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## Witt group of hyperelliptic curves

R. PARIMALA and R. SUJATHA

### Introduction

Let  $k$  be a perfect field of characteristic  $\neq 2$ . Let  $X$  be a smooth projective curve over  $k$ . Let  $W(k(X), \Omega_{k(X)})$  denote the Witt group of the function field of  $X$  with values in the module of differentials  $\Omega_{k(X)}$  of  $k(X)$ . A residue homomorphism

$$\partial : W(k(X), \Omega_{k(X)}) \rightarrow \bigoplus_{x \in X} W(k(x))$$

was defined in [7],  $k(x)$  denoting the residue field at points  $x \in X$  and a residue theorem was proved; namely the composite

$$W(k(X), \Omega_{k(X)}) \xrightarrow{\partial} \bigoplus_{x \in X} W(k(x)) \xrightarrow{\text{trace}} W(k)$$

is zero. Thus image  $\partial$  is contained in the subgroup  $(\bigoplus_{x \in X} W(k(x)))^0$  consisting of tuples  $(\mu_x)$  with  $\sum_{x \in X} \text{trace } \mu_x = 0$ . The kernel and cokernel of  $\partial$  are well understood if  $X = \mathbf{P}^1$  [13] or if  $X$  is an anisotropic conic over  $k$  [14]. To have an intrinsic description of these groups for curves of higher genus is an interesting question posed by Milnor in [13].

In this paper, we study this problem for smooth hyperelliptic curves  $X$  with a rational point of ramification over  $\mathbf{P}^1$ . Let  $\pi : X \rightarrow \mathbf{P}^1$  be a covering defined over  $\mathbf{A}^1$  by the equation  $y^2 = f(T)$ . We exhibit an exact sequence (§3)

$$0 \rightarrow W(X) \rightarrow W(k(X)) \xrightarrow{\partial^0} \left( \bigoplus_{x \in X} W(k(x)) \right)^0 \rightarrow \frac{W(k[T]_f)}{\langle 1, -f \rangle W(k)} \rightarrow W(X) \rightarrow 0.$$

where  $\partial^0$  is simply the residue map  $\partial$  through an identification of  $W(k(X), \Omega_{k(X)})$  with  $W(k(X))$  for a suitable choice of a differential as basis for  $\Omega_{k(X)}$ . We derive, as a corollary, that if all the ramification points of  $\pi$  are  $k$ -rational,  $W(X)$  is generated by one-dimensional forms. This exact sequence may be viewed in two ways: Firstly as characterising coker  $\partial^0$  as a subgroup of  $\bigoplus_{x \in S} W(k(x))$ ,  $S$  denoting the set of ramification points of  $\pi$  and secondly, as giving the defining relations for expressing  $W(X)$  as a quotient of  $\bigoplus_{x \in S} W(k(x))$ .

Using the exact sequence above, we give a more precise description of  $\text{coker } \partial^0$ . It contains a subgroup  $V_r$ , which is a quotient of  $\bigoplus_{x \in S} W(k(x))$ , which we call the *ramified part* of  $\text{coker } \partial^0$ . Under the rationality assumption that  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ ,  $\bar{k}$  denoting the algebraic closure of  $k$ , the group  $V_r$  is zero. We call  $V_{nr} = \text{coker } \partial^0 / V_r$ , the *unramified part* of  $\text{coker } \partial^0$ . This group is 2-torsion (§5). It can be computed in terms of certain cohomology groups if  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ , and supposing further that the curve  $Y = X$  or  $\mathbf{P}^1$  has the following property: ‘Graded Witt group of  $Y$  is isomorphic to the cohomology ring’; Curves over local and global fields have this property [16]. We in fact show that under these assumptions on  $X$ ,  $\text{coker } \partial^0$  is isomorphic to  $(\text{Pic } X'/2) \oplus NH^3(X')$ , where  $X' = X \setminus S$ ,  $S$  denoting the set of ramification points of  $\pi$  and  $NH^n(X')$  denotes the kernel of the map  $H_{et}^n(X', \mu_2) \rightarrow H_{et}^n(k(X'), \mu_2)$ . For a smooth projective hyperelliptic curve over a local field with good reduction, if  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ ,  $\text{coker } \partial$  is isomorphic to  $W(k) \oplus (\mathbf{Z}/2)^{4g}$  where  $g$  is the genus of the curve (Theorem 7.1). Further,  $W(X)$  is also isomorphic to the group  $(\mathbf{Z}/2)^{4g} \oplus W(k)$ ! (Theorem 7.6).

The computations yield, as a by-product, that for any smooth projective curve  $X$  over a local field with good reduction, if  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ , the classical invariants determine the class of a quadratic space in  $W(X)$ .

We record here that J. E. Shick [19] has some independent computations of  $\text{coker } \partial$  for power series fields over  $\mathbb{R}$  and of  $\mathbb{C}$ .

We thank D. S. Nagaraj for carefully going through the manuscript.

### 1. Kernel of the residue homomorphism

Let  $k$  be a perfect field of characteristic  $\neq 2$ . Let  $X$  be a smooth projective curve defined over  $k$ . For a line bundle  $\mathcal{L}$  on  $X$ , let  $W(X, \mathcal{L})$  denote the Witt group of quadratic spaces on  $X$  with values in  $\mathcal{L}$  [9]. Let  $W(X) = W(X, \mathcal{O}_X)$ .

LEMMA 1.1. *The group  $W(X, \mathcal{L})$  depends upto isomorphism, only on the class of  $\mathcal{L}$  in  $\text{Pic } X/2$ . In particular,  $W(X, \mathcal{L}^2) \simeq W(X)$ .*

*Proof.* Let  $\mathcal{M} \in \text{Pic } X$  and  $(\mathcal{E}, q)$  be a quadratic space with values in  $\mathcal{L} \otimes \mathcal{M}^2$ , i.e.,  $q : \mathcal{E} \rightarrow \mathcal{E}^* \otimes \mathcal{L} \otimes \mathcal{M}^2$ , where for any bundle  $\mathcal{F}$ ,  $\mathcal{F}^*$  denotes the dual of  $\mathcal{F}$ , is an isomorphism such that  $q' \otimes 1_{\mathcal{L} \otimes \mathcal{M}^2} = q$ . The assignment

$$(\mathcal{E}, q) \rightarrow (\mathcal{E} \otimes \mathcal{M}^*, q \otimes 1_{\mathcal{M}^*})$$

defines an isomorphism

$$W(X, \mathcal{L} \otimes \mathcal{M}^2) \simeq W(X, \mathcal{L}).$$

Let  $\Omega_X$  denote the sheaf of differentials on  $X$ . Let

$$\partial : W(k(X), \Omega_{k(X)}) \rightarrow \bigoplus_{x \in X} W(k(x))$$

be the residue homomorphism defined in [7],  $k(x)$  denoting the residue field at the closed point  $x$  of  $X$ . (Throughout, the notation  $x \in X$  stands for the set of all closed points  $x$  in  $X$ ).

LEMMA 1.2. *The kernel of the residue map*

$$\partial : W(k(X), \Omega_{k(X)}) \rightarrow \bigoplus_{x \in X} W(k(x))$$

is  $W(X, \Omega_X)$ .

*Proof.* Let  $q$  be a quadratic space over  $k(X)$  with values in  $\Omega_{k(X)}$ , whose class belongs to  $\ker \partial$ . Let  $x$  be a closed point of  $X$  and  $\pi_x$  a local parameter at  $x$ . Identifying  $W(k(X))$  with  $W(k(X), \Omega_{k(X)})$  through  $d\pi_x$ , the residue map  $\partial_x : W(k(X)) \rightarrow W(k(x))$  is simply the second residue homomorphism with respect to  $\pi_x$ . Thus  $q$  which maps to zero under  $\partial_x$  (cf. [17], p. 207) is isometric to  $q_x \otimes_{\mathcal{O}_{X,x}} k(X)$  for some  $q_x \in W(\mathcal{O}_{X,x})$ . The spaces  $q_x \cdot d\pi_x$  over  $\mathcal{O}_{X,x}$  with values in  $\Omega_{X,x}$  become isometric to  $q$  over  $k(X)$ . They patch up to yield a quadratic space  $q_X$  over  $X$  with values in  $\Omega_X$  in view of the following

LEMMA 1.3. *Let  $\mathcal{L}$  be a line bundle on  $X$ ,  $q$  a quadratic space over  $k(X)$  with values in  $\mathcal{L}_{k(X)}$ . Suppose, for every  $x \in X$ , there exists a quadratic space  $q_x$  over  $\mathcal{O}_{X,x}$  with value in  $\mathcal{L} \otimes \mathcal{O}_{X,x}$  such that  $q_x \otimes k(X) \simeq q$ . Then there exists a quadratic space  $q_X$  over  $X$  with values in  $\mathcal{L}$  such that  $q_X \otimes k(X) \simeq q$ .*

*Proof.* The proof of ([6], Corollary 2.7) in the case  $\mathcal{L} = \mathcal{O}_X$  goes through verbatim for any line bundle  $\mathcal{L}$ .

REMARK. If  $X = \mathbf{P}^1$ ,  $\ker \partial \simeq W(X) \simeq W(k)$ . ([13], Proposition 5.3). If  $X$  is an anisotropic conic,  $\ker \partial = W(X, \Omega_X)$  is computed as  $\mathcal{B}_\varphi$  in ([14], Theorem 6.2).

PROPOSITION 1.4. *Let  $X$  be a smooth hyperelliptic curve with a rational point of ramification over  $\mathbf{P}^1$ . Then  $\ker \partial \simeq W(X)$ .*

*Proof.* By (1.1) and (1.2), it suffices to show that  $\Omega_X$  is the square of a line bundle on  $X$ . Let  $\pi : X \rightarrow \mathbf{P}^1$  be a covering, defined over  $\mathbf{A}^1$  by the equation  $y^2 = f(T)$ , degree  $f = 2g + 1$ ,  $g$  being the genus of  $X$ . The divisor of the differential  $dT/y$  is  $(2g - 2)P_\infty$ ,  $P_\infty$  being the point of  $X$  lying over  $\infty$  in  $\mathbf{P}^1$ . Let  $\mathcal{L}$  be the line bundle corresponding to the divisor  $(g - 1)P_\infty$ . Then  $\Omega_X \simeq \mathcal{L}^2$ .

REMARK. As observed by M. Rost, one could define more generally, a residue map

$$\partial_{\mathcal{L}} : W(k(X), \mathcal{L}_{k(X)}) \rightarrow \bigoplus_{x \in X} (W(k(x)), (\mathcal{L} \otimes \Omega_X)(x)),$$

where  $(\mathcal{L} \otimes \Omega_X)(x)$  denotes the fibre of the line bundle  $(\mathcal{L} \otimes \Omega_X)$  at  $x$ . If  $X = \mathbf{P}^1$ , and  $\mathcal{L} = \mathcal{O}_X$ ,  $\partial_{\mathcal{O}_X} = \partial$  is the residue homomorphism discussed above, since  $\Omega_X$  is a square. If  $\mathcal{L} = \mathcal{O}_X(1)$ ,  $\partial_{\mathcal{O}_X(1)}$  is an isomorphism. In the case of an anisotropic conic,  $X$  we have  $\ker \partial_{\mathcal{O}_X} \simeq W(X) \simeq W(k) / \langle 1, -a, -b, ab \rangle W(k)$ , (cf. [1]), where  $X$  is defined by the equation  $aX^2 + bY^2 - Z^2 = 0$ . One can identify coker  $\partial$  with a subgroup of the Witt group of the residue field at the ramified point of the covering  $X \rightarrow \mathbf{P}^1$ .

**2. Some auxiliary results on trace, transfer and residue homomorphisms**

Let  $\pi : X \rightarrow \mathbf{P}^1$  be a double covering, defined over  $\mathbf{A}^1$  by the equation  $y^2 = f(T)$ , degree  $f = 2g + 1$ ,  $g$  being the genus of  $X$ . We identify  $W(k(X))$  and  $W(k(T))$  with  $W(k(X), \Omega_{k(X)})$  and  $W(k(T), \Omega_{k(T)})$  through the basis  $dT/2y$  and  $dT$  respectively. For  $y \in \mathbf{A}^1$ , if  $p \in k[T]$  is the monic irreducible polynomial which gives a parameter at  $y$ , the composite map

$$W(k(T)) \xrightarrow{dT} W(k(T), \Omega_{k(T)}) \xrightarrow{\partial_y} W(k(y))$$

is the second residue homomorphism with respect to the parameter  $pp'$ ,  $p'$  denoting the derivative of  $p$  with respect to  $T$ . Similarly, one can verify that if  $x \in X$  lies over  $y \in \mathbf{P}^1$  corresponding to  $p(T)$ , and  $x$  unramified over  $y$ , on choosing  $p(T)$  again as the parameter at  $y$ , the composite

$$W(k(X)) \xrightarrow{dT/2y} W(k(X), \Omega_{k(X)}) \xrightarrow{\partial_x} W(k(x))$$

is the second residue homomorphism with respect to the parameter  $2pp'y$ . We again denote by  $\partial$  this residue map.

For any finite separable extension  $L/K$ , let  $tr : W(L) \rightarrow W(K)$  be the map induced by the linear map trace  $L \rightarrow K$  and  $i : W(K) \rightarrow W(L)$  the map induced by the inclusion of  $K$  in  $L$ . Let  $s : W(k(X)) \rightarrow W(k(T))$  be the transfer homomorphism induced by the linear map  $s : k(X) \rightarrow k(T)$  defined by  $s(1) = 0$ ,  $s(y) = 1$  where  $\{1, y\}$  is a basis for  $k(X)$  over  $k(T)$  ([17], p. 47).

LEMMA 2.1. *The diagram*

$$\begin{array}{ccc} W(k(X)) & \xrightarrow{(\partial_x)} & \bigoplus_{x/y} W(k(x)) \\ s \downarrow & & \downarrow tr \\ W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \end{array}$$

is commutative.

*Proof.* Since the diagram

$$\begin{array}{ccc} W(\Omega_{k(X)}) & \xrightarrow{\partial} & \bigoplus_{x/y} W(k(x)) \\ tr \downarrow & & \downarrow tr \\ W(\Omega_{k(T)}) & \xrightarrow{\partial} & W(k(y)) \end{array}$$

is commutative ([7], §1), it suffices to show that the diagram

$$\begin{array}{ccc} W(k(X)) & \xrightarrow{dT/2y} & W(k(X), \Omega_{k(X)}) \\ s \downarrow & & \downarrow tr \\ W(k(T)) & \xrightarrow{dT} & W(k(T), \Omega_{k(T)}) \end{array}$$

is commutative. It is enough to check that

$$tr(\langle h_0 + h_1 y \rangle \cdot dT/2y) = s(\langle h_0 + h_1 y \rangle) \cdot dT,$$

for  $h_0, h_1 \in k(T)$ . We have,

$$\begin{aligned} tr(\langle h_0 + h_1 y \rangle \cdot dT/2y) &= \langle tr((h_0 + h_1 y)/2y) \rangle \cdot dT \\ &= \begin{pmatrix} h_1 & h_0 \\ h_0 & h_1 f \end{pmatrix} \cdot dT \\ &= s(\langle h_0 + h_1 y \rangle) \cdot dT. \end{aligned}$$

LEMMA 2.2. *The diagram*

$$\begin{array}{ccc} W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \\ i \downarrow & & \downarrow i' \\ W(k(X)) & \xrightarrow{\partial_x} & \bigoplus_{x/y} W(k(x)) \end{array}$$

is commutative if  $y$  is an unramified point for  $\pi$  and  $i'$  is the composite

$$W(k(y)) \xrightarrow{i} W(k(x)) \xrightarrow{\overline{2y^{-1}}} W(k(x)).$$

If  $x$  is a ramified point for  $\pi$ ,  $\partial_x \circ i$  is zero.

*Proof.* Let  $\langle h \rangle \in W(k(T))$  and  $x \in X$  such that  $x$  is an unramified point for  $\pi$  with  $\pi(x) = y$ . Let  $p \in k[T]$  be the monic polynomial corresponding to  $y$ . Suppose  $v_y(h) = 0$ . Then  $\partial_y(\langle h \rangle) = 0$  and  $\partial_x \circ i(\langle h \rangle) = \partial_x(\langle h \rangle) = 0$ , since  $v_x(h) = v_y(h) = 0$ . Suppose  $h = up$  with  $v_y(u) = 0$ . Since  $\partial_x$  is the second residue map with respect to the parameter  $2pp'y$  and  $\partial_y$  the second residue map with respect to  $pp'$ , we have

$$i' \circ \partial_y(\langle up \rangle) = i' \circ \partial_y(\langle (u/p') \cdot pp' \rangle) = i' \langle \overline{u/p'} \rangle = \langle \overline{u/p'2y} \rangle$$

and

$$\partial_x \circ i(\langle h \rangle) = \partial_x(\langle (u/2p'y) \cdot 2pp'y \rangle) = \langle \overline{u/2p'y} \rangle$$

Suppose  $x \in X$  is a ramified point, lying over  $y \in \mathbf{P}^1$ . For  $h \in k(T)$ ,  $v_x(ih) \equiv 0 \pmod 2$ , since  $x$  has ramification index 2, and we have  $\partial_x \circ i(\langle h \rangle) = 0$ .

**LEMMA 2.3.** *Let  $x/y$  be an unramified point for  $\pi$ . Then the diagram*

$$\begin{array}{ccc} W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \\ \langle 1, -f \rangle \downarrow & & \downarrow \langle 1, -f \rangle \\ W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \end{array}$$

is commutative.

*Proof.* Clear.

We repeatedly use the following lemma which is a consequence of the Lam's exact triangle ([17], Chapter 2, 5.10).

LEMMA 2.4. *The following triangles are exact.*

$$\begin{array}{ccc}
 W(k(T)) & \xrightarrow{i} & W(k(X)) \\
 \langle 1, -f \rangle \swarrow & & \swarrow s \\
 & & W(k(T)) \\
 \\ 
 W(k(y)) & \xrightarrow{i'} & W(k(x)) \\
 \langle 1, -f \rangle \swarrow & & \swarrow r \\
 & & W(k(y))
 \end{array}$$

where, in the second triangle,  $y$  is unramified for  $\pi$  and  $\pi(x) = y$ ; if  $y$  splits in  $X$ , we mean by  $W(k(x))$ , the direct sum  $W(k(x_1)) \oplus W(k(x_2))$  with  $\pi(x_i) = y$ .

### 3. An exact sequence

Let  $\pi : X \rightarrow \mathbf{P}^1$  be a hyperelliptic curve defined over  $\mathbf{A}^1$  by the equation  $y^2 = f(T)$ , degree  $f = 2g + 1$ ,  $g$  being the genus of  $X$ . Let

$$\partial^0 : W(k(X)) \rightarrow \left( \bigoplus_{x \in X} W(k(x)) \right)^0$$

be the residue homomorphism as defined in §2, identifying  $W(k(X))$  with  $W(k(X), \Omega_{k(X)})$  through the basis  $dT/2y$ ,  $(\bigoplus_{x \in X} W(k(x)))^0$  denoting the kernel of the trace map  $\bigoplus_{x \in X} W(k(x)) \xrightarrow{tr} W(k)$ . We fix the following notation:  $S$  = set of ramification points for  $\pi$ ,  $X' = X \setminus S$ ,  $Y = \mathbf{P}^1$ ,  $Y' = Y \setminus \pi(S)$ . We have the following commutative diagram with exact rows and columns, in view of (2.1), (2.3) and (2.4) and ([13], Theorem 5.3).

$$\begin{array}{ccccccc}
 & & W(k(X)) & \xrightarrow{\partial} & \bigoplus_{x \in X} W(k(x)) & & \\
 & & \downarrow s & & \downarrow tr & & \\
 0 & \longrightarrow & W(k) & \longrightarrow & W(k(T)) & \xrightarrow{\partial} & \bigoplus_{y \in Y'} W(k(y)) \oplus \bigoplus_{y \in \pi(S)} W(k(y)) \xrightarrow{\text{trace}} W(k) \longrightarrow 0 \\
 & & \downarrow & & \langle 1, -f \rangle \downarrow & & \downarrow \langle 1, -f \rangle \\
 0 & \longrightarrow & W(Y') & \longrightarrow & W(k(T)) & \xrightarrow{\partial} & \bigoplus_{y \in Y'} W(k(y)).
 \end{array}$$

We define a homomorphism  $\alpha : (\bigoplus_{x \in X} W(k(x)))^0 \rightarrow W(A)/\langle 1, -f \rangle \cdot W(k)$ , where  $A = k[T]_f$  as follows. Let  $\theta \in (\bigoplus_{x \in X} W(k(x)))^0$ . Then there exists  $q \in W(kT)$  with

$\partial(q) = tr \theta$ . We have for  $y \in Y'$

$$\begin{aligned} \partial_y(\langle 1, -f \rangle \cdot q) &= \langle 1, -\bar{f} \rangle \partial_y(q) \\ &= \langle 1, -\bar{f} \rangle tr(\theta_x) \\ &= 0. \end{aligned}$$

Hence  $\langle 1, -f \rangle q \in W(Y') = W(A)$ . Let  $\alpha(\theta)$  denote its class in  $W(A)/\langle 1, -f \rangle \cdot W(k)$ . If  $q_1, q_2$  and two lifts of  $tr\theta$  in  $W(k(T))$ ,  $q_1 - q_2 \in W(k)$  and  $\langle 1, -f \rangle q_1$  and  $\langle 1, -f \rangle q_2$  define the same class in  $W(A)/\langle 1, -f \rangle W(k)$ . Thus  $\alpha$  is well-defined.

**LEMMA 3.1.**  $\ker \alpha = \partial^0(W(k(X)))$ .

*Proof.* Since  $\partial \circ s = tr \circ \partial$  and  $\langle 1, -f \rangle \circ s = 0$ , we have

$$\partial(W(k(X))) \subset \ker \alpha.$$

Let  $\theta \in \bigoplus_{x \in X} W(k(x))^0$  with  $\alpha(\theta) = 0$ . Let  $q_1 \in W(k(T))$  be such that  $\partial(q_1) = tr\theta$ . Then  $\langle 1, -f \rangle q_1 \in \langle 1, -f \rangle W(k)$ . Replacing  $q_1$  by  $q_1 - q_0$  for a suitable  $q_0 \in W(k)$ , we assume that  $\langle 1, -f \rangle q_1 = 0$ . Thus, by (2.4), there exists  $q_2 \in W(k(X))$  such that  $s(q_2) = q_1$ . We have  $tr(\theta - \partial q_2) = tr\theta - \partial s q_2 = tr\theta - \partial q_1 = 0$ . The fact that  $\theta - \partial q_2 \in \partial W(k(X))$  follows from the following

**LEMMA 3.2.** Let  $\theta \in (\bigoplus_{x \in X} W(k(x)))^0$  with  $tr\theta = 0$  in  $(\bigoplus_{y \in Y} W(k(y)))^0$ . Then  $\theta \in \partial \circ i(W(k(T)))$ .

**SUBLEMMA 3.3.** Let  $(\mu_x) \in \bigoplus_{x \in X'} W(k(x))$  be such that  $tr(\mu_x) = 0$  in  $\bigoplus_{y \in Y'} W(k(y))$ . Then there exists  $q \in W(k(T))$  such that  $\partial_x(i(q)) = \mu_x$ , for  $x \in X'$ .

*Proof.* By (2.4), there exists  $(v_y) \in \bigoplus_{y \in Y'} W(k(y))$  such that  $i'(v_y) = \mu_x$ . Since  $Y' \subset \mathbb{A}^1$ , the residue map  $\partial : W(k(T)) \rightarrow \bigoplus_{y \in Y'} W(k(y))$  is surjective. Let  $q \in W(k(T))$  be such that  $\partial_y(q) = v_y$  for  $y \in Y'$ . Then, by (2.2),  $\partial \circ i(q) = i' \circ \partial_y(q) = \mu_x$  for  $x \in X'$ .

*Proof of 3.2.* By (3.3), there exists  $q \in W(k(T))$  such that  $\partial_x \circ i(q) = \theta_x$  for  $x \in X'$ . Further, by (2.2),  $\partial_x \circ i(q) = 0$  for  $x \in S = X \setminus X'$ . Since for  $x \in S$ ,  $\theta_x = tr \theta_x = 0$ , we have,  $\partial(i(q)) = \theta$ .

Let  $A = k[T]_f$ ,  $B = (k[T, y]/(y^2 - f))_f$  be the co-ordinate rings of  $Y'$  and  $X'$  respectively. Since for  $x \in S$ ,  $q \in W(A)$ ,  $\partial_x \circ i(q) = 0$  by (2.2), the natural map  $W(A) \xrightarrow{i} W(B)$  has its image contained in  $W(X)$ . This map vanishes on  $\langle 1, -f \rangle \cdot W(k)$  and induces a map  $\beta : W(A)/\langle 1, -f \rangle W(k) \rightarrow W(X)$ .

**THEOREM 3.4.** *The sequence*

$$0 \longrightarrow W(X) \xrightarrow{i} W(k(X)) \xrightarrow{\partial^0} \left( \bigoplus_{x \in X} W(k(x)) \right)^0 \xrightarrow{\alpha} W(A)/\langle 1, -f \rangle W(k) \xrightarrow{\beta} W(X) \longrightarrow 0$$

is exact.

*Proof.* Exactness at  $W(X)$  (left) and  $W(k(X))$  are proved in ([10], p. 277) noting that  $\partial^0$  is the second residue homomorphism at all points  $x \in X$ . The exactness at  $(\bigoplus_{x \in X} W(k(x)))^0$  is proved in (3.1). That  $\beta \circ \alpha = 0$  follows from the fact that  $i \circ \langle 1, -f \rangle = 0$ , (2.4). We now prove the surjectivity of  $\beta$ . We identify  $W(X)$  with the subgroup of  $W(k(X))$  which is the kernel of  $\partial^0$ . Let  $q \in W(X)$ . Then  $\partial \circ s(q) = tr \circ \partial q = 0$  so that  $s(q) \in W(k)$ . Further  $\langle 1, -f \rangle s(q) = 0$  (2.4). This implies that  $s(q) = 0$  in view of the fact that for any anisotropic quadratic space  $q$  over  $k$ ,  $q \not\cong g \cdot q$  for any odd degree polynomial  $g$ . Thus, there exists  $q_1 \in W(k(T))$  with  $i(q_1) = q$ . We have  $i' \circ \partial_y(q_1) = \partial_x \circ i(q_1) = 0$  for  $y \in Y'$ . There exists  $\mu_y \in W(k(y))$  such that  $\langle 1, -\bar{f} \rangle(\mu_y) = \partial_y(q_1)$ . Since  $\partial : W(k(T)) \rightarrow \bigoplus_{y \in Y'} W(k(y))$  is surjective, there exists  $q_2 \in W(k(T))$  such that  $\partial_y(q_2) = \mu_y$  for every  $y \in Y'$ . We have  $\partial_y(q_1 - \langle 1, -f \rangle q_2) = \langle 1, -\bar{f} \rangle \mu_y - \langle 1, -\bar{f} \rangle \partial_y(q_2) = 0$  for  $y \in Y'$  so that  $q_1 - \langle 1, -f \rangle q_2 \in W(A)$  and maps to  $q$  under  $\beta$ . We now prove exactness at  $W(A)/\langle 1, -f \rangle \cdot W(k)$ . Let  $q \in W(A)$  be such that  $\beta(\bar{q}) = 0$  in  $W(k(X))$ . By (2.4), there exists  $q_1 \in W(k(T))$  such that  $\langle 1, -f \rangle \cdot q_1 = q$ . Since  $\langle 1, -\bar{f} \rangle \partial_y(q_1) = 0$  for  $y \in Y'$ , there exists  $\mu_x \in W(k(x))$ ,  $x/y$  such that  $tr(\mu_x) = \partial_y(q_1)$ . For  $x \in S$ , we set  $\mu_x = \partial_y(q_1)$ . Clearly  $(\mu_x) \in (\bigoplus_{x \in X} W(k(x)))^0$  and  $\alpha((\mu_x)) = q$ .

**COROLLARY 3.5.** *If all ramification points of  $X$  are defined over  $k$ , then  $W(X)$  is generated by discriminants.*

*Proof.* Suppose  $f = \prod_i (T - \alpha_i)$ ,  $\alpha_i \in k$ . An immediate consequence of the Milnor sequence ([13] Theorem 5.3) is that  $W(k[T]_f)$  is generated by  $\langle \lambda(T - \alpha_i) \rangle$  and  $\langle \mu \rangle$ ,  $\mu \in k^*$ ,  $1 \leq i \leq 2g + 1$ . Since  $\beta$  is surjective, their images under  $\beta$ , which are precisely the discriminants of  $W(X)$ , generate  $W(X)$ .

#### 4. Some computations for hyperelliptic curves

Let  $X$  be a smooth hyperelliptic curve defined over  $k$ . We assume throughout that  $X$  has a rational point of ramification. Let  $\pi : X \rightarrow \mathbf{P}^1$  be a double covering as

in §3. If genus  $X > 1$ , since any two double coverings  $\pi_1, \pi_2 : X \rightarrow \mathbf{P}^1$  differ by an automorphism of  $X$ , the space  $X' = X \setminus S$ ,  $S$  denoting the set of ramification points of the covering  $\pi : X \rightarrow \mathbf{P}^1$  determines and is determined by  $X$ . Following notations of §3, let  $A = k[T]_f$  and  $B = (k[T, y]/(y^2 - f))_f$  be the co-ordinate rings of  $Y'$  and  $X'$  respectively.

**LEMMA 4.1.** *The unit group  $U(B)$  is generated by  $k^*$ ,  $y$ , and divisors of  $f$ . If  $f$  splits into linear factors over  $k$ ,  $U(B) \simeq k^* \times \mathbf{Z}^{2g+1}$ .*

*Proof.* Let  $h \in U(B)$ . Then  $\text{div } h = \sum n_i x_i$ ,  $x_i \in S$ ,  $\text{div } h$  denoting the divisor of  $h$ . Let  $\sigma$  denote the nontrivial automorphism of  $k(X)$  over  $k(T)$ . Then  $\sigma x_i = x_i$ , so that  $\text{div } \sigma h = \text{div } h$ . Thus  $h = \lambda \sigma h$ ,  $\lambda \in k^*$ . We have,  $h^2 = \lambda(h\sigma h) \in U(A)$ . Thus  $h\sigma h$  is upto a scalar from  $k^*$ , a power product of divisors of  $f$ . On the other hand, the only non-square in  $k(T)$  which becomes a square in  $k(X)$  is  $f$ . It follows that  $h^2 = \mu^2 \prod_i h_i^{2m_i} f$  or  $h^2 = \mu^2 \prod_i h_i^{2m_i}$ ,  $m_i \in \mathbf{Z}$ ,  $h_i$  divisors of  $f$  in  $k[T]$ . Thus,  $h = \pm \mu(\prod h_i^{m_i})y$  or  $h = \pm \mu(\prod h_i^{m_i})$ . Further, if  $f = \prod_{1 \leq i \leq 2g+1} (T - \alpha_i)$ ,  $\alpha_i \in k^*$ , the homomorphism  $k^* \times \mathbf{Z}^{2g+1} \rightarrow U(B)$ , defined by

$$(\lambda, (n_i)) \rightarrow \lambda(T - \alpha_1)^{n_1} \cdots (T - \alpha_{2g})^{n_{2g}} y^{n_{2g+1}}$$

is surjective, by the above remarks. Suppose

$$\lambda(T - \alpha_1)^{n_1} \cdots (T - \alpha_{2g})^{n_{2g}} \cdot y^{n_{2g+1}} = 1$$

is a relation. Then the divisor

$$\sum_{1 \leq i \leq 2g} 2n_i x_i + n_{2g+1} \left( \sum_{1 \leq i \leq 2g+1} x_i \right) - \left( \sum 2n_i + n_{2g+1}(2g+1) \right) x_\infty = 0,$$

where  $x_i \in S$  lie over  $T - \alpha_i$  and  $x_\infty$  lies over  $\infty$ . This implies that  $n_i = 0$ ,  $1 \leq i \leq 2g + 1$  and  $\lambda = 1$ . Thus we have an isomorphism  $k^* \times \mathbf{Z}^{2g+1} \simeq U(B)$ .

**LEMMA 4.2.** *Suppose  $f$  splits into linear factors over  $k$ . Then the map  $\text{Pic } X' \rightarrow \text{Pic } X'_k$  is injective.*

*Proof.* Since the divisor classes of degree zero supported on the ramification locus  $S$  are precisely the elements of  ${}_2\text{Pic } X$ , we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}_2\text{Pic } X & \longrightarrow & \text{Pic}^0 X & \longrightarrow & \text{Pic } X' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & {}_2\text{Pic } X_k & \longrightarrow & \text{Pic}^0 X_k & \longrightarrow & \text{Pic } X'_k \longrightarrow 0. \end{array}$$

Here  $\text{Pic}^0 X$  is the group of divisor classes of degree zero. The first two vertical maps are natural injections. Since  $f$  splits into a product of linear factors,  ${}_2\text{Pic } X = {}_2\text{Pic } X_{\bar{k}}$ . Since  $\text{Pic}^0 X \hookrightarrow \text{Pic}^0 X_{\bar{k}}$  is an injection, it follows that  $\text{Pic } X' \rightarrow \text{Pic } X'_{\bar{k}}$  is injective.

**LEMMA 4.3.** *The group  ${}_2\text{Pic } X' \simeq (\mathbf{Z}/2)^l$  where  $l \leq 2g$  and  $l = 2g$  if and only if  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ .*

*Proof.* We have an exact sequence

$$0 \rightarrow {}_2\text{Pic } X \rightarrow {}_4\text{Pic } X \rightarrow {}_2\text{Pic } X' \rightarrow 0.$$

Let  ${}_2\text{Pic } X \simeq (\mathbf{Z}/2)^l$ ,  $l \leq 2g$ . Let  $m$  elements in  ${}_2\text{Pic } X$  admit a square root over  $k$ . Then  $|{}_4\text{Pic } X| = m \cdot 2^l$ , with  $m \leq 2^l \leq 2^{2g}$ . Therefore  $|{}_2\text{Pic } X'| = m \leq 2^{2g}$  and equality holds if and only if  $m = 2^l = 2^{2g}$ ; i.e., if and only if  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ .

**PROPOSITION 4.4.** *Let  $\text{Disc}$  denote the discriminant group of a scheme. Let  $f$  split into linear factors over  $k$ . Then the composite map  $\text{Disc } X' \xrightarrow{N} \text{Disc } Y' \rightarrow \text{Disc } Y'/\text{Disc } k$  is surjective if and only if  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ ,  $N$  denoting the norm map.*

*Proof.* Since  $X'/Y'$  is étale quadratic, we have an exact sequence in étale cohomology groups with  $\mu_2$  coefficients ([12], p. 92),

$$0 \longrightarrow H^0(Y') \xrightarrow{\cup_{X_f}} H^1(Y') \xrightarrow{i} H^1(X') \xrightarrow{tr} H^1(Y')$$

Here,  $H^i(-)$  denotes  $H_{\text{ét}}^i(-, \mu_2)$ . The group  $H^1(-)$  is simply the discriminant group so that we have an exact sequence

$$1 \longrightarrow \text{Disc } Y'/\langle f \rangle \longrightarrow \text{Disc } X' \xrightarrow{N} \text{Disc } Y'.$$

Since the only square class in  $k(T)$  which becomes trivial in  $k(X)$  is  $\langle f \rangle$ , this sequence yields the following exact sequence

$$1 \rightarrow \text{Disc } Y'/\langle f \rangle \text{Disc } k \rightarrow \text{Disc } X'/\text{Disc } k \rightarrow \text{Disc } Y'/\text{Disc } k.$$

We denote  $U(B)$  and  $U(A)$  by  $U(X')$  and  $U(Y')$  respectively. By our hypothesis on  $f$ ,  $\text{Disc } Y' \simeq U(Y')/2 \simeq (\mathbf{Z}/2)^{2g+1} \times \text{Disc } k$  so that  $\text{Disc } Y'/\langle f \rangle \text{Disc } k \simeq (\mathbf{Z}/2)^{2g}$  and  $\text{Disc } Y'/\text{Disc } k \simeq (\mathbf{Z}/2)^{2g+1}$ . Further, the exact sequence

$$1 \rightarrow U(X')/2 \rightarrow \text{Disc } X' \rightarrow {}_2\text{Pic } X' \rightarrow 0$$

gives, by (4.1) and (4.3) that  $\text{Disc } X'/\text{Disc } k \simeq (\mathbf{Z}/2)^{2g+1+l}$ , where  ${}_2(\text{Pic } X') \simeq (\mathbf{Z}/2)^l$ . Clearly, the map  $\text{Disc } X'/\text{Disc } k \rightarrow \text{Disc } Y'/\text{Disc } k$  is surjective if and only if  $l = 2g$ ; i.e., if and only if  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ .

**5. Ramified and unramified parts of coker  $\partial^0$**

Let  $(\bigoplus_{x \in S} W(k(x)))^0$  denote the subgroup of  $(\bigoplus_{x \in X} W(k(x)))^0$  with non-zero entries only at  $x \in S$ . Let  $V_r$  be the subgroup of coker  $\partial^0$ , defined by

$$\begin{aligned} V_r &= \left( \bigoplus_{x \in S} W(k(x)) \right)^0 / \left( \partial W(k(X)) \cap \left( \bigoplus_{x \in S} W(k(x)) \right)^0 \right) \\ &= \left( \bigoplus_{x \in S} W(k(x)) \right)^0 / \left( \partial W(X') \right) \end{aligned}$$

We define  $V_{nr} = \text{coker } \partial^0 / V_r$ . If  $p : (\bigoplus_{x \in X} W(k(x)))^0 \rightarrow \bigoplus_{x \in X'} W(k(x))$  denotes the restriction of the projection,  $p$  is surjective, since  $S = X \setminus X'$  contains a rational point. Thus,

$$V_{nr} \simeq \bigoplus_{x \in X'} W(k(x)) / (p \circ \partial) W(k(X)).$$

**LEMMA 5.1.** *The map  $\alpha : \text{coker } \partial^0 \rightarrow W(A)/\langle 1, - \rangle W(k)$  maps  $V_r$  onto  $\langle 1, -f \rangle W(A)/\langle 1, -f \rangle W(k)$ .*

*Proof.* Let  $\theta \in (\bigoplus_{x \in S} W(k(x)))^0$ . Let  $q \in W(k(T))$  be such that  $\partial(q) = \text{tr } \theta$ .

Since  $\partial_y((q) = \text{tr}(\theta_y) = 0$  for  $y \notin \pi(S)$ ,  $q \in W(A)$  and  $\alpha(\bar{\theta}) = \langle 1, -f \rangle q \in \langle 1, -f \rangle W(A)/\langle 1, -f \rangle W(k)$ .

We now show that  $\alpha(V_r) = \langle 1, -f \rangle W(A)/\langle 1, -f \rangle W(k)$ . Let  $q \in W(A)$ . Let  $\mu = (\mu_x) \in (\bigoplus_{x \in X} W(k(X)))^0$  be defined by  $\mu_x = 0$  for  $x \in X'$ ,  $\mu_x = \partial_y(q)$ , for  $x \in S$ ,  $\pi(x) = y$ . Then  $\mu \in (\bigoplus_{x \in S} W(k(x)))^0$  and  $\alpha(\bar{\mu}) = \langle 1, -f \rangle q$  in  $\langle 1, -f \rangle W(A)/\langle 1, -f \rangle W(k)$ . We thus have an exact sequence

$$0 \rightarrow V_{nr} \xrightarrow{\alpha} W(A)/\langle 1, -f \rangle W(A) \xrightarrow{\beta} W(X) \rightarrow 0.$$

**PROPOSITION 5.2.** *The group  $V_{nr}$  is 2-torsion.*

*Proof.* Let  $\theta \in \bigoplus_{x \in X'} W(k(x))$ . Since  $\pi(S)$  has a rational point of ramification, there exists  $q \in W(k(T))$  such that  $\partial(q) = \text{tr } \theta$ . We have,  $\langle 1, -f \rangle (\langle 1, f \rangle q) = 0$  so

that there exists  $q_1 \in W(k(X))$  with  $s(q_1) = (\langle 1, f \rangle q)$ . Since for  $x \in X'$  with  $\pi(x) = y$ ,

$$\partial_y(\langle 1, -f \rangle q) = \langle 1, -\bar{f} \rangle \partial_y(q) = \langle 1, -\bar{f} \rangle \text{tr}(\theta_x) = 0,$$

we have  $\text{tr}(2\theta) = \partial(\langle 1, f \rangle q) = \partial(s(q_1)) = \text{tr}(\partial(q_1))$ . Thus, by (3.3), there exists  $q_2 \in W(k(T))$  such that  $\partial_x \circ i(q_2) = 2\theta_x - \partial_x q_1$  for  $x \in X'$  and  $\partial_x \circ i(q_2) = 0$  for  $x \in S$ . Thus  $2\theta - \partial(q_1 - i(q_2)) \in (\bigoplus_{x \in S} W(k(x)))^0$  and its image under the projection map  $p$  is zero. Thus the class of  $2\theta$  in  $V_{nr}$  is zero.

Therefore  $\text{coker } \partial^0$  is an extension of  $V_r$  by the 2-torsion group  $V_{nr}$ . We now show that under the rationality assumption  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ ,  $V_r = 0$ . We observe that  ${}_4\text{Pic } X_{\bar{k}}$  being a finite group, there exists a finite separable extension  $l/k$  such that  $V_r = 0$  for  $X_l$ .

**PROPOSITION 5.3.** *Suppose  ${}_2\text{Pic } X = {}_2\text{Pic } X_{\bar{k}}$ . Then the group  $V_r = 0$  if and only if  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ .*

*Proof.* We show that the map  $\partial : W(X') \rightarrow (\bigoplus_{x \in S} W(k(x)))^0$  is surjective if and only if  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ . In view of the commutative diagram

$$\begin{array}{ccccccc} W(X') & \longrightarrow & \left( \bigoplus_{x \in S} W(k(x)) \right)^0 & & & & \\ & & \downarrow s & & \parallel & & \\ 0 & \longrightarrow & W(k) & \longrightarrow & W(Y') & \longrightarrow & \left( \bigoplus_{y \in \pi(S)} W(k(y)) \right)^0 \longrightarrow 0 \end{array} \quad (**)$$

with (\*\*) exact, we need to show that  $s : W(X') \rightarrow W(Y')/W(k)$  is surjective if and only if  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ . By our assumption  ${}_2\text{Pic } X = {}_2\text{Pic } X_{\bar{k}}$ ,  $f$  splits as a product  $\prod_{1 \leq i \leq 2g+1} (T - \alpha_i)$  over  $k$ . Suppose  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ . The exact sequence (\*\*) with each  $W(k(y)) \simeq W(k)$  for  $y \in \pi(S)$  implies that  $W(Y')$  is generated by  $\text{Disc } k$  and  $\langle \lambda(T - \alpha_i) \rangle$ ,  $\lambda \in k^*$ ,  $1 \leq i \leq 2g + 1$ . It is therefore enough to show that given  $\langle \lambda(T - \alpha_i) \rangle$ ,  $\lambda \in k^*$ , there exists  $\mu \in k^*$  such that  $\langle \mu, \lambda(T - \alpha_i) \rangle \in s(W(X'))$ . By (4.4), there exists  $\tilde{z} \in \text{Disc } X'$  such that  $N(\tilde{z}) = \langle \nu(T - \alpha_i) \rangle$  for some  $\nu \in k^*$ . We have,  $s(\tilde{z}) = z_1 \langle 1, -\nu(T - \alpha_i) \rangle$  for some  $z_1 \in k(T)$ . Thus,

$$s(-z_1^{-1} \nu^{-1} \cdot \lambda \cdot \tilde{z}) = \langle -\nu^{-1} \lambda, \lambda(T - \alpha_i) \rangle.$$

Conversely, suppose  $W(X') \rightarrow W(Y')/W(k)$  is surjective. Then the map restricted to the ideal  $I(X')$  of even dimensional forms surjects onto  $I(Y')/I(k)$ . In view of the commutative diagram

$$\begin{array}{ccc} I(X') & \xrightarrow{s} & I(Y')/I(k) \\ \downarrow & & \downarrow \\ \text{Disc } X' & \xrightarrow{N} & \text{Disc } Y'/\text{Disc } k \end{array}$$

with the vertical maps surjective, it follows that  $N : \text{Disc } X' \rightarrow \text{Disc } Y'/\text{Disc } k$  is surjective. This implies, by (4.4) that  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ .

In the next section, under certain assumptions on  $k$  and  $X$ , we describe the unramified part  $V_{nr}$  of coker  $\partial^0$  cohomologically.

**6. The unramified part of coker  $\partial^0$**

Let  $Y$  be any scheme over  $k$ . Let the properties  $PQ(1)$ ,  $PQ(2)$  for  $Y$  be the following.

**PQ(1):** For every geometric point  $y \in Y$ , the invariant theorem for quadratic spaces,  $I^n(k(y))/I^{n+1}(k(y)) \xrightarrow{\sim} H_{et}^n(k(y), \mu_2)$  holds for all  $n \geq 0$ .

**PQ(2):**  $Y$  satisfies  $PQ(1)$  and the maps  $e_n : I_n(Y) \rightarrow \Gamma(Y, \mathcal{H}^n)$  defined in ([15, §1]) are surjective for  $n \geq 0$ .

Here,  $\mathcal{H}^n$  denotes the Zariski sheaf associated to the presheaf  $U \rightarrow H_{et}^n(U, \mu_2)$ . The class of schemes which satisfy  $PQ(2)$  include all smooth quasi projective curves over local fields, in view of [2] and [16]. Conjecturally, all smooth projective curves over any field satisfy  $PQ(2)$ .

We follow the same notations as in §4 and denote by  $\pi : X \rightarrow \mathbf{P}^1$  a double cover,  $X$  being a smooth hyperelliptic curve with a rational point of ramification. Under the assumptions that  $X' = X \setminus S$ ,  $Y' = Y \setminus \pi(S)$  satisfy  $PQ(2)$ , we shall describe  $V_{nr}$  as a certain cohomology group.

**LEMMA 6.1.** *Let  $Y \subseteq \mathbf{P}^1$  be any subscheme. Then  $Y$  satisfies  $PQ(2)$  if  $Y$  satisfies  $PQ(1)$ .*

*Proof.* We have the following commutative diagram (cf. [5], [10])

$$\begin{array}{ccccccc}
 & & & 0 & & 0 & \\
 & & & \downarrow & & \downarrow & \\
 & & & I_n(Y) & \xrightarrow{e_n} & \Gamma(Y, \mathcal{H}^n) & \\
 & & & \downarrow & & \downarrow & \\
 0 \longrightarrow & I^{n+1}(k(T)) & \longrightarrow & I^n(k(T)) & \longrightarrow & H^n(k(T)) & \longrightarrow 0 \\
 & \partial \downarrow & & \partial \downarrow & & \partial \downarrow & \\
 0 \longrightarrow & \left(\bigoplus_{y \in Y} I^n(k(y))\right)^0 & \longrightarrow & \left(\bigoplus_{y \in Y} I^{n-1}(k(y))\right)^0 & \longrightarrow & \left(\bigoplus_{y \in Y} H^{n-1}(k(y))\right)^0 & 
 \end{array}$$

Here  $(\bigoplus_{y \in Y} I^m(k(y)))^0$  (resp.  $(\bigoplus_{y \in Y} H^m(k(y)))^0$ ) denotes the subgroup consisting of trace zero elements. The two vertical columns are exact, by ([10], p. 277) and [5]. By the assumption on  $Y$ , the two rows are exact. The surjectivity of  $e_n: I_n(Y) \rightarrow \Gamma(Y, \mathcal{H}^n)$  follows from the surjectivity of the residue map  $\partial: I^{n+1}(k(T)) \rightarrow (\bigoplus_{y \in Y} I^n(k(y)))^0$  [13], Theorem 5.3).

LEMMA 6.2. Suppose  $\mathbf{P}^1$  and  $X$  satisfy  $PQ(1)$ . Then the sequence

$$I_n(A) \xrightarrow{i} I_n(B) \xrightarrow{s} I_n(A)$$

is exact for  $n \geq 0$ .

*Proof.* Since  $B/A$  is unramified, by (2.1), (2.2) and (2.3), we have the following commutative diagram:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 \\
 & & \downarrow & & \downarrow & & \downarrow \\
 & & I_n(A) & \longrightarrow & I_n(B) & \longrightarrow & I_n(A) \\
 & & \downarrow & & \downarrow & & \downarrow \\
 I^{n-1}(k(T)) & \xrightarrow{\langle 1, -f \rangle} & I^n(k(T)) & \longrightarrow & I^n(k(X)) & \longrightarrow & I^n(k(T)) \\
 \partial \downarrow & & \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\
 \bigoplus_{y \in Y'} I^{n-2}(k(y)) & \xrightarrow{\langle 1, -\bar{f} \rangle} & \bigoplus_{y \in Y'} I^{n-1}(k(y)) & \longrightarrow & \bigoplus_{x \in X'} I^{n-1}(k(x)) & \longrightarrow & \bigoplus_{y \in Y'} I^{n-1}(k(y)).
 \end{array}$$

The vertical columns are exact by ([10], p. 277). Exactness of the rows is a consequence of the assumption  $PQ(1)$  for  $X$  and  $\mathbf{P}^1$  [3]. Exactness of the top row follows from the surjectivity of  $\partial: I^{n-1}(k(T)) \rightarrow \bigoplus_{y \in Y'} I^{n-2}(k(y))$ ,  $Y'$  being contained in  $\mathbf{A}^1$ .

LEMMA 6.3. Suppose  $X'$ , and  $Y'$  satisfy  $PQ(2)$ . Then

$$\langle \langle 1, -f \rangle W(A) \rangle \cap I_n(A) \simeq \langle 1, -f \rangle I_{n-1}(A).$$

*Proof.* We assume, by induction, that

$$\langle \langle 1, -f \rangle W(A) \rangle \cap I_m(A) = \langle 1, -f \rangle I_{m-1}(A)$$

for  $m \leq n - 1$ . Let  $q \in (\langle \langle 1, -f \rangle W(A) \rangle \cap I_n(A))$ . By induction, we may write  $q = \langle 1, -f \rangle q_1$ ,  $q_1 \in I_{n-2}(A)$ . Since  $X', Y'$  satisfy  $PQ(1)$ , and  $B/A$  is étale quadratic,

we have the following commutative diagram

$$\begin{array}{ccccc}
 I_{n-2}(B) & \xrightarrow{s} & I_{n-2}(A) & \xrightarrow{\langle 1, -f \rangle} & I_{n-1}(A) \\
 & & \downarrow e_{n-2} & & \downarrow e_{n-1} \\
 H^{n-2}(B) & \xrightarrow{tr} & H^{n-2}(A) & \xrightarrow{\cup \chi_f} & H^{n-1}(A)
 \end{array}$$

with the bottom row exact. Since

$$\langle 1, -f \rangle q_1 = q \in I_n(A), e_{n-1}(q) = 0; \text{ i.e., } \chi_f \cup e_{n-2}(q_1) = 0.$$

Therefore, there exists  $\theta \in H^{n-2}(B)$  such that  $tr\theta = e_{n-2}(q_1)$ . Let  $\tilde{\theta} \in \Gamma(B, \mathcal{H}^{n-2})$  be the image of  $\theta$  in  $H^{n-2}(k(X))$ . By the assumption that  $X'$  satisfies  $PQ(2)$ , there exists  $q_2 \in I_{n-2}(B)$  such that  $e_{n-2}(q_2) = \tilde{\theta}$ . The diagram

$$\begin{array}{ccc}
 I_{n-2}(B) & \xrightarrow{s} & I_{n-2}(A) \\
 \downarrow e_{n-2} & & \downarrow e_{n-2} \\
 \Gamma(B, \mathcal{H}^{n-2}) & \xrightarrow{tr} & \Gamma(A, \mathcal{H}^{n-2}) = H^{n-2}(A)
 \end{array}$$

can be verified to be commutative, so that  $e_{n-2}(q_1 - sq_2) = 0$ . Thus  $q_1 - sq_2 \in I_{n-1}(A)$  and  $\langle 1, -f \rangle(q_1 - sq_2) = \langle 1, -f \rangle q_1 \in \langle 1, -f \rangle I_{n-1}(A)$ . This proves the lemma.

We now assume that  $X'$  and  $Y'$  satisfy  $PQ(2)$ . The group  $V_{nr} = \ker(W(A)/\langle 1, -f \rangle W(A) \xrightarrow{i} W(X))$  has a filtration induced by the filtration  $\{I_m(A)\}$  on  $W(A)$ . Since the map  $W(X) \rightarrow W(B)$  is injective and since  $i$  preserves filtration, by (6.3), we have,

$$\begin{aligned}
 (V_{nr})_m &= \ker(I_m(A)/(\langle 1, -f \rangle W(A) \cap I_m(A)) \xrightarrow{i} I_m(B)) \\
 &= \ker(I_m(A)/\langle 1, -f \rangle I_{m-1}(A) \xrightarrow{i} I_m(B)).
 \end{aligned}$$

We now define a map  $\eta_m : (V_{nr})_m \rightarrow NH^m(B) = \ker(H^m(B) \rightarrow \Gamma(B, \mathcal{H}^m))$  as follows. Consider the following commutative diagram:

$$\begin{array}{ccc}
 I_m(A) & \xrightarrow{i} & I_m(B) \\
 \downarrow e_m & \nearrow 0 \searrow NH^m(B) & \downarrow e_m \\
 & & H^m(B) \\
 H^m(A) & \xrightarrow{\quad} & \Gamma(B, \mathcal{H}^m) \rightarrow 0
 \end{array}$$

Let  $x \in I_m(A)$  be such that  $i(x) = 0$ . Then the element  $i(e_m(x)) \in H^m(B)$  maps to zero in  $\Gamma(B, \mathcal{H}^m)$ , by the commutativity of the above diagram. Hence  $i(e_m(x)) \in NH^m(B)$ . We define  $\eta_m(\bar{x}) = i \circ e_m(x)$ . To show that  $\eta_m$  is well-defined, we need to check that for  $x \in \langle 1, -f \rangle I_{m-1}(A)$ ,  $\eta_m(\bar{x}) = 0$ . Let  $x = \langle 1, -f \rangle x'$ ,  $x' \in I_{m-1}(A)$ . We have,  $i(e_m(x)) = i(\chi_f \cup e_{m-1}(x')) = \chi_{i(f)} \cup i \circ e_{m-1}(x') = 0$  since  $f$  is a square in  $B$ . Thus we have a well-defined homomorphism

$$\eta_m : (V_{nr})_m \rightarrow NH^m(B).$$

LEMMA 6.4.  $\text{Ker } \eta_m = (V_{nr})_{m+1}$ .

*Proof.* Let  $\eta_m(\bar{x}) = 0$  with  $x \in I_m(A)$ . Then  $ie_m(x) = 0$  and the exactness of the sequence

$$H^{m-1}(A) \xrightarrow{\cup \chi_f} H^m(A) \xrightarrow{i} H_m(B) \xrightarrow{ir} H^m(A) \tag{***}$$

implies that there exists  $y \in H^{m-1}(A)$  such that  $\chi_f \cup y = e_m(x)$ . By (6.1), there exists  $z \in I_{m-1}(A)$  such that  $e_{m-1}(z) = y$ . We have,  $e_m(x - \langle 1, -f \rangle \cdot z) = 0$  so that  $x - \langle 1, -f \rangle \cdot z \in I_{m+1}(A)$  and its class in  $(V_{nr})_{m+1}$  is simply the class of  $x$ .

We thus have a filtration  $\{(V_{nr})_m\}$  on  $V_{nr}$  with successive quotients  $(V_{nr})_m / (V_{nr})_{m+1}$  injecting into  $NH^m(B)$ .

**THEOREM 6.5.** *Under the assumption that  $X'$  and  $Y'$  have  $PQ(2)$ ,  $V_{nr} \xrightarrow{\sim} \bigoplus_{m \geq 2} NH^m(B)$ .*

*Proof.* Since by (5.2),  $V_{nr}$  is a 2-torsion group, it is enough to show that  $\eta_m$  maps  $(V_{nr})_m$  onto  $NH^m(B)$ . Let  $x \in NH^m(B)$ . Since  $NH^n(A) = 0 \forall n$ ,  $trx = 0$ , and the exact sequence (\*\*\*) implies that there exists  $y \in H^m(A)$  with  $i(y) = x$ . By (6.1), there exists  $z \in I_m(A)$  with  $e_m(z) = y$ . Then  $e_m \circ i(z) = \text{class of } x \text{ in } \Gamma(B, \mathcal{H}^m)$  which is zero since  $x \in NH^m(B)$ . Thus  $i(z) \in I_{m+1}(B)$  and  $s \circ i(z) = 0$ . By (6.2), there exists  $z' \in I_{m+1}(A)$  with  $i(z') = i(z)$ . Replacing  $z$  by  $z - z'$  which again maps to  $y$  under  $e_m$ , we have  $i(z) = 0$ ; i.e.,  $\bar{z} \in (V_{nr})_m$  with  $\eta_m(\bar{z}) = x$ .

### 7. An example

**THEOREM 7.1.** *Let  $X$  be a smooth projective hyperelliptic curve defined over a local field  $k$  with residue field characteristic  $\neq 2$ . Suppose  $X$  has a rational point of ramification,  $X$  has good reduction and  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ . Then*

$$\text{coker } \partial \xrightarrow{\sim} W(k) \oplus (\mathbb{Z}/2)^{4g},$$

$g$  being the genus of  $X$ .

In view of results of [2], any curve over a local field satisfies  $PQ(1)$ . It is shown in [16] that any such curve also satisfies  $PQ(2)$ . Therefore by our assumption  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ , we have,  $\text{coker } \partial \simeq W(k) \oplus (\oplus_{m \geq 2} NH^m(X'))$ . Let  $G = G(\bar{k}/k)$ ,  $\bar{k}$  denoting the algebraic closure of  $k$ . Then  $cd_2 k \leq 2$  [18] and  $cd_2 X_{\bar{k}} \leq 1$ ,  $X_{\bar{k}}$  being affine. The spectral sequence ([12], p. 105)

$$H^i(G, H^j(X'_{\bar{k}})) \Rightarrow H^n(X')$$

yields  $H^n(X') = 0$  for  $n \geq 4$ . Thus  $\text{coker } \partial^0 \simeq NH^2(X') \oplus NH^3(X')$ . We shall now compute these groups.

**LEMMA 7.2.** *Let  $X$  be any smooth projective curve of genus  $g$  (not necessarily hyperelliptic) over a local field  $k$  with residue field characteristic  $\neq 2$  and such that  $X(k) \neq \emptyset$  and  ${}_2\text{Pic } X = {}_2\text{Pic } X_{\bar{k}}$ . Then  $H^3(X) \simeq (\mathbf{Z}/2)^{2g+2}$  and  $\Gamma(X, \mathcal{H}^3) = 0$ .*

*Proof.* The only two non-zero terms in the above spectral sequence contributing to  $H^3(X)$  are  $H^1(G, H^2(X_{\bar{k}}))$  and  $H^2(G, H^1(X_{\bar{k}}))$ . The only possible non-zero differential  $H^0(G, H^2(X_{\bar{k}})) \rightarrow H^2(G, H^1(X_{\bar{k}}))$  is zero,  $X(k)$  being non-empty, since  $H^2(X) \rightarrow H^0(G, H^2(X_{\bar{k}}))$  is surjective. Therefore

$$\begin{aligned} H^3(X) &\simeq H^2(G, H^1(X_{\bar{k}})) \oplus H^1(G, H^2(X_{\bar{k}})) \\ &\simeq (\mathbf{Z}/2)^{2g} \oplus (\mathbf{Z}/2)^2. \end{aligned}$$

In fact the action of  $G$  on  $H^1(X_{\bar{k}}) \simeq {}_2\text{Pic } X_{\bar{k}} \simeq (\mathbf{Z}/2)^{2g}$  is trivial by our assumption and  $H^2(X_{\bar{k}}) \simeq \text{Pic } X_{\bar{k}}/2 \simeq \mathbf{Z}/2$  with trivial action again. Further,  $k$  being a local field,  $H^2(G, \mathbf{Z}/2) \simeq {}_2\text{Br}(k) \simeq \mathbf{Z}/2$  and  $H^1(G, \mathbf{Z}/2) \simeq k^*/k^{*2} \simeq \mathbf{Z}/2 \times \mathbf{Z}/2$ . In view of [4],  $NH^3(X) \simeq k^*/k^{*2} \times J(k)/2J(k)$ . Since  $k$  is a local field, by [11],  $J(k)$  contains a subgroup  $\mathcal{M}$  isomorphic to copies of the valuation ring such that  $J(k)/\mathcal{M}$  is finite. The 2-primary part of  $J(k)/\mathcal{M}$  is isomorphic to  $\prod_{1 \leq j \leq l} (\mathbf{Z}/2^j)$ , where  $l = \dim_{\mathbf{Z}/2}({}_2\text{Pic } X) = 2g$  by our assumption. Therefore  $J(k)/2J(k) \simeq (\mathbf{Z}/2)^{2g}$ , so that  $NH^2(X) \simeq (\mathbf{Z}/2)^{2g+2}$ . Thus  $NH^3(X) = H^3(X)$  and  $\Gamma(X, \mathcal{H}^3) = 0$ .

**COROLLARY 7.3.** *Let  $X$  be a smooth projective curve over a local field  $k$  with residue field characteristic  $\neq 2$ . Suppose  $X$  has good reduction and  ${}_2\text{Pic } X = {}_2\text{Pic } X_{\bar{k}}$ . Then the classical invariants uniquely determine the class of a quadratic space in  $W(X)$ .*

*Proof.* In view of ([15], §1), we have injections  $rk : W(X)/I(X) \hookrightarrow \mathbf{Z}/2$ ,  $\text{disc} : I(X)/I_2(X) \hookrightarrow H^1(X)$ ,  $c : I_2(X)/I_3(X) \hookrightarrow {}_2\text{Br}(X) = \Gamma(X, \mathcal{H}^2)$ , where  $rk, \text{disc}$

and  $c$  stand for rank, discriminant and Hasse–Witt invariant maps. Since  $I_4(X) \hookrightarrow I^4(k(X)) = 0$  [2] and  $I_3(X)$  injects into  $\Gamma(X, \mathcal{H}^3) = 0$  by (7.2), we have,  $rk$ , disc and  $c$  uniquely determine an element in  $W(X)$ .

**LEMMA 7.4.** *Let  $X$  be a hyperelliptic curve. Then  $NH^2(X') \simeq (\mathbf{Z}/2)^{2g}$ , under the assumptions of (7.1) on  $X$ .*

*Proof.* We have  $NH^2(X') \simeq \text{Pic } X'/2$ . The exact sequence

$$0 \rightarrow {}_2\text{Pic } X \rightarrow \text{Pic}^0 X \rightarrow \text{Pic } X' \rightarrow 0$$

yields the following long exact sequence

$$0 \rightarrow {}_2\text{Pic } X \rightarrow {}_2\text{Pic}^0 X \rightarrow {}_2\text{Pic } X' \rightarrow {}_2\text{Pic } X/2 \rightarrow \text{Pic}^0 X/2 \rightarrow \text{Pic } X'/2 \rightarrow 0.$$

We have  ${}_2\text{Pic } X = {}_2\text{Pic}^0 X$ ,  ${}_2\text{Pic } X' \simeq (\mathbf{Z}/2)^{2g}$  (4.3),  ${}_2\text{Pic } X/2 \simeq (\mathbf{Z}/2)^{2g}$  and  $\text{Pic}^0 X/2 = J(k)/2J(k) \simeq (\mathbf{Z}/2)^{2g}$ , in view of (7.2). We therefore have  $\text{Pic } X'/2 \simeq (\mathbf{Z}/2)^{2g}$ .

**LEMMA 7.5.** *Let  $X$  be a hyperelliptic curve. Then  $NH^3(X') \simeq (\mathbf{Z}/2)^{2g}$ , under the assumptions of (7.1) on  $X$ .*

*Proof.* We have an exact sequence

$$0 \rightarrow U(X'_k)/2 \rightarrow H^1(X'_k) \rightarrow {}_2\text{Pic } X'_k \rightarrow 0.$$

By (4.1) and (4.3),  $U(X'_k)/2 \simeq (\mathbf{Z}/2)^{2g+1}$  and  ${}_2\text{Pic } X'_k \simeq (\mathbf{Z}/2)^{2g}$ . Therefore  $H^1(X'_k) \simeq (\mathbf{Z}/2)^{4g+1}$ . Further, since  $U(X'_k)/2$  is generated by  $\{y, T - \alpha_i\}$ ,  $1 \leq i \leq 2g$ , which are defined over  $k$ , and  ${}_2\text{Pic } X'_k$  is also defined over  $k$  under the assumption  ${}_4\text{Pic } X' = {}_4\text{Pic } X'_k$ , the action of  $G$  on  $H^1(X'_k)$  is trivial. The only non-zero terms in the spectral sequence

$$H^i(G, H^j(X'_k)) \Rightarrow H^n(X')$$

contributing to  $H^3(X')$  is  $H^2(G, H^1(X'_k))$  with all the differentials vanishing, as before. We therefore have

$$H^3(X') \simeq H^2(G, H^1(X'_k)) \simeq (\mathbf{Z}/2)^{4g+1}.$$

We shall now compute  $\Gamma(X', \mathcal{H}^3)$ . The sequence

$$H^3(k(X)) \xrightarrow{tr} H^3(k(T)) \xrightarrow{\cup_{X'}^{\cup_{Y'}}} H^4(k(T))$$

is exact and since  $cd_2(k) \leq 2, cd_2(k(T)) \leq 3, H^4(k(T)) = 0$ . Thus

$$tr : H^3(k(X)) \rightarrow H^3(k(T))$$

is surjective. It induces a map

$$tr : \Gamma(X', \mathcal{H}^3) \rightarrow \Gamma(Y', \mathcal{H}^3) \xrightarrow{\sim} H^3(Y').$$

We show that this map is surjective. Let  $\lambda \in H^3(Y')$  and  $\mu \in H^3(k(X))$  be such that  $tr \mu = \lambda$ , identifying  $H^3(Y')$  with a subgroup of  $H^3(k(T))$ . In view of the commutative diagram

$$\begin{array}{ccccc} H^3(k(T)) & \xrightarrow{i} & H^3(k(X)) & \xrightarrow{tr} & H^3(k(T)) \\ \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\ \bigoplus_{y \in Y'} H^2(k(y)) & \xrightarrow{i} & \bigoplus_{x \in X'} H^2(k(x)) & \xrightarrow{tr} & \bigoplus_{y \in Y'} H^2(k(y)) \end{array}$$

with exact rows,  $tr \circ \partial \mu = \partial \circ tr \mu = \partial(\lambda) = 0$  and hence there exists  $v \in \bigoplus_{y \in Y'} H^2(k(y))$  with  $i(v) = \partial(\mu)$ . Since  $Y' \subset A^1, \partial : H^3(k(T)) \rightarrow \bigoplus_{y \in Y'} H^2(k(y))$  is surjective and hence there exists  $\tilde{v} \in H^3(k(T))$  with  $\partial(\tilde{v}) = v$ . We have  $\partial(\mu - iv) = 0$  so that  $(\mu - iv) \in \Gamma(X', \mathcal{H}^3)$  and maps to  $\lambda \in \Gamma(Y', \mathcal{H}^3) = H^3(A)$ . We thus have a surjection  $tr : \Gamma(X', \mathcal{H}^3) \rightarrow \Gamma(Y', \mathcal{H}^3)$ . We now compute its kernel. Since  $H^3(k) = 0$ , the map  $\partial : H^3(A) \rightarrow (\bigoplus_{y \in \pi(S)} H^2(k(y)))^0$  is an isomorphism. Since the square

$$\begin{array}{ccc} \Gamma(X', \mathcal{H}^3) & \xrightarrow{\partial} & \left( \bigoplus_{x \in S} H^2(k(x)) \right)^0 \\ tr \downarrow & & \parallel \\ H^3(A) & \xrightarrow{\sim} & \left( \bigoplus_{y \in \pi(S)} H^2(k(y)) \right)^0 \end{array}$$

is commutative, we have,  $\ker tr = \ker \partial = \Gamma(X, \mathcal{H}^3) = 0$ , by [5] and (7.2). Thus,  $\Gamma(X', \mathcal{H}^3) \xrightarrow{\sim} H^3(A) \xrightarrow{\sim} (\mathbf{Z}/2)^{2g+1}$ . Therefore  $NH^3(X') \xrightarrow{\sim} (\mathbf{Z}/2)^{2g}$ .

This completes the proof of Theorem 7.1. Finally, we use the exact sequence (§3) to compute the defining relations for  $W(X)$  as a quotient of  $\bigoplus_{x \in S} W(k(x))$ . More precisely, we have the following

**THEOREM 7.6.** *Under the same hypothesis as in (7.1),*

$$W(X) \simeq (\mathbf{Z}/2)^{4g} \oplus W(k).$$

*Proof.* In view of (3.4) and (7.1), we have an exact sequence

$$0 \rightarrow (\mathbf{Z}/2)^{4g} \rightarrow W(A)/(\langle 1, -f \rangle W(k)) \rightarrow W(X) \rightarrow 0 \tag{*}$$

The residue map  $\partial : W(A) \rightarrow \bigoplus_{1 \leq i \leq 2g+1} W(k)$  is surjective, with kernel  $W(k)$ . We have, in  $W(k(T))$ ,  $(W(k) \cap \langle 1, -f \rangle \cdot W(k)) = 0$ . In fact, for  $q \in W(k) \cap \langle 1, -f \rangle \cdot W(k)$ ,  $q$  extends to zero in  $W(X)$ . Since  $X(k)$  is non-empty, specialising at a rational point yields  $q = 0$  in  $W(k)$ . We thus have an exact sequence

$$0 \rightarrow W(k) \rightarrow W(A)/(\langle 1, -f \rangle \cdot W(k)) \rightarrow \bigoplus_{2g+1} W(k)/\partial(\langle 1, -f \rangle W(k)) \rightarrow 0.$$

The image of the map  $\eta : W(k) \rightarrow \bigoplus_{2g+1} W(k)$  defined by

$$\eta(q) = (-f'(\alpha_1)q, -f'(\alpha_2)q, \dots, -f'(\alpha_{2g+1})q)$$

is precisely  $\partial(\langle 1, -f \rangle \cdot W(k))$ . The map  $\eta$  is injective, since for  $q \in W(k)$ ,  $\eta(q) = 0$  implies that  $\partial(\langle 1, -f \rangle q) = 0$ ; i.e.,  $\langle 1, -f \rangle q \in W(k) \cap \langle 1, -f \rangle W(k) = 0$  and  $q \simeq fq$ . Since degree  $f$  is odd,  $q = 0$ . Clearly  $\eta$  is a split injection, a section  $t$  being given by  $t(q_1, q_2, \dots, q_{2g+1}) = -f'(\alpha_1) \cdot q_1$ . We thus have an isomorphism

$$\tilde{\eta} : W(A)/(\langle 1, -f \rangle W(k)) \rightarrow W(k) \oplus \left( \bigoplus_{2g} W(k) \right)$$

given by  $\tilde{\eta}(\bar{q}) = (\tilde{q}, (\partial_{x_i} q))$ ,  $2 \leq i \leq 2g + 1$ ,  $x_i \in S$ ,  $\tilde{q}$  denoting specialisation at  $\infty$ . If  $\bar{q} \in W(A)/\langle 1, -f \rangle W(k)$ , maps to zero in  $W(X)$ , specialising at  $\infty$ , we see that  $\tilde{q} = 0$ , so that in the sequence (\*),  $(\mathbf{Z}/2)^{4g}$  injects into the factor  $\bigoplus_{2g} W(k) \simeq \bigoplus_{4g} W(F)$  where  $F$  denotes the residue field of  $k$ . If  $-1$  is a square in  $F$ ,  $W(F) \simeq (\mathbf{Z}/2)^2$  and if  $-1$  is not a square in  $F$ ,  $W(F) \simeq \mathbf{Z}/4$ . Therefore,

$$\begin{aligned} W(X) &\simeq W(k) \oplus W(F)^{4g}/(\mathbf{Z}/2)^{4g} \\ &\simeq W(k) \oplus (\mathbf{Z}/2)^{4g}. \end{aligned}$$

The above theorem leads one to the following natural questions.

**QUESTION 1.** *For a smooth hyperelliptic curve  $X$  over an arbitrary ground field  $k$ , (with  ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$ ), is  $W(X)$  isomorphic to  $W(k) \oplus (\mathbf{Z}/2)^{4g}$ ?*

A positive answer to this question will also provide evidence to an affirmative answer to the following more general

QUESTION. (Scharlau) *Let  $X$  be a smooth projective curve over a field  $k$ . If  $W(k)$  is finitely generated, is  $W(X)$  finitely generated?*

QUESTION 2. *For a smooth projective curve  $X$  over  $k$  with  $X(k) \neq \emptyset$ , is  $\text{coker } \partial \xrightarrow{\simeq} W(X)$ ?*

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