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Witt group of hyperelliptic curves

R. PARIMALA and R. SUJATHA

Introduction

Let k be a perfect field of characteristic $\neq 2$. Let X be a smooth projective curve over k . Let $W(k(X), \Omega_{k(X)})$ denote the Witt group of the function field of X with values in the module of differentials $\Omega_{k(X)}$ of $k(X)$. A residue homomorphism

$$\partial : W(k(X), \Omega_{k(X)}) \rightarrow \bigoplus_{x \in X} W(k(x))$$

was defined in [7], $k(x)$ denoting the residue field at points $x \in X$ and a residue theorem was proved; namely the composite

$$W(k(X), \Omega_{k(X)}) \xrightarrow{\partial} \bigoplus_{x \in X} W(k(x)) \xrightarrow{\text{trace}} W(k)$$

is zero. Thus image ∂ is contained in the subgroup $(\bigoplus_{x \in X} W(k(x)))^0$ consisting of tuples (μ_x) with $\sum_{x \in X} \text{trace } \mu_x = 0$. The kernel and cokernel of ∂ are well understood if $X = \mathbf{P}^1$ [13] or if X is an anisotropic conic over k [14]. To have an intrinsic description of these groups for curves of higher genus is an interesting question posed by Milnor in [13].

In this paper, we study this problem for smooth hyperelliptic curves X with a rational point of ramification over \mathbf{P}^1 . Let $\pi : X \rightarrow \mathbf{P}^1$ be a covering defined over \mathbf{A}^1 by the equation $y^2 = f(T)$. We exhibit an exact sequence (§3)

$$0 \rightarrow W(X) \rightarrow W(k(X)) \xrightarrow{\partial^0} \left(\bigoplus_{x \in X} W(k(x)) \right)^0 \rightarrow \frac{W(k[T]_f)}{\langle 1, -f \rangle W(k)} \rightarrow W(X) \rightarrow 0.$$

where ∂^0 is simply the residue map ∂ through an identification of $W(k(X), \Omega_{k(X)})$ with $W(k(X))$ for a suitable choice of a differential as basis for $\Omega_{k(X)}$. We derive, as a corollary, that if all the ramification points of π are k -rational, $W(X)$ is generated by one-dimensional forms. This exact sequence may be viewed in two ways: Firstly as characterising coker ∂^0 as a subgroup of $\bigoplus_{x \in S} W(k(x))$, S denoting the set of ramification points of π and secondly, as giving the defining relations for expressing $W(X)$ as a quotient of $\bigoplus_{x \in S} W(k(x))$.

Using the exact sequence above, we give a more precise description of $\text{coker } \partial^0$. It contains a subgroup V_r which is a quotient of $\bigoplus_{x \in S} W(k(x))$, which we call the *ramified part* of $\text{coker } \partial^0$. Under the rationality assumption that ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$, \bar{k} denoting the algebraic closure of k , the group V_r is zero. We call $V_{nr} = \text{coker } \partial^0 / V_r$, the *unramified part* of $\text{coker } \partial^0$. This group is 2-torsion (§5). It can be computed in terms of certain cohomology groups if ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$, and supposing further that the curve $Y = X$ or \mathbf{P}^1 has the following property: ‘Graded Witt group of Y is isomorphic to the cohomology ring’; Curves over local and global fields have this property [16]. We in fact show that under these assumptions on X , $\text{coker } \partial^0$ is isomorphic to $(\text{Pic } X'/2) \oplus NH^3(X')$, where $X' = X \setminus S$, S denoting the set of ramification points of π and $NH^n(X')$ denotes the kernel of the map $H_{et}^n(X', \mu_2) \rightarrow H_{et}^n(k(X'), \mu_2)$. For a smooth projective hyperelliptic curve over a local field with good reduction, if ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$, $\text{coker } \partial$ is isomorphic to $W(k) \oplus (\mathbb{Z}/2)^{4g}$ where g is the genus of the curve (Theorem 7.1). Further, $W(X)$ is also isomorphic to the group $(\mathbb{Z}/2)^{4g} \oplus W(k)!$ (Theorem 7.6).

The computations yield, as a by-product, that for any smooth projective curve X over a local field with good reduction, if ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$, the classical invariants determine the class of a quadratic space in $W(X)$.

We record here that J. E. Shick [19] has some independent computations of $\text{coker } \partial$ for power series fields over \mathbb{R} and of \mathbb{C} .

We thank D. S. Nagaraj for carefully going through the manuscript.

1. Kernel of the residue homomorphism

Let k be a perfect field of characteristic $\neq 2$. Let X be a smooth projective curve defined over k . For a line bundle \mathcal{L} on X , let $W(X, \mathcal{L})$ denote the Witt group of quadratic spaces on X with values in \mathcal{L} [9]. Let $W(X) = W(X, \mathcal{O}_X)$.

LEMMA 1.1. *The group $W(X, \mathcal{L})$ depends upto isomorphism, only on the class of \mathcal{L} in $\text{Pic } X/2$. In particular, $W(X, \mathcal{L}^2) \simeq W(X)$.*

Proof. Let $\mathcal{M} \in \text{Pic } X$ and (\mathcal{E}, q) be a quadratic space with values in $\mathcal{L} \otimes \mathcal{M}^2$, i.e., $q : \mathcal{E} \rightarrow \mathcal{E}^* \otimes \mathcal{L} \otimes \mathcal{M}^2$, where for any bundle \mathcal{F} , \mathcal{F}^* denotes the dual of \mathcal{F} , is an isomorphism such that $q' \otimes 1_{\mathcal{L} \otimes \mathcal{M}^2} = q$. The assignment

$$(\mathcal{E}, q) \rightarrow (\mathcal{E} \otimes \mathcal{M}^*, q \otimes 1_{\mathcal{M}^*})$$

defines an isomorphism

$$W(X, \mathcal{L} \otimes \mathcal{M}^2) \simeq W(X, \mathcal{L}).$$

Let Ω_X denote the sheaf of differentials on X . Let

$$\partial : W(k(X), \Omega_{k(X)}) \rightarrow \bigoplus_{x \in X} W(k(x))$$

be the residue homomorphism defined in [7], $k(x)$ denoting the residue field at the closed point x of X . (Throughout, the notation $x \in X$ stands for the set of all closed points x in X).

LEMMA 1.2. *The kernel of the residue map*

$$\partial : W(k(X), \Omega_{k(X)}) \rightarrow \bigoplus_{x \in X} W(k(x))$$

is $W(X, \Omega_X)$.

Proof. Let q be a quadratic space over $k(X)$ with values in $\Omega_{k(X)}$, whose class belongs to $\ker \partial$. Let x be a closed point of X and π_x a local parameter at x . Identifying $W(k(X))$ with $W(k(X), \Omega_{k(X)})$ through $d\pi_x$, the residue map $\partial_x : W(k(X)) \rightarrow W(k(x))$ is simply the second residue homomorphism with respect to π_x . Thus q which maps to zero under ∂_x (cf. [17], p. 207) is isometric to $q_x \otimes_{\mathcal{O}_{X,x}} k(X)$ for some $q_x \in W(\mathcal{O}_{X,x})$. The spaces $q_x \cdot d\pi_x$ over $\mathcal{O}_{X,x}$ with values in $\Omega_{X,x}$ become isometric to q over $k(X)$. They patch up to yield a quadratic space q_X over X with values in Ω_X in view of the following

LEMMA 1.3. *Let \mathcal{L} be a line bundle on X , q a quadratic space over $k(X)$ with values in $\mathcal{L}_{k(X)}$. Suppose, for every $x \in X$, there exists a quadratic space q_x over $\mathcal{O}_{X,x}$ with value in $\mathcal{L} \otimes \mathcal{O}_{X,x}$ such that $q_x \otimes k(X) \simeq q$. Then there exists a quadratic space q_X over X with values in \mathcal{L} such that $q_X \otimes k(X) \simeq q$.*

Proof. The proof of ([6], Corollary 2.7) in the case $\mathcal{L} = \mathcal{O}_X$ goes through verbatim for any line bundle \mathcal{L} .

REMARK. If $X = \mathbf{P}^1$, $\ker \partial \simeq W(X) \simeq W(k)$. ([13], Proposition 5.3). If X is an anisotropic conic, $\ker \partial = W(X, \Omega_X)$ is computed as \mathcal{B}_θ in ([14], Theorem 6.2).

PROPOSITION 1.4. *Let X be a smooth hyperelliptic curve with a rational point of ramification over \mathbf{P}^1 . Then $\ker \partial \simeq W(X)$.*

Proof. By (1.1) and (1.2), it suffices to show that Ω_X is the square of a line bundle on X . Let $\pi : X \rightarrow \mathbf{P}^1$ be a covering, defined over \mathbf{A}^1 by the equation $y^2 = f(T)$, degree $f = 2g + 1$, g being the genus of X . The divisor of the differential dT/y is $(2g - 2)P_\infty$, P_∞ being the point of X lying over ∞ in \mathbf{P}^1 . Let \mathcal{L} be the line bundle corresponding to the divisor $(g - 1)P_\infty$. Then $\Omega_X \simeq \mathcal{L}^2$.

REMARK. As observed by M. Rost, one could define more generally, a residue map

$$\partial_{\mathcal{L}} : W(k(X), \mathcal{L}_{k(X)}) \rightarrow \bigoplus_{x \in X} (W(k(x)), (\mathcal{L} \otimes \Omega_X)(x)),$$

where $(\mathcal{L} \otimes \Omega_X)(x)$ denotes the fibre of the line bundle $(\mathcal{L} \otimes \Omega_X)$ at x . If $X = \mathbf{P}^1$, and $\mathcal{L} = \mathcal{O}_X$, $\partial_{\mathcal{O}_X} = \partial$ is the residue homomorphism discussed above, since Ω_X is a square. If $\mathcal{L} = \mathcal{O}_X(1)$, $\partial_{\mathcal{O}_X(1)}$ is an isomorphism. In the case of an anisotropic conic, X we have $\ker \partial_{\mathcal{O}_X} \simeq W(X) \simeq W(k)/\langle 1, -a, -b, ab \rangle W(k)$, (cf. [1]), where X is defined by the equation $aX^2 + bY^2 - Z^2 = 0$. One can identify coker ∂ with a subgroup of the Witt group of the residue field at the ramified point of the covering $X \rightarrow \mathbf{P}^1$.

2. Some auxiliary results on trace, transfer and residue homomorphisms

Let $\pi : X \rightarrow \mathbf{P}^1$ be a double covering, defined over \mathbf{A}^1 by the equation $y^2 = f(T)$, degree $f = 2g + 1$, g being the genus of X . We identify $W(k(X))$ and $W(k(T))$ with $W(k(X), \Omega_{k(X)})$ and $W(k(T), \Omega_{k(T)})$ through the basis $dT/2y$ and dT respectively. For $y \in \mathbf{A}^1$, if $p \in k[T]$ is the monic irreducible polynomial which gives a parameter at y , the composite map

$$W(k(T)) \xrightarrow{dT} W(k(T), \Omega_{k(T)}) \xrightarrow{\partial_y} W(k(y))$$

is the second residue homomorphism with respect to the parameter pp' , p' denoting the derivative of p with respect to T . Similarly, one can verify that if $x \in X$ lies over $y \in \mathbf{P}^1$ corresponding to $p(T)$, and x unramified over y , on choosing $p(T)$ again as the parameter at y , the composite

$$W(k(X)) \xrightarrow{dT/2y} W(k(X), \Omega_{k(X)}) \xrightarrow{\partial_x} W(k(x))$$

is the second residue homomorphism with respect to the parameter $2pp'y$. We again denote by ∂ this residue map.

For any finite separable extension L/K , let $tr : W(L) \rightarrow W(K)$ be the map induced by the linear map trace $L \rightarrow K$ and $i : W(K) \rightarrow W(L)$ the map induced by the inclusion of K in L . Let $s : W(k(X)) \rightarrow W(k(T))$ be the transfer homomorphism induced by the linear map $s : k(X) \rightarrow k(T)$ defined by $s(1) = 0$, $s(y) = 1$ where $\{1, y\}$ is a basis for $k(X)$ over $k(T)$ ([17], p. 47).

LEMMA 2.1. *The diagram*

$$\begin{array}{ccc} W(k(X)) & \xrightarrow{(\partial_x)} & \bigoplus_{x/y} W(k(x)) \\ s \downarrow & & \downarrow tr \\ W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \end{array}$$

is commutative.

Proof. Since the diagram

$$\begin{array}{ccc} W(\Omega_{k(X)}) & \xrightarrow{\partial} & \bigoplus_{x/y} W(k(x)) \\ tr \downarrow & & \downarrow tr \\ W(\Omega_{k(T)}) & \xrightarrow{\partial} & W(k(y)) \end{array}$$

is commutative ([7], §1), it suffices to show that the diagram

$$\begin{array}{ccc} W(k(X)) & \xrightarrow{dT/2y} & W(k(X), \Omega_{k(X)}) \\ s \downarrow & & \downarrow tr \\ W(k(T)) & \xrightarrow{dT} & W(k(T), \Omega_{k(T)}) \end{array}$$

is commutative. It is enough to check that

$$tr(\langle h_0 + h_1 y \rangle \cdot dT/2y) = s(\langle h_0 + h_1 y \rangle) \cdot dT,$$

for $h_0, h_1 \in k(T)$. We have,

$$\begin{aligned} tr(\langle h_0 + h_1 y \rangle \cdot dT/2y) &= \langle tr((h_0 + h_1 y)/2y) \rangle \cdot dT \\ &= \begin{pmatrix} h_1 & h_0 \\ h_0 & h_1 f \end{pmatrix} \cdot dT \\ &= s(\langle h_0 + h_1 y \rangle) \cdot dT. \end{aligned}$$

LEMMA 2.2. *The diagram*

$$\begin{array}{ccc} W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \\ i \downarrow & & \downarrow i' \\ W(k(X)) & \xrightarrow{\partial_x} & \bigoplus_{x/y} W(k(x)) \end{array}$$

is commutative if y is an unramified point for π and i' is the composite

$$W(k(y)) \xrightarrow{i} W(k(x)) \xrightarrow{\overline{2y}^{-1}} W(k(x)).$$

If x is a ramified point for π , $\partial_x \circ i$ is zero.

Proof. Let $\langle h \rangle \in W(k(T))$ and $x \in X$ such that x is an unramified point for π with $\pi(x) = y$. Let $p \in k[T]$ be the monic polynomial corresponding to y . Suppose $v_y(h) = 0$. Then $\partial_y(\langle h \rangle) = 0$ and $\partial_x \circ i(\langle h \rangle) = \partial_x(\langle h \rangle) = 0$, since $v_x(h) = v_y(h) = 0$. Suppose $h = up$ with $v_y(u) = 0$. Since ∂_x is the second residue map with respect to the parameter $2pp'y$ and ∂_y the second residue map with respect to pp' , we have

$$i' \circ \partial_y(\langle up \rangle) = i' \circ \partial_y(\langle u/p' \rangle \cdot pp') = i' \langle \overline{u/p'} \rangle = \langle \overline{u/p'2y} \rangle$$

and

$$\partial_x \circ i(\langle h \rangle) = \partial_x(\langle (u/2p'y) \cdot 2pp'y \rangle) = \langle \overline{u/2p'y} \rangle$$

Suppose $x \in X$ is a ramified point, lying over $y \in \mathbf{P}^1$. For $h \in k(T)$, $v_x(ih) \equiv 0 \pmod{2}$, since x has ramification index 2, and we have $\partial_x \circ i(\langle h \rangle) = 0$.

LEMMA 2.3. *Let x/y be an unramified point for π . Then the diagram*

$$\begin{array}{ccc} W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \\ \langle 1, -f \rangle \downarrow & & \downarrow \langle 1, -f \rangle \\ W(k(T)) & \xrightarrow{\partial_y} & W(k(y)) \end{array}$$

is commutative.

Proof. Clear.

We repeatedly use the following lemma which is a consequence of the Lam's exact triangle ([17], Chapter 2, 5.10).

LEMMA 2.4. *The following triangles are exact.*

$$\begin{array}{ccc}
 W(k(T)) & \xrightarrow{i} & W(k(X)) \\
 \langle 1, -f \rangle \swarrow & & \swarrow s \\
 & W(k(T)) & \\
 W(k(y)) & \xrightarrow{i'} & W(k(x)) \\
 \langle 1, -f \rangle \swarrow & & \swarrow tr \\
 & W(k(y)) &
 \end{array}$$

where, in the second triangle, y is unramified for π and $\pi(x) = y$; if y splits in X , we mean by $W(k(x))$, the direct sum $W(k(x_1)) \oplus W(k(x_2))$ with $\pi(x_i) = y$.

3. An exact sequence

Let $\pi : X \rightarrow \mathbf{P}^1$ be a hyperelliptic curve defined over \mathbf{A}^1 by the equation $y^2 = f(T)$, degree $f = 2g + 1$, g being the genus of X . Let

$$\partial^0 : W(k(X)) \rightarrow \left(\bigoplus_{x \in X} W(k(x)) \right)^0$$

be the residue homomorphism as defined in §2, identifying $W(k(X))$ with $W(k(X), \Omega_{k(X)})$ through the basis $dT/2y$, $(\bigoplus_{x \in X} W(k(x)))^0$ denoting the kernel of the trace map $\bigoplus_{x \in X} W(k(x)) \xrightarrow{tr} W(k)$. We fix the following notation: S = set of ramification points for π , $X' = X \setminus S$, $Y = \mathbf{P}^1$, $Y' = Y \setminus \pi(S)$. We have the following commutative diagram with exact rows and columns, in view of (2.1), (2.3) and (2.4) and ([13], Theorem 5.3).

$$\begin{array}{ccccccc}
 W(k(X)) & \xrightarrow{\partial} & \bigoplus_{x \in X} W(k(x)) & & & & \\
 s \downarrow & & \downarrow tr & & & & \\
 0 \longrightarrow W(k) & \longrightarrow W(k(T)) & \xrightarrow{\partial} \bigoplus_{y \in Y'} W(k(y)) & \bigoplus_{y \in \pi(S)} W(k(y)) & \xrightarrow{\text{trace}} W(k) & \longrightarrow 0 \\
 \downarrow & \langle 1, -f \rangle \downarrow & \downarrow \langle 1, -f \rangle & & & & \\
 0 \longrightarrow W(Y') & \longrightarrow W(k(T)) & \xrightarrow{\partial} \bigoplus_{y \in Y'} W(k(y)) & & & &
 \end{array}$$

We define a homomorphism $\alpha : (\bigoplus_{x \in X} W(k(x)))^0 \rightarrow W(A)/\langle 1, -f \rangle \cdot W(k)$, where $A = k[T]_f$ as follows. Let $\theta \in (\bigoplus_{x \in X} W(k(x)))^0$. Then there exists $q \in W(k(T))$ with

$\partial(q) = \text{tr } \theta$. We have for $y \in Y'$

$$\begin{aligned}\partial_y(\langle 1, -f \rangle \cdot q) &= \langle 1, -\bar{f} \rangle \partial_y(q) \\ &= \langle 1, -\bar{f} \rangle \text{tr}(\theta_x) \\ &= 0.\end{aligned}$$

Hence $\langle 1, -f \rangle q \in W(Y') = W(A)$. Let $\alpha(\theta)$ denote its class in $W(A)/\langle 1, -f \rangle \cdot W(k)$. If q_1, q_2 and two lifts of $\text{tr} \theta$ in $W(k(T))$, $q_1 - q_2 \in W(k)$ and $\langle 1, -f \rangle q_1$ and $\langle 1, -f \rangle q_2$ define the same class in $W(A)/\langle 1, -f \rangle W(k)$. Thus α is well-defined.

LEMMA 3.1. $\ker \alpha = \partial^0(W(k(X)))$.

Proof. Since $\partial \circ s = \text{tr} \circ \partial$ and $\langle 1, -f \rangle \circ s = 0$, we have

$$\partial(W(k(X))) \subset \ker \alpha.$$

Let $\theta \in \bigoplus_{x \in X} W(k(x))^0$ with $\alpha(\theta) = 0$. Let $q_1 \in W(k(T))$ be such that $\partial(q_1) = \text{tr} \theta$. Then $\langle 1, -f \rangle q_1 \in \langle 1, -f \rangle W(k)$. Replacing q_1 by $q_1 - q_0$ for a suitable $q_0 \in W(k)$, we assume that $\langle 1, -f \rangle q_1 = 0$. Thus, by (2.4), there exists $q_2 \in W(k(X))$ such that $s(q_2) = q_1$. We have $\text{tr}(\theta - \partial q_2) = \text{tr} \theta - \partial s q_2 = \text{tr} \theta - \partial q_1 = 0$. The fact that $\theta - \partial q_2 \in \partial W(k(X))$ follows from the following

LEMMA 3.2. Let $\theta \in (\bigoplus_{x \in X} W(k(x)))^0$ with $\text{tr} \theta = 0$ in $(\bigoplus_{y \in Y} W(k(y)))^0$. Then $\theta \in \partial \circ i(W(k(T)))$.

SUBLEMMA 3.3. Let $(\mu_x) \in \bigoplus_{x \in X'} W(k(x))$ be such that $\text{tr}(\mu_x) = 0$ in $\bigoplus_{y \in Y'} W(k(y))$. Then there exists $q \in W(k(T))$ such that $\partial_x(i(q)) = \mu_x$, for $x \in X'$.

Proof. By (2.4), there exists $(v_y) \in \bigoplus_{y \in Y'} W(k(y))$ such that $i'(v_y) = \mu_x$. Since $Y' \subset \mathbb{A}^1$, the residue map $\partial : W(k(T)) \rightarrow \bigoplus_{y \in Y'} W(k(y))$ is surjective. Let $q \in W(k(T))$ be such that $\partial_y(q) = v_y$ for $y \in Y'$. Then, by (2.2), $\partial \circ i(q) = i' \circ \partial_y(q) = \mu_x$ for $x \in X'$.

Proof of 3.2. By (3.3), there exists $q \in W(k(T))$ such that $\partial_x \circ i(q) = \theta_x$ for $x \in X'$. Further, by (2.2), $\partial_x \circ i(q) = 0$ for $x \in S = X \setminus X'$. Since for $x \in S$, $\theta_x = \text{tr } \theta_x = 0$, we have, $\partial(i(q)) = \theta$.

Let $A = k[T]_f$, $B = (k[T, y]/(y^2 - f))_f$ be the co-ordinate rings of Y' and X' respectively. Since for $x \in S$, $q \in W(A)$, $\partial_x \circ i(q) = 0$ by (2.2), the natural map $W(A) \xrightarrow{i} W(B)$ has its image contained in $W(X)$. This map vanishes on $\langle 1, -f \rangle \cdot W(k)$ and induces a map $\beta : W(A)/\langle 1, -f \rangle W(k) \rightarrow W(X)$.

THEOREM 3.4. *The sequence*

$$0 \longrightarrow W(X) \xrightarrow{i} W(k(X)) \xrightarrow{\partial^0} \left(\bigoplus_{x \in X} W(k(x)) \right)^0 \xrightarrow{\alpha} W(A)/\langle 1, -f \rangle W(k) \xrightarrow{\beta} W(X) \longrightarrow 0$$

is exact.

Proof. Exactness at $W(X)$ (left) and $W(k(X))$ are proved in ([10], p. 277) noting that ∂^0 is the second residue homomorphism at all points $x \in X$. The exactness at $(\bigoplus_{x \in X} W(k(x)))^0$ is proved in (3.1). That $\beta \circ \alpha = 0$ follows from the fact that $i \circ \langle 1, -f \rangle = 0$, (2.4). We now prove the surjectivity of β . We identify $W(X)$ with the subgroup of $W(k(X))$ which is the kernel of ∂^0 . Let $q \in W(X)$. Then $\partial \circ s(q) = tr \circ \partial q = 0$ so that $s(q) \in W(k)$. Further $\langle 1, -f \rangle s(q) = 0$ (2.4). This implies that $s(q) = 0$ in view of the fact that for any anisotropic quadratic space q over k , $q \not\rightarrow g \cdot q$ for any odd degree polynomial g . Thus, there exists $q_1 \in W(k(T))$ with $i(q_1) = q$. We have $i' \circ \partial_y(q_1) = \partial_x \circ i(q_1) = 0$ for $y \in Y'$. There exists $\mu_y \in W(k(y))$ such that $\langle 1, -\bar{f} \rangle(\mu_y) = \partial_y(q_1)$. Since $\partial : W(k(T)) \rightarrow \bigoplus_{y \in Y'} W(k(y))$ is surjective, there exists $q_2 \in W(k(T))$ such that $\partial_y(q_2) = \mu_y$ for every $y \in Y'$. We have $\partial_y(q_1 - \langle 1, -f \rangle q_2) = \langle 1, -\bar{f} \rangle \mu_y - \langle 1, -\bar{f} \rangle \partial_y(q_2) = 0$ for $y \in Y'$ so that $q_1 - \langle 1, -f \rangle q_2 \in W(A)$ and maps to q under β . We now prove exactness at $W(A)/\langle 1, -f \rangle \cdot W(k)$. Let $q \in W(A)$ be such that $\beta(\bar{q}) = 0$ in $W(k(X))$. By (2.4), there exists $q_1 \in W(k(T))$ such that $\langle 1, -f \rangle \cdot q_1 = q$. Since $\langle 1, -\bar{f} \rangle \partial_y(q_1) = 0$ for $y \in Y'$, there exists $\mu_x \in W(k(x))$, x/y such that $tr(\mu_x) = \partial_y(q_1)$. For $x \in S$, we set $\mu_x = \partial_y(q_1)$. Clearly $(\mu_x) \in (\bigoplus_{x \in X} W(k(x)))^0$ and $\alpha((\mu_x)) = q$.

COROLLARY 3.5. *If all ramification points of X are defined over k , then $W(X)$ is generated by discriminants.*

Proof. Suppose $f = \prod_i (T - \alpha_i)$, $\alpha_i \in k$. An immediate consequence of the Milnor sequence ([13] Theorem 5.3) is that $W(k[T]_f)$ is generated by $\langle \lambda(T - \alpha_i) \rangle$ and $\langle \mu \rangle$, $\mu \in k^*$, $1 \leq i \leq 2g + 1$. Since β is surjective, their images under β , which are precisely the discriminants of $W(X)$, generate $W(X)$.

4. Some computations for hyperelliptic curves

Let X be a smooth hyperelliptic curve defined over k . We assume throughout that X has a rational point of ramification. Let $\pi : X \rightarrow \mathbf{P}^1$ be a double covering as

in §3. If genus $X > 1$, since any two double coverings $\pi_1, \pi_2 : X \rightarrow \mathbf{P}^1$ differ by an automorphism of X , the space $X' = X \setminus S$, S denoting the set of ramification points of the covering $\pi : X \rightarrow \mathbf{P}^1$ determines and is determined by X . Following notations of §3, let $A = k[T]_f$ and $B = (k[T, y]/(y^2 - f))_f$ be the co-ordinate rings of Y' and X' respectively.

LEMMA 4.1. *The unit group $U(B)$ is generated by k^* , y , and divisors of f . If f splits into linear factors over k , $U(B) \simeq k^* \times \mathbf{Z}^{2g+1}$.*

Proof. Let $h \in U(B)$. Then $\text{div } h = \sum n_i x_i$, $x_i \in S$, $\text{div } h$ denoting the divisor of h . Let σ denote the nontrivial automorphism of $k(X)$ over $k(T)$. Then $\sigma x_i = x_i$, so that $\text{div } \sigma h = \text{div } h$. Thus $h = \lambda \sigma h$, $\lambda \in k^*$. We have, $h^2 = \lambda(h\sigma h) \in U(A)$. Thus $h\sigma h$ is upto a scalar from k^* , a power product of divisors of f . On the other hand, the only non-square in $k(T)$ which becomes a square in $k(X)$ is f . It follows that $h^2 = \mu^2 \prod_i h_i^{2m_i} f$ or $h^2 = \mu^2 \prod_i h_i^{2m_i}$, $m_i \in \mathbf{Z}$, h_i divisors of f in $k[T]$. Thus, $h = \pm \mu(\prod h_i^{m_i})y$ or $h = \pm \mu(\prod h_i^{m_i})$. Further, if $f = \prod_{1 \leq i \leq 2g+1} (T - \alpha_i)$, $\alpha_i \in k^*$, the homomorphism $k^* \times \mathbf{Z}^{2g+1} \rightarrow U(B)$, defined by

$$(\lambda, (n_i)) \rightarrow \lambda(T - \alpha_1)^{n_1} \cdots (T - \alpha_{2g})^{n_{2g}} y^{n_{2g+1}}$$

is surjective, by the above remarks. Suppose

$$\lambda(T - \alpha_1)^{n_1} \cdots (T - \alpha_{2g})^{n_{2g}} \cdot y^{n_{2g+1}} = 1$$

is a relation. Then the divisor

$$\sum_{1 \leq i \leq 2g} 2n_i x_i + n_{2g+1} \left(\sum_{1 \leq i \leq 2g+1} x_i \right) - \left(\sum 2n_i + n_{2g+1}(2g+1) \right) x_\infty = 0,$$

where $x_i \in S$ lie over $T - \alpha_i$ and x_∞ lies over ∞ . This implies that $n_i = 0$, $1 \leq i \leq 2g+1$ and $\lambda = 1$. Thus we have an isomorphism $k^* \times \mathbf{Z}^{2g+1} \simeq U(B)$.

LEMMA 4.2. *Suppose f splits into linear factors over k . Then the map $\text{Pic } X' \rightarrow \text{Pic } X'_k$ is injective.*

Proof. Since the divisor classes of degree zero supported on the ramification locus S are precisely the elements of ${}_2\text{Pic } X$, we have the following commutative diagram with exact rows

$$\begin{array}{ccccccc} 0 & \longrightarrow & {}_2\text{Pic } X & \longrightarrow & \text{Pic}^0 X & \longrightarrow & \text{Pic } X' \longrightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & {}_2\text{Pic } X_k & \longrightarrow & \text{Pic}^0 X_k & \longrightarrow & \text{Pic } X'_k \longrightarrow 0. \end{array}$$

Here $\text{Pic}^0 X$ is the group of divisor classes of degree zero. The first two vertical maps are natural injections. Since f splits into a product of linear factors, ${}_2\text{Pic } X = {}_2\text{Pic } X_{\bar{k}}$. Since $\text{Pic}^0 X \hookrightarrow \text{Pic}^0 X_{\bar{k}}$ is an injection, it follows that $\text{Pic } X' \rightarrow \text{Pic } X'_{\bar{k}}$ is injective.

LEMMA 4.3. *The group ${}_2\text{Pic } X' \xrightarrow{\sim} (\mathbb{Z}/2)^l$ where $l \leq 2g$ and $l = 2g$ if and only if ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$.*

Proof. We have an exact sequence

$$0 \rightarrow {}_2\text{Pic } X \rightarrow {}_4\text{Pic } X \rightarrow {}_2\text{Pic } X' \rightarrow 0.$$

Let ${}_2\text{Pic } X \xrightarrow{\sim} (\mathbb{Z}/2)^l$, $l \leq 2g$. Let m elements in ${}_2\text{Pic } X$ admit a square root over k . Then $|{}_4\text{Pic } X| = m \cdot 2^l$, with $m \leq 2^l \leq 2^{2g}$. Therefore $|{}_2\text{Pic } X'| = m \leq 2^{2g}$ and equality holds if and only if $m = 2^l = 2^{2g}$; i.e., if and only if ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$.

PROPOSITION 4.4. *Let Disc denote the discriminant group of a scheme. Let f split into linear factors over k . Then the composite map $\text{Disc } X' \xrightarrow{N} \text{Disc } Y' \rightarrow \text{Disc } Y'/\text{Disc } k$ is surjective if and only if ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$, N denoting the norm map.*

Proof. Since X'/Y' is étale quadratic, we have an exact sequence in étale cohomology groups with μ_2 coefficients ([12], p. 92),

$$0 \longrightarrow H^0(Y') \xrightarrow{\cup_{X_f}} H^1(Y') \xrightarrow{i} H^1(X') \xrightarrow{tr} H^1(Y')$$

Here, $H^i(-)$ denotes $H^i_{\text{ét}}(-, \mu_2)$. The group $H^1(-)$ is simply the discriminant group so that we have an exact sequence

$$1 \longrightarrow \text{Disc } Y'/\langle f \rangle \longrightarrow \text{Disc } X' \xrightarrow{N} \text{Disc } Y'.$$

Since the only square class in $k(T)$ which becomes trivial in $k(X)$ is $\langle f \rangle$, this sequence yields the following exact sequence

$$1 \rightarrow \text{Disc } Y'/\langle f \rangle \text{Disc } k \rightarrow \text{Disc } X'/\text{Disc } k \rightarrow \text{Disc } Y'/\text{Disc } k.$$

We denote $U(B)$ and $U(A)$ by $U(X')$ and $U(Y')$ respectively. By our hypothesis on f , $\text{Disc } Y' \xrightarrow{\sim} U(Y')/2 \xrightarrow{\sim} (\mathbb{Z}/2)^{2g+1} \times \text{Disc } k$ so that $\text{Disc } Y'/\langle f \rangle \text{Disc } k \xrightarrow{\sim} (\mathbb{Z}/2)^{2g}$ and $\text{Disc } Y'/\text{Disc } k \xrightarrow{\sim} (\mathbb{Z}/2)^{2g+1}$. Further, the exact sequence

$$1 \rightarrow U(X')/2 \rightarrow \text{Disc } X' \rightarrow {}_2\text{Pic } X' \rightarrow 0$$

gives, by (4.1) and (4.3) that $\text{Disc } X'/\text{Disc } k \xrightarrow{\sim} (\mathbb{Z}/2)^{2g+1+l}$, where ${}_2(\text{Pic } X') \xrightarrow{\sim} (\mathbb{Z}/2)^l$. Clearly, the map $\text{Disc } X'/\text{Disc } k \rightarrow \text{Disc } Y'/\text{Disc } k$ is surjective if and only if $l = 2g$; i.e., if and only if ${}_4\text{Pic } X = {}_4\text{Pic } X_K$.

5. Ramified and unramified parts of $\text{coker } \partial^0$

Let $(\bigoplus_{x \in S} W(k(x)))^0$ denote the subgroup of $(\bigoplus_{x \in X} W(k(x)))^0$ with non-zero entries only at $x \in S$. Let V_r be the subgroup of $\text{coker } \partial^0$, defined by

$$\begin{aligned} V_r &= \left(\bigoplus_{x \in S} W(k(x)) \right)^0 / \left(\partial W(k(X)) \cap \left(\bigoplus_{x \in S} W(k(x)) \right)^0 \right) \\ &= \left(\bigoplus_{x \in S} W(k(x)) \right)^0 / \left(\partial W(X') \right) \end{aligned}$$

We define $V_{nr} = \text{coker } \partial^0 / V_r$. If $p : (\bigoplus_{x \in X} W(k(x)))^0 \rightarrow \bigoplus_{x \in X'} W(k(x))$ denotes the restriction of the projection, p is surjective, since $S = X \setminus X'$ contains a rational point. Thus,

$$V_{nr} \xrightarrow{\sim} \bigoplus_{x \in X'} W(k(x)) / (p \circ \partial) W(k(X)).$$

LEMMA 5.1. *The map $\alpha : \text{coker } \partial^0 \rightarrow W(A)/\langle 1, - \rangle W(k)$ maps V_r onto $\langle 1, -f \rangle W(A)/\langle 1, -f \rangle W(k)$.*

Proof. Let $\theta \in (\bigoplus_{x \in S} W(k(x)))^0$. Let $q \in W(k(T))$ be such that $\partial(q) = \text{tr } \theta$.

Since $\partial_y((q) = \text{tr}(\theta_y) = 0$ for $y \notin \pi(S)$, $q \in W(A)$ and $\alpha(\bar{\theta}) = \langle 1, -f \rangle q \in \langle 1, -f \rangle W(A)/\langle 1, -f \rangle W(k)$.

We now show that $\alpha(V_r) = \langle 1, -f \rangle W(A)/\langle 1, -f \rangle W(k)$. Let $q \in W(A)$. Let $\mu = (\mu_x) \in (\bigoplus_{x \in X} W(k(X)))^0$ be defined by $\mu_x = 0$ for $x \in X'$, $\mu_x = \partial_y(q)$, for $x \in S$, $\pi(x) = y$. Then $\mu \in (\bigoplus_{x \in S} W(k(x)))^0$ and $\alpha(\bar{\mu}) = \langle 1, -f \rangle q$ in $\langle 1, -f \rangle W(A)/\langle 1, -f \rangle W(k)$. We thus have an exact sequence

$$0 \rightarrow V_{nr} \xrightarrow{\alpha} W(A)/\langle 1, -f \rangle W(A) \xrightarrow{\beta} W(X) \rightarrow 0.$$

PROPOSITION 5.2. *The group V_{nr} is 2-torsion.*

Proof. Let $\theta \in \bigoplus_{x \in X'} W(k(x))$. Since $\pi(S)$ has a rational point of ramification, there exists $q \in W(k(T))$ such that $\partial(q) = \text{tr } \theta$. We have, $\langle 1, -f \rangle (\langle 1, f \rangle q) = 0$ so

that there exists $q_1 \in W(k(X))$ with $s(q_1) = (\langle 1, f \rangle q)$. Since for $x \in X'$ with $\pi(x) = y$,

$$\partial_y(\langle 1, -f \rangle q) = \langle 1, -\bar{f} \rangle \partial_y(q) = \langle 1, -\bar{f} \rangle \text{tr}(\theta_x) = 0,$$

we have $\text{tr}(2\theta) = \partial(\langle 1, f \rangle q) = \partial(s(q_1)) = \text{tr}(\partial(q_1))$. Thus, by (3.3), there exists $q_2 \in W(k(T))$ such that $\partial_x \circ i(q_2) = 2\theta_x - \partial_x q_1$ for $x \in X'$ and $\partial_x \circ i(q_2) = 0$ for $x \in S$. Thus $2\theta - \partial(q_1 - i(q_2)) \in (\oplus_{x \in S} W(k(x)))^0$ and its image under the projection map p is zero. Thus the class of 2θ in V_{nr} is zero.

Therefore $\text{coker } \partial^0$ is an extension of V_r by the 2-torsion group V_{nr} . We now show that under the rationality assumption ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$, $V_r = 0$. We observe that ${}_4\text{Pic } X_{\bar{k}}$ being a finite group, there exists a finite separable extension l/k such that $V_r = 0$ for X_l .

PROPOSITION 5.3. *Suppose ${}_2\text{Pic } X = {}_2\text{Pic } X_{\bar{k}}$. Then the group $V_r = 0$ if and only if ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$.*

Proof. We show that the map $\partial : W(X') \rightarrow (\oplus_{x \in S} W(k(x)))^0$ is surjective if and only if ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$. In view of the commutative diagram

$$\begin{array}{ccccccc} W(X') & \longrightarrow & \left(\bigoplus_{x \in S} W(k(x)) \right)^0 \\ & \searrow s \downarrow & \parallel \\ 0 \longrightarrow W(k) & \longrightarrow & W(Y') \longrightarrow \left(\bigoplus_{y \in \pi(S)} W(k(y)) \right)^0 \longrightarrow 0 \end{array} \quad (**)$$

with (**) exact, we need to show that $s : W(X') \rightarrow W(Y')/W(k)$ is surjective if and only if ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$. By our assumption ${}_2\text{Pic } X = {}_2\text{Pic } X_{\bar{k}}$, f splits as a product $\prod_{1 \leq i \leq 2g+1} (T - \alpha_i)$ over k . Suppose ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$. The exact sequence (**) with each $W(k(y)) \simeq W(k)$ for $y \in \pi(S)$ implies that $W(Y')$ is generated by $\text{Disc } k$ and $\langle \lambda(T - \alpha_i) \rangle$, $\lambda \in k^*$, $1 \leq i \leq 2g+1$. It is therefore enough to show that given $\langle \lambda(T - \alpha_i) \rangle$, $\lambda \in k^*$, there exists $\mu \in k^*$ such that $\langle \mu, \lambda(T - \alpha_i) \rangle \in s(W(X'))$. By (4.4), there exists $\tilde{z} \in \text{Disc } X'$ such that $N(\tilde{z}) = \langle \nu(T - \alpha_i) \rangle$ for some $\nu \in k^*$. We have, $s(\tilde{z}) = z_1 \langle 1, -\nu(T - \alpha_i) \rangle$ for some $z_1 \in k(T)$. Thus,

$$s(-z_1^{-1} \nu^{-1} \cdot \lambda \cdot \tilde{z}) = \langle -\nu^{-1} \lambda, \lambda(T - \alpha_i) \rangle.$$

Conversely, suppose $W(X') \rightarrow W(Y')/W(k)$ is surjective. Then the map restricted to the ideal $I(X')$ of even dimensional forms surjects onto $I(Y')/I(k)$. In view of the commutative diagram

$$\begin{array}{ccc} I(X') & \xrightarrow{s} & I(Y')/I(k) \\ \downarrow & & \downarrow \\ \text{Disc } X' & \xrightarrow{N} & \text{Disc } Y'/\text{Disc } k \end{array}$$

with the vertical maps surjective, it follows that $N : \text{Disc } X' \rightarrow \text{Disc } Y' / \text{Disc } k$ is surjective. This implies, by (4.4) that ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$.

In the next section, under certain assumptions on k and X , we describe the unramified part V_{nr} of coker ∂^0 cohomologically.

6. The unramified part of coker ∂^0

Let Y be any scheme over k . Let the properties $PQ(1)$, $PQ(2)$ for Y be the following.

PQ(1): For every geometric point $y \in Y$, the invariant theorem for quadratic spaces, $I^n(k(y))/I^{n+1}(k(y)) \xrightarrow{\sim} H_{et}^n(k(y), \mu_2)$ holds for all $n \geq 0$.

PQ(2): Y satisfies $PQ(1)$ and the maps $e_n : I_n(Y) \rightarrow \Gamma(Y, \mathcal{H}^n)$ defined in ([15, §1]) are surjective for $n \geq 0$.

Here, \mathcal{H}^n denotes the Zariski sheaf associated to the presheaf $U \rightarrow H_{et}^n(U, \mu_2)$. The class of schemes which satisfy $PQ(2)$ include all smooth quasi projective curves over local fields, in view of [2] and [16]. Conjecturally, all smooth projective curves over any field satisfy $PQ(2)$.

We follow the same notations as in §4 and denote by $\pi : X \rightarrow \mathbf{P}^1$ a double cover, X being a smooth hyperelliptic curve with a rational point of ramification. Under the assumptions that $X' = X \setminus S$, $Y' = Y \setminus \pi(S)$ satisfy $PQ(2)$, we shall describe V_{nr} as a certain cohomology group.

LEMMA 6.1. *Let $Y \subseteq \mathbf{P}^1$ be any subscheme. Then Y satisfies $PQ(2)$ if Y satisfies $PQ(1)$.*

Proof. We have the following commutative diagram (cf. [5], [10])

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 & & I_n(Y) & \xrightarrow{e_n} & \Gamma(Y, \mathcal{H}^n) & & \\
 & & \downarrow & & \downarrow & & \\
 0 \longrightarrow & I^{n+1}(k(T)) & \longrightarrow & I^n(k(T)) & \longrightarrow & H^n(k(T)) & \longrightarrow 0 \\
 & \partial \downarrow & & \partial \downarrow & & \partial \downarrow & \\
 0 \longrightarrow & \left(\bigoplus_{y \in Y} I^n(k(y)) \right)^0 & \longrightarrow & \left(\bigoplus_{y \in Y} I^{n-1}(k(y)) \right)^0 & \longrightarrow & \left(\bigoplus_{y \in Y} H^{n-1}(k(y)) \right)^0 &
 \end{array}$$

Here $(\bigoplus_{y \in Y} I^m(k(y)))^0$ (resp. $(\bigoplus_{y \in Y} H^m(k(y)))^0$) denotes the subgroup consisting of trace zero elements. The two vertical columns are exact, by ([10], p. 277) and [5]. By the assumption on Y , the two rows are exact. The surjectivity of $e_n: I_n(Y) \rightarrow \Gamma(Y, \mathcal{H}^n)$ follows from the surjectivity of the residue map $\partial: I^{n+1}(k(T)) \rightarrow (\bigoplus_{y \in Y} I^n(k(y)))^0$ [13], Theorem 5.3).

LEMMA 6.2. Suppose \mathbf{P}^1 and X satisfy $PQ(1)$. Then the sequence

$$I_n(A) \xrightarrow{i} I_n(B) \xrightarrow{s} I_n(A)$$

is exact for $n \geq 0$.

Proof. Since B/A is unramified, by (2.1), (2.2) and (2.3), we have the following commutative diagram:

$$\begin{array}{ccccccc} & & 0 & & 0 & & 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ & & I_n(A) & \longrightarrow & I_n(B) & \longrightarrow & I_n(A) \\ & & \downarrow & & \downarrow & & \downarrow \\ I^{n-1}(k(T)) & \xrightarrow{\langle 1, -f \rangle} & I^n(k(T)) & \longrightarrow & I^n(k(X)) & \longrightarrow & I^n(k(T)) \\ \partial \downarrow & & \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\ \bigoplus_{y \in Y'} I^{n-2}(k(y)) & \xrightarrow{\langle 1, -\bar{f} \rangle} & \bigoplus_{y \in Y'} I^{n-1}(k(y)) & \longrightarrow & \bigoplus_{x \in X'} I^{n-1}(k(x)) & \longrightarrow & \bigoplus_{y \in Y'} I^{n-1}(k(y)). \end{array}$$

The vertical columns are exact by ([10], p. 277). Exactness of the rows is a consequence of the assumption $PQ(1)$ for X and \mathbf{P}^1 [3]. Exactness of the top row follows from the surjectivity of $\partial: I^{n-1}(k(T)) \rightarrow \bigoplus_{y \in Y'} I^{n-2}(k(y))$, Y' being contained in A^1 .

LEMMA 6.3. Suppose X' , and Y' satisfy $PQ(2)$. Then

$$(\langle 1, -f \rangle W(A)) \cap I_n(A) \simeq \langle 1, -f \rangle I_{n-1}(A).$$

Proof. We assume, by induction, that

$$(\langle 1, -f \rangle W(A)) \cap I_m(A) = \langle 1, -f \rangle I_{m-1}(A)$$

for $m \leq n-1$. Let $q \in (\langle 1, -f \rangle W(A)) \cap I_n(A)$. By induction, we may write $q = \langle 1, -f \rangle q_1$, $q_1 \in I_{n-2}(A)$. Since X' , Y' satisfy $PQ(1)$, and B/A is étale quadratic,

we have the following commutative diagram

$$\begin{array}{ccccc} I_{n-2}(B) & \xrightarrow{s} & I_{n-2}(A) & \xrightarrow{\langle 1, -f \rangle} & I_{n-1}(A) \\ & & \downarrow e_{n-2} & & \downarrow e_{n-1} \\ H^{n-2}(B) & \xrightarrow{tr} & H^{n-2}(A) & \xrightarrow{\cup \chi_f} & H^{n-1}(A) \end{array}$$

with the bottom row exact. Since

$$\langle 1, -f \rangle q_1 = q \in I_n(A), e_{n-1}(q) = 0; \quad \text{i.e., } \chi_f \cup e_{n-2}(q_1) = 0.$$

Therefore, there exists $\theta \in H^{n-2}(B)$ such that $tr\theta = e_{n-2}(q_1)$. Let $\tilde{\theta} \in \Gamma(B, \mathcal{H}^{n-2})$ be the image of θ in $H^{n-2}(k(X))$. By the assumption that X' satisfies $PQ(2)$, there exists $q_2 \in I_{n-2}(B)$ such that $e_{n-2}(q_2) = \tilde{\theta}$. The diagram

$$\begin{array}{ccc} I_{n-2}(B) & \xrightarrow{s} & I_{n-2}(A) \\ \downarrow e_{n-2} & & \downarrow e_{n-2} \\ \Gamma(B, \mathcal{H}^{n-2}) & \xrightarrow{tr} & \Gamma(A, \mathcal{H}^{n-2}) = H^{n-2}(A) \end{array}$$

can be verified to be commutative, so that $e_{n-2}(q_1 - sq_2) = 0$. Thus $q_1 - sq_2 \in I_{n-1}(A)$ and $\langle 1, -f \rangle(q_1 - sq_2) = \langle 1, -f \rangle q_1 \in \langle 1, -f \rangle I_{n-1}(A)$. This proves the lemma.

We now assume that X' and Y' satisfy $PQ(2)$. The group $V_{nr} = \ker (W(A) / \langle 1, -f \rangle W(A) \xrightarrow{i} W(X))$ has a filtration induced by the filtration $\{I_m(A)\}$ on $W(A)$. Since the map $W(X) \rightarrow W(B)$ is injective and since i preserves filtration, by (6.3), we have,

$$\begin{aligned} (V_{nr})_m &= \ker (I_m(A) / (\langle 1, -f \rangle W(A) \cap I_m(A)) \xrightarrow{i} I_m(B)) \\ &= \ker (I_m(A) / \langle 1, -f \rangle I_{m-1}(A) \xrightarrow{i} I_m(B)). \end{aligned}$$

We now define a map $\eta_m : (V_{nr})_m \rightarrow NH^m(B) = \ker (H^m(B) \rightarrow \Gamma(B, \mathcal{H}^m))$ as follows. Consider the following commutative diagram:

$$\begin{array}{ccc} I_m(A) & \xrightarrow{i} & I_m(B) \\ & \searrow 0 \nearrow NH^m(B) & \\ & & H^m(B) \\ e_m \downarrow & \nearrow & \downarrow e_m \\ H^m(A) & \xrightarrow{\quad} & \Gamma(B, \mathcal{H}^m) \rightarrow 0 \end{array}$$

Let $x \in I_m(A)$ be such that $i(x) = 0$. Then the element $i(e_m(x)) \in H^m(B)$ maps to zero in $\Gamma(B, \mathcal{H}^m)$, by the commutativity of the above diagram. Hence $i(e_m(x)) \in NH^m(B)$. We define $\eta_m(\bar{x}) = i \circ e_m(x)$. To show that η_m is well-defined, we need to check that for $x \in \langle 1, -f \rangle I_{m-1}(A)$, $\eta_m(\bar{x}) = 0$. Let $x = \langle 1, -f \rangle x'$, $x' \in I_{m-1}(A)$. We have, $i(e_m(x)) = i(\chi_f \cup e_{m-1}(x')) = \chi_{i(f)} \cup i \circ e_{m-1}(x') = 0$ since f is a square in B . Thus we have a well-defined homomorphism

$$\eta_m : (V_{nr})_m \rightarrow NH^m(B).$$

LEMMA 6.4. $\text{Ker } \eta_m = (V_{nr})_{m+1}$.

Proof. Let $\eta_m(\bar{x}) = 0$ with $x \in I_m(A)$. Then $ie_m(x) = 0$ and the exactness of the sequence

$$H^{m-1}(A) \xrightarrow{\cup \chi_f} H^m(A) \xrightarrow{i} H_m(B) \xrightarrow{ir} H^m(B) \quad (***)$$

implies that there exists $y \in H^{m-1}(A)$ such that $\chi_f \cup y = e_m(x)$. By (6.1), there exists $z \in I_{m-1}(A)$ such that $e_{m-1}(z) = y$. We have, $e_m(x - \langle 1, -f \rangle \cdot z) = 0$ so that $x - \langle 1, -f \rangle \cdot z \in I_{m+1}(A)$ and its class in $(V_{nr})_{m+1}$ is simply the class of x .

We thus have a filtration $\{(V_{nr})_m\}$ on V_{nr} with successive quotients $(V_{nr})_m / (V_{nr})_{m+1}$ injecting into $NH^m(B)$.

THEOREM 6.5. Under the assumption that X' and Y' have $PQ(2)$, $V_{nr} \xrightarrow{\sim} \bigoplus_{m \geq 2} NH^m(B)$.

Proof. Since by (5.2), V_{nr} is a 2-torsion group, it is enough to show that η_m maps $(V_{nr})_m$ onto $NH^m(B)$. Let $x \in NH^m(B)$. Since $NH^n(A) = 0 \forall n$, $tr x = 0$, and the exact sequence (***) implies that there exists $y \in H^m(A)$ with $i(y) = x$. By (6.1), there exists $z \in I_m(A)$ with $e_m(z) = y$. Then $e_m \circ i(z) = \text{class of } x \text{ in } \Gamma(B, \mathcal{H}^m)$ which is zero since $x \in NH^m(B)$. Thus $i(z) \in I_{m+1}(B)$ and $s \circ i(z) = 0$. By (6.2), there exists $z' \in I_{m+1}(A)$ with $i(z') = i(z)$. Replacing z by $z - z'$ which again maps to y under e_m , we have $i(z) = 0$; i.e., $\bar{z} \in (V_{nr})_m$ with $\eta_m(\bar{z}) = x$.

7. An example

THEOREM 7.1. Let X be a smooth projective hyperelliptic curve defined over a local field k with residue field characteristic $\neq 2$. Suppose X has a rational point of ramification, X has good reduction and ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$. Then

$$\text{coker } \partial \xrightarrow{\sim} W(k) \oplus (\mathbb{Z}/2)^{4g},$$

g being the genus of X .

In view of results of [2], any curve over a local field satisfies $PQ(1)$. It is shown in [16] that any such curve also satisfies $PQ(2)$. Therefore by our assumption ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$, we have, $\text{coker } \partial \simeq W(k) \oplus (\oplus_{m \geq 2} NH^m(X'))$. Let $G = G(\bar{k}/k)$, \bar{k} denoting the algebraic closure of k . Then $cd_2 k \leq 2$ [18] and $cd_2 X'_{\bar{k}} \leq 1$, $X'_{\bar{k}}$ being affine. The spectral sequence ([12], p. 105)

$$H^i(G, H^j(X'_{\bar{k}})) \Rightarrow H^n(X')$$

yields $H^n(X') = 0$ for $n \geq 4$. Thus $\text{coker } \partial^0 \simeq NH^2(X') \oplus NH^3(X')$. We shall now compute these groups.

LEMMA 7.2. *Let X be any smooth projective curve of genus g (not necessarily hyperelliptic) over a local field k with residue field characteristic $\neq 2$ and such that $X(k) \neq \emptyset$ and ${}_2\text{Pic } X = {}_2\text{Pic } X_{\bar{k}}$. Then $H^3(X) \simeq (\mathbb{Z}/2)^{2g+2}$ and $\Gamma(X, \mathcal{H}^3) = 0$.*

Proof. The only two non-zero terms in the above spectral sequence contributing to $H^3(X)$ are $H^1(G, H^2(X_{\bar{k}}))$ and $H^2(G, H^1(X_{\bar{k}}))$. The only possible non-zero differential $H^0(G, H^2(X_{\bar{k}})) \rightarrow H^2(G, H^1(X_{\bar{k}}))$ is zero, $X(k)$ being non-empty, since $H^2(X) \rightarrow H^0(G, H^2(X_{\bar{k}}))$ is surjective. Therefore

$$\begin{aligned} H^3(X) &\simeq H^2(G, H^1(X_{\bar{k}})) \oplus H^1(G, H^2(X_{\bar{k}})) \\ &\simeq (\mathbb{Z}/2)^{2g} \oplus (\mathbb{Z}/2)^2. \end{aligned}$$

In fact the action of G on $H^1(X_{\bar{k}}) \simeq {}_2\text{Pic } X_{\bar{k}} \simeq (\mathbb{Z}/2)^{2g}$ is trivial by our assumption and $H^2(X_{\bar{k}}) \simeq \text{Pic } X_{\bar{k}}/2 \simeq \mathbb{Z}/2$ with trivial action again. Further, k being a local field, $H^2(G, \mathbb{Z}/2) \simeq {}_2\text{Br}(k) \simeq \mathbb{Z}/2$ and $H^1(G, \mathbb{Z}/2) \simeq k^*/k^{*2} \simeq \mathbb{Z}/2 \times \mathbb{Z}/2$. In view of [4], $NH^3(X) \simeq k^*/k^{*2} \times J(k)/2J(k)$. Since k is a local field, by [11], $J(k)$ contains a subgroup \mathcal{M} isomorphic to copies of the valuation ring such that $J(k)/\mathcal{M}$ is finite. The 2-primary part of $J(k)/\mathcal{M}$ is isomorphic to $\prod_{1 \leq j \leq l} (\mathbb{Z}/2^j)$, where $l = \dim_{\mathbb{Z}/2}({}_2\text{Pic } X) = 2g$ by our assumption. Therefore $J(k)/2J(k) \simeq (\mathbb{Z}/2)^{2g}$, so that $NH^2(X) \simeq (\mathbb{Z}/2)^{2g+2}$. Thus $NH^3(X) = H^3(X)$ and $\Gamma(X, \mathcal{H}^3) = 0$.

COROLLARY 7.3. *Let X be a smooth projective curve over a local field k with residue field characteristic $\neq 2$. Suppose X has good reduction and ${}_2\text{Pic } X = {}_2\text{Pic } X_{\bar{k}}$. Then the classical invariants uniquely determine the class of a quadratic space in $W(X)$.*

Proof. In view of ([15], §1), we have injections $rk : W(X)/I(X) \hookrightarrow \mathbb{Z}/2$, $\text{disc} : I(X)/I_2(X) \hookrightarrow H^1(X)$, $c : I_2(X)/I_3(X) \hookrightarrow {}_2\text{Br}(X) = \Gamma(X, \mathcal{H}^2)$, where rk , disc

and c stand for rank, discriminant and Hasse–Witt invariant maps. Since $I_4(X) \hookrightarrow I^4(k(X)) = 0$ [2] and $I_3(X)$ injects into $\Gamma(X, \mathcal{H}^3) = 0$ by (7.2), we have, rk , $disc$ and c uniquely determine an element in $W(X)$.

LEMMA 7.4. *Let X be a hyperelliptic curve. Then $NH^2(X') \simeq (\mathbf{Z}/2)^{2g}$, under the assumptions of (7.1) on X .*

Proof. We have $NH^2(X') \simeq \text{Pic } X'/2$. The exact sequence

$$0 \rightarrow {}_2\text{Pic } X \rightarrow \text{Pic}^0 X \rightarrow \text{Pic } X' \rightarrow 0$$

yields the following long exact sequence

$$0 \rightarrow {}_2\text{Pic } X \rightarrow {}_2\text{Pic}^0 X \rightarrow {}_2\text{Pic } X' \rightarrow {}_2\text{Pic } X/2 \rightarrow \text{Pic}^0 X/2 \rightarrow \text{Pic } X'/2 \rightarrow 0.$$

We have ${}_2\text{Pic } X = {}_2\text{Pic}^0 X$, ${}_2\text{Pic } X' \simeq (\mathbf{Z}/2)^{2g}$ (4.3), ${}_2\text{Pic } X/2 \simeq (\mathbf{Z}/2)^{2g}$ and $\text{Pic}^0 X/2 = J(k)/2J(k) \simeq (\mathbf{Z}/2)^{2g}$, in view of (7.2). We therefore have $\text{Pic } X'/2 \simeq (\mathbf{Z}/2)^{2g}$.

LEMMA 7.5. *Let X be a hyperelliptic curve. Then $NH^3(X') \simeq (\mathbf{Z}/2)^{2g}$, under the assumptions of (7.1) on X .*

Proof. We have an exact sequence

$$0 \rightarrow U(X'_k)/2 \rightarrow H^1(X'_k) \rightarrow {}_2\text{Pic } X'_k \rightarrow 0.$$

By (4.1) and (4.3), $U(X'_k)/2 \simeq (\mathbf{Z}/2)^{2g+1}$ and ${}_2\text{Pic } X'_k \simeq (\mathbf{Z}/2)^{2g}$. Therefore $H^1(X'_k) \simeq (\mathbf{Z}/2)^{4g+1}$. Further, since $U(X'_k)/2$ is generated by $\{y, T - \alpha_i\}$, $1 \leq i \leq 2g$, which are defined over k , and ${}_2\text{Pic } X'_k$ is also defined over k under the assumption ${}_4\text{Pic } X' = {}_4\text{Pic } X'_k$, the action of G on $H^1(X'_k)$ is trivial. The only non-zero terms in the spectral sequence

$$H^i(G, H^j(X'_k)) \Rightarrow H^n(X')$$

contributing to $H^3(X')$ is $H^2(G, H^1(X'_k))$ with all the differentials vanishing, as before. We therefore have

$$H^3(X') \simeq H^2(G, H^1(X'_k)) \simeq (\mathbf{Z}/2)^{4g+1}.$$

We shall now compute $\Gamma(X', \mathcal{H}^3)$. The sequence

$$H^3(k(X)) \xrightarrow{tr} H^3(k(T)) \xrightarrow{\cup x_f} H^4(k(T))$$

is exact and since $cd_2(k) \leq 2$, $cd_2(k(T)) \leq 3$, $H^4(k(T)) = 0$. Thus

$$tr : H^3(k(X)) \rightarrow H^3(k(T))$$

is surjective. It induces a map

$$tr : \Gamma(X', \mathcal{H}^3) \rightarrow \Gamma(Y', \mathcal{H}^3) \xrightarrow{\sim} H^3(Y').$$

We show that this map is surjective. Let $\lambda \in H^3(Y')$ and $\mu \in H^3(k(X))$ be such that $tr \mu = \lambda$, identifying $H^3(Y')$ with a subgroup of $H^3(k(T))$. In view of the commutative diagram

$$\begin{array}{ccccc} H^3(k(T)) & \xrightarrow{i} & H^3(k(X)) & \xrightarrow{tr} & H^3(k(T)) \\ \partial \downarrow & & \partial \downarrow & & \partial \downarrow \\ \bigoplus_{y \in Y'} H^2(k(y)) & \xrightarrow{i} & \bigoplus_{x \in X'} H^2(k(x)) & \xrightarrow{tr} & \bigoplus_{y \in Y'} H^2(k(y)) \end{array}$$

with exact rows, $tr \circ \partial \mu = \partial \circ tr \mu = \partial(\lambda) = 0$ and hence there exists $v \in \bigoplus_{y \in Y'} H^2(k(y))$ with $i(v) = \partial(\mu)$. Since $Y' \subset \mathbb{A}^1$, $\partial : H^3(k(T)) \rightarrow \bigoplus_{y \in Y'} H^2(k(y))$ is surjective and hence there exists $\tilde{v} \in H^3(k(T))$ with $\partial(\tilde{v}) = v$. We have $\partial(\mu - i\tilde{v}) = 0$ so that $(\mu - i\tilde{v}) \in \Gamma(X', \mathcal{H}^3)$ and maps to $\lambda \in \Gamma(Y', \mathcal{H}^3) = H^3(A)$. We thus have a surjection $tr : \Gamma(X', \mathcal{H}^3) \rightarrow \Gamma(Y', \mathcal{H}^3)$. We now compute its kernel. Since $H^3(k) = 0$, the map $\partial : H^3(A) \rightarrow (\bigoplus_{y \in \pi(S)} H^2(k(y)))^0$ is an isomorphism. Since the square

$$\begin{array}{ccc} \Gamma(X', \mathcal{H}^3) & \xrightarrow{\partial} & \left(\bigoplus_{x \in S} H^2(k(x)) \right)^0 \\ tr \downarrow & & \parallel \\ H^3(A) & \xrightarrow{\sim} & \left(\bigoplus_{y \in \pi(S)} H^2(k(y)) \right)^0 \end{array}$$

is commutative, we have, $\ker tr = \ker \partial = \Gamma(X, \mathcal{H}^3) = 0$, by [5] and (7.2). Thus, $\Gamma(X', \mathcal{H}^3) \xrightarrow{\sim} H^3(A) \xrightarrow{\sim} (\mathbb{Z}/2)^{2g+1}$. Therefore $NH^3(X') \xrightarrow{\sim} (\mathbb{Z}/2)^{2g}$.

This completes the proof of Theorem 7.1. Finally, we use the exact sequence (§3) to compute the defining relations for $W(X)$ as a quotient of $\bigoplus_{x \in S} W(k(x))$. More precisely, we have the following

THEOREM 7.6. *Under the same hypothesis as in (7.1),*

$$W(X) \simeq (\mathbf{Z}/2)^{4g} \oplus W(k).$$

Proof. In view of (3.4) and (7.1), we have an exact sequence

$$0 \rightarrow (\mathbf{Z}/2)^{4g} \rightarrow W(A)/(\langle 1, -f \rangle W(k)) \rightarrow W(X) \rightarrow 0 \quad (*)$$

The residue map $\partial : W(A) \rightarrow \bigoplus_{1 \leq i \leq 2g+1} W(k)$ is surjective, with kernel $W(k)$. We have, in $W(k(T))$, $(W(k) \cap \langle 1, -f \rangle \cdot W(k)) = 0$. In fact, for $q \in W(k) \cap \langle 1, -f \rangle \cdot W(k)$, q extends to zero in $W(X)$. Since $X(k)$ is non-empty, specialising at a rational point yields $q = 0$ in $W(k)$. We thus have an exact sequence

$$0 \rightarrow W(k) \rightarrow W(A)/(\langle 1, -f \rangle \cdot W(k)) \rightarrow \bigoplus_{2g+1} W(k)/\partial(\langle 1, -f \rangle W(k)) \rightarrow 0.$$

The image of the map $\eta : W(k) \rightarrow \bigoplus_{2g+1} W(k)$ defined by

$$\eta(q) = (-f'(\alpha_1)q, -f'(\alpha_2)q, \dots, -f'(\alpha_{2g+1})q)$$

is precisely $\partial(\langle 1, -f \rangle \cdot W(k))$. The map η is injective, since for $q \in W(k)$, $\eta(q) = 0$ implies that $\partial(\langle 1, -f \rangle q) = 0$; i.e., $\langle 1, -f \rangle q \in W(k) \cap \langle 1, -f \rangle W(k) = 0$ and $q \simeq fq$. Since degree f is odd, $q = 0$. Clearly η is a split injection, a section t being given by $t(q_1, q_2, \dots, q_{2g+1}) = -f'(\alpha_1) \cdot q_1$. We thus have an isomorphism

$$\tilde{\eta} : W(A)/(\langle 1, -f \rangle W(k)) \rightarrow W(k) \oplus \left(\bigoplus_{2g} W(k) \right)$$

given by $\tilde{\eta}(\bar{q}) = (\tilde{q}, (\partial_{x_i} q))$, $2 \leq i \leq 2g+1$, $x_i \in S$, \tilde{q} denoting specialisation at ∞ . If $\bar{q} \in W(A)/\langle 1, -f \rangle W(k)$, maps to zero in $W(X)$, specialising at ∞ , we see that $\tilde{q} = 0$, so that in the sequence $(*)$, $(\mathbf{Z}/2)^{4g}$ injects into the factor $\bigoplus_{2g} W(k) \simeq \bigoplus_{4g} W(F)$ where F denotes the residue field of k . If -1 is a square in F , $W(F) \simeq (\mathbf{Z}/2)^2$ and if -1 is not a square in F , $W(F) \simeq \mathbf{Z}/4$. Therefore,

$$\begin{aligned} W(X) &\simeq W(k) \oplus W(F)^{4g}/(\mathbf{Z}/2)^{4g} \\ &\simeq W(k) \oplus (\mathbf{Z}/2)^{4g}. \end{aligned}$$

The above theorem leads one to the following natural questions.

QUESTION 1. *For a smooth hyperelliptic curve X over an arbitrary ground field k , (with ${}_4\text{Pic } X = {}_4\text{Pic } X_{\bar{k}}$), is $W(X)$ isomorphic to $W(k) \oplus (\mathbf{Z}/2)^{4g}$?*

A positive answer to this question will also provide evidence to an affirmative answer to the following more general

QUESTION. (Scharlau) *Let X be a smooth projective curve over a field k . If $W(k)$ is finitely generated, is $W(X)$ finitely generated?*

QUESTION 2. *For a smooth projective curve X over k with $X(k) \neq \emptyset$, is $\text{coker } \partial \xrightarrow{\sim} W(X)$?*

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