.

*Notation:*

- (a) Let  $S$  be a Riemann surface. Then  $S(N)$  (respectively  $S(D)$ ) denotes the eigenvalue problem on  $S$  with respect to Neumann boundary conditions (respectively with respect to Dirichlet boundary conditions). If we have an eigenvalue problem on  $S$  with respect to mixed boundary conditions (on one portion  $D$  of the boundary we have Dirichlet boundary conditions, on the other part we have Neumann boundary conditions), then we write  $S(M; D)$ .
- (b) Let  $H$  be a right-angled hexagon. Then there are three pairs of opposite sides which we denote by  $a/x, b/y, c/z$ , such that among  $a, b, c$  there are no neighbors.

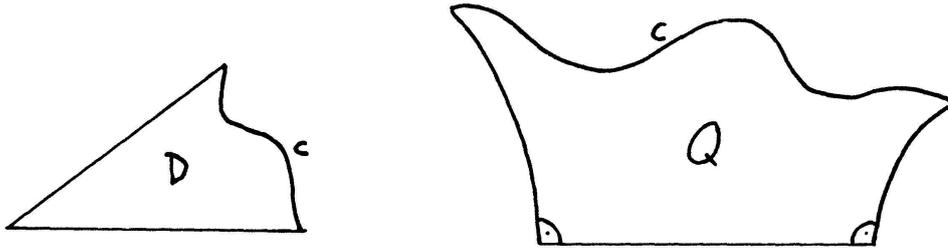
## II. Basic Lemmas

All domains are supposed to be in the hyperbolic plane  $\mathbb{H}^2$ . We refer the reader to [1] or [5] for results concerning hyperbolic trigonometry.

(a) *Right-angled hexagons*

We need two Lemmas from [4] and the Cheeger inequality. Proofs are found in [4] or [9].

LEMMA a. *Let  $D$  be a “triangle” of the following kind: two sides of  $D$  are geodesic segments, the third one a piecewise smooth curve  $c$ . Then  $L(c) > Ar(D)$ . ( $L = \text{length}$ ,  $Ar = \text{area}$ )*



LEMMA b. *Let  $Q$  be a “quadrilateral” of the following kind: three sides of  $Q$  are geodesic segments, which enclose right angles. The fourth side is a piecewise smooth curve  $c$ . Then*

$$L(c) > Ar(Q)$$

This Lemma has the following generalization.

LEMMA b'. *The claim of Lemma b holds if one replaces the two right angles of  $Q$  by angles  $\alpha$  and  $\delta$  with  $\alpha + \delta = \pi$ .*

*Proof.* This change of  $Q$  affects neither  $L(c)$  nor  $Ar(Q)$ .

THEOREM (Cheeger inequality). *Let  $M$  be a Riemann surface and let  $\lambda$  be the smallest nonzero eigenvalue of  $M$ . Then  $\lambda \geq \frac{1}{4}h^2$ , where  $h$  is the isoperimetric constant of Cheeger.*

REMARK. With respect to Neumann boundary conditions,  $h(M)$  is defined as follows:

$$h(M) = \inf \frac{L(\Omega)}{\min \{Ar(M_1), Ar(M_2)\}},$$

where the infimum is with respect to all piecewise smooth curves  $\Omega$  which divide  $M$

into two disjoint subsurfaces  $M_1$  and  $M_2$  with  $\Omega$  as common boundary. With respect to Dirichlet boundary conditions,  $h(M)$  is defined as follows:

$$h(M) = \inf \frac{L(\Omega)}{Ar(M_1)}$$

where  $\Omega$  is as above with  $\partial M_1 \cap \partial M = \phi$ . With respect to Neumann boundary conditions, these results of [9] follow:

LEMMA c. *A geodesic triangle has no nonzero small eigenvalue.*

*Proof.* The Cheeger constant  $h$  is greater than 1, by Lemma a.

LEMMA d. *A geodesic quadrilateral has at most two small eigenvalues.*

*Proof.* Lemma c and principle of monotonicity.

LEMMA e. *A right-angled pentagon has no nonzero small eigenvalue.*

*Proof.* The Cheeger constant  $h$  is greater than 1, by Lemmas a and b.

LEMMA f. *A right-angled hexagon  $H$  has at most two small eigenvalues. Moreover, if  $H$  has two small eigenvalues, then the nodal line of an eigenfunction of  $\lambda_2$  connects two opposite sides of  $H$ .*

*Proof.* Lemma e and principle of monotonicity.

### (b) Symmetry-Lemma

SYMMETRY-LEMMA. *Let  $M$  be a compact Riemann surface with a (nontrivial) involution  $\Psi$  and a symmetrical axis  $t$  (composed by geodesic segments) which divides  $M$  into two isometric parts  $A$  and  $B$  and which is composed by fixed points with respect to  $\Psi$ . The eigenvalues on  $M(N)$  we denote by  $\lambda_i$ . The eigenvalues on  $A(N)$  and the eigenvalues on  $A(M; t)$  we order in a list and label them  $\mu_i$ . Then  $\lambda_i = \mu_i$ , for every  $i = 1, 2, 3, \dots$  Moreover, every eigenfunction on  $A(N)$  or on  $A(M; t)$  is a restriction of an eigenfunction on  $M(N)$ .*

*Proof.* It is easy to show ([9]) that every eigenspace on  $M(N)$  has an orthogonal basis of eigenfunctions which are either symmetric or antisymmetric with respect to  $\Psi$ . In the following, we suppose that we have on  $M(N)$  such an orthogonal basis of eigenfunctions of this kind.

(i) Let  $\phi$  be a symmetric eigenfunction on  $M(N)$ . Then  $\phi|_A$  is an eigenfunction on  $A(N)$ . If  $\psi$  is another symmetric eigenfunction on  $M(N)$ , then

$(\phi | A, \psi | A) = 0$ . Similarly, antisymmetric eigenfunctions  $\phi^*$  and  $\psi^*$  on  $M(N)$ , restricted to  $A$ , are eigenfunctions on  $A(M; t)$  and  $(\phi^* | A, \psi^* | A) = 0$ .

(ii) Now let  $\phi_1, \dots, \phi_n$  be an orthogonal basis of the eigenspace of an eigenvalue  $\lambda$  on  $A(N)$ ,  $n \geq 1$ . Let  $\phi'_1, \dots, \phi'_n$  be the corresponding symmetric functions on  $M$  which are produced by reflection with respect to  $t$  of the  $\phi_j$ . The  $\phi'_j$  are pairwise orthogonal and are also orthogonal to all antisymmetric eigenfunctions on  $M(N)$ . Thus there are symmetric eigenfunctions  $\psi'_1, \dots, \psi'_n$  on  $M(N)$ , for which  $(\phi'_j, \psi'_j) \neq 0, j = 1, \dots, n$ . We define  $\psi_j := \psi'_j | A$ . Then the  $\psi_j$  are eigenfunctions on  $A(N)$ . Moreover, they are eigenfunctions of the eigenvalue  $\lambda$ , since otherwise  $(\phi_j, \psi_j) = (\phi'_j, \psi'_j) = 0, j = 1, \dots, n$ . Thus, the  $\psi_j$  form an orthogonal basis of the eigenspace of the eigenvalue  $\lambda$  on  $A(N)$  and the  $\phi_j$  can be represented in this basis. It follows that the  $\phi'_j$  can be represented in the  $\psi'_j$  and are therefore eigenfunctions on  $M(N)$ .

The proof is analogous for eigenfunctions on  $A(M; t)$ . □

**COROLLARY.** *Let  $H$  be a right-angled hexagon and let  $H(N)$  have two small eigenvalues. Let the nodal line  $t$  of an eigenfunction  $\phi$  of  $\lambda_2$  connect the two opposite sides  $c$  and  $z$  of  $H$ . Reflect  $H$  with respect to one of the other four sides of  $H$ , producing an octagon  $A$ . Then  $A(N)$  has three small eigenvalues.*

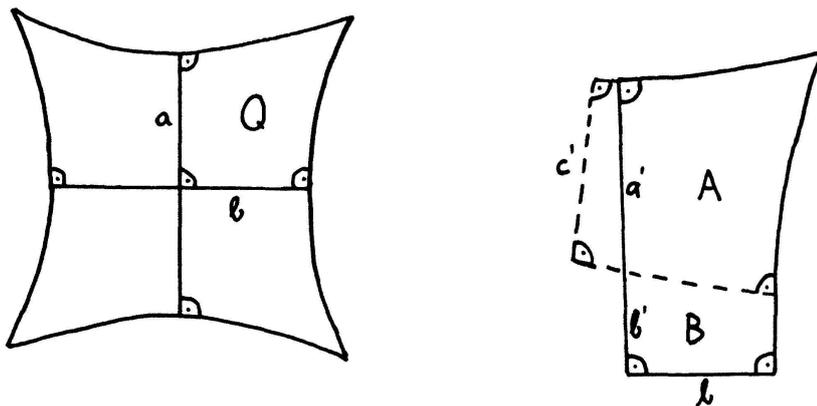
*Proof.*  $A$  is composed of two isometric hexagons  $H$  and  $H'$ . Define the function  $\phi'$  on  $H'$  as the reflection of  $\phi$ . Define the function  $\psi$  on  $A$  as follows:  $\psi | H = \phi, \psi | H' = \phi'$ . Then  $\psi$  is an eigenfunction on  $A$  with three nodal domains. The corollary then follows by Courant's Nodal Domain Theorem. □

(c) *Quadrilateral-Lemma*

**QUADRILATERAL-LEMMA.** *Let  $Q$  be a geodesic quadrilateral with three right angles. Let  $a$  and  $b$  be neighbouring sides, each between two right angles. Let  $L(a) \geq L(b)$ . Then  $Q(M; a)$  has no small eigenvalue.*

*Proof.* Let  $Q(M; a)$  have a small eigenvalue  $\lambda$ .

(i) Suppose that  $L(a) = L(b)$ . We reflect  $Q$  with respect to the side  $a$ , defining a new quadrilateral  $Q'$  which we reflect with respect to the prolonged side  $b$ , defining a quadrilateral  $A$ .  $A(N)$  has two small eigenvalues (because we have also reflected the eigenfunctions). Then, since  $A$  has different axes of symmetry,  $A(N)$  has three small eigenvalues, contradicting Lemma d in IIa.



(ii) Now suppose that  $L(a) > L(b)$ . We symmetrize  $Q$  into a quadrilateral  $Q'$  as in the figure:  $Q'$  has two sides  $c$  and  $c'$  with  $L(c) = L(c')$ .  $Q$  is divided by  $Q'$  into two parts  $A$  and  $B$ . Side  $a$  is divided by  $Q'$  into two parts  $a' \subset A$  and  $b' \subset B$ . Either  $A(M; a')$  or  $B(M; b')$  must have a small eigenvalue. This is impossible for  $B(M; b')$  because of Lemma b of IIa:  $B(M; b')$  has Cheeger constant  $h > 1$ . Thus  $A(M; a')$  has a small eigenvalue with eigenfunction  $\phi$ .

Define a function  $\phi'$  on  $Q'$  by continuing  $\phi$  on  $Q' \setminus Q$  by 0. The Rayleigh-Quotient of  $\phi'$  is less than  $\frac{1}{4}$  and thus there is a small eigenvalue on  $Q'(M; c')$ , contradicting part (i) of this proof. □

REMARK. The Rayleigh-Quotient of  $f$  (on a surface  $M$ ) is defined as

$$\frac{(\text{grad } f, \text{grad } f)}{(f, f)},$$

where  $(,)$  denotes the inner product on the Hilbert space  $L^2(M)$ .

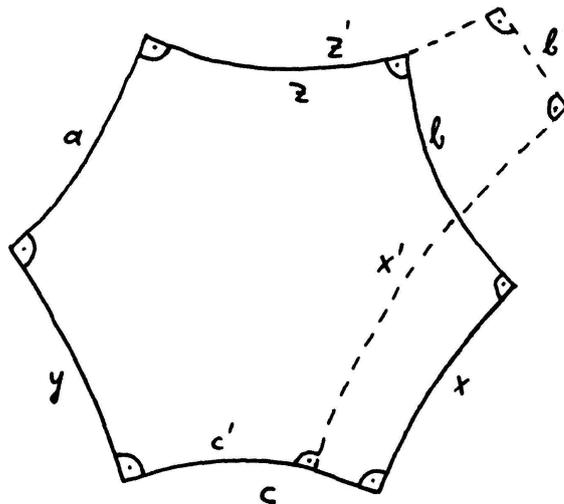
REMARK. The Quadrilateral-Lemma has the following generalization. Its claim holds if the right angle between the sides  $a$  and  $b$  is replaced by another angle. The proof is similar.

COROLLARY 1. *Let  $Q$  be an “infinite” quadrilateral, that is, a quadrilateral with four vertices on  $\partial \mathbb{H}^2$ . Let  $a$  and  $b$  be the common orthogonals between opposite sides of  $Q$ . Let  $L(a) > L(b)$ . Let  $Q(N)$  have two small eigenvalues. Then the nodal line  $t$  of an eigenfunction of  $\lambda_2$  lies on  $b$ . Moreover  $L(b) < 2 \sinh^{-1}(1)$ .*

*Proof.* It follows from hyperbolic trigonometry that  $a$  and  $b$  are orthogonal and are symmetrical axes of  $Q$ ; moreover  $L(b) < 2 \sinh^{-1}(1)$ . The Symmetry-Lemma asserts that  $t$  lies either on  $a$  or on  $b$ . The Quadrilateral-Lemma now proves the claim. □

**COROLLARY 2.** *Let  $Q$  be a quadrilateral with two right angles, with a side  $c$  between these two angles and with two vertices on  $\partial\mathbb{H}^2$ . Let  $L(c) \leq 2 \sinh^{-1}(1)$ . Then  $Q(N)$  has no nonzero small eigenvalue.*

**COROLLARY 3.** *Let  $H$  be a right-angled hexagon. Let  $H(M; a, b, c)$  have a small eigenvalue  $\lambda$ . Let  $H'$  be another right-angled hexagon with sides  $a', b', c', x', y', z'$ . Let  $a = a', b > b', c > c', y' = y$ . Then  $H'(M; a', b', c')$  has a small eigenvalue  $\lambda' < \lambda$ .*



*Proof.* Superimpose the two hexagons as shown in the figure. The proof is now the same as the proof of the Quadrilateral-Lemma. □

**PENTAGON-LEMMA.** *Let  $P$  be a right-angled pentagon. Let  $a$  be a side of  $P$ . Let  $P(M; a)$  have a small eigenvalue. Then  $L(a) < \sinh^{-1}(1)$ . (Proof [9].)*

### III. Proof of the main theorem

Every  $Y$ -piece  $M$  is composed of two isometric right-angled hexagons  $H_M$ . The symmetrical axis (composed by three geodesic segments  $a, b, c$  which are each a common orthogonal between two boundary components of  $M$ ) induces an involution  $\Psi$  on  $M$ .

*Proof of the main theorem.* Let  $M$  be a  $Y$ -piece and assume that  $M(N)$  have three small eigenvalues.

Let  $H := H_M$ . Let  $\phi$  and  $\psi$  be (mutually orthogonal) eigenfunctions of the two nonzero small eigenvalues of  $M$  and suppose that  $\phi$  and  $\psi$  are symmetric or antisymmetric with respect to the involution  $\Psi$ .

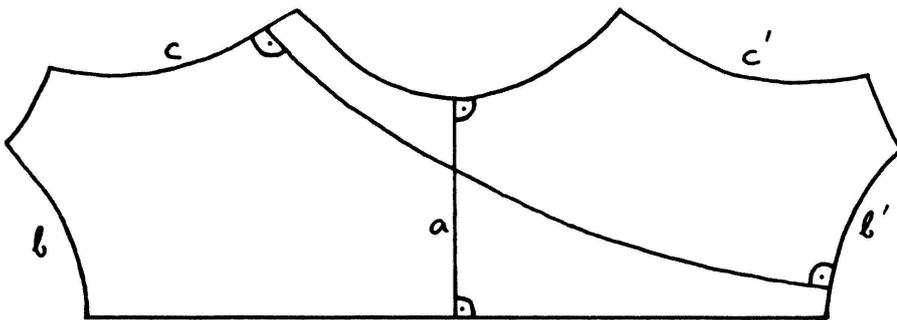
(i)  $\phi$  and  $\psi$  cannot both be symmetric with respect to  $\Psi$ . Otherwise, by the Symmetry-Lemma, the hexagon  $H$  would have three small eigenvalues (with respect to Neumann boundary conditions), contradicting Lemma f of IIa.

(ii)  $\phi$  and  $\psi$  cannot both be antisymmetric. Otherwise,  $\phi$  and  $\psi$  would have an even number of nodal domains, by antisymmetry, and hence two nodal domains, by Courant's Nodal Domain Theorem. Then the nodal lines of  $\phi$  and  $\psi$  would be identically the symmetrical axis of  $M$  and  $\phi$  and  $\psi$  could not be orthogonal.

It follows that we may assume that  $\phi$  is symmetric and  $\psi$  antisymmetric.

(iii) Claim. We can assume without loss of generality that two sides of  $S$  are arbitrary small.

*Proof.* The Symmetry-Lemma says that  $H(N)$  has two small eigenvalues and that  $H(M; a, b, c)$  has one small eigenvalue. These two conditions we denote by condition  $N$  and condition  $M$  for  $H$ . Let the nodal line of  $\phi$  on  $M$  connect the sides  $c$  and  $z$  of  $H$ . We now reflect  $H$  with respect to the side  $a$ , the result being an octagon  $A$  (figure). This we cut along the common orthogonal between the sides  $c$  and  $b'$  (the reflected  $b$ ) and the result is two right-angled hexagons,  $H_1$  and  $H_2$ . By Corollary 3 of IIc, condition  $M$  holds for these two hexagons. By the corollary of IIb,  $A(N)$  has three small eigenvalues. Thus, condition  $N$  holds for one of the two hexagons by the principle of monotonicity. We now select that hexagon for which the conditions  $M$  and  $N$  both hold and repeat the process. Thereby, two of the three sides  $a, b, c$  are reduced each time. It is easy to show ([9]) that in this way one can make two of the three sides arbitrarily small.



(iv) Thus, supposing the sides  $a$  and  $b$  of  $H$  to be very small, we reflect  $H$  with respect to the side  $c$ , defining an octagon  $Q$ . By the Symmetry-Lemma  $Q(N)$  has three small eigenvalues.  $Q$  has four very small sides  $a, b, a', b'$  where  $a', b'$  are reflected sides  $a, b$ . We cut  $Q$  along the common orthogonal between  $a$  and  $b'$ ,

defining two right-angled hexagons. Both have three very small sides, so that their Cheeger constant  $h$  satisfies  $h > 1$ , by IIa. Thus the hexagons have no nonzero small eigenvalue with respect to Neumann boundary conditions. It follows by the principle of monotonicity that  $Q(N)$  has *at most two small eigenvalues*, contradicting the conclusion of the Symmetry-Lemma of above. So the  $Y$ -piece  $M$  has at most two small eigenvalues.  $\square$

**COROLLARY 1.** *Let  $H$  be a right-angled hexagon and let  $H(N)$  have two small eigenvalues. Then  $H$  has a pair of opposite sides which are both strictly longer than  $2 \sinh^{-1}(1)$ .*

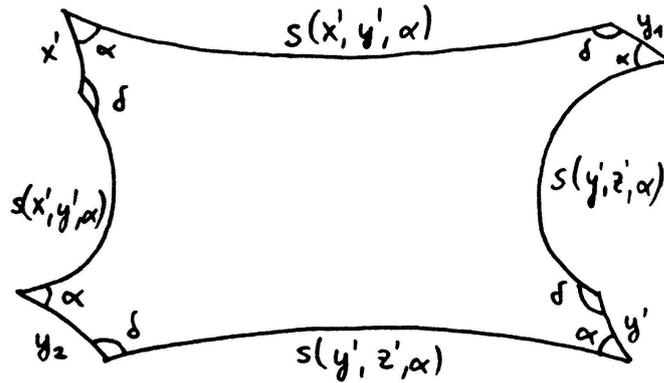
*Proof.* We iterate the process of the above proof. During this, the three sides  $a, b, c$  of  $H$  do not get longer. Repeating the process arbitrarily often, the hexagon  $H$  converges into a quadrilateral  $Q$  with two vertices on  $\partial\mathbb{H}^2$ , as the quadrilateral in Corollary 2 of IIc. By [7], the small eigenvalues of  $H$  converge into small eigenvalues of  $Q$ , so by the mentioned corollary, the basic side of  $Q$  must be longer than  $2 \sinh^{-1}(1)$ . Thus one of the three sides  $a, b, c$  is longer than  $2 \sinh^{-1}(1)$ .

Analogously, we show that one of the sides  $x, y, z$  of  $H$  must be longer than  $2 \sinh^{-1}(1)$ . The claim follows now by hyperbolic trigonometry which states that in  $H$  the longest side of the triple  $a, b, c$  is opposite the longest side of the triple  $x, y, z$ .  $\square$

*Second proof of the main theorem.* Let  $M$  be a  $Y$ -piece with boundary components  $x', y', z'$ . Let  $P$  be the center of the common orthogonal between  $x'$  and  $y'$ . Let  $s(x', y', \alpha)$  be a non self-intersecting geodesic on  $M$  passing through  $P$  such that one end point lies on  $x'$  and the other lies on  $y'$ , and this geodesic intersects  $x'$  and  $y'$  by an angle  $\alpha \in [0, \pi/2]$ . If  $\alpha = \pi/2$ , then  $s(x', y', \alpha)$  is the common orthogonal between  $x'$  and  $y'$  and is unique. In the other cases, there are two different geodesics both of which we denote by  $s(x', y', \alpha)$  and which are symmetric with respect to the involution  $\Psi$  of  $M$ . If  $\alpha = 0$ , we call  $s(x', y', \alpha)$  the common asymptotic geodesic of  $x'$  and  $y'$ . In this case, of course,  $s(x', y', 0)$  does not intersect  $x'$  or  $y'$ . We now fix  $\alpha \in [0, \pi/2]$  and cut  $M$  along a geodesic  $s(x', y', \alpha)$ . Denote the new surface by  $M'$  and cut  $M'$  along the geodesic  $s(y', z', \alpha)$ , producing an octagon  $A$  with four angles  $\alpha$  and four angles  $\delta = \pi - \alpha$  such that  $\alpha$  and  $\delta$  are always neighbouring angles. Now, of course,  $s(y', z', \alpha)$  is unique on  $M'$ . The geodesic  $y'$  has been cut into two parts  $y_1$  and  $y_2$  which are both sides of  $A$ . The other sides of  $A$  are  $x'$  and  $z'$ , twice  $s(x', y', \alpha)$  and twice  $s(y', z', \alpha)$ .

We now cut  $A$  along the geodesic  $s(x', z', \alpha)$  into two hexagons  $H_1$  and  $H_2$ .

Select  $\alpha$  very small. Then, the two hexagons  $H_1$  and  $H_2$  have three (pairwise non-neighbouring) sides which are very small. It follows that the Cheeger constant



$h$  for  $H_1(N)$  and for  $H_2(N)$  is greater than 1 (compare with Lemma  $b'$  of IIa). Thus these hexagons have no nonzero small eigenvalues. Then by the principle of monotonicity,  $M$  has at most two small eigenvalues.  $\square$

REMARK. Let  $\alpha$  converge to 0. Then the octagon  $A$  in the proof above converges to an “infinite” quadrilateral  $Q$ . By [7] the small eigenvalues of  $A$  tend to small eigenvalues of  $Q$ . Since a quadrilateral has at most two small eigenvalues, by Corollary d of IIa, the claim of the main theorem follows once more.

COROLLARY 2. A closed Riemann surface  $M$  of genus  $g$  can be cut into  $4g - 4$  geodesic triangles.

*Proof.* We cut  $M$  into  $2g - 2$   $Y$ -pieces. Each  $Y$ -piece we cut by asymptotic geodesics  $s(x', y', 0)$ ,  $s(y', z', 0)$  and  $s(x', z', 0)$  into two geodesic triangles (notation as above). Of course, the vertices of these triangles all lie on  $\partial\mathbb{H}^2$ .  $\square$

REMARK. The number  $4g - 4$  in Corollary 2 is minimal; a closed Riemann surface  $M$  of genus  $g$  cannot be cut into less than  $4g - 4$  geodesic triangles since the volume of  $M$  is  $(4g - 4)\pi$  and the volume of a geodesic triangle is at most  $\pi$ .

#### IV. Classification of the $Y$ -pieces

DEFINITION. Let  $M$  be a  $Y$ -piece with hexagon  $H := H_M$  and involution  $\Psi$ . We define the following classification:

TYPE  $S$ .  $M(N)$  has two small eigenvalues. The eigenfunctions of  $\lambda_2(M(N))$  are symmetric with respect to  $\Psi$ .

**TYPE A.**  $M(N)$  has two small eigenvalues. The eigenfunctions of  $\lambda_2(M(N))$  are antisymmetric with respect to  $\Psi$ .

**TYPE D.**  $M(D)$  has a small eigenvalue.

**TYPE K.**  $M(D)$  has no small eigenvalue,  $M(N)$  has no nonzero small eigenvalue.

**PROPOSITION.** *Every Y-piece  $M$  belongs to exactly one of the four types, and there exist Y-pieces of each type.*

*Proof.* Let  $x', y', z'$  be the boundary components of  $M$  and let  $H := H_M$  be the hexagon of  $M$  such that the sides  $x, y, z$  are half of  $x', y', z'$ .

(i)  $H(a, b, c)$  and  $H(x, y, z)$  cannot both have a small eigenvalue. If the Cheeger constant of  $H(a, b, c)$  is  $< 1$ , then  $a + b + c < \pi$ .

But also  $a + b + c + x + y + z > 2\pi$ , and the claim follows.

(ii) The following relations hold:

$M$  is of type  $S \Leftrightarrow H(N)$  has two small eigenvalues.

$M$  is of type  $A \Leftrightarrow H(a, b, c)$  has a small eigenvalue.

$M$  is of type  $D \Leftrightarrow H(x, y, z)$  has a small eigenvalue.

By the main theorem and by part (i), it follows that  $M$  belongs to exactly one of the four types.

(iii) Let  $\varepsilon > 0$ . As  $a, b, c$  (respectively  $x, y, z$ ) can be made arbitrarily small, there are right-angled hexagons  $H$  such that the lowest eigenvalue of  $H(a, b, c)$  (respectively of  $H(x, y, z)$ ) is less than  $\varepsilon$ .

Furthermore, one of the common orthogonals between two opposite sides of a right-angled hexagon can be made arbitrary small, and thus there are hexagons  $H$  such that the smallest nonzero eigenvalue of  $H(N)$  is lesser than  $\varepsilon$ .

As an example of type  $K$ , we may take a right-angled hexagon  $H$  such that all six sides of  $H$  have the same length. □

The following is a criterion for distinguishing between type  $S$  and type  $A$ .

**LEMMA.** *Let  $M$  be a Y-piece with hexagon  $H := H_M$  and let  $M(N)$  have two small eigenvalues. If  $H$  has a pair of opposite sides which are both strictly longer than  $2 \sinh^{-1}(1)$ , then  $M$  is of type  $S$ .*

*Otherwise,  $M$  is of type  $A$ .*

*Proof.* Compare with Corollary 1 of III.

## V. Outlook

The question of the existence of closed Riemann surfaces of genus  $g$  with more than  $2g - 2$  small eigenvalues is still an open question. To this, we will add a few remarks.

(a) In the proof of Theorem 1 of the introduction, if the required surface is constructed of  $Y$ -pieces of type  $D$ , it follows by the proposition of IV that this surface has no more than  $2g - 2$  small eigenvalues.

(b) Naturally, one tries to cut a closed Riemann surface into  $Y$ -pieces of type  $D$  or of type  $K$ . To do so, one needs criteria which indicate when a  $Y$ -piece is of one of these types. Hence the following is crucial. Let  $Q$  be an "infinite" quadrilateral with symmetrical axes  $a$  and  $b$  and  $L(b) < L(a)$ . Let  $Q(N)$  have two small eigenvalues. What is the upper bound for the length of  $b$ ?

Corollary 1 of IIa says that  $L(b) < 2 \sinh^{-1}(1) = 1,76 \dots$ , but this reflects only the fact  $L(b) < L(a)$ . On the other hand, our numerical experiments indicate that  $L(b) < 0,9$ . It should be possible to improve theoretically the upper bound for the length of  $b$ .

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