Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 65 (1990)

Artikel: Geometric actions of surface groups on ...-trees.

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DOI: https://doi.org/10.5169/seals-49740

Comment. Math. Helvetici 65 (1990) 519–533
0010-2571/90/040519-15$1.50 + 0.20/0

Geometric actions of surface groups on Λ-trees

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In this paper we characterize actions of surface groups on Λ-trees which are dual to Λ-measured geodesic laminations. Here Λ is an ordered abelian group—for example, $\mathbf{Z}$, $\mathbf{Q} \oplus \sqrt{2}\mathbf{Q}$, $\mathbf{R}$ with the usual ordering or $\mathbf{Z} \oplus \mathbf{Z}$ with the lexicographical ordering.

Our interest in this problem stems from the work of J. W. Morgan and P. B. Shalen [Mo–Sh1], [Mo]. Let Γ be a finitely generated group which is not virtually abelian and $\{\rho_i\}_i$ a sequence of discrete faithful representations of Γ into the group of isometries of $\mathbf{H}^n$. They show that there is a subsequence $\{\rho_{i_j}\}_j$ either converging to a discrete faithful representation or converging to an action on a Λ-tree. For such an action the stabilizer of each segment is virtually abelian.

The convergence to the Λ-tree is in the following sense. Let l_{ρ_i} be the length function of the representation ρ_i and l the length function of the action on the limit Λ-tree. There is a natural ratio $\cdot/\cdot : \Lambda^{<0} \times \Lambda^{<0} \to [0, +\infty]$. If $\lim_{j\to\infty} l_{\rho_{i_j}}(g) = \lim_{j\to\infty} l_{\rho_{i_j}}(h) = \infty$, for some $g, h \in \Gamma$, then

$$\lim_{j\to\infty} \frac{l_{\rho_{i_j}}(g)}{l_{\rho_{i_j}}(h)} = \frac{-l(g)}{-l(h)}.$$

Their work on the degeneration of hyperbolic structures has inspired other views. G. Brumfiel has explained this in terms of the real spectrum compactification of algebraic varieties [Br1], [Br2], [Br3] and M. Bestvina [Be] and F. Paulin [Pa] have explained it in purely geometric terms.

Let F be a compact surface with or without boundary and having negative Euler characteristic. Any complete hyperbolic structure on F determines a discrete, faithful representation of $\pi_1 F$ into the group of isometries of $\mathbf{H}^2$ which is unique upto conjugation. If one applies the above result to a sequence of such representations, then one obtains an action on a Λ-tree with the following properties:

[1]Supported in part by a SUNY Research Department Grant and by an NSF Postdoctoral Research Fellowship.

(*) the stabilizer of each segment is cyclic; and

(**) for all $g, h \in \pi_1 F$, the axes of g and h in T intersect, whenever g and h are hyperbolic in T, and g and h are hyperbolic in $\mathbf{H}^2$ and the axes in $\mathbf{H}^2$ intersect transversely in one point.

The first property follows from above and the fact that virtually abelian subgroups of $\pi_1 F$ are cyclic. The second property was proved later in [Mo–Ot]. Notice it is independent of the hyperbolic structure on F.

In [Mo–Sh1] they apply this to compactify the Teichmüller space of F and reprove a result of W. P. Thurston [Th] that the boundary is homeomorphic to the space of projective $\mathbf{R}$-measured compact geodesic laminations on F. Their proof is indirect. This left open the problem of understanding the actions on the Λ-trees and finding a more tree theoretic proof of Thurston's result [Sha].

Morgan and J.-P. Otal [Mo–Ot] proved the following. Let $\pi_1 F \times T \to T$ be a minimal action on a Λ-tree, such that each peripheral element of $\pi_1 F$ fixes a point. Then the action is dual to a Λ-measured compact geodesic lamination[2] if and only if it satisfies (*) and (**). Also they asked whether (*) or (**) implies the other?

Here is our main result.

(3.3) THEOREM. *Let F be a complete hyperbolic surface of finite area. A minimal action on a Λ-tree $\pi_1 F \times T \to T$ is dual to a Λ-measured geodesic lamination if and only if it has a length function of a general type and it satisfies* (**).

Under the additional assumption about peripheral elements the lamination would necessarily be compact. Also if an action satisfies (*), then its length function is of a general type. Thus our result implies the result in [Mo–Ot].

Theorem (3.3) answers half of the question of Morgan and Otal. Namely having a length function of a general type and (**) implies (*). The example of §4 shows that (**) alone is insufficient. The other half of the question is answered in [Sk2] where the following is proved. Let $\pi_1 F \times T \to T$ be a minimal action on an $\mathbf{R}$-tree, such that each peripheral element of $\pi_1 F$ fixes a point. The action is dual to an $\mathbf{R}$-measured compact geodesic lamination if and only if it satisfies (*). In particular, under these additional hypotheses (*) implies (**). And there is an example of an action of a closed surface group on a non-Archimedean Λ-tree with a length

[2]Their definitions of *measured lamination* and *dual* differ from ours. The translation is made as follows. Suppose a minimal action on an Λ-tree is dual in their sense to a Λ-measured (*topological*) lamination. Then each path component of the pre-image in $\mathbf{H}^2$ of the lamination determines two distinct endpoints on the circle at infinity. If each component of the lamination is replaced by a geodesic with the same endpoints, then what results is a Λ-measured geodesic lamination which is dual in our sense to the action.

function of a general type and satisfying (*), but not dual to a Λ-measured lamination. This shows that (*) alone is insufficient.

The idea of the proof of Theorem (3.3) is simple. It is obvious that if an action is dual to a Λ-measured geodesic lamination, then the action is of a general type and the action satisfies (**). To prove the converse we study the relationship between the ends to the Λ-tree and the ends of $\mathbf{H}^2$. Observe that if the action were dual to a Λ-measured geodesic lamination, then the lamination would necessarily determine a map from the space of ends of the Λ-tree to the circle at infinity of $\mathbf{H}^2$. Conversely, this map must determine the Λ-measured geodesic lamination. Thus we use the two hypotheses to find this map.

§1 contains the definitions surrounding trees and laminations. And §2 reviews the definition of ends of a Λ-tree. Then we define a topology on the tree union its ends. We show that the ends act like the circle at infinity of $\mathbf{H}^2$. The main proposition is a simultaneous version of a well known result for actions on $\mathbf{H}^2$ and the analogous result for Λ-trees. In §3 we prove the main theorem.

In §4 there are two examples demonstrating the importance of the hypotheses in Theorem (3.3). In §5 we prove the applications. We show from Theorem (3.3) that actions on Λ-trees which are limits of hyperbolic structures are dual to Λ-measured geodesic laminations. To do this we show that such actions have a length function of a general type and then we reveiw the argument from [Mo–Ot] that the action must satisfy (**). And for completeness we prove directly that the boundary of the compactification of Teichmüller space is homeomorphic to the space of projective $\mathbf{R}$-measured compact geodesic laminations.

The work on degenerations of hyperbolic structures suggests the question which actions on Λ-trees are limits of hyperbolic structures? And are there characterizations as in Theorem (3.3) for groups other than surface groups?

I would like to thank Charles Livingston and Peter Shalen for their help. Also I am grateful to Marc Culler for his many suggestions for improving this paper.

1. Definitions

The definitions of this section are taken from [Mo–Sh1], [A–B] (cp. [Cu–Mo]), and [Mo–Ot]. A pair $(\Lambda, <)$ is an *ordered abelian group* if Λ is an abelian group and $<$ is a total ordering, such that $a < b$, $c \leq d$ implies $a + c < b + d$. A subgroup A is *convex* if $x, y \in A$ implies $[x, y] \subseteq A$. The convex subgroups are totally ordered by inclusion. The *rank* of Λ is the cardinality of the set of nontrivial convex subgroups. A rank zero or one group is called *Archimedean*. An Archimedean group admits an order preserving embedding into $\mathbf{R}$ which is unique upto multiplication by a positive number.

Let $\Lambda^{<0}$ be the negative elements of Λ. The *natual ratio* $\cdot/\cdot : \Lambda^{<0} \times \Lambda^{<0} \to [0, +\infty]$ is defined as follows. Let $x, y \in \Lambda^{<0}$. Let A be the smallest convex subgroup containing x, y and let A_0 be the subgroup which is the union of all convex subgroups properly contained in A. So either x or y is not contained in A_0. It follows A/A_0 is rank one. Define $x/y = \iota(x)/\iota(y)$, where $\iota : A/A_0 \to \mathbf{R}$ is any embedding.

A *Λ-metric space* is a pair (X, d), where $d : X \times X \to \Lambda^{0\leq}$ satisfies

(i) for all $x, y \in X$, $d(x, y) = d(y, x)$;
(ii) for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$; and
(iii) for all $x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$.

A function $g : X \to X$ is an *isometry* if $d(g(x), g(y)) = d(x, y)$, for all $x, y \in X$. A (*closed*) *segment* in X is an arc isometric to a (closed) interval in Λ.

A *Λ-tree* is a non-empty, Λ-metric space (T, d) satisfying

(i) any two points in T are joined by a unique closed segment;
(ii) if two closed segments have a common endpoint, then their intersection is a closed segment; and
(iii) if two closed segments meet in exactly one common endpoint, then their union is a closed segment.

A Λ-tree is *linear* if it is isometric to a subset of Λ, i.e. isometric to a segment. Let $[x, y]$ denote the unique closed segment from x to y.

Let S be a subset of a Λ-tree T. A *component* of S is a maximal subtree of S. A point $x \in T$ is an *endpoint* if $T - \{x\}$ has no more than one component and a *vertex* if it has more than two components.

Let $g : T \to T$ be an isometry. The isometry g is *elliptic* if it has a fixed point. The *characteristic set* of g is $T_g = \{x \mid g(x) = x\}$. This is a Λ-tree.

The isometry g is an *inversion* if it is not elliptic but it stabilizes a segment. Its *characteristic set* is $T_g = \varnothing$. Notice that g^2 has a fixed point.

And g is *hyperbolic* if it is neither an inversion nor elliptic and it has a unique non-empty, maximal, invariant linear subtree. Its *characteristic set* T_g is this unique maximal invariant linear subtree. The action on T_g is non-trivial translation. Every isometry is either elliptic, an inversion or hyperbolic [Al–Ba].

Define its *length* $l(g) \in \Lambda^{0\leq}$ to be 0 if g is either an inversion or elliptic; and $\min_{x\in T} \{d(x, g(x))\}$ if g is hyperbolic.

A group action on a tree $G \times T \to T$ is understood to be by isometries. The *length function* is $l : G \to \Lambda^{0\leq}$.

Set $\beta(s, t) = l(st) - l(s) - l(t)$. The length function is *abelian* if $\beta \le 0$. The length function is *dihedral* if it is not abelian and $\beta(s, t) \le 0$, for all hyperbolic s, t. And the length function is of a *general type* if it is neither abelian nor dihedral.

If s, t are not inversions, then $d(T_s, T_t) = \max\{\frac{1}{2}\beta(s, t), 0\}$. It follows that the length function is of a general type if and only if there exist hyperbolic s, t, such that $T_s \cap T_t = \varnothing$. Also if the length function is of a general type, then for any hyperbolic s, there is a hyperbolic t, such that $T_s \cap T_t = \varnothing$.

The action $G \times T \to T$ is *trivial* if there is a fixed point. The action is *minimal* if there is no invariant proper, non-empty subtree.

Let F be a compact surface with or without boundary and having negative Euler characteristic. For convenience fix a complete hyperbolic structure of finite area. Let $\mathbf{H}^2 \to F$ be the universal covering and let $\pi_1 F$ act on $\mathbf{H}^2$ by covering transformations. A *geodesic lamination* $\mathscr{L}$ is a non-empty, closed subset of F, such that each path component is a simple geodesic. Since the hyperbolic structure has finite area, no component is properly homotopic to infinity.

Recall if γ, γ' are two paths, then $\gamma\gamma'$ denotes their product whenever it is defined. A path in F with its endpoints in the complement of $\mathscr{L}$ is *transverse* to $\mathscr{L}$ if it may be written as a product $\gamma_1 \cdots \gamma_n$, where for each γ_i there is a neighborhood U of γ_i, a closed 0-dimensional subset Z of $(0, 1)$ and a homeomorphism $(U, U \cap \mathscr{L}, U \cap \gamma_i) \to ([0, 1] \times [0, 1], [0, 1] \times Z, \{\frac{1}{2}\} \times [0, 1])$. A *$\Lambda$-measure* on a lamination $\mathscr{L}$ is a function μ from the set of paths transverse to $\mathscr{L}$ to $\Lambda^{0\le}$, such that (i) if γ, γ' are homotopic through a 1-parameter family of transverse paths, then $\mu(\gamma) = \mu(\gamma')$; and (ii) if the product $\gamma\gamma'$ is defined, then $\mu(\gamma\gamma') = \mu(\gamma) + \mu(\gamma')$.

A Λ-measured lamination $(\mathscr{L}, \mu)$ in F has a lift $(\tilde{\mathscr{L}}, \tilde{\mu})$ in $\mathbf{H}^2$. Say the action $\pi_1 F \times T \to T$ is *dual* to a measured lamination $(\mathscr{L}, \mu)$ if there is an equivariant bijection p from the connected components of $\mathbf{H}^2 - \tilde{\mathscr{L}}$ to the vertices of T, such that $\tilde{\mu}(\gamma) = d(p(x), p(y))$ for all transverse paths $\gamma : [0, 1] \to \mathbf{H}^2$ from x to y meeting each leaf of $\tilde{\mathscr{L}}$ at most once.

Notice if an action on a Λ-tree with length function l is dual to $(\mathscr{L}, \mu)$, then for each $g \in \pi_1 F$, there is a closed path γ whose free hopotopy class is conjugate to g and $l(g) = \mu(\gamma)$.

Every Λ-measured geodesic lamination is dual to a unique minimal action on a Λ-tree. This is proved in [Hat], [Mo–Sh4] for $\Lambda = \mathbf{R}$, and in [Mo–Ot] for general Λ. Conversely, as long as the complete hyperbolic structure of finite area is fixed, every minimal action is dual to at most one Λ-measured geodesic lamination.

2. Ends of Λ-trees

In this section we review the definition of ends of a Λ-tree from [Al–Ba] and [Ba]. Then we define a topology on T union its ends. We also prove some theorems about the ends. The main proposition is a simultaneous version of a well known result for actions on $\mathbf{H}^2$ and the analogous result for Λ-trees. This allows the translation of (**) into a statement about the ends of T and the ends of $\mathbf{H}^2$.

Recall the following. Given $\mathbf{H}^2$ its circle at infinity S^∞ is the set of endpoints of complete geodesics in $\mathbf{H}^2$. The set $\mathbf{H}^2 \amalg S^\infty$ is topologized to be homeomorphic to a 2-ball. An isometry of $\mathbf{H}^2$ extends to S^∞. This is a model for defining the ends of a Λ-tree.

Let T be a Λ-tree and $x \in T$. A *ray from* x is a linear subtree with x as an endpoint. A *T-ray from* x is a maximal ray from x. Let L be a T-ray from x and L' a T-ray from x'. Say that L and L' are *equivalent* if $L \cap L'$ is a T-ray from x'', for some x''. This defines an equivalence relation on the set of T-rays. An *end* of T is an equivalence class of T-rays. An end is *closed* if it is an endpoint of T, otherwise it is *open*. If ϵ is an end of T, then let $[x, \epsilon)$ denote the unique T-ray from x in the equivalence class ϵ. The set of all ends is denoted $Ends(T)$.

We now define a topology on $T \cup Ends(T)$. Let $S \subseteq T$ be a finite set of points and U a component of $T - S$. A *basic open neighborhood*, denoted $\hat{U}$, is U union the set of all ends of T contained in U. Let $\partial U = \hat{U} - U$. Endow $\hat{T} = T \cup Ends(T)$ with the topology generated by all such $\hat{U}$. The set $Ends(T)$ is not necessarily closed. It is easy to see that this topology is Hausdorff and that the empty set together with all basic open neighborhoods is a basis. Any isometry of T extends uniquely to a homeomorphism of $\hat{T}$.

Also recall a *Fuchsian group* G is a discrete group of isometries of $\mathbf{H}^2$. A *limit point* is a point $z \in S^\infty$, satisfying for all $x \in \mathbf{H}^2 \amalg S^\infty$ and neighborhood V of z, there exists $g \in G$, such that $g(x) \in V$. The set of limit points is the *limit set* $L \subseteq S^\infty$. If G is co-compact, then $L = S^\infty$.

The first two propositions show that the ends of a Λ-tree share some of the properties of the circle at infinity of $\mathbf{H}^2$. These will not be used in this paper.

(2.1) PROPOSITION. *Let $G \times T \to T$ be an action on a Λ-tree. If $G \times T \to T$ is minimal, then for every point $x \in T$ and neighborhood $\hat{U}$ of an end, there is $g \in G$, such that $g(x) \in \hat{U}$.*

Proof. Suppose $G \times T \to T$ is minimal. Let $x \in T$ and $\hat{U}$ be a neighborhood of an end. Let S be convex span of $G(x)$. So S is the minimal Λ-tree containing $G(x)$ and S is invariant. Since the action is minimal, $S = T$ and $S \cap \hat{U} \neq \emptyset$. Therefore there is $g \in G$, such that $g(x) \in \hat{U}$. □

Given two distinct ends ϵ, ϵ' there is a unique linear sub-tree with ends ϵ, ϵ' [Ba].

(2.2) PROPOSITION. *Let $G \times T \to T$ be an action on an Λ-tree. If $G \times T \to T$ is minimal and has a length function of a general type, then for every point $\epsilon \in Ends(T)$ and neighborhood $\hat{U}$ of an end, there is $g \in G$, such that $g(\epsilon) \in \hat{U}$.*

Proof. Suppose $G \times T \to T$ is minimal and has a length function of a general type. Let ϵ be an end and $\hat{U}$ be a neighborhood of an end. Let S be convex span of $G(\epsilon)$. Since the action has a length function of a general type, $S \neq \varnothing$. So S is the minimal Λ-tree containing $G(\epsilon)$ and S is invariant. Since the action is minimal, $S = T$ and $S \cap \hat{U} \neq \varnothing$. Therefore there is $g \in G$, such that $g(\epsilon) \in \hat{U}$. □

The non-trivial isometries of $\mathbf{H}^2$ are classified as elliptic, parabolic and hyperbolic. If g is a hyperbolic isometry, its unique, maximal invariant geodesic is called the *axis* A_g. Its fixed points on the circle at infinity are the two ends of A_g. Denote these A_g^- and A_g^+, which are expansive and attractive, respectively.

This has an analog for Λ-trees. Let g be a hyperbolic isometry of T. Then g fixes exactly two ends of T which are the ends of T_g [Al–Ba]. We denote these ends T_g^- and T_g^+, which are expansive and attractive, respectively.

(2.3) LEMMA. *Let $G \times T \to T$ be an action on a Λ-tree. If the action is minimal and has a length function of a general type, then for every hyperbolic $g \in G$ and neighborhood $\hat{U}$ of an end, there is $f \in G$, such that $f(T_g) \subseteq \hat{U}$.*

Proof. Suppose the action is minimal and has a length function of a general type. Let $g \in G$ be hyperbolic and $\hat{U}$ be a neighborhood of an end. Since the action has a length function of a general type, there is an $h \in G$, such that $T_g \cap T_h = \varnothing$. Let B be the shortest segment from T_g to T_h. Let S be convex span of $G(B)$. So S is the minimal Λ-tree containing $G(B)$ and S is invariant. Since the action is minimal, $S = T$ and $S \cap \hat{U} \neq \varnothing$. Thus there is $f \in G$, such that $f(T_g) \subseteq \hat{U}$. □

The next lemma says that hyperbolics are easy to construct.

(2.4) LEMMA. *Let $G \times T \to T$ be an action on an Λ-tree and $G \times \mathbf{H}^2 \to \mathbf{H}^2$ be an action. If $g \in G$ is hyperbolic in T and $h \in G$ is hyperbolic in $\mathbf{H}^2$, then there exists $m, n \geq 0$, such that $g^m h^n$ is hyperbolic in both T and $\mathbf{H}^2$.*

Proof. Suppose $g \in G$ is hyperbolic in T and $h \in G$ is hyperbolic in $\mathbf{H}^2$. If g is hyperbolic in $\mathbf{H}^2$, then $g^1 h^0$ is hyperbolic in both T and $\mathbf{H}^2$. Similarly, if h is hyperbolic in T, then $g^0 h^1$ is hyperbolic in both T and $\mathbf{H}^2$.

Now suppose g is either trivial, elliptic or parabolic in $\mathbf{H}^2$ and h is either elliptic or an inversion in T. It is easy to see that for some $q > 0$, that $g^m h^{qn}$ is hyperbolic in T for all $m, n > 0$. Finally, $g^m h^{qn}$ is hyperbolic in $\mathbf{H}^2$ for some $m, n > 0$. □

The next result extends Lemma (2.4) and implies that there are many elements which are hyperbolic in both T and $\mathbf{H}^2$. Recall a Fuchsian group is *elementary* if the limit set L has two or fewer points.

(2.5) LEMMA. *Let $G \times T \to T$ be an action on a Λ-tree and $G \times \mathbf{H}^2 \to \mathbf{H}^2$ be an action. If the action on T is minimal and has a length function of a general type and the action on $\mathbf{H}^2$ is non-elementary, then there are $g, h \in G$ which are hyperbolic in both T and $\mathbf{H}^2$ and satisfy $T_g \cap T_h = \emptyset = A_g \cap A_h$.*

Proof. Suppose the action on T is minimal and has a length function of a general type and the action on $\mathbf{H}^2$ is non-elementary. By Lemma (2.4) there is $g_0 \in G$ which is hyperbolic in both T and $\mathbf{H}^2$. By taking translates one finds $g_1, \dots, g_5 \in G$ which are hyperbolic in both T and $\mathbf{H}^2$. Furthermore, without loss of generality suppose for distinct i, j, k, that $T_{g_i} \cap T_{g_j} = \emptyset$ and T_{g_i} does not separate T_{g_j} from T_{g_k}. Similarly there are distinct $h_1, \dots, h_5 \in G$ which are hyperbolic in both T and $\mathbf{H}^2$. Also suppose for all distinct i, j, k, that $A_{h_i} \cap A_{h_j} = \emptyset$ and A_{h_i} does not separate A_{g_j} from A_{h_k}.

So for each i, there are at most two j's, such that $T_{g_i} \cap T_{h_j} \neq \emptyset$. Similarly, for each j, there are at most two i's, such that $A_{g_i} \cap A_{h_j} \neq \emptyset$. But there are 25 pairs g_i, h_j. Now simple counting shows that for some i, j, the elements $g = g_i$ and $h = h_j$ have disjoint axes in both T and $\mathbf{H}^2$. □

If a Fuchsian group G with limit set L is non-elementary, then $\{(A_s^-, A_s^+)\}_s$ is dense in $L \times L$ [Gre] (cp. [Gro]). The next proposition is a simultaneous version of this result and the corresponding result for Λ-trees. Some of the difficulty in proving this proposition is that Λ may be non-Archimedean.

(2.6) PROPOSITION. *Let $G \times T \to T$ be an action on a Λ-tree and $G \times \mathbf{H}^2 \to \mathbf{H}^2$ be an action. If the action on T is minimal and has a length function of a general type and the action on $\mathbf{H}^2$ is non-elementary, then $\{(T_s^-, T_s^+, A_s^-, A_s^+)\}_s$ is dense in $\{(T_g^+, T_h^+, A_g^+, A_h^+)\}_{g,h}$, where s, g, h range over all elements which are hyperbolic in both T and $\mathbf{H}^2$.*

Proof. Suppose the action on T is minimal and has a length function of a general type and the action on $\mathbf{H}^2$ is non-elementary. Let $g, h \in \pi_1 F$ and be hyperbolic in both T and in $\mathbf{H}^2$. Let $\hat{U}^- \times \hat{U}^+ \times V^- \times V^+$ be a neighborhood of $(T_g^+, T_h^+, A_g^+, A_h^+)$. It suffices to find an s, such that $(T_s^-, T_s^+, A_s^-, A_s^+) \in \hat{U}^- \times \hat{U}^+ \times V^- \times V^+$. Suppose there are $j, k \in G$ hyperbolic in both T and $\mathbf{H}^2$, such that $T_j \subseteq \hat{U}^-$, $A_j \subseteq V^-$, $T_k \subseteq \hat{U}^+$, $A_k \subseteq V^+$. Furthermore, T_j^-, T_j^+ are distinct from T_k^-, T_k^+ and A_j^-, A_j^+ are distinct from A_k^-, A_k^+. Then take $s = k^n j^n$ for large enough n.

Thus it suffices to find such j, k. By Lemmas (2.3) and (2.5) we may find j hyperbolic in both T and in $\mathbf{H}^2$, such that $T_j \subseteq \hat{U}^-$ and T_j^-, T_j^+ are distinct from T_g^-, T_g^+ and A_j^-, A_j^+ are distinct from A_g^-, A_g^+. Similarly there is k hyperbolic in both T and in $\mathbf{H}^2$, such that $T_k \subseteq \hat{U}^+$ and T_k^-, T_k^+ are distinct from T_h^-, T_h^+ and A_k^-, A_k^+ are distinct from A_h^-, A_h^+. Now replace j, k by $g^n j g^{-n}, h^m k h^{-m}$ for sufficiently large m, n. □

It is an interesting question whether Proposition (2.6) has a generalization for a group acting simultaneously on more than two spaces?

3. Main Theorem

In this section we prove the main theorem by simultaneously constructing a continuous function from the space of ends of T to the circle at infinity S^∞. It is interesting to note that this is exactly the opposite of the construction in [Mo–Ot]. There they construct a map from $\mathbf{H}^2$ to T.

Let F be a compact surface with or without boundary and having negative Euler characteristic. For convenience fix a complete, hyperbolic structure of finite area and let $\pi_1 F \times \mathbf{H}^2 \to \mathbf{H}^2$ be the covering action. Let Λ be an ordered abelian group and T a Λ-tree and let $\pi_1 F \times T \to T$ be a minimal action.

Define the relation $R \subseteq Ends(T) \times S^\infty$ as $R = \{(T_g^+, A_g^+)\}_g$, where g ranges over all elements of $\pi_1 F$ which are hyperbolic in both $\mathbf{H}^2$ and T. For any $X \subseteq Ends(T)$ let $R(X) = \{A_g^+\}$, where g ranges over all elements of $\pi_1 F$, such that g is hyperbolic in both T and $\mathbf{H}^2$, and $T_g^+ \in X$. Similarly, for any $Y \subseteq S^\infty$ let $R^{-1}(Y) = \{T_g^+\}$, where g ranges over all elements of $\pi_1 F$, such that g is hyperbolic in both T and $\mathbf{H}^2$, and $A_g^+ \in Y$.

It will be a consequence of Theorem (3.3) that under the correct hypotheses R extends to a continuous function $Ends(T) \to S^\infty$.

(3.1) LEMMA. *Let $p \in T$ and $\{U_\alpha\}_\alpha$ components of $T - \{p\}$ and $Z_\alpha = R(\partial U_\alpha)$. If the action on T has a length function of a general type and satisfies* (**), *then for all α no two points in Z_α are separated on S^∞ by any two points in $\cup_{\alpha \neq \beta} Z_\beta$.*

Proof. Suppose the action $\pi_1 F \times T \to T$ has a length function of a general type and satisfies (**). We argue by contradiction.

Fix α and let $x, y \in Z_\alpha, z \in Z_\beta, w \in Z_\gamma$, where $\alpha \neq \beta, \gamma$. Suppose x, y are separated on S^∞ by z, w. By definition there are g, h, such that $T_g^+, T_h^+ \in \partial U_\alpha$ and $A_g^+ = x, A_h^+ = y$. Similarly there are j, k, such that $T_j^+ \in \partial U_\beta, T_k^+ \in \partial U_\gamma$ and $A_j^+ = z, A_k^+ = w$.

By assumption the action on T is minimal and has a length function of a general type. By Proposition (2.6) there exist $s, t \in \pi_1 F$, such that $(T_s^-, T_s^+, A_s^-, A_s^+)$ approximates $(T_g^+, T_h^+, A_g^+, A_h^+)$ and $(T_t^-, T_t^+, A_t^-, A_t^+)$ approximates $(T_j^+, T_k^+, A_j^+, A_k^+)$. More precisely, there are s, t, such that

(i) $T_s^-, T_s^+ \in \partial U_\alpha$;

(ii) $T_t^- \in \partial U_\beta$, $T_t^+ \in \partial U_\gamma$; and

(iii) A_s^-, A_s^+ are separated on S^∞ by A_t^-, A_t^+.

So $T_s \subseteq U_\alpha$ and $T_t \subseteq U_\beta \cup \{p\} \cup U_\gamma$. In particular $T_s \cap T_t = \varnothing$. But A_s and A_t intersect transversely in one point. This contradicts the supposition (**). □

(3.2) LEMMA. *let $p \in T$ and $\{U_\alpha\}_\alpha$ be components of $T - \{p\}$. If the action on T has a length function of a general type and satisfies* (**), *then there are unique closed intervals $I_\alpha \subseteq S^\infty$, such that*

(i) $\overline{\cup I_\alpha} = S^\infty$;

(ii) *for all* $\alpha \neq \beta$, $\mathring{I}_\alpha \cap \mathring{I}_\beta = \varnothing$; *and*

(iii) *for all* α, $R^{-1}(\mathring{I}_\alpha) \subseteq \partial U_\alpha \subseteq R^{-1}(I_\alpha)$.

Proof. Suppose the action $\pi_1 F \times T \to T$ has a length function of a general type and satisfies (**). Let $Z_\alpha = R(\partial U_\alpha)$. Set $I_\alpha = \bar{Z}_\alpha$. By Lemma (3.1) no pair of points in the same Z_α are separated on S^∞ by a pair of points in $\cup_{\alpha \neq \beta} Z_\beta$. And by hypothesis $\cup Z_\alpha$ is dense in S^∞. So I_α is connected. Minimality implies that for any α, the set $Z_\alpha \neq \varnothing$. Furthermore, there is some $\beta \neq \alpha$. Therefore I_α is an interval. It is routine to check (i), (ii) and (iii). □

(3.3) THEOREM. *Let F be a complete hyperbolic surface of finite area. A minimal action on a Λ-tree $\pi_1 F \times T \to T$ is dual to a Λ-measured geodesic lamination if and only if it has a length function of a general type and it satisfies* (**).

Proof. It is clear if the action is geometric, then it is it has a length function of a general type and satisfies (**). Now suppose the action has a length function of a general type and satisfies (**). It will be shown that the action is geometric.

By hypothesis the covering action is non-elementary. Let $p \in T$ and let $\{U_\alpha\}$ be the collection of componenets of $T - \{p\}$. Let $\{I_\alpha\}_\alpha$ be as in Lemma (3.2). Define X_p to be the convex closure of $\cup(S^\infty - \mathring{I}_\alpha)$. The frontier of X_p in $\mathbf{H}^2$ is a union of disjoint complete geodesics.

Now let p, q be distinct points in T. Let C_α and C_β be the component of $T - \{p\}$ containing q and the component of $T - \{q\}$ containing p, respectively. Since $I_\alpha \cup I_\beta = S^\infty$, the set $X_p \cap X_q$ is either empty or a single geodesic.

Define $\tilde{\mathscr{L}}$ to be the closure of $\cup Fr(X_p)$. Clearly $\tilde{\mathscr{L}}$ is a geodesic lamination on $\mathbf{H}^2$. We now show $X_p \mapsto p$ defines a bijection from the components of $\mathbf{H}^2 - \tilde{\mathscr{L}}$ to the vertices of T.

It suffices to see the function is well defined. Let X be a component of $\mathbf{H}^2 - \tilde{\mathscr{L}}$. Since $\overline{\cup I_\alpha} = S^\infty$, the space $\mathbf{H}^2 - X$ has at least 3 compontents. So there are 3 distinct points a, b, c in T, such that X_a, X_b, X_c lie in distinct components of $\mathbf{H}^2 - X$. Notice that X_b and X_c lie in the same component of $\mathbf{H}^2 - X_a$. Thus b and c lie in the same component of $T - \{a\}$. A similar statement holds for b and c. Thus there is a unique point p in T that separates a, b, c. By construction X_p must separate X_a, X_b, X_c. Therefore $X_p = X$. Furthermore, from above if $X_p = X_q$, then $p = q$.

Finally define a Λ-measure $\tilde{\mu}$ as follows. First consider only paths γ in $\mathbf{H}^2$ transverse to $\tilde{\mathscr{L}}$ and meeting each component of $\tilde{\mathscr{L}}$ at most once. Then γ has endpoints in X_p, X_q, for some p, q. Define $\tilde{\mu}(\gamma) = d(p, q)$.

For such paths we need to check conditions (i) and (ii) in the definition of Λ-measure. Suppose γ, γ' are transverse and each meets each component of $\tilde{\mathscr{L}}$ at most once. If γ, γ' are homotopic through a 1-parameter family of transverse paths, then they have their endpoints in the same complementary components of the lamination. By construction of the measure $\mu(\gamma) = \mu(\gamma')$. Now suppose $\gamma\gamma'$ is defined and $\gamma\gamma'$ meets each component of $\tilde{\mathscr{L}}$ at most once. It follows from the definition of Λ-tree that $\mu(\gamma\gamma') = \mu(\gamma) + \mu(\gamma')$.

Extend $\tilde{\mu}$ to all transverse paths additively. It is easy to see that $\tilde{\mu}$ is a Λ-measure. The construction of $(\tilde{\mathscr{L}}, \tilde{\mu})$ is equivariant, so it determines a Λ-measured lamination $(\mathscr{L}, \mu)$ to which the action is dual. □

Notice the assumption that F has finite area is used only to conclude that the limit set of $\pi_1 F$ is S^∞ and the I_α's are unique. So the hypothesis of finite area may be replaced by limit set S^∞.

Actually the theorem holds if the finite area hypothesis is dropped altogether. And with the correct definitions the above proofs go through for F an orbifold.

If F is a surface of non-negative Euler characteristic, then the action of $\pi_1 F$ on Λ-trees are abelian and therefore satisfy (**). These actions are easily classified and are all dual to Λ-measured *topological* laminations.

4. Examples

In this section we give examples which show the necessity in Theorem (3.3) of the hypotheses that the action has a length function of a general type and satisfies (**).

(4.1) EXAMPLE. Let F be a closed hyperbolic surface of genus three. There is a $\mathbf{Z}$-tree T and minimal action $\pi_1 F \times T \to T$ with length function of a general type and not dual to a $\mathbf{Z}$-measured geodesic lamination.

Take F_1 and F_2 to be compact subsurfaces each homeomorphic to a once-punctured genus two surface and intersecting in a twice-punctured genus one surface F_3. By Van Kampen's Theorem $\pi_1 F = \pi_1(F_1) *_{\pi_1(F_3)} \pi_1(F_2)$. This corresponds to an action $\pi_1 F \times T \to T$, where T is a $\mathbf{Z}$-tree [Ser]. The vertex stabilizers are the conjugates of $\pi_1(F_1)$ and $\pi_1(F_2)$. Clearly the length function is of a general type. However, the action is not geometric, for some of the edge stabilizers are conjugate to $\pi_1(F_3)$ which is non-cyclic. □

By Theorem (3.2) the above action does not satisfy (**). The reader should play with the above example to see explicitly how property (**) is violated. See [Sk1] for another example of an $\mathbf{R}$-tree T and a minimal action $\pi_1 F \times T \to T$ with length function of a general type which is not dual to a $\mathbf{R}$-measured geodesic lamination.

(4.2) EXAMPLE. Let F be a closed hyperbolic surface of genus two. There is a minimal action $\pi_1 F \times T \to T$ on a $\mathbf{Z}$-tree which satisfies (**) and which is not dual to a $\mathbf{Z}$-measured geodesic lamination.

Just let $\pi_1 F$ act on $\mathbf{Z}$ by non-trivial translation. The action satisfies (**). Since it is abelian, it is not dual to a $\mathbf{Z}$-measured geodesic lamination. □

5. Applications

In this section we prove that Theorem (3.3) implies that if a minimal action on a Λ-tree is the limit of a sequence of representations coming from complete hyperbolic structures, then the action is dual to a Λ-measured geodesic lamination. And we prove directly the boundary of the compactification of Teichmüller space is homeomorphic to the space of projective $\mathbf{R}$-measured compact geodesic laminations.

Let F be a surface of negative Euler characteristic. Fix a complete hyperbolic structure of finite area on F. Let $\pi_1 F \times T \to T$ be an action on a Λ-tree. Suppose $\{\rho_i\}_i$ is a sequence of representations of $\pi_1 F$ coming from complete hyperbolic structures on F which converges to the action. We will show the action has a length function of a general type and satisfies (**).

First recall that if ρ is a representation of $\pi_1 F$ into the group of isometries of $\mathbf{H}^2$, its *length function* $l_\rho : \pi_1 F \to [0, \infty)$ sends g to the translation length of $\rho(g)$.

Since F has negative Euler characteristic, there is some $g \in \pi_1 F$, such that $\lim_{i \to \infty} l_{\rho_i}(g) = \infty$. It follows that $l(g) > 0$ and g is hyperbolic in T. Also there is some conjugate of g, say h, such that $\mathbf{H}^2$ divided out by $\langle g, h \rangle$ is a twice-punctured disk with g, h represented by disjoint simple loops. Then the commutator $ghg^{-1}h^{-1}$ satisfies $l_{\rho_i}(g) + l_{\rho_i}(h) < l_{\rho_i}(ghg^{-1}h^{-1})$, for all i. It follows that $l(g) + l(h) \leq l(ghg^{-1}h^{-1})$ and $T_g \cap T_h$ is a segment of length no greater than $\min\{l(g), l(h)\}$. Therefore the action has a length function of a general type.

The argument that the action satisfies (**) is from [Mo–Ot]. Suppose g and h are elements of $\pi_1 F$ which are hyperbolic in T. Also suppose g and h are hyperbolic in $\mathbf{H}^2$ and A_g and A_h intersect transversely in one point. Then an easy calculation shows that $l_{\rho_i}(gh) \leq l_{\rho_i}(g) + l_{\rho_i}(h)$, for all i. Without loss of generality suppose $l(g) \leq l(h)$ and rewrite the above as

$$\frac{l_{\rho_i}(gh)}{l_{\rho_i}(h)} \leq \frac{l_{\rho_i}(g)}{l_{\rho_i}(h)} + 1.$$

Then taking limits yields

$$\frac{-l(gh)}{-l(h)} \leq \frac{-l(g)}{-l(h)} + 1.$$

This implies $l(gh) \leq l(g) + l(h)$ and $T_g \cap T_h \neq \varnothing$. Therefore the action satisfies (**).

So Theorem (3.3) implies the following.

(5.1) THEOREM. *If a minimal action on a Λ-tree $\pi_1 F \times T \to T$ is the limit of a sequence of representations coming from complete hyperbolic structures on F, then the action is dual to a Λ-measured geodesic lamination.* □

If the sequence $\{\rho_i\}_i$ comes from complete hyperbolic structures of finite area, then $\rho_i(g)$ is parabolic for all peripheral elements g. This means that $l(g) = l_{\rho_i}(g) = 0$. Therefore g fixes a point of T and the dual lamination is necessarily compact. This special case of Theorem (5.1) where the sequence comes from complete hyperbolic structures of finite area also follows from [Mo–Ot].

We now review the compactification of [Mo–Sh1]. The *Teichmüller space* of F is the space of complete hyperbolic structures of finite area on F, denoted $\mathscr{T}_F$. Recall every complete hyperbolic structure of finite area on F comes from a discrete, faithful representation ρ of $\pi_1 F$ into the group of isometries of $\mathbf{H}^2$. Define an embedding $\mathscr{T}_F \to ([0, \infty)^{\pi_1 F} - \mathbf{0})/\mathbf{R}^+$ by $\rho \mapsto [l_\rho]$. The closure of $\mathscr{T}_F$, denoted $\hat{\mathscr{T}}(F)$, is compact.

Each point of $\partial\mathscr{T}(F) = \hat{\mathscr{T}}(F) - \mathscr{T}(F)$ is the projective length function of a minimal action on an $\mathbf{R}$-tree.

The above arguments work equally well to show these actions also have length functions of a general type and satisfy (**). Thus by Theorem (3.3) the set $\partial\mathscr{T}(F)$ is contained in the space of projective **R**-measured compact geodesic laminations. Finally we need the fact that the space of projective **R**-measured compact geodesic laminations is contained in $\partial\mathscr{T}(F)$. This is proved in Thurston's program [Th] and may also be demonstrated directly [Mo–Sh1]. Together these imply the following.

(5.2) THEOREM. [Thurston] *The space $\partial\mathscr{T}(F)$ is homeomorphic to the space of projective* **R***-measured compact geodesic laminations.* □

Alternate proofs of the above result are given in both [Mo–Sh1] and [Mo–Ot].

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Received November 11, 1988