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A priori bounds of Castelnuovo type for cohomological Hilbert functions

M. BRODMANN

1. Introduction

In 1893 Catelnuovo [10] proved the following result: Given a smooth curve Y in the projective space $\mathbb{P}^3$, there is an integer r, such that for any $n \geq r$ the surfaces of degree n in $\mathbb{P}^3$ cut out of a complete linear system on the curve Y. Thereby, one may choose $r = \deg(Y) - 2$.

In other words, Castelnuovos result says that the maps

$$H^0(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) \to H^0(Y, \mathcal{O}_Y(n))$$

become surjective for $n \geq r$. Denoting the vanishing ideal of Y in $\mathcal{O}_{\mathbb{P}^3}$ by $\mathcal{J}$, applying cohomology to the sequences $0 \to \mathcal{J}(n) \to \mathcal{O}_{\mathbb{P}^3}(n) \to \mathcal{O}_Y(n) \to 0$ and observing that $H^1(\mathbb{P}^3, \mathcal{O}_{\mathbb{P}^3}(n)) = 0$, we see that the previous statement is equivalent to

$$H^1(\mathbb{P}^3, \mathcal{J}(n)) = 0, \quad \forall n \geq r.$$

Meanwhile, the vanishing of higher cohomology of projective varieties with coefficients in positively twisted coherent sheaves has become a prominent subject in algebraic geometry.

In one respect, Serre [39] generalized Castelnuovos result to the maximally possible extent, by showing, that for any coherent sheaf $\mathcal{F}$ over any projective variety X, there is an integer r, such that $H^i(X, \mathcal{F}(n))$ vanishes for all $n \geq r$ and all $i > 0$.

But contrary to Castelnuovos result (which, in the special case it refers to, gives an explicit value for r) Serre's result is not of quantitative nature. Nevertheless, there are a lot of results, which give upper bounds on Serre's number r for specific coherent sheaves $\mathcal{F}$. The common idea of these results is to bound r by only finitely many simple invariants of the corresponding sheaf $\mathcal{F}$.

So, Mumford [34] gave a quantitative approach to Castelnuovo's problem for arbitrary coherent sheaves of ideals $\mathcal{J} \subseteq \mathcal{O}_{\mathbb{P}^d}$ over a given projective space $\mathbb{P}^d$. He namely introduced the *Castelnuovo-regularity* reg $(\mathscr{J})$ of such a sheaf $\mathscr{J}$ of ideals as the minimal number $m \in \mathbb{Z}$ for which

$$H^i(\mathbb{P}^d, \mathscr{J}(n)) = 0 \quad \text{for all } i > 0 \quad \text{and all } n \geq m - i.$$

Then he proved, that reg $(\mathscr{J})$ has an upper bound which depends only on the (finitely many) coefficients of the Hilbert-polynomial of $\mathscr{J}$. Gotzmann [14] later gave a refinement of Mumfords result and applied it to study certain subschemes of Hilbert schemes (s. [15]).

Meanwhile Castelnuovos original result was extended to the vanishing ideals $\mathscr{J} \subseteq \mathcal{O}_{\mathbb{P}^d}$ of specific closed subvarieties $Y \subseteq \mathbb{P}^d$. So, first of all, Gruson–Lazarsfeld–Peskine [20] generalized Castelnuovos bound to arbitrary reduced curves. Pinkham gave a further extension to smooth surfaces in $\mathbb{P}^4$ and $\mathbb{P}^5$ (cf. [38]), whereas Lazarsfeld [27] finally settled the case of arbitrary smooth non-degenerate surfaces $Y \subseteq \mathbb{P}^d$, by proving reg $(\mathscr{J}) \leq \deg(Y) + 2 - d$. Apparently the inequality reg $(\mathscr{J}) \leq \deg(Y) + \dim(Y) - d$ is expected to hold true in general for the vanishing ideal sheaf $\mathscr{J} \subseteq \mathcal{O}_{\mathbb{P}^d}$ of a nondegenerate, smooth closed subvariety $Y \subseteq \mathbb{P}^d$. Note, that Bayer–Mumford [2] have established the weaker estimate

$$\operatorname{reg}(\mathscr{J}) \leq (\dim(Y) + 1)(\deg(Y) - 2) + 1 \text{ in the described general situation.}$$

The case of vanishing ideals of varieties $Y \subseteq \mathbb{P}^d$ with certain arithmetic properties (for example the property of being arithmetically of Buchsbaum-type) has been studied extensively by Nagel–Vogel [35], by Stückrad–Vogel [41, 42, 43] and—in a related situation—by Miro–Roig [32].

The search of Castelnuovo bounds for vanishing ideals is related to the search of bounds on the degrees of the defining equations of projective varieties. This subject recently has been studied by several authors, too. We only mention a few of them: Ballico [1], Geramita [13], Maroscia–Vogel [28], Maroscia–Vogel–Stückrad [29], Treger [44], Trung–Valla [45].

In the present paper we will show, that the cohomological Hilbert-functions

$$n \mapsto h^i(\mathscr{F}(n)) := \dim H^i(X, \mathscr{F}(n)), \quad (i > 0)$$

of a coherent sheaf $\mathscr{F}$ over a projective variety $X \subseteq \mathbb{P}^d$ are bounded in the range $n \geq -i$ by finitely many invariants of $\mathscr{F}$. In particular, the Castelnuovo-regularity reg $(\mathscr{F})$ of $\mathscr{F}$, which is defined in the same way as done previously for sheaves of ideals, is bounded only by these invariants.

To formulate our main result, we have to introduce a few notations. So, let $X \subseteq \mathbb{P}^d$ be a closed subscheme of the projective space $\mathbb{P}^d_k$. Let $\mathscr{F}$ be a coherent sheaf of $\mathcal{O}_X$-modules. As done already above, we write $h^i(\mathscr{F}(n))$ instead of the k-vector space dimension of the Serre-cohomology group $H^i(X, \mathscr{F}(n))$ instead of the k-vector space dimension of the Serre-cohomology group $H^i(X, \mathscr{F}(n))$. Thereby, twisting is understood with respect to the given embedding $X \hookrightarrow \mathbb{P}^d$.

Moreover we introduce the reduced linear subdimension of $\mathscr{F}$ as the invariant

$$\operatorname{lsdim}^{(0)}(\mathscr{F}) = \min \{\dim \langle \overline{\{x\}} \rangle \mid x \in \operatorname{Ass}(\mathscr{F}),\ x \text{ non closed}\},$$

(where $\langle \overline{\{x\}} \rangle$ denotes the linear span of $\overline{\{x\}}$) and the reduced global subdepth as being the number

$$\delta^{(0)}(\mathscr{F}) = \min \{\operatorname{depth}(\mathscr{F}_x) \mid x \in X \text{ closed},\ x \notin \operatorname{Ass}(\mathscr{F})\},$$

thereby assuming that $\dim(\mathscr{F}) := \dim \operatorname{supp}(\mathscr{F}) > 0$.

Using these notations, we may formulate our main result as follows (cf. (6.11), (6.12)):

(1.1) THEOREM. *Let $0 < i \leq e$ be integers. Then, there are functions*

$$B_{e,i} : \mathbb{N}^2 \times \mathbb{N}_0^{e-i+1} \times \mathbb{Z}_{\geq -i} \to \mathbb{N}_0,$$

$$C_{e,i} : \mathbb{N}^2 \times \mathbb{N}_0^{e-i+1} \to \mathbb{Z},$$

such that for any coherent sheaf $\mathscr{F}$ over an arbitrary closed subscheme X of $\mathbb{P}^d$ with $0 < \dim(\mathscr{F}) \leq e$ the following statements hold true:

(i) $h^i(\mathscr{F}(n)) \leq B_{e,i}(\operatorname{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}); h^i(\mathscr{F}(-i)), \ldots, h^e(\mathscr{F}(-e)); n); \forall n \geq -i$

(ii) $h^i(\mathscr{F}(n)) = 0$ *for all* $n \geq C_{e,i}(\operatorname{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F});$
$h^i(\mathscr{F}(-i)), \ldots, h^e(\mathscr{F}(-e)))$. □

So, the cohomological Hilbert functions $n \mapsto h^i(\mathscr{F}(n))$ $(i > 0)$ of a coherent sheaf $\mathscr{F}$ over a closed subscheme X of $\mathbb{P}^d$ are bounded (in the range $n \geq -i$) by the invariants $\operatorname{lsdim}^{(0)}(\mathscr{F})$, $\delta^{(0)}(\mathscr{F})$, $h^1(\mathscr{F}(-1)), \ldots, h^{\dim(\mathscr{F})}(\mathscr{F}(-\dim(\mathscr{F})))$. In particular the same holds true for the Castelnuovo-regularity $\operatorname{reg}(\mathscr{F})$ of $\mathscr{F}$.

Let us compare (1.1) with Mumfords regularity bound [34] for sheaves of ideals $\mathscr{I} \subseteq \mathcal{O}_{\mathbb{P}^d}$. The statement (1.1)(ii) may be viewed as a kind of extension of Mumfords result to arbitrary coherent sheaves over projective varieties: It namely bounds the Castelnuovo-regularity of such sheaves in terms of only finitely many invariants. In view of statement (i), our result also may be considered as a refinement of Mumford's: It namely does not only bound the regularity of a sheaf $\mathscr{F}$, but in addition its cohomological Hilbert functions. Clearly there is also an essential difference between Mumfords result and Theorem (1.1): Namely, our bounding invariants are by no means related directly to the coefficients of the

Hilbert polynomial of the occurring sheaf $\mathcal{F}$. But, this difference is not surprising, as (even for direct sums of line bundles over $\mathbb{P}^d$) the regularity of coherent sheaves is not bounded in general in terms of their Hilbert polynomial (cf. [34]). The only serious attempt to extend Mumfords result directly has been made by Kleiman [19, Exp XIII] who gave a regularity bound for subsheaves of trivial bundles over $\mathbb{P}^d$ in terms of their Hilbert polynomial.

It turns out, that our system of bounding invariants is too large in some sense. The cohomological Hilbert functions in question are bounded in fact already by the numbers $h^1(\mathcal{F}(-1)), \ldots, h^{\dim(\mathcal{F})}(\mathcal{F}(-\dim(\mathcal{F})))$. We namely shall prove (cf. (6.14), (6.15)):

(1.2) THEOREM. *Let $0 < i \le e$ be integers. Then, there are functions*

$$G_{e,i} : \mathbb{N}_0^{e-i+1} \times \mathbb{Z}_{\ge -i} \to \mathbb{N}_0; \quad F_{e,i} : \mathbb{N}_0^{e-i+1} \to \mathbb{Z}$$

such that for any coherent sheaf $\mathcal{F}$ over an arbitrary closed subscheme X of $\mathbb{P}^d$ with $0 < \dim(\mathcal{F}) \le e$ the following statements hold true:

(i) $h^i(\mathcal{F}(n)) \le G_{e,i}(h^i(\mathcal{F}(-i)), \ldots, h^e(\mathcal{F}(-e)); n), \quad \forall n \ge -i$

(ii) $h^i(\mathcal{F}(n)) = 0$ *for all* $n \ge F_{e,i}(h^i(\mathcal{F}(-i)), \ldots, h^e(\mathcal{F}(-e)))$. □

Thereby the functions $G_{e,i}$ and $F_{e,i}$ are defined by

$$G_{e,i}(c_i, \ldots, c_e; n) := B_{e,i}(1, 1; c_i, \ldots, c_e; n);$$

$$F_{e,i}(c_i, \ldots, c_e) := C_{e,i}(1, 1, c_i, \ldots, c_e),$$

where $B_{e,i}$ and $C_{e,i}$ are the bounding functions occurring in (1.1).

Obviously, the bounds given in (1.2) are weaker than the corresponding bounds of (1.1). Moreover it is easy to see that $h^i(\mathcal{F}(-i)), \ldots, h^e(\mathcal{F}(-e))$ is a minimal system of bounding invariants for the cohomological Hilbert functions of $\mathcal{F}$ (cf. (7.14)).

Nevertheless, for particular classes of sheaves, cohomology may be bounded by shorter systems of invariants. So, generalizing a result of Elencwajg–Forster [12] we proved in [8], that the cohomological Hilbert functions of a vector bundle $\mathcal{E}$ over $\mathbb{P}^d$ are bounded only by the rank, the first two Chern numbers and the span of the generic splitting type of $\mathcal{E}$. For invertible sheaves over projective varieties there are similar bounds of regularity, which depend only on few invariants (cf. [19, Exp. XIII], [24]).

To use the cohomology dimensions $h^1(\mathcal{F}(-1)), \ldots, h^e(\mathcal{F}(-e))$ "on the diagonal" as a system of invariants to bound the cohomology of a coherent sheaf $\mathcal{F}$ of dimension $\leq e$ in the positive range is fairly natural. It namely is an easy observation that $h^1(\mathscr{F}(-1)) = h^2(\mathscr{F}(-2)) = \cdots = h^e(\mathscr{F}(-e)) = 0$ induces $h^i(\mathscr{F}(n)) = 0$ for all $n \geq -i$ $(i = 1, \ldots, e)$ (cf. [34]). Thus, if $h^i(\mathscr{F}(-i)) = 0$ for all $i > 0$, then in particular $\operatorname{reg}(\mathscr{F}) = 0$. So, in some sense, (1.1) and especially (1.2) extend this observation to the case of non-vanishing cohomology dimensions $h^i(\mathscr{F}(-i))$. Moreover, having defined the bounding functions $B_{e,i}$, $C_{e,i}$, $G_{e,i}$ and $F_{e,i}$ we will see that (1.1) and (1.2) give back the previous observation, (cf. (7.11)).

Obviously, (1.1) and (1.2) furnish regularity bounds for arbitrary coherent sheaves. More precisely, choosing the bounding functions $C_{e,i}$ and $F_{e,i}$ as in (1.1) resp. (1.2), we have (cf. (7.9), (7.10)):

(1.3) THEOREM. *Let $e \in \mathbb{N}$. Then, for any coherent sheaf $\mathscr{F}$ over an arbitrary closed subscheme X of $\mathbb{P}^d$ with* $0 < \dim(\mathscr{F}) \leq e$:

$$\operatorname{reg}(\mathscr{F}) \leq C_{e,1}(\operatorname{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}); h^1(\mathscr{F}(-1)), \ldots, h^e(\mathscr{F}(-e))) + 1. \qquad \square$$

(1.4) THEOREM. *Let $e \in \mathbb{N}$. Then, for any $\mathscr{F}$ as in* (1.3)

$$\operatorname{reg}(\mathscr{F}) \leq F_{e,1}(h^1(\mathscr{F}(-1)), \ldots, h^e(\mathscr{F}(-e))) + 1. \qquad \square$$

Our approach is fairly different from the methods of other authors. As Hilbert polynomials do not occur in our estimate, our arguments differ in an essential way from the ones found in Mumford [34]—which latter base on techniques developed by Kleiman [24], Matsusaka [31] and Nakai [36]. As free resolutions do not enter into our considerations, we will not use the syzygetic approach of Eisenbud–Goto [11], Ooishi [37] and Bayer–Stillmann [3]. Dealing with arbitrary coherent sheaves, we cannot use the methods applied by numerous authors to study the behaviour of cohomological Hilbert functions in special cases ([22], [26], [28], [29], [33], [35], [41–45]).

What we use instead is the hyperplane section method, which we applied (in some special cases) already in [5], [6], [7], [8], [9]. What we use of this method is essentially the following fact, which was shown in [8]: Let $X \subseteq \mathbb{P}^d$ be a closed subscheme and let $\mathscr{F}$ be a coherent sheaf over X. Let $\mathfrak{H}$ be a linear system of hyperplane sections of X of positive dimension N. Assume that $H \cap \operatorname{Ass}(\mathscr{F}) = \emptyset$ for all $H \in \mathfrak{H}$. Let $i \in \mathbb{N}$ and let $\mu \in \mathbb{Z}$ such that $H^i(H, \mathscr{F} \restriction H(n)) = 0$ for all $H \in \mathfrak{H}$ and all $n \geq \mu$. Then in the range $n \geq \mu$ the cohomological Hilbert function $h^i(\mathscr{F}(n))$ decreases in steps of at least N, until it reaches the value 0.

This observation—together with some information on the possible choices of N—will furnish an inductive procedure, which allows to define recursively the bounding functions, that occur in (1.1)–(1.4).

Besides the previous general results, we consider a very special situation, too. Namely, using the Kodaira vanishing theorem, we prove that the cohomology of a smooth, closed, non-degenerate subvariety X of $\mathbb{P}^d_{\mathbb{C}}$ is bounded only by the embedding dimension d, the dimension e of X and the invarient $h^e(\mathcal{O}_X(-e))$. More precisely, we consider the functions $B_{e,i}$, $C_{e,i}$ of (1.1) and use them to define new functions

$$B^*, \hat{B}_i : \mathbb{N}^2 \times \mathbb{N}_0^2 \to \mathbb{N}_0; \quad C^*, \hat{C}_i : \mathbb{N}^2 \times \mathbb{N}_0 \to \mathbb{Z}$$

by setting

$$\hat{B}_i(a, b, c; n) := B_{b,i}(a, b; 0, \ldots, 0, c; n),$$

$$\hat{C}_i(a, b, c) := C_{b,\, i}(a, b; 0, \ldots, 0, c).$$

$$B^*(a, b, c; n) := B_{a,1}(a, b; 0, \ldots, 0, \overset{b}{\hat{c}}, 0, \ldots, 0; n)$$

$$C^*(a, b, c) := C_{a,1}(a, b; 0, \ldots, 0, \quad c, 0, \ldots, 0; n)$$

Then, we prove (cf. (8.15)):

(1.5) THEOREM. *Let X be a smooth, closed, non-degenerate subvariety of the complex projective space $\mathbb{P}^d_{\mathbb{C}}$. Put $e = \dim(X)$, and let $\mathcal{J}_X \subseteq \mathcal{O}_{\mathbb{P}^d}$ be the sheaf of vanishing ideals of X. Then, for $0 < i \leq e$:*

(i) $h^i(\mathcal{O}_X(n)) \leq \hat{B}_i(d, e, h^e(\mathcal{O}_X(-e)); n), \quad \forall n \geq 0$
(ii) $h^i(\mathcal{O}_X(n)) = 0, \quad \forall n \geq \hat{C}_i(d, e, h^e(\mathcal{O}_X(-e)))$.
(iii) $\operatorname{reg}(\mathcal{O}_X) \leq \hat{C}_1(d, e; h^e(\mathcal{O}_X(-e))) + 1$.
(iv) $h^1(\mathcal{J}_X(n)) \leq B^*(d, e+1, h^e(\mathcal{O}_X(-e)); n-1), \quad \forall n \geq 0$.
(v) $h^1(\mathcal{J}_X(n)) = 0, \quad \forall n \geq C^*(d, e+1, h^e(\mathcal{O}_X(-e)))$. □

Originally, the hyperplane section method only works over algebraically closed fields. (Its failure in the real case may be deduced from the counterexample (2.3) of [6]). But, as cohomological Hilbert functions are not affected by base field extensions, our theorems (1.2) and (1.4) are valid for projective varieties over arbitrary fields.

Finally, let us say a few words about the organization of the present paper, which is divided up into eight sections. In Section 2 we introduce the notion of the (linear) dimension spectrum of a coherent sheaf and note a few basic facts about the global subdepth. These preliminaries are needed to understand the invariants $\delta^{(0)}(\mathcal{F})$ and $\operatorname{lsdim}^{(0)}(\mathcal{F})$ occurring in our theorems (1.1) and (1.3). Section 3 is devoted to present what we shall use from the hyperplane section method. Section

4 still has auxiliary character. It namely gives some results of Bertini-type for the invariants introduced in Section 2. In Section 5 all these preliminaries are combined to give a recursive procedure to bound cohomological Hilbert functions by the invariants occurring in (1.1). Next—in Section 6—we use the preceding results to introduce the bounding functions $B_{e,i}$, $C_{e,i}$, $G_{e,i}$ and $F_{e,i}$ and to prove our theorems (1.1) and (1.2). In Section 7 we investigate Castelnuovo-regularities for arbitrary coherent sheaves and prove in particular (1.3) and (1.4). Finally, Section 8 is concerned with the study of the cohomology of smooth, projective complex varieties. There, our last theorem (1.5) will be established.

As for the unexplained terminology and notations, we refer to [17], [21] and [30].

It should be noted, that all our results stay valid in the complex analytic case. (cf. [40], [16]).

2. Dimension spectra and global subdepth

Let k be an algebraically closed field. We write $S = k \oplus S_1 \oplus S_2 \oplus \cdots$ for the polynomial ring $k[z_0, \ldots, z_d]$, thereby considering S as a graded k-algebra in the canonical way. Moreover we consider the projective space $\mathbb{P}^d := \operatorname{Proj}(S)$ and fix a coherent sheaf $\mathscr{F}$ of $\mathcal{O}_{\mathbb{P}^d}$-modules.

(2.1) DEFINITION. The *dimension spectrum* $\underline{\dim}(\mathscr{F})$ of $\mathscr{F}$ is defined as follows:

$$\underline{\dim}(\mathscr{F}) := \begin{cases} \{\dim \overline{\{x\}} \mid x \in \operatorname{Ass}(\mathscr{F})\}, & \text{if } \mathscr{F} \neq 0, \\ \{-1\}, & \text{if } \mathscr{F} = 0. \end{cases}$$

The *subdimension* $\operatorname{sdim}(\mathscr{F})$ is defined by

$$\operatorname{sdim}(\mathscr{F}) := \min \underline{\dim}(\mathscr{F}). \qquad \bigcirc$$

Now, let $X \subseteq \mathbb{P}^d$ be a non-empty closed subset. We write $\langle X \rangle$ for the *linear span* of X. So, $\langle X \rangle$ is the intersection of all linear projective subspaces $\mathbb{P}^s \subseteq \mathbb{P}^d$ which contain X.

Now, the *linear dimension* $\operatorname{ldim} X$ of X is defined as the dimension of the linear span of X:

$$\operatorname{ldim} X := \dim \langle X \rangle.$$

(2.2) DEFINITION. The *linear dimension spectrum* $\underline{\mathrm{ldim}}\,(\mathscr{F})$ of the sheaf $\mathscr{F}$ is defined as follows:

$$\underline{\mathrm{ldim}}\,(\mathscr{F}) := \begin{cases} \{\mathrm{ldim}\,\overline{\{x\}} \mid x \in \mathrm{Ass}\,(\mathscr{F})\}, & \text{if } \mathscr{F} \neq 0, \\ \{-1\}, & \text{if } \mathscr{F} = 0. \end{cases}$$

The *linear subdimension* $\mathrm{lsdim}\,(\mathscr{F})$ is defined by

$$\mathrm{lsdim}\,(\mathscr{F}) := \min \underline{\mathrm{ldim}}\,(\mathscr{F}). \qquad \bigcirc$$

Now, we want to introduce the notion of reduced dimension spectra.

(2.3) DEFINITION. Let r be an integer ≥ 0. Then, the r-th *reduction* of $\mathrm{Ass}\,(\mathscr{F})$ is defined as:

$$\mathrm{Ass}^{(r)}\,(\mathscr{F}) := \{x \in \mathrm{Ass}\,(\mathscr{F}) \mid \dim \overline{\{x\}} \geq r\}.$$

The *r-reduced subdimension spectrum* $\underline{\dim}^{(r)}\,(\mathscr{F})$ of $\mathscr{F}$ is defined by:

$$\underline{\dim}^{(r)}\,(\mathscr{F}) := \begin{cases} \{\dim \overline{\{x\}} - r \mid x \in \mathrm{Ass}^{(r)}\,(\mathscr{F})\}, & \text{if } \mathrm{Ass}^{(r)}\,(\mathscr{F}) \neq \emptyset \\ \{-1\}, & \text{if } \mathrm{Ass}^{(r)}\,(\mathscr{F}) = \emptyset \end{cases}$$

The *r-reduced dimension* $\mathrm{sdim}^{(r)}\,(\mathscr{F})$ of $\mathscr{F}$ is defined by:

$$\mathrm{sdim}^{(r)}\,(\mathscr{F}) := \min \underline{\dim}^{(r)}\,(\mathscr{F}).$$

The *r-reduced linear subdimension spectrum* $\underline{\mathrm{ldim}}^{(r)}\,(\mathscr{F})$ of $\mathscr{F}$ correspondingly is defined as:

$$\underline{\mathrm{ldim}}^{(r)}\,(\mathscr{F}) := \begin{cases} \{\mathrm{ldim}\,\overline{\{x\}} - r \mid x \in \mathrm{Ass}^{(r)}\,(\mathscr{F})\}, & \text{if } \mathrm{Ass}^{(r)}\,(\mathscr{F}) \neq \emptyset \\ \{-1\}, & \text{if } \mathrm{Ass}^{(r)}\,(\mathscr{F}) \neq \emptyset. \end{cases}$$

The *r-reduced linear subdimension* $\mathrm{lsdim}^{(r)}\,(\mathscr{F})$ of $\mathscr{F}$ is given by

$$\mathrm{lsdim}^{(r)}\,(\mathscr{F}) := \min \underline{\mathrm{ldim}}^{(r)}\,(\mathscr{F}).$$

(2.4) REMARKS. (A) As the closure of $\mathrm{Ass}\,(\mathscr{F})$ coincides with the support $\mathrm{supp}\,(\mathscr{F})$ of $\mathscr{F}$, we have

(i) $\dim\,(\mathscr{F}) = \dim \mathrm{supp}\,(\mathscr{F}) = \max \underline{\dim}\,(\mathscr{F})$.

Thereby we use the convention, that $\dim \emptyset = -1$. In particular—observing in addition that $\dim X \leq \operatorname{ldim} X$—we get:

(ii) $\operatorname{sdim}(\mathscr{F}) \leq \dim(\mathscr{F})$, $\quad \operatorname{sdim}(\mathscr{F}) \leq \operatorname{lsdim}(\mathscr{F})$

(iii) $\mathscr{F} = 0 \Leftrightarrow \operatorname{sdim}(\mathscr{F}) = -1 \Leftrightarrow \dim(\mathscr{F}) = -1 \Leftrightarrow \operatorname{lsdim}(\mathscr{F}) = -1$.

(B) Let r be a non-negative integer. Then

(iv) $\operatorname{sdim}^{(r)}(\mathscr{F}) \leq \operatorname{lsdim}^{(r)}(\mathscr{F})$.

Observing that $\operatorname{Ass}^{(r)}(\mathscr{F}) \neq \emptyset$ if $\dim(\mathscr{F}) > r$, we get:

(v) (a) $\dim(\mathscr{F}) > r \Rightarrow \operatorname{sdim}^{(r)}(\mathscr{F}), \quad \operatorname{lsdim}^{(r)}(\mathscr{F}) > 0$

(b) $\dim(\mathscr{F}) \leq r \Leftrightarrow \operatorname{sdim}^{(r)}(\mathscr{F}) = -1 \Leftrightarrow \operatorname{lsdim}^{(r)}(\mathscr{F}) = -1$.

Note also the following statement, which follows immediately from the definitions:

(vi) *If* $\dim(\mathscr{F}) > r + 1$, *then*:

(a) $\operatorname{sdim}^{(r+1)}(\mathscr{F}) \geq \operatorname{sdim}^{(r)}(\mathscr{F}) - 1$,

(b) $\operatorname{lsdim}^{(r+1)}(\mathscr{F}) \geq \operatorname{lsdim}^{(r)}(\mathscr{F}) - 1$.

If $\operatorname{sdim}^{(r)}(\mathscr{F}) > 1$, then, equality holds in (a) and (b).

By induction we get from (vi):

(vii) *If* $\dim(\mathscr{F}) > r$, *then*:

$$\operatorname{sdim}^{(r)}(\mathscr{F}) \geq \operatorname{sdim}(\mathscr{F}) - r, \quad \operatorname{lsdim}^{(r)}(\mathscr{F}) \geq \operatorname{lsdim}^{(0)}(\mathscr{F}) - r.$$

If $\operatorname{sdim}(\mathscr{F}) > r$, then equality holds in both places.

(C) For later use we notice:

(iix) $$\underline{\dim}^{(0)}(\mathscr{F}) = \begin{cases} \{\dim \overline{\{x\}} \mid x \in \operatorname{Ass}(\mathscr{F}),\, x \textit{ non closed}\}, & \textit{if } \dim(\mathscr{F}) > 0, \\ \{-1\}, & \textit{if } \dim(\mathscr{F}) \leq 0. \end{cases}$$

(ix) $$\underline{\operatorname{ldim}}^{(0)}(\mathscr{F}) = \begin{cases} \{\operatorname{ldim} \overline{\{x\}} \mid x \in \operatorname{Ass}(\mathscr{F}),\, x \textit{ non closed}\}, & \textit{if } \dim(\mathscr{F}) > 0, \\ \{-1\}, & \textit{if } \dim(\mathscr{F}) \leq 0. \end{cases} \qquad \bigcirc$$

Besides the previous dimension related invariants, we also will introduce an invariant of $\mathscr{F}$ which is related to the depths of the stalks of $\mathscr{F}$.

(2.5) DEFINITION. The *global subdepth* $\delta(\mathscr{F})$ of the sheaf $\mathscr{F}$ is defined by

$$\delta(\mathscr{F}) := \begin{cases} \min\{\operatorname{depth}(\mathscr{F}) \mid x \in \mathbb{P}^d,\, x \text{ closed}\}, & \text{if } \mathscr{F} \neq 0. \\ -1, & \text{if } \mathscr{F} = 0. \end{cases}$$

The 0-*reduced global subdepth* $\delta^{(0)}(\mathscr{F})$ of $\mathscr{F}$ is defined as:

$$\delta^{(0)}(\mathscr{F}) := \begin{cases} \min\{\operatorname{depth}(\mathscr{F}) \mid x \in \mathbb{P}^d,\ x \text{ closed},\ x \notin \operatorname{Ass}(\mathscr{F})\}, & \text{if } \dim(\mathscr{F}) > 0, \\ -1, & \text{if } \dim(\mathscr{F}) \leq 0, \end{cases}$$

(2.6) REMARK. Note the following obvious relations:

(i) $\delta(\mathscr{F}) \leq \operatorname{sdim}(\mathscr{F}),\ \delta^{(0)}(\mathscr{F}) \leq \operatorname{sdim}^{(0)}(\mathscr{F})$

(ii) $\delta(\mathscr{F}) > 0 \Leftrightarrow \operatorname{sdim}(\mathscr{F}) > 0 \Leftrightarrow \delta(\mathscr{F}) = \delta^{(0)}(\mathscr{F}) \geq 0$

(iii) $\dim(\mathscr{F}) > 0 \Rightarrow \delta^{(0)}(\mathscr{F}) \geq \delta(\mathscr{F})$. ○

To compare the reduced invariants to the non-reduced ones, we introduce a special sheaf $\bar{\bar{\mathscr{F}}}$, the so called reduction of $\mathscr{F}$.

We begin with defining the *torsion-subsheaf* $T(\mathscr{F})$ of $\mathscr{F}$. $T(\mathscr{F})$ is the subsheaf of sections whose support consists of only finitely many points. $T(\mathscr{F})$ is nothing else than the maximal subsheaf of $\mathscr{F}$ which is of finite length. $T(\mathscr{F})$ also may be described as the maximal subsheaf of $\mathscr{F}$ whose support consists exactly of the closed members of $\operatorname{Ass}(\mathscr{F})$.

(2.7) DEFINITION. The *reduction* $\bar{\bar{\mathscr{F}}}$ of $\mathscr{F}$ is defined as the coherent sheaf $\mathscr{F}/T(\mathscr{F})$. ○

(2.8) REMARK. From the definition of $\bar{\bar{\mathscr{F}}}$ the following statements are obvious:

(i) $\operatorname{Ass}(\bar{\bar{\mathscr{F}}}) = \operatorname{Ass}^{(0)}(\mathscr{F})$.

(ii) $\bar{\bar{\mathscr{F}}}_x = \mathscr{F}_x$ for all $x \notin \operatorname{Ass}(\mathscr{F}) - \operatorname{Ass}^{(0)}(\mathscr{F})$.

(iii) $\bar{\bar{\mathscr{F}}} = 0 \Leftrightarrow T(\mathscr{F}) = \mathscr{F} \Leftrightarrow \dim(\mathscr{F}) \leq 0$.

(iv) $\bar{\bar{\mathscr{F}}} = 0 \Leftrightarrow T(\mathscr{F}) = 0 \Leftrightarrow \operatorname{sdim}(\mathscr{F}) > 0$, (for $\mathscr{F} \neq 0$). ○

Now, from (2.8)(i) and (ii) we may draw the following immediate conclusion:

(2.9) PROPOSITION. *Let* $\bar{\bar{\mathscr{F}}}$ *be the reduction of* $\mathscr{F}$. *Then*:

(i) $\underline{\dim}^{(0)}(\mathscr{F}) = \underline{\dim}(\bar{\bar{\mathscr{F}}})$.

(ii) $\underline{\operatorname{ldim}}^{(0)}(\mathscr{F}) = \underline{\operatorname{ldim}}(\bar{\bar{\mathscr{F}}})$.

(iii) $\operatorname{sdim}^{(0)}(\mathscr{F}) = \operatorname{sdim}(\bar{\bar{\mathscr{F}}})$.

(iv) $\operatorname{lsdim}^{(0)}(\mathscr{F}) = \operatorname{lsdim}(\bar{\bar{\mathscr{F}}})$.

(v) $\delta^{(0)}(\mathscr{F}) = \delta(\bar{\bar{\mathscr{F}}})$. □

The previous proposition (2.9) later on often will be used to express 0-reduced invariants of $\mathscr{F}$ by the corresponding non reduced invariants of the reduction $\bar{\bar{\mathscr{F}}}$ of $\mathscr{F}$. As we then will be treating higher cohomology, this replacement is justified by the following observation.

(2.10) REMARK. Consider the twisted short exact sequences $0 \to T(\mathcal{F})(n) \to \mathcal{F}(n) \to \bar{\mathcal{F}}(n) \to 0$. As $T(\mathcal{F}(n))$ is of finite support, $H^i(\mathbb{P}^d, T(\mathcal{F})(n))$ vanishes for all $i > 0$. So applying cohomology to the above sequence, we get natural isomorphisms.

(i) $H^i(\mathbb{P}^d, \mathcal{F}(n)) \cong H^i(\mathbb{P}^d, \bar{\mathcal{F}}(n)), \quad \forall i > 0, \quad \forall n \in \mathbb{Z}$.

By (2.8)(i) we get $\mathrm{Ass}^{(r)}(\bar{\mathcal{F}}) = \mathrm{Ass}^{(r)}(\mathcal{F})$ for all $r \geq 0$. Consequently:

(ii) $\underline{\dim}^{(r)}(\bar{\mathcal{F}}) = \underline{\dim}^{(r)}(\mathcal{F}), \quad \mathrm{sdim}^{(r)}(\bar{\mathcal{F}}) = \mathrm{sdim}^{(r)}(\mathcal{F})$;

(iii) $\underline{\mathrm{ldim}}^{(r)}(\bar{\mathcal{F}}) = \underline{\mathrm{ldim}}^{(r)}(\mathcal{F}), \quad \mathrm{lsdim}^{(r)}(\bar{\mathcal{F}}) = \mathrm{lsdim}^{(r)}(\mathcal{F})$. ○

Finally, we introduce the following notation:

(2.11) DEFINITION. Let r be a non-negative integer. Then, the *r-reduced global subdepth* $\delta^{(r)}(\mathcal{F})$ is defined by:

$$\delta^{(r)}(\mathcal{F}) := \begin{cases} \max\{1, \delta^{(0)}(\mathcal{F}) - r\}, & \text{if } \dim(\mathcal{F}) > r. \\ -1, & \text{if } \dim(\mathcal{F}) \leq r. \end{cases}$$ ○

(2.12) REMARK. In view of (2.6)(i), (2.4)(ii), (vii) we have:

(i) $\dim(\mathcal{F}) > r \Rightarrow 0 < \delta^{(r)}(\mathcal{F}) \leq \mathrm{lsdim}^{(r)}(\mathcal{F})$.

Moreover (2.9)(v) shows that

(ii) $\delta^{(r)}(\bar{\mathcal{F}}) = \delta^{(r)}(\mathcal{F})$. ○

3. Restrictions to general hyperplanes

As in the previous section we write $\mathbb{P}^d = \mathrm{Proj}(S)$, where $S = \bigoplus_{n \leq 0} S_n$ is the polynomial ring $k[z_0, \ldots, z_d]$ in the indeterminates $z_0, \ldots, z_d$ over the algebraically closed ground field k. Moreover we fix a coherent sheaf $\mathcal{F}$ over $\mathbb{P}^d$. We suppose $d > 0$.

Now, let $f \in S_1 - \{0\}$ be a non-trivial linear form in S. We write H_f for the hyperplane defined in $\mathbb{P}^d$ by f:

$$H_f := \mathrm{Proj}(S/fS) \cong \mathbb{P}^{d-1} \subseteq \mathbb{P}^d.$$

If $M \subseteq S_1$ is a set of linear forms, we write $\mathfrak{H}(M)$ for the set of all hyperplanes defined by M:

$$\mathfrak{H}(M) := \{H_f \mid f \in M - \{0\}\}.$$

The set $\mathfrak{H}(S_1)$ of all hyperplanes $H \subseteq \mathbb{P}^d$ will be denoted by $\mathfrak{G}$.

If $L \subseteq S_1$ is a k-vector-space of dimension $N > 0$, then $\mathfrak{H}(L)$ is a *linear system* of dimension $N - 1$.

Now, let $H \subseteq \mathbb{P}^d$ be a hyperplane. H is said to be *general with respect to the sheaf* $\mathscr{F}$, if H avoids all points associated to $\mathscr{F}$:

$$H \text{ general with respect to } \mathscr{F} :\Leftrightarrow H \cap \operatorname{Ass}(\mathscr{F}) = \emptyset$$

Moreover a linear form $f \in S_1 - \{0\}$ is called *general with respect to* $\mathscr{F}$, if the corresponding hyperplane H_f is general with respect to $\mathscr{F}$.

A set $\mathfrak{H} \subseteq \mathfrak{G}$ of hyperplanes is called *general with respect to* $\mathscr{F}$ if all $H \in \mathfrak{H}$ are general with respect to $\mathscr{F}$. Correspondingly a set $M \subseteq S_1$ of linear forms is called general with respect to $\mathscr{F}$, if the set $\mathfrak{H}(M)$ is.

In the present paper, the "hyperplane section method" presented in [8] will play an important role. Therefore we need to know about the existence of large general linear systems. The following result will be useful in this respect:

(3.1) PROPOSITION. *Let* $\mathscr{F} \neq 0$, *and let* $\mathfrak{H} \subseteq \mathfrak{G}$ *be a linear system of hyperplanes, which is general with respect to* $\mathscr{F}$. *Then, there is a linear system* $\mathfrak{L} \subseteq \mathfrak{G}$ *which is general with respect to* $\mathscr{F}$ *and such that*

$$\mathfrak{H} \subseteq \mathfrak{L}, \quad \dim(\mathfrak{L}) = \operatorname{lsdim}(\mathscr{F}).$$

Proof. Put $\{x_1, \ldots, x_r\} = \operatorname{Ass}(\mathscr{F})$, thereby considering x_i as a homogenous prime ideal of S. We write $V_i = S_1 \cap x_i$, and let $M \subseteq S_1$ be the k-space of linear forms which defines $\mathfrak{H}$. Then $M \cap V_i = 0$ for $i = 1, \ldots, r$. Moreover $\mu := \min_{i=1}^r \dim(S_1/V_i)$ equals $\operatorname{lsdim}(\mathscr{F}) + 1$.

So, as k is infinite, we find a linear subspace $L \subseteq S_1$ with the following three properties:

$$M \subseteq L: \quad L \cap V_i = 0 \ (i = 1, \ldots, r); \quad \dim_k(L) = \mu.$$

Setting $\mathfrak{L} := \mathfrak{H}(L)$, we thus get a linear system with the requested properties. □

(3.2) COROLLARY. *Let* $\mathscr{F} \neq 0$. *Then*:

(i) *There is a linear system* $\mathfrak{H}$ *of hyperplanes general with respect to* $\mathscr{F}$ *and such that* $\dim(\mathfrak{H}) = \operatorname{lsdim}(\mathscr{F})$.

(ii) *Any linear system* $\mathfrak{L}$ *of hyperplanes which is general with respect to* $\mathscr{F}$ *satisfies* $\dim(\mathfrak{L}) \leq \operatorname{lsdim}(\mathscr{F})$. □

As already said above, a crucial technical point of the present paper is to apply the hyperplane section method of [8]. We now will present what will be used of this method. We begin with introducing some notations.

If $H = H_f = \operatorname{Proj}(S/fS)$ is a hyperplane, we will have to look at the restriction

$$\mathscr{F} \upharpoonright H = \mathcal{O}_H \otimes \mathscr{F}$$

of $\mathscr{F}$ to H.

If n is an integer, we write for $i = 0, 1, \ldots$

$$\text{(3.3)} \quad \text{(i)}\ h^i(\mathscr{F}(n)) := \dim_k H^i(\mathbb{P}^d, \mathscr{F}(n)),$$
$$\text{(ii)}\ h^i(\mathscr{F} \upharpoonright H(n)) := \dim_k H^i(H, (\mathscr{F} \upharpoonright H)(n)).$$

Thereby twisting on H is defined by means of the canonical embedding $H \hookrightarrow \mathbb{P}^d$. The function $n \mapsto h^i(\mathscr{F}(n))$ is called the i-th *cohomological Hilbert function of* $\mathscr{F}$. Next, we define the i-th *right-regularity* of $\mathscr{F}$ by

$$\text{(3.4)} \quad \mu^i_{\mathscr{F}} := \sup\{n \in \mathbb{Z} \mid h^i(\mathscr{F}(n-1)) \neq 0\}, \quad (i > 0),$$

thereby using the convention that $\sup \emptyset = -\infty$. It is well known, that $\mu^i_{\mathscr{F}} < \infty$ for all $i > 0$ (cf. [3, No. 63 Prop. 3]). It will be one of our main tasks to give upper bounds for the right-regularities $\mu^i_{\mathscr{F}}$ of $\mathscr{F}$ for $i = 1, 2, \ldots$.

Using the hyperplane section method, such bounds may be obtained from the Hilbert functions $h^i(\mathscr{F} \upharpoonright H(n))$, where H runs through an appropriate linear system of hyperplanes.

To give precise statements, we have to introduce some more notations. So, let $\mathfrak{H} \subseteq \mathfrak{G}$ be a linear system of hyperplanes in $\mathbb{P}^d$. We then put

$$\text{(3.5)} \quad \text{(i)}\ h^i(\mathscr{F} \upharpoonright \mathfrak{H}(n)) := \sup\{h^i(\mathscr{F} \upharpoonright H(n)) \mid H \in \mathfrak{H}\},$$
$$\text{(ii)}\ \bar{h}^i(\mathscr{F} \upharpoonright \mathfrak{H}(n)) := \inf\{h^i(\mathscr{F} \upharpoonright H(n)) \mid H \in \mathfrak{H}\}.$$

$$\text{(3.6)} \quad \text{(i)}\ \mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}} := \sup\{\mu^i_{\mathscr{F} \upharpoonright H} \mid H \in \mathfrak{H}\}.$$
$$\text{(ii)}\ \bar{\mu}^i_{\mathscr{F} \upharpoonright \mathfrak{H}} := \inf\{\mu^i_{\mathscr{F} \upharpoonright H} \mid H \in \mathfrak{H}\}.$$

(3.7) REMARK. Fix n. Then, there is a non-empty open subset U of $\mathfrak{H}$ (with respect to the canonical Zariski-topology on the Grassmannian of all hyperplanes in $\mathbb{P}^d$) such that $h^i(\mathscr{F} \upharpoonright H(n)) = \bar{h}_i(\mathscr{F} \upharpoonright \mathfrak{H}(n))$ for all $H \in U$. Therefore $\bar{h}_i(\mathscr{F} \upharpoonright \mathfrak{H}(n))$ is referred to as the *generic value* of $h^i(\mathscr{F} \upharpoonright H(n))$ for $H \in \mathfrak{H}$.

Similarly $\bar{\mu}^i_{\mathscr{F} \upharpoonright \mathfrak{H}}$ is attained by $\mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}}$ for all H which belong to a non-empty open set $U \subseteq \mathfrak{H}$. Therefore $\bar{\mu}^i_{\mathscr{F} \upharpoonright \mathfrak{H}}$ is called the generic i-th *right regularity* of $\mathscr{F} \upharpoonright H$ for $H \in \mathfrak{H}$. ○

Now, the hyperplane section method gives the following result (cf. [8, (3.11)]).

(3.8) PROPOSITION. *Let* $\mathfrak{H} \subseteq \mathfrak{G}$ *be a linear system of hyperplanes which is general with respect to* $\mathscr{F}$. *Assume, that* $\mathfrak{H}$ *is of positive dimension, and let* $i > 0$, $n_0 \in \mathbb{Z}$. *Put* $l^i(n) := h^i(\mathscr{F}(n_0)) + \Sigma_{n_0 < m \leq n} \bar{h}^i(\mathscr{F} \upharpoonright \mathfrak{H}(n))$. *Then*:

$$\text{(i)}\ \mu^i_{\mathscr{F}} \leq \max\{n_0, \mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}}\} + \frac{l^i(\bar{\mu}^i_{\mathscr{F} \upharpoonright \mathfrak{H}})}{\dim(\mathfrak{H})} - 1.$$

$$\text{(ii)}\ h^i(\mathscr{F}(n)) \leq \begin{cases} l^i(n), & \textit{for } n_0 \leq n < \bar{\mu}^i_{\mathscr{F} \upharpoonright \mathfrak{H}} \\ l^i(\bar{\mu}^i_{\mathscr{F} \upharpoonright \mathfrak{H}}), & \textit{for } \bar{\mu}^i_{\mathscr{F} \upharpoonright \mathfrak{H}} \leq n < \mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}} \\ l^i(\bar{\mu}^i_{\mathscr{F} \upharpoonright \mathfrak{H}}) - (n - \mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}} + 1)\dim(\mathfrak{H}), & \textit{for } \mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}} \leq n < \mu^i_{\mathscr{F}} \\ 0, & \textit{for } n \geq \mu^i_{\mathscr{F}} \end{cases} \qquad \square$$

(3.9) REMARK. The previous result provides us with an inductive method for finding upper bounds on the cohomological Hilbert function $h^i(\mathscr{F}(n))$. Thereby, we will proceed as follows:

(i) *Assume that* $\dim(\mathscr{F}) > 0$ *and* $\operatorname{lsdim}(\mathscr{F}) > 0$. *Assume moreover given an upper bound on* $h^i(\mathscr{F}(n_0))$ *for some* $n_0 \in \mathbb{Z}$.

(ii) *Suppose given some uniform upper bound on* $\mu^i_{\mathscr{F} \upharpoonright H}$, H *running through all hyperplanes which are general with respect to* $\mathscr{F}$.

(iii) *Assume moreover, that upper bounds are known on* $h^i(\mathscr{F} \upharpoonright \bar{H}(n))$ $(n > n_0)$ *and on* $\mu^i_{\mathscr{F} \upharpoonright \bar{H}}$ *for some special hyperplane* $\bar{H}$ *which is general with respect to* $\mathscr{F}$.

(iv) *Choose a linear system* $\mathfrak{H}$ *of hyperplanes, which contains* $\bar{H}$, *which is general with respect to* $\mathscr{F}$ *and whose dimension has the largest possible value* $\operatorname{lsdim}(\mathscr{F})$.

(v) *Apply* (3.8) *after having replaced* $h^i(\mathscr{F}(n_0))$, $\mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}}$, $\bar{h}^i(\mathscr{F} \upharpoonright \mathfrak{H}(n))$ *and* $\mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}}$ *by the corresponding upper bounds given in* (i), (ii), (iii) *respectively*. ○

If $\dim(\mathscr{F}) > 1$, the bounds requested in (3.9)(ii), (iii) already are obtained by applying the described induction procedure to all pairs $(H, \mathscr{F} \upharpoonright H)$, where $H \cong \mathbb{P}^{d-1}$ runs through all hyperplanes which are general with respect to $\mathscr{F}$. Therefore information is needed about the invariants $\operatorname{lsdim}(\mathscr{F} \upharpoonright H)$ for all such H. To get the uniform bounds mentioned in (ii), we have to look for a uniform lower bound on the numbers $\operatorname{lsdim}(\mathscr{F} \upharpoonright H)$. In view of the inequalities (2.4)(ii) and (2.6)(i), it suffices to find uniform lower bounds on $\delta(\mathscr{F} \upharpoonright H)$ (resp. on $\delta^{(0)}(\mathscr{F} \upharpoonright H)$). As we will see in a moment, finding these latter bounds is easy.

To get the "generic" bounds mentioned in (iii), we need to know the maximal value of $\operatorname{lsdim}(\mathscr{F} \upharpoonright H)$ (resp. of $\operatorname{lsdim}^{(0)}(\mathscr{F} \upharpoonright H)$). This needs a more detailed insight into the behaviour of the linear dimension spectrum $\underline{\mathrm{ldim}}(\mathscr{F} \restriction H)$ for a generic hyperplane H. A detailed study of this latter problem will be given in the next section.

Before looking at the (reduced) global subdepths $\delta(\mathscr{F} \restriction H)$ (resp. $\delta^{(0)}(\mathscr{F} \restriction H)$) of the restrictions $\mathscr{F} \restriction H$ we prove the following result:

(3.10) LEMMA. *Let $y \in H$, where $H \subseteq \mathbb{P}^d$ is a hyperplane which is general with respect to $\mathscr{F}$. Then $(\mathscr{F} \restriction H)_y \subseteq \mathscr{F}_y/a\mathscr{F}_y$, where $a \in \mathfrak{m}_{\mathbb{P}^d,y}$ is a non-zero-divisor with respect to $\mathscr{F}_y$.*

Proof. The local vanishing ideal $\mathscr{I}_y \subseteq \mathcal{O}_{\mathbb{P}^d,y}$ of H at y is a proper principal ideal, hence of the form $\mathscr{I}_y = a\mathcal{O}_{\mathbb{P}^d,y}$, with $a \in \mathfrak{m}_{\mathbb{P}^d,y}$. As H is general with respect to $\mathscr{F}$, a has to avoid all members of $\mathrm{Ass}(\mathscr{F}_y)$, and hence is a non-zero divisor with respect to $\mathscr{F}_y$. As $(\mathscr{F} \restriction H)_y = \mathcal{O}_{H,y} \otimes \mathscr{F}_y = \mathcal{O}_{\mathbb{P}^d,y}/a\mathcal{O}_{\mathbb{P}^d,y} \otimes \mathscr{F}_y = \mathscr{F}_y/a\mathscr{F}_y$ we get our claim. □

(3.11) LEMMA. *Let $H \subseteq \mathbb{P}^d$ be a hyperplane. Then, for the reductions introduced in* (2.7) *we have*

$$\overline{(\mathscr{F} \restriction H)} = \overline{(\mathscr{F} \restriction H)}.$$

Proof. Immediate from the observation that the kernel of the canonical map $\mathscr{F} \restriction H \to \mathscr{F} \restriction H$ is of finite length hence contained in the torsion subsheaf $T(\mathscr{F} \restriction H)$. □

(3.12) PROPOSITION. *Let $\mathscr{F} \neq 0$ and let $H \subseteq \mathbb{P}^d$ be a hyperplane which is general with respect to $\mathscr{F}$. Then*

(i) $\delta(\mathscr{F} \restriction H) \geq \delta(\mathscr{F}) - 1$.

(ii) *If* $\dim(\mathscr{F}) > 1$, *then* $\delta^{(0)}(\mathscr{F} \restriction H) \geq \delta^{(0)}(\mathscr{F}) - 1$.

(iii) *If* $\dim(\mathscr{F}) > r + 1$, *then* $\delta^{(r)}(\mathscr{F} \restriction H) \geq \delta^{(r+1)}(\mathscr{F}) > 0$.

Proof. (i) The case $\delta(\mathscr{F}) = 0$ is obvious. So let $\delta(\mathscr{F}) > 0$. Choose $y \in H$. Then, by (3.10), $\mathrm{depth}((\mathscr{F} \restriction H)_y) = \mathrm{depth}(\mathscr{F}_y) - 1$. Making y run through all closed points of H, we get our claim.

(ii) Apply (i) to $\overline{\mathscr{F}}$, thereby observing (3.11), (2.6) and (2.9)(v).

(iii) Immediate from (i), observing the definition (2.11). □

We close this section by a lemma, which will be used later.

(3.13) LEMMA. *Let $\mathscr{F} \neq 0$ and let $H \subseteq \mathbb{P}^d$ be a hyperplane which is general with respect to $\mathscr{F}$. Let $x \in \mathrm{Ass}(\mathscr{F})$. Then, any generic point y of $\overline{\{x\}} \cap H$ belongs to* $\mathrm{Ass}(\mathscr{F} \restriction H)$.

Proof. According to (3.11) we may write $(\mathcal{F} \upharpoonright H)_y = \mathcal{F}_y/a\mathcal{F}_y$, where $a \in \mathfrak{m}_{\mathbb{P}^d,y}$ is a non-zero-divisor with respect to $\mathcal{F}_y$. Consequently depth $((\mathcal{F} \upharpoonright H)_y) =$ depth $(\mathcal{F}_y) - 1$. As $\mathcal{F}_y$ admits the non-zero divisior $a \in \mathfrak{m}_{\mathbb{P}^d,y}$, we have depth $(\mathcal{F}_y) \geq 1$. As $x \in \mathrm{Ass}(\mathcal{F})$, $y \in \overline{\{x\}}$ and $\mathrm{codim}_{\overline{\{x\}}}\{\overline{\{y\}}\} \leq 1$ we have depth $(\mathcal{F}_y) \leq 1$. Therefore depth $(\mathcal{F}_y) = 1$, and consequently depth $((\mathcal{F} \upharpoonright H)_y) = 0$, hence $y \in \mathrm{Ass}(\mathcal{F} \upharpoonright H)$. □

(3.14) REMARK. The previous lemma will be used in the next section. For the moment we notice, that it immediately proves the following easy facts:

(i) *If* $\mathcal{F} \neq 0$, *then* $\dim(\mathcal{F} \upharpoonright H) = \dim(\mathcal{F}) - 1$.

(ii) *If* $\dim(\mathcal{F}) > 0$, *then* $\mathrm{sdim}(\mathcal{F} \upharpoonright H) \leq \mathrm{sdim}^{(0)}(\mathcal{F}) - 1$ *and* $\mathrm{lsdim}(\mathcal{F} \upharpoonright H) \leq \mathrm{lsdim}^{(0)}(\mathcal{F}) - 1$.

4. Generic hyperplanes

In the sequel let X be an arbitrary noetherian scheme, and let $\mathcal{G}$ denote a coherent sheaf of $\mathcal{O}_X$-modules.

(4.1) DEFINITION. A point $z \in X$ is called *critical* with respect to $\mathcal{G}$, if it satisfies the following properties.

(i) depth $(\mathcal{G}_z) = 1$.

(ii) $\mathrm{codim}_{\overline{\{x\}}}(\overline{\{z\}}) > 1$ for all $x \in \mathrm{Ass}(\mathcal{G})$ with $z \in \overline{\{x\}}$.

Moreover we put:

$$\mathfrak{C}_X(\mathcal{G}) = \mathfrak{C}(\mathcal{G}) := \{z \in X \mid z \text{ critical with respect to } \mathcal{G}\} \qquad \bigcirc$$

First we want to show that $\mathfrak{C}(\mathcal{G})$ is finite, whenever the scheme X is excellent. Obviously, for an open subset U of X we have

$$(4.2) \quad \mathfrak{C}(\mathcal{G}) \cap U = \mathfrak{C}(\mathcal{G} \upharpoonright U).$$

So, as X admits a finite affine open covering, we may put our attention to the case where X is affine. To treat this particular case, we define:

(4.3) DEFINITION. Let R be a noetherian ring, and let M be a finitely generated R-Module. A prime ideal $\mathfrak{p} \in \mathrm{Spec}(R)$ is called *critical* with respect to M, if:

(i) depth $(M_{\mathfrak{p}}) = 1$.

(ii) $ht(\mathfrak{p}/\mathfrak{q}) > 1$ *for all* $\mathfrak{q} \in \mathrm{Ass}(M)$ with $\mathfrak{q} \subseteq \mathfrak{p}$.

Moreover we put

$C(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathfrak{p}$ critical with respect to $M\}$. ○

(4.4) REMARK. Keeping the notations of (4.3) and denoting by $\tilde{M}$ the coherent sheaf induced by M over Spec (R), we obviously have

$$\mathfrak{C}_{\operatorname{Spec}(R)}(\tilde{M}) = C_R(M).$$

So, our task is reduced to study the sets $C_R(M)$ for finitely generated modules M over a noetherian excellent Ring R. To do so, we introduce ideal-transforms in the sense of Grothendieck [18] and Brodmann [4].

(4.5) DEFINITION. Let R be a noetherian ring, and let $\mathbb{J}$ be a multiplicative filter of ideals of R. (So $\mathbb{J}$ a set of ideals of R such that $J, L \in \mathbb{J}$ always induces $JL \in \mathbb{J}$). Then, the $\mathbb{J}$-transform is the covariant, left-exact functor on the category of R-modules, which is defined by:

$$M \mapsto \varinjlim_{J \in \mathbb{J}} \operatorname{Hom}_R(J, M) := D_{\mathbb{J}}(M). \qquad ○$$

(4.6) DEFINITION. Let R and $\mathbb{J}$ be as in (4.5), and let M be a finitely generated R-module. Then we put

$$P_{\mathbb{J}}(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \operatorname{depth}(M_{\mathfrak{p}}) = 1 \wedge \exists J \in \mathbb{J} : J \subseteq \mathfrak{p}\} \qquad ○$$

The following finiteness-criterion for the set $P_{\mathbb{J}}(M)$ will play a certain role at a later instance.

(4.7) LEMMA. *Let R be a noetherian ring, let $\mathbb{J}$ be a multiplicative filter of ideals of R, and let M be a finitely generated R-module such that $J \not\subseteq \mathfrak{q}$ for any $J \in \mathbb{J}$ and any $\mathfrak{q} \in \operatorname{Ass}(M)$. Assume moreover that $D_{\mathbb{J}}(M)$ is finitely generated as an R-module. Then the set $P_{\mathbb{J}}(M)$ is finite.*

Proof. See [4, (3.2)]. □

(4.7) will be used together with the following finiteness-criterion for $\mathbb{J}$-transforms, which is given by [4, (4.9)].

(4.8) PROPOSITION. *Let R be a noetherian excellent ring, let $\mathbb{J}$ be a multiplicative filter of ideals of R, and let M be a finitely generated R-module. Assume that $ht((J + \mathfrak{q})/\mathfrak{q}) \neq 1$ for all $J \in \mathbb{J}$ and all $\mathfrak{q} \in \operatorname{Ass}(M)$. Then $D_{\mathbb{J}}(M)$ is a finitely generated R-module.* □

Now, we may prove crucial finiteness result.

(4.9) PROPOSITION. *Let R be an excellent noetherian ring, and let M be a finitely generated R-module. Then the set $C_R(M)$ is finite.*

Proof. Let us assume to the contrary, that $C_R(M)$ is infinite. For each $\mathfrak{p} \in C_R(M)$ let $S(\mathfrak{p}) = \{\mathfrak{q} \in \operatorname{Ass}(M) \mid \mathfrak{q} \subseteq \mathfrak{p}\}$.

$S(\mathfrak{p})$ always is a non-empty finite subset of the finite set $\operatorname{Ass}(M)$. In particular $S(\mathfrak{p})$ takes only finitely many different values, if $\mathfrak{p}$ runs through $C_R(M)$. So, there is an infinite subset $C \subseteq C_R(M)$, such that $S(\mathfrak{p})$ takes the same value $S \subseteq \operatorname{Ass}(M)$ for all $\mathfrak{p} \in C$.

Let $S' = \operatorname{Ass}(M) - S$. If $S' \neq \emptyset$ put $I = \bigcap_{\mathfrak{p} \in S'} \mathfrak{p}$ and set $\bar{M} = M/T_I(M)$, where $T_I(M)$ denotes the I-torsion $\{m \in M \mid \exists n \in \mathbb{N} : I^n m = 0\}$. As $V(I) \cap S' = \emptyset$, we have $\operatorname{Ass}(\bar{M}) = S$. As $V(I) \cap C = \emptyset$ we have $\bar{M}_\mathfrak{p} = M_\mathfrak{p}$ for all $\mathfrak{p} \in C$. Therefore $C \subseteq C_R(\bar{M})$ and $\mathfrak{q} \subseteq \mathfrak{p}$ for all $\mathfrak{q} \in \operatorname{Ass}(\bar{M})$ and all $\mathfrak{p} \in C$. Thus, replacing M by $\bar{M}$, we may assume that $\mathfrak{q} \subseteq \mathfrak{p}$ for all $\mathfrak{q} \in \operatorname{Ass}(M)$ and all $\mathfrak{p} \in C$.

Now, let $\mathbb{J}$ be the set of all finite products $J = \mathfrak{p}_1 \cdots \mathfrak{p}_r$, whose factors $\mathfrak{p}_i$ belong to C. Then it is obvious by our construction, that $C \subseteq P_\mathbb{J}(M)$. So $P_\mathbb{J}(M)$ is infinite.

Now, let $\mathfrak{q} \in \operatorname{Ass}(M)$ and let $J \in \mathbb{J}$. We may write $J = \mathfrak{p}_1 \cdots \mathfrak{p}_r$ with $\mathfrak{p}_1, \ldots, \mathfrak{p}_r \in C$. As $\mathfrak{q} \subseteq \mathfrak{p}_i$ and $ht(\mathfrak{p}_i/\mathfrak{q}) > 1$ for $i = 1, \ldots, r$ we have $ht((J + \mathfrak{q})/\mathfrak{q}) > 1$. So, by (4.8) $D_\mathbb{J}(M)$ is finitely generated. Moreover $J \not\subseteq \mathfrak{q}$ for any $J \in \mathbb{J}$ and any $\mathfrak{q} \in \operatorname{Ass}(M)$. So, by (4.7) we get the contradiction that $P_\mathbb{J}(M)$ is finite. □

(4.10) COROLLARY. *Let X be a noetherian excellent scheme, and let $\mathscr{G}$ be a coherent sheaf of $\mathcal{O}_X$-modules. Then, the set $\mathfrak{C}(\mathscr{G})$ is finite.* □

Now, we return back to our original task—namely the study of the restrictions $\mathscr{F} \restriction H$ of a coherent sheaf $\mathscr{F}$ over $\mathbb{P}^d$ to a general hyperplane $H \subseteq \mathbb{P}^d$. First we prove the following complement to (3.13).

(4.11) LEMMA. *Let $H \subseteq \mathbb{P}^d$ be a hyperplane such that $H \cap (\operatorname{Ass}(\mathscr{F}) \cup \mathfrak{C}_{\mathbb{P}^d}(\mathscr{F})) = \emptyset$. Then H is general with respect to $\mathscr{F}$. Moreover, for $\overline{\text{any}}$ $y \in \operatorname{Ass}(\mathscr{F} \restriction H)$ there is an $x \in \operatorname{Ass}(\mathscr{F})$ such that y is a generic point of $H \cap \overline{\langle x \rangle}$.*

Proof. As $H \cap \operatorname{Ass}(\mathscr{F}) = \phi$, H is general with respect to the sheaf $\mathscr{F}$.

Now, let $y \in \operatorname{Ass}(\mathscr{F} \restriction H)$. Using the Lemma (3.10), we may write $(\mathscr{F} \restriction H)_y = \mathscr{F}_y / a\mathscr{F}_y$, where $a \in \mathfrak{m}_{\mathbb{P}^d, y}$ is a non-zero-divisor with respect to $\mathscr{F}_y$ and a non-unit in $\mathcal{O}_{\mathbb{P}^d, y}$. Consequently $\operatorname{depth}(\mathscr{F} \restriction H)_y = \operatorname{depth}(\mathscr{F}_y) - 1$. As $y \in \operatorname{Ass}(\mathscr{F} \restriction H)$, we have $\operatorname{depth}(\mathscr{F} \restriction H)_y = 0$ and thus obtain $\operatorname{depth}(\mathscr{F}_y) = 1$.

As $y \in H$ and $H \cap \mathfrak{C}_{\mathbb{P}^d}(\mathscr{F}) = \emptyset$, we have $y \notin \mathfrak{C}_{\mathbb{P}^d}(\mathscr{F})$. Observing that $\operatorname{depth}(\mathscr{F}_y) = 1$ we thus find a point $x \in \operatorname{Ass}(\mathscr{F})$ with $y \in \overline{\{x\}}$ and $\operatorname{codim}_{\overline{\{x\}}} \overline{\{y\}} = 1$. Clearly y will be a generic point of $H \cap \overline{\{x\}}$. □

We say, that a property holds for a *generic hyperplane* $H \subseteq \mathbb{P}^d$, if it is satisfied on a non-empty open set of such hyperplanes. (Thereby the Grassmannian of all hyperplanes $H \subseteq \mathbb{P}^d$ is furnished with its natural Zariski-topology.)

We now want to compare the reduced dimension spectra of $\mathscr{F}$ and $\mathscr{F} \upharpoonright H$ for a generic hyperplane. To do so, we first prove:

(4.12) LEMMA. *Let* $x \in \mathbb{P}^d$ *be such that* $\dim \overline{\{x\}} > 1$. *Then, for a generic hyperplane* $H \subseteq \mathbb{P}^d$ *the intersection* $\overline{\{x\}} \cap H$ *is irreducible and satisfies*

$$\operatorname{ldim}(\overline{\{x\}} \cap H) = \operatorname{ldim}(\overline{\{x\}}) - 1.$$

Proof. We write $\overline{\{x\}} = X = \operatorname{Proj}(T)$, where T is a graded homomorphic image domain of the polynomial ring $k[z_0, \ldots, z_d] = S$. By Bertini's theorem there is a non-empty open set $U \subseteq S_1$ of linear forms such that $H_f \cap X = \operatorname{Proj}(T/\bar{f}T)$ is an integral scheme of codimension 1 in X for all $f \in U$ (cf. [23, (6.11)]).

It remains to show, that $\operatorname{ldim}(H_f \cap X) = \operatorname{ldim}(X) - 1$ for all such f. So, fix f, let $\bar{f} \in T$ be its canonical image and let z be the generic point of $H_f \cap X$. Considering z as a homogeneous prime ideal of T, we want to show that $\dim_k (T_1 \cap z) = 1$. As $k\bar{f}$ is a non-zero subspace of $T_1 \cap z$, we are left with proving $z \cap T_1 \subseteq k\bar{f}$.

As $\operatorname{Proj}(T/\bar{f}T) = H_f \cap X$ is an integral scheme, there is an $r \in \mathbb{N}$ with $\mathfrak{m}^r z \subseteq \bar{f}T$, where $\mathfrak{m}$ denotes the homogeneous maximal ideal of T. Writing Q for the total ring of fractions of T we thus may write $z \subseteq \bar{f}\Gamma$, with $\Gamma = \{q \in Q \mid \exists t \in \mathbb{N} : \mathfrak{m}^t q \in T\}$. But the ring Γ is nothing else than the total ring of sections $\bigoplus_{n \in \mathbb{Z}} \Gamma(X, \mathcal{O}_X(n))$ of $X = \operatorname{Proj}(T)$. As X is integral, $\Gamma_0 = \Gamma(X, \mathcal{O}_X) = k$. So we get

$$z \cap T_1 \subseteq \bar{f}\Gamma \cap T_1 = \bar{f}\Gamma_0 \cap T_1 = \bar{f}_k. \qquad \square$$

(4.13) PROPOSITION. *Let* $\mathscr{F} \neq 0$. *Then, for a generic hyperplane* $H \subseteq \mathbb{P}^d$ *we have*:

(i) *H is general with respect to* $\mathscr{F}$.

(ii) *For each* $y' \in \operatorname{Ass}(\mathscr{F} \upharpoonright H)$ *there is a* $x \in \operatorname{Ass}(\mathscr{F})$ *such that y is a generic point of* $H \cap \overline{\{x\}}$.

(iii) *If* $y \in \operatorname{Ass}(\mathscr{F} \upharpoonright H)$ *is a non-closed point and if* $x \in \operatorname{Ass}(\mathscr{F})$ *is as in* (ii), *then y is the unique generic point of* $H \cap \overline{\{x\}}$ *and*

$$\operatorname{ldim}(\overline{\{y\}}) = \operatorname{ldim}(\overline{\{x\}}) - 1.$$

Proof. To avoid the finitely many points of $\operatorname{Ass}(\mathscr{F}) \cup \mathfrak{C}(\mathscr{F})$ (cf. (4.10)) is a generic property of hyperplanes. So, by (4.11), (i) and (ii) are satisfied for generic hyperplanes $H \subseteq \mathbb{P}^d$. Applying (4.12) to any of the (finitely many) points $x \in \operatorname{Ass}(\mathscr{F})$ with $\dim \overline{\{x\}} > 1$, and observing (ii), we obtain (iii). $\square$

Now, using the notations introduced in (2.3), we may conclude:

(4.14) COROLLARY. *Let* $\dim(\mathscr{F}) > r + 1$. *Then, for a generic hyperplane* $H \subseteq \mathbb{P}^d$ *we have*:

(i) *H is general with respect to $\mathscr{F}$.*

(ii) $\underline{\dim}^{(r)}(\mathscr{F} \restriction H) = \underline{\dim}^{(r+1)}(\mathscr{F})$.

(iii) $\underline{\mathrm{ldim}}^{(r)}(\mathscr{F} \restriction H) = \underline{\mathrm{ldim}}^{(r+1)}(\mathscr{F})$.

(iv) $\mathrm{sdim}^{(r)}(\mathscr{F} \restriction H) = \mathrm{sdim}^{(r+1)}(\mathscr{F})$.

(v) $\mathrm{lsdim}^{(r)}(\mathscr{F} \restriction H) = \mathrm{lsdim}^{(r+1)}(\mathscr{F})$.

Proof. All statements follow from (4.13) and the definitions. □

5. Recursive bounds

In this section we perform the induction procedure described in (3.9). Again let $\mathscr{F}$ be a coherent sheaf over the projective space $\mathbb{P}^d = \mathbb{P}^d_k$, where k is an algebraically closed field.

More precisely, we want to give bounds on the functions $-i \leq n \mapsto h^i(\mathscr{F}(n))$, which depend only on the dimensions $h^j(\mathscr{F}(-j))(j = i, \ldots, \dim(\mathscr{F}))$, the linear dimension spectrum $\underline{\mathrm{ldim}}(\mathscr{F})$ and the 0-reduced global subdepth $\delta^{(0)}(\mathscr{F})$ of $\mathscr{F}$.

First of all we introduce the following notation:

$$(5.1)\quad h_i^{(r)}(\mathscr{F}) := \sum_{j=0}^{r} \binom{r}{j} h^{i+j}(\mathscr{F}(-i-j));$$

obviously $h_i^{(0)}(\mathscr{F}) = h^i(\mathscr{F}(-i))$.

(5.2) LEMMA. *Let H be a hyperplane, which is general with respect to $\mathscr{F}$. Then, for all $r \geq 0$*

$$h_i^{(r)}(\mathscr{F} \restriction H) \leq h_i^{(r+1)}(\mathscr{F}).$$

Proof. Let $\imath : H \to \mathbb{P}^d$ be the inclusion map and consider the sequences

$$0 \to \mathcal{O}_{\mathbb{P}^d}(n-1) \to \mathcal{O}_{\mathbb{P}^d}(n) \to \imath_* \mathcal{O}_H(n) \to 0.$$

As H is general with respect to $\mathscr{F}$, taking tensor products with $\mathscr{F}$ leaves this sequence exact, hence furnishes exact sequences.

$$0 \to \mathscr{F}(n-1) \to \mathscr{F}(n) \to \imath_* \mathscr{F} \restriction H(n) \to 0.$$

Applying cohomology and observing that $\imath$ is a closed immersion (which allows to replace $H^j(\mathbb{P}^d, \imath_* \mathscr{F} \upharpoonright H(n))$ by $H^j(\mathscr{F} \upharpoonright H(n))$ we thus get exact sequences.

$$H^{i+j}(\mathscr{F}(-i-j)) \to H^{i+j}(\mathscr{F} \upharpoonright H(-i-j)) \to H^{i+j+1}(\mathscr{F}(-i-j-1)).$$

These allow to write

$$h^{i+j}(\mathscr{F} \upharpoonright H(-i-j)) \leq h^{i+j}(\mathscr{F}(-i-j)) + h^{i+j+1}(\mathscr{F}(-i-j-1).$$

Consequently

$$\begin{aligned} h_i^{(r)}(\mathscr{F} \upharpoonright H) &= \sum_{j=0}^{r} \binom{r}{j} h^{i+j}(\mathscr{F} \upharpoonright H(-i-j)) \\ &\leq \sum_{j=0}^{r} \binom{r}{j} [h^{i+j}(\mathscr{F}(-i-j)) + h^{i+j+1}(\mathscr{F}(-i-j-1))] \\ &= h^i(\mathscr{F}(-i)) + \sum_{j=0}^{r} \left[\binom{r}{j} + \binom{r}{j-1}\right] h^{i+j}(\mathscr{F}(-i-j)) \\ &\quad + h^{i+r+1}(\mathscr{F}(-i-r-1)) = \sum_{j=0}^{r+1} \left(\binom{r+1}{j} h^{i+j}(\mathscr{F}(-i-j)\right) \\ &= h_i^{(r+1)}(\mathscr{F}). \end{aligned}$$ □

Now, we write $e = \dim(\mathscr{F})$, let $i > 0$, $r \geq 0$ and define by descending induction on r certain integers $\mu_i^{(r)}, \bar{\mu}_i^{(r)} > -i$ and certain functions $n \mapsto s_i^{(r)}(n)$, $n \mapsto \bar{s}_i^{(r)}(n)$, $(n \geq -i)$.

Moreover, in (5.5)(i), (ii), the symbol $[\cdot]^+$ shall be used to denote *least upper natural bounds*:

$$[a]^+ := \min\{n \in \mathbb{Z} \mid a \leq n, n > 0\}; \quad (a \in \mathbb{R}).$$

The functions $t_i^{(r)}$ and $\bar{t}_i^{(r)}$ which shall be defined in (5.4) have auxiliary character only.

Now, the mentioned functions and invariants are defined as follows:

(5.3) (i) $\mu_i^{(r)} = \bar{\mu}_i^{(r)} = -i+1$; *for* $r > e - i$.

(ii) $s_i^{(r)}(n) = \bar{s}_i^{(r)}(n) = 0, (n \geq i)$; *for* $r > e - i$.

(5.4) (i) $t_i^{(r)}(n) := h_i^{(r)}(\mathscr{F}) + \sum_{m=-i+1}^{n} s_i^{(r+1)}(m), (-i \leq n \leq \mu_i^{(r+1)})$; *for* $r \leq e - i$.

(ii) $\bar{t}_i^{(r)}(n) := h_i^{(r)}(\mathscr{F}) + \sum_{m=-i+1}^{n} \bar{s}_i^{(r+1)}(m), (-i \leq n \leq \bar{\mu}_i^{(r+1)})$; *for* $r \leq e - i$.

(5.5) (i) $\mu_i^{(r)} := \mu_i^{(r+1)} + \left[\dfrac{t_i^{(r)}(\mu_i^{(r+1)})}{\delta^{(r)}(\mathscr{F})}\right]^+ - 1$; *for* $r \leq e - i$.

(ii) $\bar{\mu}_i^{(r)} := \mu_i^{(r+1)} + \left[\dfrac{\bar{t}_i^{(r)}(\bar{\mu}_i^{(r+1)})}{\operatorname{lsdim}^{(r)}(\mathscr{F})}\right]^+ - 1$; *for* $r \leq e - i$.

(5.6) *For* $r \leq e - i$, *we finally put*:

(i)

$$s_i^{(r)}(n) := \begin{cases} t_i^{(r)}(n); & (-i \leq n < \mu_i^{(r+1)}) \\ t_i^{(r)}(\mu_i^{(r+1)}) - (n + 1 - \mu_i^{(r+1)})\delta^{(r)}(\mathscr{F}); & (\mu_i^{(r+1)} \leq n < \mu_i^{(r)}) \\ 0; & (\mu_i^{(r)} \leq n) \end{cases}$$

(ii)

$$\bar{s}_i^{(r)}(n) := \begin{cases} \bar{t}_i^{(r)}(n); & (-i \leq n < \bar{\mu}_i^{(r+1)}) \\ \bar{t}_i^{(r)}(\mu_i^{(r+1)}); & (\bar{\mu}_i^{(r+1)} \leq n < \mu_i^{(r+1)}) \\ \bar{t}_i^{(r)}(\bar{\mu}_i^{(r+1)}) - (n + 1 - \mu_i^{(r+1)}) \operatorname{lsdim}^{(r)}(\mathscr{F}); & (\mu_i^{(r+1)} \leq n < \bar{\mu}_i^{(r)}) \\ 0; & (\bar{\mu}_i^{(r)} \leq n) \end{cases}$$

(5.7) REMARK. (A) If $r \leq e - i$, then $\dim(\mathscr{F}) = e \geq r + i > r$ (as $i > 0$). So, by (2.12)(i) $0 < \delta^{(r)}(\mathscr{F}) \leq \operatorname{lsdim}^{(r)}(\mathscr{F})$. Therefore the invariants $\mu_i^{(r)}$ and $\bar{\mu}_i^{(r)}$ are well defined. Consequently the definitions (5.4) and (5.6) make sense, too.

(B) By descending induction on r, it is easy to verify, that:

(i) (a) $\mu_i^{(r+1)} \leq \bar{\mu}_i^{(r)}$.

(b) $s_i^{(r)}(n) = 0$, *for all* $n \geq \mu_i^{(r)}$.

(c) $\bar{s}_i^{(r)}(n) = 0$, *for all* $n \geq \bar{\mu}_{(r)}$. ○

The previously introduced functions $s_i^{(r)}, \bar{s}_i^{(r)}, t_i^{(r)}, \bar{t}_i^{(r)}$, as well as the invariants $\mu_i^{(r)}$ and $\bar{\mu}_i^{(r)}$ clearly depend on $\mathscr{F}$. To formulate the results to come, we have to observe this dependence. Therefore we write:

(5.8) (i) $t_{i,\mathscr{F}}^{(r)} := t_i^{(r)}$; $\bar{t}_{i,\mathscr{F}}^{(r)} := \bar{t}_i^{(r)}$; $s_{i,\mathscr{F}}^{(r)} := s_i^{(r)}$; $\bar{s}_{i,\mathscr{F}}^{(r)} := \bar{s}_i^{(r)}$.

(ii) $\mu_{i,\mathscr{F}}^{(r)} := \mu_i^{(r)}$; $\bar{\mu}_{i,\mathscr{F}}^{(r)} := \bar{\mu}_i^{(r)}$.

(5.9) LEMMA. *Let $i > 0$, $\mathscr{F} \neq 0$, and let $H \subseteq \mathbb{P}^d$ be a hyperplane which is general with respect to $\mathscr{F}$. Then*:

(i) $\mu^{(r)}_{i,\mathscr{F}\restriction H} \leq \mu^{(r+1)}_{i,\mathscr{F}}$.
(ii) $s^{(r)}_{i,\mathscr{F}\restriction H}(n) \leq s^{(r+1)}_{i,\mathscr{F}\restriction H}(n)$, *for all* $n \geq -i$.

Proof. Yet writing $e = \dim(\mathscr{F})$, we have $\dim(\mathscr{F} \restriction H) = e - 1$, (cf. (3.14)). We proceed by descending induction on r. Let $r > e - 1 - i$. Then $r + 1 > e - i$. So, the invariants occurring in (i) coincide with $-i + 1$ whereas the functions in (ii) vanish (cf. (5.3)).

So, let $r \leq e - 1 - i$, thus $r + 1 \leq e - i$. Then, by induction we may assume that $\mu^{(r+1)}_{i,\mathscr{F}\restriction H} \leq \mu^{(r+2)}_{i,\mathscr{F}}$ and that $s^{(r+1)}_{i,\mathscr{F}\restriction H}(n) \leq s^{(r+2)}_{i,\mathscr{F}}(n)$, $\forall n \geq -i$. By (5.2) we moreover have $h^{(r)}_i(\mathscr{F} \restriction H) \leq h^{(r+1)}_i(\mathscr{F})$.

Consequently

$$\begin{aligned} t^{(r)}_{i,\mathscr{F}\restriction H}(n) &= h^{(r)}_i(\mathscr{F} \restriction H) + \sum_{m=-i+1}^{n} s^{(r+1)}_{i,\mathscr{F}\restriction H}(m) \\ &\leq h^{(r+1)}_i(\mathscr{F}) + \sum_{m=-i+1}^{n} s^{(r+2)}_{i,\mathscr{F}}(m) = t^{(r+1)}_{i,\mathscr{F}}(n) \quad \text{for } -i \leq n \leq \mu^{(r+1)}_{i,\mathscr{F}\restriction H}. \end{aligned}$$

As $t^{(r+1)}_{i,\mathscr{F}}(n)$ is non-decreasing in the range $-i \leq n \leq \mu^{(r+2)}_{i,\mathscr{F}\restriction H}$, we get in particular $t^{(r)}_{i,\mathscr{F}\restriction H}(\mu^{(r+1)}_{i,\mathscr{F}\restriction H}) \leq t^{(r+1)}_{i,\mathscr{F}}(\mu^{(r+2)}_{i,\mathscr{F}})$. As $\dim(\mathscr{F}) = e \geq r + 1 + i > r + 1$, (3.12)(iii) induces $\delta^{(r)}(\mathscr{F} \restriction H) \geq \delta^{(r+1)}(\mathscr{F}) > 0$. Therefore we obtain

$$\begin{aligned} \mu^{(r)}_{i,\mathscr{F}\restriction H} &= \mu^{(r+1)}_{i,\mathscr{F}\restriction H} + \left[\frac{t^{(r+1)}_{i,\mathscr{F}\restriction H}(\mu^{(r+1)}_{i,\mathscr{F}\restriction H})}{\delta^{(r)}(\mathscr{F} \restriction H)}\right]^+ - 1 \\ &\leq \mu^{(r+2)}_{i,\mathscr{F}} + \left[\frac{t^{(r+2)}_{i,\mathscr{F}}(\mu^{(r+2)}_{i,\mathscr{F}})}{\delta^{(r+2)}(\mathscr{F})}\right]^+ - 1 = \mu^{(r+1)}_{i,\mathscr{F}}. \end{aligned}$$

This proves (i).

Now, the proof of (ii) is easy. For $-i \leq n \leq \mu^{(r+1)}_{i,\mathscr{F}\restriction H}$ we already have $s^{(r)}_{i,\mathscr{F}\restriction H}(n) \leq t^{(r+1)}_{i,\mathscr{F}}(n) = s^{(r+1)}_{i,\mathscr{F}}(n)$, where the last equality follows, in view of (5.6)(i), as $\mu^{(r+1)}_{i,\mathscr{F}\restriction H} \leq \mu^{(r+2)}_{i,\mathscr{F}}$. In the range $n \geq \mu^{(r+1)}_{i,\mathscr{F}\restriction H}$ the function $s^{(r)}_{i,\mathscr{F}\restriction H}$ is non-increasing, whereas the function $s^{(r+1)}_{i,\mathscr{F}}$ is non-decreasing in the range $-i \leq n < \mu^{(r+2)}_{i,\mathscr{F}}$ (cf. (5.6)(i)). Consequently $s^{(r+1)}_{i,\mathscr{F}\restriction H}(n)$ for $-i \leq n < \mu^{(r+2)}_{i,\mathscr{F}}$.

At $n = \mu^{(r+2)}_{i,\mathscr{F}} - 1$ the function $s^{(r+1)}_{i,\mathscr{F}}$ starts decreasing linearly with slope $\delta^{(r+1)}(\mathscr{F}) > 0$ until it reaches the value 0, (cf. (5.6)(i)). The function $s^{(r)}_{i,\mathscr{F}\restriction H}$ starts to decrease linearly already at the place $n = \mu^{(r+1)}_{i,\mathscr{F}\restriction H} - 1(\leq \mu^{(r+2)}_{i,\mathscr{F}} - 1)$ until its value reaches 0. Thereby, the slope is $\delta^{(r)}(\mathscr{F} \restriction H) \geq \delta^{(r)}(\mathscr{F})$, hence not less than the slope in the previous case. Consequently $s^{(r)}_{i,\mathscr{F}\restriction H}(n) \leq s^{(r+1)}_{i,\mathscr{F}}(n)$ is true for all $n \geq -i$. $\square$

(5.10) LEMMA. *Let $i > 0, \mathscr{F} \neq 0$. Then, for a generic hyperplane $H \subseteq \mathbb{P}^d$*
(i) $\bar{\mu}^{(r)}_{i,\mathscr{F}\restriction H} \leq \bar{\mu}^{(r+1)}_{i,\mathscr{F}}$.
(ii) $\bar{s}^{(r)}_{i,\mathscr{F}\restriction H}(n) \leq \bar{s}^{(r+1)}_{i,\mathscr{F}}(n)$, *for all $n \geq -i$.*

Proof. We proceed in the same way as in the previous proof. Thereby we observe that H is general with respect to $\mathscr{F}$, which, by (5.9), already induces $\mu^{(r+1)}_{i,\mathscr{F}\restriction H} \leq \mu^{(r+2)}_{i,\mathscr{F}}$ and $\mu^{(r)}_{i,\mathscr{F}\restriction H} \leq \mu^{(r+1)}_{i,\mathscr{F}}$.

Here, again both statements are obvious for $r > e-1-i$. If $r \geq e-1-i$, we get exactly in the same way as before.

$$\bar{t}^{(r)}_{i,\mathscr{F}\restriction H}(n) \leq \bar{t}^{(r+1)}_{i,\mathscr{F}}(n) \quad \text{for } -i \leq n \leq \bar{\mu}^{(r+1)}_{i,\mathscr{F}\restriction H} \leq \bar{\mu}^{(r+2)}_{i,\mathscr{F}}$$

and

$$\bar{t}^{(r)}_{i,\mathscr{F}\restriction H}(\bar{\mu}^{(r+1)}_{i,\mathscr{F}\restriction H}) \leq \bar{t}^{(r+1)}_{i,\mathscr{F}}(\bar{\mu}^{(r+2)}_{i,\mathscr{F}}).$$

Now, $\bar{\mu}^{(r)}_{i,\mathscr{F}\restriction H} \leq \bar{\mu}^{(r+1)}_{i,\mathscr{F}}$ follows immediately from the definition (5.5)(ii).

Statement (ii) again is shown similarly as in the proof of (5.9). Again we know already that the requested inequality holds in the range $-i \leq n \leq \bar{\mu}^{(r+1)}_{i,\mathscr{F}\restriction H}$. Again by (5.6)(ii) $\bar{s}^{(r)}_{i,\mathscr{F}\restriction H}$ is non-increasing in the range $n \geq \bar{\mu}^{(r+1)}_{i,\mathscr{F}\restriction H}$, whereas $\bar{s}^{(r+1)}_{i,\mathscr{F}}$ is non-decreasing in the range $-i \leq n < \mu^{(r+2)}_{i,\mathscr{F}}$ by (5.6)(ii). Consequently (ii) holds true for $-i \leq n < \mu^{(r+2)}_{i,\mathscr{F}}$. Now, we may conclude as both functions drop linearly with the same slope $\operatorname{lsdim}^{(r)}(\mathscr{F} \restriction H) = \operatorname{lsdim}^{(r)}(\mathscr{F})$, $\bar{s}^{(r)}_{i,\mathscr{F}\restriction H}$ beginning to do so at an earlier instance than $\bar{s}^{(r+1)}_{i,\mathscr{F}}$. □

(5.11) THEOREM. *Let $\mathscr{F} \neq 0$, and let $i > 0$. Then:*
(i) $h^i(\mathscr{F}(n)) \leq \bar{s}^{(0)}_{i,\mathscr{F}}(n)$ *for all $n \geq i$.*
(ii) $\mu^i_{\mathscr{F}} \leq \bar{\mu}^{(0)}_{\mathscr{F}}$.

Proof. We proceed by introduction on $e = \dim(\mathscr{F})$. If $e = 0$, $h^i(\mathscr{F}(n)) = 0$ for all n, hence our claim is obvious.

So, let $e > 0$. Let $\tilde{\mathscr{F}}$ be the reduction of $\mathscr{F}$, as it was introduced in (2.7). Then $h^{i+j}(\tilde{\mathscr{F}}(n)) = h^{i+j}(\mathscr{F}(n))$ for all $j \geq 0$ (cf. (2.10)(ii)) shows that $h^{(r)}_i(\tilde{\mathscr{F}}) = h^{(r)}_i(\mathscr{F})$, for all $r \geq 0$ (cf. (5.11)). By (2.10)(iii) we have $\operatorname{lsdim}^{(r)}(\tilde{\mathscr{F}}) = \operatorname{lsdim}^{(r)}(\mathscr{F})$ for all $r \geq 0$. Finally, by (2.12)(ii) $\delta^{(r)}(\tilde{\mathscr{F}}) = \delta^{(r)}(\mathscr{F})$ for all $r \geq 0$. Altogether we see, that the functions $\bar{s}^{(r)}_i$ and the invariants $\mu^{(r)}_i, \bar{\mu}^{(r)}_i$ are not affected, if we replace $\mathscr{F}$ by $\tilde{\mathscr{F}}$. Thus we may assume that $\mathscr{F} = \tilde{\mathscr{F}}$. Then $\operatorname{lsdim}^{(0)}(\mathscr{F}) = \operatorname{lsdim}(\mathscr{F}) > 0$.

Now, let $H \subseteq \mathbb{P}^d$ be a generic hyperplane. Then H is general with respect to $\mathscr{F}$, (cf. (4.14)). Moreover, by (3.1) H is member of a linear system $\mathfrak{H}$ of hyperplanes general with respect to $\mathscr{F}$ and such that $\dim(\mathfrak{H}) = \operatorname{lsdim}(\mathscr{F}) = \operatorname{lsdim}^{(0)}(\mathscr{F})$.

As $\dim(\mathscr{F} \upharpoonright H) = e - 1$, we may assume by induction, that $h^i(\mathscr{F} \upharpoonright H(n)) \leq \bar{s}^{(0)}_{i,\mathscr{F} \upharpoonright H}$ for all $n \geq -i$. In view of (5.10) we thus obtain $h^i(\mathscr{F} \upharpoonright H(n)) \leq \bar{s}^{(1)}_{i,\mathscr{F}}(n)$ for all $n \geq -i$.

By our choice of $\mathfrak{H}$, the generic values $\bar{h}^i(\mathscr{F} \upharpoonright H(n))$ as they were introduced in (3.5)(ii) do not exceed $h^i(\mathscr{F} \upharpoonright H(n))$. So we get, $\bar{h}^i(\mathscr{F} \upharpoonright \mathfrak{H}(n)) \leq \bar{s}^{(1)}_{i,\mathscr{F}}(n)$ for all $n \geq -i$.

Now, we write

$$l^i(n) = h^i(\mathscr{F}(-i)) + \sum_{-i < m \leq n} \bar{h}^i(\mathscr{F} \upharpoonright H(n)).$$

As $h^{(0)}_i(\mathscr{F}) = h^i(\mathscr{F}(-i))$, we obtain

$$l^i(n) = h^{(0)}_i(\mathscr{F}) + \sum_{m=-i+1}^{n} \bar{s}^{(1)}_{i,\mathscr{F}}(n).$$

For $-i \leq n < \mu^{(1)}_{i,\mathscr{F}}$, the right hand expression coincides with $\bar{s}^{(0)}_{i,\mathscr{F}}(n)$, (cf. (5.6)(ii), (5.7)(ii)(c)). Thus, in view of (3.8) we obtain $h^i(\mathscr{F}(n)) \leq \bar{s}^{(0)}_{i,\mathscr{F}}(n)$ for $-i \leq n < \mu^{(1)}_{i,\mathscr{F}}$.

Moreover $\bar{s}^{(0)}_{i,\mathscr{F}}$ begins to decrease linearly with slope $\operatorname{lsdim}^{(0)}(\mathscr{F}) = \dim(\mathfrak{H})$ at the place $n = \mu^{(1)}_{i,\mathscr{F}}$, until it reaches 0. By (3.8)

$$h^i(\mathscr{F}(n)) \leq \max\{0, l^i(\bar{\mu}^i_{\mathscr{F} \upharpoonright \mathfrak{H}}) - \dim(\mathfrak{H})(n - \mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}} + 1)\} \quad \text{for } n \geq \mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}} - 1.$$

So, it remains to show the inequality $\mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}} \leq \mu^{(1)}_{i,\mathscr{F}}$.

To do so, let $L \in \mathfrak{H}$. Then (5.9) gives $\mu^{(0)}_{i,\mathscr{F} \upharpoonright L} \leq \mu^{(1)}_{i,\mathscr{F}}$. By induction, $h^i(\mathscr{F} \upharpoonright L(n)) = 0$ for all $n \geq \bar{\mu}^{(0)}_{\mathscr{F} \upharpoonright L}$ (note that $\dim(\mathscr{F} \upharpoonright L) = e - 1$), hence $\mu^i_{\mathscr{F} \upharpoonright L} \leq \bar{\mu}^{(0)}_{i,\mathscr{F} \upharpoonright L}$. By (5.7)(i)(a) $\bar{\mu}^{(0)}_{i,\mathscr{F} \upharpoonright L} \leq \mu^{(0)}_{i,\mathscr{F} \upharpoonright L}$. Consequently we obtain $\mu^i_{\mathscr{F} \upharpoonright L} \leq \mu^{(0)}_{i,\mathscr{F} \upharpoonright L}$ for all $L \in \mathfrak{H}$. This proves $\mu^i_{\mathscr{F} \upharpoonright \mathfrak{H}} \leq \mu^{(1)}_{i,\mathscr{F}}$. □

(5.12) REMARK. It is already clear from their definition, that the bounding functions $\bar{s}^{(0)}_{\mathscr{F}}$ and the bounding invariants $\bar{\mu}^{(0)}_{\mathscr{F}}$ depend only on the parameters $\delta^{(j)}(\mathscr{F})$, $\operatorname{lsdim}^{(j)}(\mathscr{F})$, $h^{i+j}(\mathscr{F}(-i-j))$, $(j = 0, \ldots, e - i;\ e = \dim(\mathscr{F}))$. In the next section, this dependence will be studied in more detail. ○

6. Bounding functions

In this section we consider the dependence of the functions $\bar{s}^{(0)}_{i,\mathscr{F}}$ and the invariants $\bar{\mu}^{(0)}_{i,\mathscr{F}}$ on the parameters $\operatorname{lsdim}^{(r)}(\mathscr{F})$, $\delta^{(r)}(\mathscr{F})$ $(r = 0, 1, \ldots, e - i)$ and on the parameters $h^j(\mathscr{F}(-j))$ $(j = i, i + 1, \ldots, e)$.

Thereby we state in a more explicit way what already has been noticed in the previous section: The cohomological Hilbert functions $n \mapsto h^i(\mathscr{F}(n))(n \geq -i)$ have upper bounds which depend only on the previously mentioned parameters, (cf. (5.12)).

First, we introduce some notations. If u, v are natural numbers, we write $\mathbb{F}^{u,v}$ for the set of all functions $f: \mathbb{N}^u \times \mathbb{N}_0^v \to \mathbb{Z}$ with the following property:

(6.1) $f(a_1, \ldots, a_u; b_1, \ldots, b_v) \leq f(a'_1, \ldots, a'_u; b'_1, \ldots, b'_v)$ *if* $a_i \geq a'_i > 0$ *for* $i = 1, \ldots, u$ *and* $b'_j \geq b_j \geq 0$ *for* $j = 1, \ldots, v$.

Moreover we write $\mathbb{F}_l^{u,v}(l \in \mathbb{Z})$ for the set of all functions $g: \mathbb{N}^u \times \mathbb{N}_0^v \times \mathbb{Z}_{\geq l} \to \mathbb{N}_0$ such that

(6.2) $\forall n \geq l : g(-, n) : \mathbb{N}^u \times \mathbb{N}_0^v \to \mathbb{N}_0$ *belongs to* $\mathbb{F}^{u,v}$.

Now, we fix integers $e \geq i > 0$. Then, mimicking the construction of the invariants $\mu_i^{(r)}, \bar{\mu}_i^{(r)}$ and of the functions $s_i^{(r)}, \bar{s}_i^{(r)}$ (cf. (5.3–6)) we define functions

$$M_{e,i}^{(r)}, \bar{M}_{e,i}^{(r)} : \mathbb{N}^{2(e-i+1)} \times \mathbb{N}_0^{e-i+1} \to \mathbb{Z}$$

$$S_{e,i}^{(r)}, \bar{S}_{e,i}^{(r)} : \mathbb{N}^{2(e-i+1)} \times \mathbb{N}_0^{e-i+1} \times \mathbb{Z}_{\geq -i} \to \mathbb{N}_0$$

as will be done below. Thereby we assume that

$$(\underline{a}, n) := (a_1^{(0)}, \ldots, a_1^{(e-i)}; a_2^{(0)}, \ldots, a_2^{(e-i)}; a_3^{(i)}, \ldots, a_3^{(e)}; n)$$

belongs to $\mathbb{N}^{2(e-i+1)} \times \mathbb{N}_0^{e-i+1} \times \mathbb{Z}_{\geq -i}$.

If $r > e - i$, we put:

(6.3) (i) $M_{e,i}^{(r)}(\underline{a}) = \bar{M}_{e,i}^{(r)}(\underline{a}) := -i + 1.$
(ii) $S_{e,i}^{(r)}(\underline{a}, n) = \bar{S}_{e,i}^{(r)}(\underline{a}, n) := 0.$

If $0 \leq r \leq e - i$, we introduce auxiliary functions $T_{e,i}^{(r)}, \bar{T}_{e,i}^{(r)}$ by

(6.4) (i) $$T_{e,i}^{(r)}(\underline{a}, n) := \sum_{j=0}^{r} \binom{r}{j} a_3^{(i+j)} + \sum_{m=-i+1}^{n} S_{e,i}^{(r+1)}(\underline{a}, m),$$

(ii) $$\bar{T}_{e,i}^{(r)}(\underline{a}, n) := \sum_{j=0}^{r} \binom{r}{j} a_3^{(i+j)} + \sum_{m=-i+1}^{n} \bar{S}_{e,i}^{(r+1)}(\underline{a}, m).$$

Then, we define

$$(6.5)\quad \text{(i)}\ M_{e,i}^{(r)}(\underline{a}) := M_{e,i}^{(r+1)}(\underline{a}) + \left[\frac{T_{e,i}^{(r)}(\underline{a}, M_{e,i}^{(r+1)}(\underline{a}))}{a_2^{(r)}}\right]^+ - 1.$$

$$\text{(ii)}\ \bar{M}_{e,i}^{(r)}(\underline{a}) := M_{e,i}^{(r+1)}(\underline{a}) + \left[\frac{\bar{T}_{e,i}^{(r)}(\underline{a}; \bar{M}_{e,i}^{(r+1)}(\underline{a}))}{a_1^{(r)}}\right]^+ - 1.$$

(6.6) (i)

$$S_{e,i}^{(r)}(\underline{a}, n) := \begin{cases} T_{e,i}^{(r)}(\underline{a}, n); & (-i \le n < M_{e,i}^{(r+1)}(\underline{a})). \\ T_{e,i}^{(r)}(\underline{a}, M_{e,i}^{(r+1)}(\underline{a})) - (n + 1 - M_{e,i}^{(r+1)}(\underline{a}))a_2^{(r)}; & (M_{e,i}^{(r+1)}(\underline{a}) \le n < M_{e,i}^{(r)}(\underline{a})). \\ 0; & (M_{e,i}^{(r)}(\underline{a}) \le n). \end{cases}$$

(ii)

$$\bar{S}_{e,i}^{(r)}(\underline{a}, n) := \begin{cases} \bar{T}_{e,i}^{(r)}(\underline{a}, n); & (-i \le n < \bar{M}_{e,i}^{(r+1)}(\underline{a})). \\ \bar{T}_{e,i}^{(r)}(\underline{a}, \bar{M}_{e,i}^{(r+1)}(\underline{a})); & (\bar{M}_{e,i}^{(r+1)}(\underline{a}) \le n < M_{e,i}^{(r+1)}(\underline{a})). \\ \bar{T}_{e,i}^{(r)}(\underline{a}, \bar{M}_{e,i}^{(r+1)}(\underline{a})) - (n + 1 - M_{e,i}^{(r+1)}(\underline{a}))a_1^{(r)}; & (M_{e,i}^{(r+1)}(\underline{a}) \le n < \bar{M}_{e,i}^{(r)}(\underline{a})). \\ 0; & (\bar{M}_{e,i}^{(r)}(\underline{a}) \le n). \end{cases}$$

(6.7) REMARK. (A) From the definitions of the above functions, the monotony-properties (6.1) and (6.2) are easily verified. So we may conclude:

(i) $M_{e,i}^{(r)}, \bar{M}_{e,i}^{(r)} \in \mathbb{F}^{2(e-i+1),e-i+1}$.

(ii) $S_{e,i}^{(r)}, \bar{S}_{e,i}^{(r)} \in \mathbb{F}_{-i}^{2(e-i+1),e-i+1}$.

(B) The following comparison statements are immediate from the definitions:

(iii) (a) $a_1^{(j)} \ge a_2^{(j)}$ *for all* $j \le e - i \Rightarrow \bar{M}_{e,i}^{(r)}(\underline{a}) \le M_{e,i}^{(r)}(\underline{a}), \bar{S}_{e,i}^{(r)}(\underline{a}, n) \le S_{e,i}^{(r)}(\underline{a}, n)$.

(b) $a_1^{(j)} \le a_2^{(j)}$ *for all* $j \le e - i \Rightarrow \bar{M}_{e,i}^{(r)}(\underline{a}) \ge M_{e,i}^{(r)}(\underline{a}), \bar{S}_{e,i}^{(r)}(\underline{a}, n) \ge S_{e,i}(\underline{a}, n)$.

Concerning the vanishing of the functions $S_{e,i}^{(r)}$ and $\bar{S}_{e,i}^{(r)}$ we have

(iv) (a) $S_{e,i}^{(r)}(\underline{a}, n) = 0, \quad \forall n \ge M_{e,i}^{(r)}(\underline{a})$.

(b) $\bar{S}_{e,i}^{(r)}(\underline{a}, n) = 0, \quad \forall n \ge \bar{M}_{e,i}^{(r)}(\underline{a})$.

In addition, we see from our definitions:

(v) *The values* $M_{e,i}^{(r)}(\underline{a}), \bar{M}_{e,i}^{(r)}(\underline{a})$ *and the functions* $S_{e,i}^{(r)}(\underline{a}, n)$ *and* $\bar{S}_{e,i}^{(r)}(\underline{a}, n)$ *are independent from the parameters* $a_1^{(j)}, a_2^{(j)}$ *with* $j < r$.

(C) Now, for an integer e' with $e \geq e' \geq i$ we put

$$\underline{\tilde{a}} := (a_1^{(0)}, \ldots, a_1^{(e-i)}; a_2^{(0)}, \ldots, a_2^{(e-i)}; a_3^{(i)}, \ldots, a_3^{(e')}, 0, \ldots, 0)$$
$$(\in \mathbb{N}^{2(e-i-1)} \times \mathbb{N}_0^{e-i+1})$$
$$\underline{a}' := (a_1^{(0)}, \ldots, a_1^{(e'-i)}; a_2^{(0)}, \ldots, a_2^{(e'-1)}; a_3^{(i)}, \ldots, a_3^{(e')}).$$

Then, from our definitions we get (by descending induction on r) the following relations:

(vi) (a) $M_{e,i}^{(r)}(\underline{\tilde{a}}) \geq M_{e',i}^{(r)}(\underline{a}')$; $\bar{M}_{e,i}^{(r)}(\underline{\tilde{a}}) \geq \bar{M}_{e',i}^{(r)}(\underline{a}')$.

(b) $S_{e,i}^{(r)}((\underline{\tilde{a}}; n) \geq S_{e',i}^{(r)}(\underline{a}'; n)$; $\bar{S}_{e,i}^{(r)}(\underline{\tilde{a}}, n) \geq \bar{S}_{e',i}^{(r)}(\underline{a}', n)$.

(D) We want to express the invariants $\mu_{i,\mathscr{F}}^{(r)}, \bar{\mu}_{i,\mathscr{F}}^{(r)}$ and the bounding functions $s_{i,\mathscr{F}}^{(r)}, \bar{s}_{i,\mathscr{F}}^{(r)}$ of the previous section by means of the functions $M_{e,i}^{(r)}, \bar{M}_{e,i}^{(r)}$ resp. $S_{e,i}^{(r)}, \bar{S}_{e,i}^{(r)}$. To do so, we introduce the following notations, in which $\mathscr{F}$ is a coherent sheaf over $\mathbb{P}^d$:

$$\underline{l}_{i,\mathscr{F}} := (\mathrm{lsdim}^{(0)}(\mathscr{F}), \ldots, \mathrm{lsdim}^{(e-i)}(\mathscr{F})),$$
$$\underline{\delta}_{i,\mathscr{F}} := (\delta^{(0)}(\mathscr{F}), \ldots, \delta^{(e-i)}(\mathscr{F})),$$
$$\underline{h}_{i,\mathscr{F}} := (h^i(\mathscr{F}(-i)), \ldots, h^e(\mathscr{F}(-e))).$$

Then, we have for any coherent sheaf $\mathscr{F}$ of dimension e over $\mathbb{P}^d$:

(vii) $\mu_{i,\mathscr{F}}^{(r)} = M_{e,i}^{(r)}(\underline{l}_{i,\mathscr{F}}, \underline{\delta}_{i,\mathscr{F}}, h_{i,\mathscr{F}})$,

(viii) $\bar{\mu}_{i,\mathscr{F}}^{(r)} = \bar{M}_{e,i}^{(r)}(\underline{l}_{i,\mathscr{F}}, \underline{\delta}_{i,\mathscr{F}}, \underline{h}_{i,\mathscr{F}})$,

(ix) $s_{i,\mathscr{F}}^{(r)}(n) = S_{e,i}^{(r)}(\underline{l}_{i,\mathscr{F}}, \underline{\delta}_{i,\mathscr{F}}, \underline{h}_{i,\mathscr{F}}; n)$, $(n \geq -i)$,

(x) $\bar{s}_{i,\mathscr{F}}^{(r)}(n) = \bar{S}_{e,i}^{(r)}(\underline{l}_{i,\mathscr{F}}, \underline{\delta}_{i,\mathscr{F}}, \underline{h}_{i,\mathscr{F}}; n)$, $(n \geq -i)$.

(E) Finally, it should be noted that the functions $M_{e,i}^{(r)}$ and $S_{e,i}^{(r)}$ may be expressed by the functions $\bar{M}_{e,i}^{(r)}$ and $\bar{S}_{e,i}^{(r)}$ in the following way

(xi) (a) $M_{e,i}^{(r)}(\underline{a}_1; \underline{a}_2; \underline{a}_3; n) = \bar{M}_{e,i}^{(r)}(\underline{a}_2; \underline{a}_2; \underline{a}_3; n)$.

(b) $S_{e,i}^{(r)}(\underline{a}_1; \underline{a}_2; \underline{a}_3) = \bar{S}_{e,i}^{(r)}(\underline{a}_2; \underline{a}_2; \underline{a}_3)$.

Thereby, we use the notation

$$\underline{a}_j := (a_j^{(0)}, \ldots, a_j^{(e-i)}), (j = 1, 2); \quad \underline{a}_3 := (a_3^{(i)}, \ldots, a_3^{(e)}). \qquad \bigcirc$$

Now, to simplify matters (may be to the cost of the quality of our bounds) we introduce functions

$$C_{e,i}^{(r)} : \mathbb{N}^2 \times \mathbb{N}_0^{e-i+1} \to \mathbb{Z},$$
$$B_{e,i}^{(r)} : \mathbb{N}^2 \times \mathbb{N}_0^{e-i+1} \times \mathbb{Z}_{\geq -i} \to \mathbb{Z},$$

as it will be done below. Thereby, for $a \in \mathbb{Z}$ we write

$$a^{[r]} := \max\{a - r, 1\}.$$

Then, for

$$(*)\ (a, b; \underline{c}; n) = (a, b; c^{(i)}, \ldots, c^{(e)}; n) \in \mathbb{N}^2 \times \mathbb{N}_0^{(e-i+1)} \times \mathbb{Z}_{\geq -i}$$

we put

$$(6.8)\ C_{e,i}^{(r)}(a, b; \underline{c}; n) := \bar{M}_{e,i}^{(r)}(a^{[0]}, \ldots, a^{[e-i]}, b^{[0]}, \ldots, b^{[e-i]}; \underline{c}),$$

$$(6.9)\ B_{e,i}^{(r)}(a, b; \underline{c}; n) := \bar{S}_{e,i}^{(r)}(a^{[0]}, \ldots, a^{[e-i]}, b^{[0]}, \ldots, b^{[e-i]}; \underline{c}; n).$$

(6.10) REMARK. (A) Again, it follows easily from the previous definitions that (cf. (6.7)):

(i) $C_{e,i}^{(r)} \in \mathbb{F}^{2,e-i+1}$,

(ii) $B_{e,i}^{(r)} \in \mathbb{F}_{-i}^{2,e-i+1}$.

From (6.7)(iii) and (iv) we get the following comparison and vanishing statements (in which $(a, b; \underline{c}; n)$ is defined by $(*)$):

(iii) $a \geq b \Rightarrow C_{e,i}^{(r)}(a, b; \underline{c}) \leq C_{e,i}^{(r)}(b, b; \underline{c})$ *and* $B_{e,i}^{(r)}(a, b; \underline{c}, n) \leq B_{e,i}^{(r)}(b, b; \underline{c}; n)$.

(iv) $B_{e,i}^{(r)}(a, b; \underline{c}; n) = 0, \forall n \geq C_{e,i}^{(r)}(a, b; \underline{c})$.

(B) Now, let e' be an integer with $e \geq e' \geqslant i$. We put

$$\tilde{\underline{c}} = (c^{(i)}, \ldots, c^{(e')}, 0, \ldots, 0)(\in \mathbb{N}^{e-i+1}), \quad \underline{c}' = (c^{(i)}, \ldots, c^{(e')}).$$

Then—from (6.7)(vi)—we obtain:

(v) (a) $C_{e,i}^{(r)}(a, b; \tilde{\underline{c}}) \geq C_{e',i}^{(r)}(a, b; \underline{c}')$,

(b) $B_{e,i}^{(r)}(a, b; \tilde{\underline{c}}; n) \geq B_{e',i}^{(r)}(a, b; \underline{c}'; n)$.

(C) Obviously, the functions $C_{e,i}^{(r)}$ and $B_{e,i}^{(r)}$ may be defined by the following recursive procedure (cf. (6.5), (6.6), (6.7)(xi))

(vi) (a) $C_{e,i}^{(r)}(a, b; \underline{c}; n) = -i + 1$, *for* $r > e - i$.

(b) $B_{e,i}^{(r)}(a, b; \underline{c}; n) \equiv 0$, *for* $r > e - i$.

If $0 \leq r \leq e - i$, then we define the auxiliary function

$$(\text{vii})\ W_{e,i}^{(r)}(a, b; \underline{c}; n) := \sum_{j=0}^{r} \binom{r}{j} c^{(i+j)} + \sum_{m=-i+1}^{n} B_{e,i}^{(r+1)}(a, b; \underline{c}; m).$$

Now, using these intermediate functions, we put

(viii) $$C^{(r)}_{e,i}(a, b; \underline{c}) := C^{(r+1)}_{e,i}(b, b; \underline{c}) + \left[\frac{W^{(r)}_{e,i}(a, b; \underline{c}; C^{(r+1)}_{e,i}(a, b; \underline{c}))}{a^{[r]}}\right]^{+} - 1.$$

Then, finally we set (for $0 \le r \le e - i$)
(ix)

$$B^{(r)}_{e,i}(a, b; \underline{c}; n) = \begin{cases} W^{(r)}_{e,i}(a, b; \underline{c}; n); & (-i \le n < C^{(r+1)}_{e,i}(a, b; \underline{c})). \\ W^{(r)}_{e,i}(a, b; \underline{c}; C^{(r+1)}_{e,i}(a, b; \underline{c})); & (C^{(r+1)}_{e,i}(a, b; \underline{c}) \le n < C^{(r+1)}_{e,i}(b, b; \underline{c})). \\ W^{(r)}_{e,i}(a, b; \underline{c}; C^{(r+1)}_{e,i}(a, b; \underline{c})) - (n+1) - C^{(r+1)}_{e,i}(b, b; \underline{c}))a^{[r]}; & \\ & (C^{(r+1)}_{e,i}(b, b; \underline{c})) \le n < C^{(r)}_{e,i}(a, b; \underline{c})). \\ 0; & (C^{(r)}_{e,i}(a, b; \underline{c}) \le n). \end{cases}$$ ○

Now, if we fix integers $e \ge i > 0$, the previously defined functions bound the i-th Hilbert function of coherent sheaves over $\mathbb{P}^d$ as follows:

(6.11) PROPOSITION. *Let* $0 < \dim(\mathscr{F}) \le e$. *Then*:
(i) $h^i(\mathscr{F}(n)) \le B^{(0)}_{e,i}(\mathrm{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}); h^i(\mathscr{F}(-i)), \ldots, h^e(\mathscr{F}(-e)); n)$, $\forall n \ge -i$.
(ii) $h^i(\mathscr{F}(n)) = 0, \quad \forall n \ge C^{(0)}_{e,i}(\mathrm{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}); h^i(\mathscr{F}(-i)), \ldots, h^e(\mathscr{F}(-e)))$.

Proof. Assume first, that $\dim(\mathscr{F}) = e$. Then, as $C^{(r)}_{e,i} \in \mathbb{F}^{2,e-i+1}$ and $B^{(r)}_{e,i} \in \mathbb{F}^{2,e-i+1}_{-i}$ (cf. (6.10)(i), (ii)) and as

$$\mathrm{lsdim}^{(r)}(\mathscr{F}) \ge \mathrm{lsdim}^{(0)}(\mathscr{F}) - r = \mathrm{lsdim}^{(0)}(\mathscr{F})^{[r]} \text{ for } r = 0, 1, \ldots, e - i (< \dim(\mathscr{F}))$$

(cf. (2.4)(vii), (2.11)) we get (in the notations of (6.7)(D)) the relations (cf. (6.7)(viii), (x)):

$$\begin{aligned} \bar{s}^{(0)}_{i,\mathscr{F}}(n) &= \bar{S}^{(0)}_{e,i}(\underline{l}_{i,\mathscr{F}}, \underline{\delta}_{i,\mathscr{F}}, \underline{h}_{i,\mathscr{F}}; n) \le B^{(0)}_{e,i}(\mathrm{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}), \underline{h}_{i,\mathscr{F}}; n) \\ &= B^{(0)}_{e,i}(\mathrm{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}); h^i(\mathscr{F}(-i)), \ldots, h^e(\mathscr{F}(-e)); n), (n \ge -i), \\ \bar{\mu}^{(0)}_{i,\mathscr{F}} &= \bar{M}^{(0)}_{e,i}(\underline{l}_{i,\mathscr{F}}; \underline{\delta}_{i,\mathscr{F}}; \underline{h}_{i,\mathscr{F}}) \le C^{(0)}_{e,i}(\mathrm{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}); \underline{h}_{i,\mathscr{F}}; n) \\ &= C^{(0)}_{e,i}(\mathrm{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}); h^i(\mathscr{F}(-i)), \ldots, h^e(\mathscr{F}(-e))). \end{aligned}$$

In view of (5.11) this proves our claim if $\dim(\mathscr{F}) = e$.

Now, let $\dim(\mathscr{F}) =: e' < e$. If $i > e'$, our claim is obvious, as then $h^i(\mathscr{F}(n)) = 0$ for all $n \in \mathbb{Z}$. If $i \le e'$, we conclude by the formulas (6.10)(v), thereby observing that $h^j(\mathscr{F}(-j)) = 0$ for all $j > e'$. □

(6.12) REMARK. Let $X \subseteq \mathbb{P}^d$ be a closed subscheme of $\mathbb{P}^d$, and let $\mathscr{F}$ be a coherent sheaf over X. Denoting the inclusion map $X \hookrightarrow \mathbb{P}^d$ be $\imath$, we have (cf. [16])

$$H^i(X, \mathscr{F}(n)) = H^i(\mathbb{P}^d, \imath_* \mathscr{F}(n)), \quad \forall n \in \mathbb{Z}.$$

(Thereby, twisting of $\mathscr{F}$ is understood wth respect to the embedding $\imath$). Now, applying (6.11) to $\imath_* \mathscr{F}$ and writing

$$B_{e,i} := B^{(0)}_{e,i}, \qquad C_{e,i} := C^{(0)}_{e,i}, \qquad h^i(\mathscr{F}(n)) = \dim_k H^i(X, \mathscr{F}(n)),$$

we get:

$$h^i(\mathscr{F}(n)) \le B_{e,i}(\operatorname{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}); h^i(\mathscr{F}(-i)), \dots, h^e(\mathscr{F}(-e)); n), \ (n \ge -i),$$

$$h^i(\mathscr{F}(n)) = 0, \quad \forall n \ge C_{e,i}(\operatorname{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}); h^i(\mathscr{F}(-i)), \dots, h^e(\mathscr{F}(-e))),$$

whenever $0 \le \dim(\mathscr{F}) \le e$. This obviously proves (1.1). ○

Finally, we put (for $(\underline{c}; n) = (c^{(i)}, \dots, c^{(e)}; n) \in \mathbb{N}_0^{e-i+1} \times \mathbb{Z}_{\ge -i}$):

(6.13) (i) $F_{e,i}(c^{(i)}, \dots, c^{(e)}) := C^{(0)}_{e,i}(1, 1; c^{(i)}, \dots, c^{(e)})$,
(ii) $G_{e,i}(c^{(i)}, \dots, c^{(e)}; n) := B^{(0)}_{e,i}(1, 1; c^{(i)}, \dots, c^{(e)}; n)$.

Then, as $1 \le \operatorname{lsdim}^{(0)}(\mathscr{F})^{[0]}$, and by the monotony property of the functions $C^{(0)}_{e,i}$ and $B^{(0)}_{e,i}$, we immediately get from (6.11):

(6.14) COROLLARY. *Let* $\dim(\mathscr{F}) \le e$. *Then*:
(i) $h^i(\mathscr{F}(n)) \le G_{e,i}(h^i(\mathscr{F}(-i)), \dots, h^e(\mathscr{F}(-e)); n), \quad \forall n \ge -i.$
(ii) $h^i(\mathscr{F}(n)) = 0$ *for all* $n \ge F_{e,i}(h^i(\mathscr{F}(-i)), \dots, h^e(\mathscr{F}(-e)))$. □

(6.15) REMARK. (A) Obviously, now (1.2) follows from (6.14) in the same way as (1.1) follows from (6.11).

(B) The previously introduced bounding functions

$$F_{e,i} : \mathbb{N}_0^{e-i+1} \to \mathbb{Z}; \quad G_{e,i} : \mathbb{N}_0^{e-i+1} \times \mathbb{Z}_{\ge -i} \to \mathbb{N}_0$$

may be defined by the following procedure (cf. (6.10)(C)): (Again we assume $(\underline{c}; n) = (c^{(i)}, \ldots, e^{(e)}; n) \in N_0^{e-i+1} \times \mathbb{Z}_{\geq -i}$)

(i) $F_{e,i} := F_{e,i}^{(0)}$; $G_{e,i} := G_{e,i}^{(0)}$.

(ii) $F_{e,i}^{(r)}(\underline{c}) = -i + 1$; $G_{e,i}^{(r)}(\underline{c}; n) = 0$, for $r > e - i$.

If $0 \leq r \leq e - i$, then we intermediately put:

$$\text{(iii)}\quad U_{e,i}^{(r)}(\underline{c}; n) := \sum_{j=0}^{r} \binom{r}{j} c^{(i+j)} + \sum_{m=-i+1}^{r} G_{e,i}^{(r+1)}(\underline{c}; m).$$

Using these auxiliary functions $U_{e,i}^{(r)}$ we finally set:

(iv) $F_{e,i}^{(r)}(\underline{c}) = F_{e,i}^{(r+1)}(\underline{c}) + U_{e,i}^{(r)}(\underline{c}; F_{e,i}^{(r+1)}(\underline{c})) - 1$.

(v)

$$G_{e,i}^{(r)}(\underline{c}; n) = \begin{cases} U_{e,i}^{(r)}(\underline{c}); & (-i \leq n < F_{e,i}^{(r+1)}(\underline{c})). \\ U_{e,i}^{(r)}(\underline{c}; F_{e,i}^{(r+1)}(\underline{c})) - (n + 1 - F_{e,i}^{(r+1)}(\underline{c})); & (F_{e,i}^{(r+1)}(\underline{c}) \leq n < F_{e,i}^{(r)}(\underline{c})). \\ 0; & (F_{e,i}^{(r)}(\underline{c}) \leq n). \end{cases} \qquad \circ$$

7. A priori bounds for Castelnuovo regularities

Generalizing the point of view of Castelnuovos original problem, we say that a coherent $\mathcal{O}_{\mathbb{P}^d}$-sheaf $\mathcal{F}$ is *m-regular* if

(7.1) $H_i(\mathbb{P}^d, \mathcal{F}(n)) = 0, \quad \forall i > 0, \quad \forall n \geq m - i.$

This general definition goes back to Mumford [34]. It is well known, that $H^i(\mathbb{P}^d, \mathcal{F}(n)) = 0, \forall n \gg 0, \forall i > 0$ (cf. [39]). As $H^i(\mathbb{P}^d, \mathcal{F}(n)) = 0$ for all $n \in \mathbb{Z}$ and all $i > \dim(\mathcal{F})$ (cf. [39]), $\mathcal{F}$ always is m-regular for some $m \in \mathbb{Z}$. The minimally possible value among all these numbers m is called the *Castelnuovo-regularity* of $\mathcal{F}$ and denoted by $\operatorname{reg}(\mathcal{F})$:

(7.2) $\operatorname{reg}(\mathcal{F}) := \inf\{m \in \mathbb{Z} \mid H^i(\mathbb{P}^d, \mathcal{F}(n)) = 0, \forall i > 0, \forall n \geq m - i\}$.

So, in terms of the invariants $\mu_{\mathcal{F}}^i$ (cf. (3.4)) we may write

(7.3) $\operatorname{reg}(\mathcal{F}) = \sup\{\mu_{\mathcal{F}}^i + i \mid i > 0\}$.

(7.4) LEMMA. *Let $\mathcal{F} \neq 0$, and let $H \subseteq \mathbb{P}^d$ be a hyperplane which is general with respect to $\mathcal{F}$. Let $j > 1$. Then $\mu_{\mathcal{F}}^j \leq \mu_{\mathcal{F}\restriction H}^{j-1} - 1$.*

Proof. Consider the following exact sequences

$$H^{j-1}(H, \mathscr{F} \restriction H(n)) \to H^j(\mathbb{P}^d, \mathscr{F}(n-1)) \xrightarrow{\alpha_n} H^j(\mathbb{P}^d, \mathscr{F}(n)).$$

For all $n \geq \mu^{j-1}_{\mathscr{F} \restriction H}$ the left-hand space vanishes. So, the map α_n becomes injective for all such n. As $H^j(\mathbb{P}^d, \mathscr{F}(n))$ vanishes for all $n \gg 0$, $H^j(\mathbb{P}^d, \mathscr{F}(n-1))$ already has to vanish for all $n \geq \mu^{j-1}_{\mathscr{F} \restriction H}$. Therefore $\mu^j_{\mathscr{F}} \leq \mu^{j-1}_{\mathscr{F} \restriction H} - 1$. □

Now, we are ready to prove the following result:

(7.5) THEOREM. *Let $\mathscr{F} \neq 0$, and let $j \geq i > 0$. Then $H^j(\mathbb{P}^d, \mathscr{F}(n)) = 0$ for all $n \geq \bar{\mu}^{(j-i)}_{i,\mathscr{F}} - j + i$.*

Proof. We proceed by induction on $j - i$. If $j - i = 0$ we have $j = i$ and thus may conclude by (5.11)(ii). So, let $j > i$. Then, in particular $j > 1$. Now, choose a generic hyperplane $H \subseteq \mathbb{P}^d$. Then, first of all H in general with respect to $\mathscr{F}$. So, (7.4) gives $\mu^j_{\mathscr{F}} \leq \mu^{j-1}_{\mathscr{F} \restriction H} - 1$. By induction

$$\mu^{j-1}_{\mathscr{F} \restriction H} \leq \bar{\mu}^{(j-i-1)}_{i,\mathscr{F} \restriction H} - j + 1 + i.$$

In view of (5.10) we may write $\bar{\mu}^{(j-i-1)}_{i,\mathscr{F} \restriction H} \leq \bar{\mu}^{(j-i)}_{i,\mathscr{F}}$. So, altogether we obtain $\mu^j_{\mathscr{F}} \leq \mu^{(j-1)}_{i,\mathscr{F}} - j + i$. This proves our claim. □

As a first application to this we obtain.

(7.6) COROLLARY. *Let $\mathscr{F} \neq 0$, $j \geq i > 0$. Then $H^j(\mathbb{P}^d, \mathscr{F}(n)) = 0$ for all $n \geq \bar{\mu}^{(0)}_{i,\mathscr{F}} - j + i$.*

Proof. Obvious, as $\bar{\mu}^{(j-i)}_{i,\mathscr{F}} \leq \bar{\mu}^{(0)}_{i,\mathscr{F}}$. □

Applying (7.6) with $i = 1$, we get the following regularity-bound for $\mathscr{F}$:

(7.7) COROLLARY. $\operatorname{reg}(\mathscr{F}) \leq \bar{\mu}^{(0)}_{1,\mathscr{F}} + 1$. □

Now defining $C^{(0)}_{e,1} : \mathbb{N}^2 \times \mathbb{N}_0^{e-i+1} \to \mathbb{Z}$ according to (6.8), we conclude as in (6.11):

(7.8) COROLLARY. *Let* $\dim(\mathscr{F}) \leq e, j \geq i > 0$. *Then $H^j(\mathbb{P}^d, \mathscr{F}(n))$ vanishes for all $n \in \mathbb{Z}$ with*

$$n \geq C^{(0)}_{e,i}(\operatorname{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}), h^i(\mathscr{F}(-i)), \ldots, h^e(\mathscr{F}(-i)) - j + 1. \quad \square$$

From (7.8) we get in particular (cf. (7.7)):

(7.9) COROLLARY. *If* $0 < \dim(\mathscr{F}) \leq e$, *then*

$$\operatorname{reg}(\mathscr{F}) \leq C_{e,1}^{(0)}(\operatorname{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}), h^1(\mathscr{F}(-1)), \dots, h^e(\mathscr{F}(-e))) + 1. \qquad \square$$

Finally, defining $F_{e,i} : \mathbb{N}_0^e \to \mathbb{Z}$ by $F_{e,i}(c_1, \dots, c_e) := C_{e,1}^{(0)}(1, 1, c_1, \dots, c_e)$—as done already in the previous section—we get

(7.10) COROLLARY. *If* $\dim(\mathscr{F}) \leq e$, *then*

$$\operatorname{reg}(\mathscr{F}) \leq F_{e,i}(h^1(\mathscr{F}(-1)), \dots, (\mathscr{F}(-e))) + 1. \qquad \square$$

(7.11) REMARK. Now, obviously, (1.3) and (1.4) follow from (7.9) resp. (7.10) by the same arguments that were given in (6.12).

(7.12) REMARK. It is obvious from the definitions (6.4), (6.5) and (6.8), (6.9), that the functions $C_{e,i}^{(r)}$ and $B_{e,i}^{(r)}$ satisfy

(i) $C_{e,i}^{(r)}(a, b; \underline{0}) = -i + 1$

(ii) $B_{e,i}^{(r)}(a, b; \underline{0}; n) = 0, \quad \forall n \geq -i.$

Consequently we get from (7.8)

(iii) *If* $\dim(\mathscr{F}) \leq e$ *and* $h^j(\mathscr{F}(-j)) = 0$ *for* $j = i, \dots, e$, *then* $H^j(\mathbb{P}^d, \mathscr{F}(n)) = 0$ *for all* $j \in \{i, \dots, e\}$ *and all* $n \geq -j$.

So, applying this with $i = 1$ we obtain:

(iv) *If* $H^i(\mathbb{P}^d, \mathscr{F}(-i)) = 0$ *for* $i = 1, 2, \dots, \dim(\mathscr{F})$, *then* $\operatorname{reg}(\mathscr{F}) = 0$.

This statement is given in [34] and may be easily verified in a direct way. As already mentioned in the introduction, (7.8) generalizes the vanishing statement (iv) to a corresponding statement about bounds. ○

(7.13) EXAMPLE. Let $0 < j \leq d$ and let $\imath : \mathbb{P}^e \to \mathbb{P}^d$ be a linear embedding. Consider the coherent $\mathscr{O}_{\mathbb{P}^d}$-sheaves

$$\mathscr{G}_{e,t} := \imath_* \mathscr{O}_{\mathbb{P}^d}(t), \quad (t \in \mathbb{Z}).$$

Then clearly

(i) $h^i(\mathscr{G}_{e,t}(n)) = 0, \quad \forall n \in \mathbb{Z}, \quad \forall i \neq 0, e.$

$$\text{(ii)}\ h^e(\mathscr{G}_{e,t}(n)) = \begin{cases} 0; & (n \geq t - e) \\ \binom{-n + t - e - 1}{e}; & n < t - e). \end{cases}$$

(iii) $\delta^{(0)}(\mathscr{G}_{e,t}) = \operatorname{lsdim}^{(0)}(\mathscr{G}_{e,t}) = \dim(\mathscr{G}_{e,t}) = e$.
(iv) $\operatorname{reg}(\mathscr{G}_{e,t}) = t$.

Observing that

$$C^{(0)}_{e,e}(e, e; 0, \dots, 0, c) = -e + \left[\frac{c}{e}\right]^{+}$$

$$B^{(0)}_{e,e}(e, e; 0, \dots, 0, c; n) = \max\{0, c-(n+e)e\}, \quad (n \geq -e)$$

(6.11) furnishes the bounds

$$h^e(\mathscr{G}_{e,t}(n)) \leq \max\left\{0, \binom{t-1}{e} - (n+e)e\right\}; \quad (n \geq -e),$$

$$h^e(\mathscr{G}_{e,t}(n)) = 0, \quad \forall n \geq -e + \left[\binom{t-1}{e}e^{-1}\right]^{+},$$

for any $t \geq 1$. For large values of t, these bounds are very weak with respect to what we know by (i) and (ii). Consequently the regularity bounds given by (7.9) or (7.10) will heavily exceed the actual value given by (iv). This is not surprising, as the small system of bounding invariants we use may not store much information on the specific nature of a sheaf. ○

(7.14) REMARK. The previous example illustrates that the generality of our approach goes to the cost of the strength of our bounds. To get sharper bounds, we thus should at least use more bounding invariants. Moreover, the sheaves $\mathscr{G}_{e,t}$ of (7.13) may be used to show that the numbers $h^1(\mathscr{F}(-1)), \dots, h^e(\mathscr{F}(-e))$ form a minimal system of invariants for bounding the Castelnuovo regularity of arbitrary coherent sheaves of dimension $\leq e$ over $\mathbb{P}^d$. More precisely:

(i) *Let $0 < j \leq e \leq d$ be integers. Then, there is no function $R : \mathbb{N}^{e-1} \to \mathbb{Z}$ such that for any coherent sheaf $\mathscr{F}$ of dimension e*

$$\operatorname{reg}(\mathscr{F}) \leq R(h^1(\mathscr{F}(-1)), \dots, h^{j-1}(\mathscr{F}(-(j-1))), h^{j+1}(\mathscr{F}(-(j+1)), \dots, h^e(\mathscr{F}(-e))).$$

To see this, choose $t \in \mathbb{N}$ and put $\mathscr{F}_{e,j,t} := \mathscr{G}_{e,0} \oplus \mathscr{G}_{j,t}$.

According to (7.14)(iii) and (iv) this sheaf is of dimension e and satisfies $\operatorname{reg}(\mathscr{F}_{e,j,t}) = t$. By (7.14)(i), (ii) $h^s(\mathscr{F}_{e,j,t}(-s)) = 0$ for all $s \neq j$. So, assuming that R exists, we would have the contradiction

$$t \leq R(0, \dots, 0), \quad \forall t \in \mathbb{N}.$$

For fixed d the invariants $\mathrm{lsdim}^{(0)}(\mathscr{F})$ and $\delta^{(0)}(\mathscr{F})$ take only finitely many values. So (i) implies:

(ii) *Let $0 < j \leq e \leq d$ be integers. Then, there is no bound of regularity for arbitrary coherent sheaves $\mathscr{F}$ of dimension $\leq e$ over $\mathbb{P}^d$, which depends only on*

$$\mathrm{lsdim}^{(0)}(\mathscr{F}), \delta^{(0)}(\mathscr{F}), h^1(\mathscr{F}(-1)), \ldots, h^{j-1}(\mathscr{F}(-(j-1))),$$
$$h^{j+1}(\mathscr{F}(-j+1))), \ldots, h^e(\mathscr{F}(-e)).$$

Now, it is obvious that none of the systems occurring in (ii) is a bounding system for all cohomological Hilbert functions $h^i(\mathscr{F}(n))$ $(i = 1, 2, \ldots)$ for arbitrary coherent sheaves $\mathscr{F}$ of dimension $\leq e$ over $\mathbb{P}^d$. In this sense $h^1(\mathscr{F}(-1)), \ldots, h^e(\mathscr{F}(-e))$ form a minimal system of invariants bounding the cohomology of all such sheaves. ○

8. Smooth varieties in characteristic 0

Throughout this section we assume, that the ground field k is of characteristic 0 (and—as previously—algebraically closed).

Moreover we assume that $X \subseteq \mathbb{P}^d$ is a closed, smooth, connected non-degenerate subvariety of positive dimension. Writing $\imath$ for the inclusion map $X \hookrightarrow \mathbb{P}^d$ we thus have

$$(8.1)\quad \text{(i)}\ \mathrm{lsdim}^{(0)}(\imath_* \mathcal{O}_X) = \mathrm{lsdim}(\imath_* \mathcal{O}_X) = d.$$
$$\text{(ii)}\ \delta^{(0)}(\imath_* \mathcal{O}_X) = \delta(\imath_* \mathcal{O}_X) = \dim(X) := e.$$

Writing

$$(8.2)\quad h^i(\mathcal{O}_X(n)) := \dim_k H^i(X, \mathcal{O}_X(n)),$$

(where twisting is understood with respect to the embedding $\imath$), we have (cf. [20])

$$(8.3)\quad h^i(\mathcal{O}_X(n)) = h^i(\imath_* \mathcal{O}_X(n)), \quad \forall n \in \mathbb{Z}, \quad \forall i \geq 0.$$

Now, defining $C^{(0)}_{e,i} \in \mathbb{F}^{2,e-i+1}$, $B^{(0)}_{e,i} \in \mathbb{F}^{2,e-i+1}_{-i}$ according to (6.8) and (6.9), we introduce functions

$$\hat{C}_i : \mathbb{N}^2 \times \mathbb{N}_0 \to \mathbb{Z}; \quad \hat{B}_i : \mathbb{N}^2 \times \mathbb{N}_0 \times \mathbb{Z}_{\geq -i} \to \mathbb{Z}$$

by setting (with $(0, \ldots, 0, c) \in \mathbb{N}_0^{b-i+1}$):

(8.4) (i) $\hat{C}_i(a, b, c) := C_{b,i}^{(0)}(a, b; 0, \ldots, 0, c)$
(ii) $\hat{B}_i(a, b, c; n) = B_{b,i}^{(0)}(a, b; 0, \ldots, 0, c; n)$

(8.5) REMARK: (A) It is obvious from the definitions of $\hat{C}_i$ and $\hat{B}_i$ that
(i) $\hat{C}_i \in \mathbb{F}^{2,1}$: $\hat{B}_i \in \mathbb{F}^{2,1}_{-i}$.
(ii) $\hat{B}_i(a, b, c; n) = 0, \quad \forall n \geq \hat{C}_i(a, b, c)$.
(iii) $\hat{C}_i(a, b, 0) = -i + 1$.
(iv) $\hat{B}_i(a, b, 0; n) = 0, \quad \forall n \geq -i$.
(B) By (6.10)(C), the above functions may be described as follows. First we put

(v) $\hat{C}_i := \hat{C}_i^{(0)}; \quad \hat{B}_i := \hat{B}_i^{(0)}; \quad (i = 1, 2, \ldots)$.

Then, by descending induction on r, define functions

$$\hat{C}_i^{(r)} : \mathbb{N}^2 \times \mathbb{N}_0 \to \mathbb{Z}; \quad \hat{B}_i^{(r)} : \mathbb{N}^2 \times \mathbb{N}_0 \times \mathbb{Z}_{\geq -i} \to \mathbb{N}_0$$

according to the following formulas, in which $(a, b, c; n) \in \mathbb{N}^2 \times \mathbb{N}_0 \times \mathbb{Z}_{\geq -i}$:
(vi) (a) $\hat{C}_i^{(r)}(a, b, c) = -i + 1, \quad$ *for* $r > b - i$.
(b) $\hat{B}_i^{(r)}(a, b, c; n) = 0, \quad$ *for* $r > b - i$.
In the range $0 \leq r \leq b - i$ we first define auxiliary functions
(vii)

$$\hat{V}_i^{(r)}(a, b, c; n) := \begin{cases} c; & (r = b - i). \\ \displaystyle\sum_{m=-i+1}^{n} \hat{B}_i^{(r+1)}(a, b, c; m); & (r < b - i). \end{cases}$$

Using these functions, we ultimately put:

$$\text{(iix)} \quad \hat{C}_i^{(r)}(a, b, c) := \hat{C}_i^{(r+1)}(b, b, c) + \left[\frac{\hat{V}_i^{(r)}(a, b, \hat{C}_i^{(r+1)}(a, b, c))}{a^{[r]}}\right]^+ - 1.$$

(ix)

$$\hat{B}_i^{(r)}(a, b, c; n) := \begin{cases} \hat{V}_i^{(r)}(a, b, c; n); & (-i \leq n < \hat{C}_i^{(r+1)}(a, b, c)). \\ \hat{V}_i^{(r)}(a, b, \hat{C}_i^{(r+1)}(a, b, c)); & (\hat{C}_i^{(r+1)}(a, b, c) \leq n < \hat{C}_i(b, b, c)). \\ \hat{V}_i^{(r)}(a, b, \hat{C}_i^{(r+1)}(a, b, c)) - (n + 1 - \hat{C}_i^{(r+1)}(a, b, c))a^{[r]}; & \\ & (\hat{C}_i^{(r+1)}(b, b, c) \leq n < \hat{C}_i^{(r)}(a, b, c)). \\ 0; & (\hat{C}_i^{(r)}(a, b, c) \leq n). \end{cases}$$

○

Now, using the above functions, we get the following bounds on the cohomological Hilbert functions $h^i(\mathcal{O}_X(n))$:

(8.6) PROPOSITION. *Let* $0 < i \leq e, j < -e + i$. *Then*:
(i) $h^i(\mathcal{O}_X(n)) \leq \hat{B}_i(d, e, h^e(\mathcal{O}_X(j)); \quad n - j - e), \quad \forall n \geq i + j + e$.
(ii) $h^i(\mathcal{O}_X(n)) = 0, \quad \forall n \geq \hat{C}_i(d, e, h^e(\mathcal{O}_X(j))) + j + e$.

Proof. In view of (8.5)(ii) it suffices to prove (i). According to the Kodaira vanishing theorem [25] we have $h^l(\mathcal{O}_X(n)) = 0$ for all $l < e$ and all $n < 0$. Let $i \leq l < e$. Then $-l + j + e < -l - e + i + e = -l + i \leq 0$ shows that $h^l(\imath_* \mathcal{O}_X(j + e)(-l)) = h^l(\mathcal{O}_X(-l + j + e)) = 0$ for $l = i, \ldots, e - 1$, (cf. (8.3)).
So, applying (6.11)(i) to $\imath_* \mathcal{O}_X(j + e)$ and observing (8.1), (8.2), (8.3) we obtain

$$\begin{aligned} & h^i(\mathcal{O}_X(j + e + n)) \\ & \quad = h^i(\imath_* \mathcal{O}_X(j + e)(n)) \leq B^{(0)}_{e,i}(d, e; 0, \ldots, 0, h^e(\imath_* \mathcal{O}_X(j + e)(-e)); n) \\ & \quad = B^{(0)}_{e,i}(d, e, 0, \ldots, 0, h^e(\mathcal{O}_X(j)); n) = \hat{B}_i(d, e, h^e(\mathcal{O}_X(j)); n) \end{aligned}$$

for all $n \geq -i$.
Consequently

$$h^i(\mathcal{O}_X(n)) \leq \hat{B}_i(d, e, h^e(\mathcal{O}_X(j)); n - j - e), \quad \forall n \geq -i + j + e \qquad \square$$

Applying (8.6) with $j = -e$, we get bounds on the cohomological Hilbert functions $n \mapsto h^i(\mathcal{O}_X(n))$, which depend only on $h^e(\mathcal{O}_X(-e))$.

(8.7) COROLLARY. *Let* $0 < i \leq e$. *Then*
(i) $h^i(\mathcal{O}_X(n)) \leq \hat{B}_i(d, e, h^e(\mathcal{O}_X(-e)); n), \quad \forall n \geq -i$
(ii) $h^i(\mathcal{O}_X(n)) = 0, \quad \forall n \geq \hat{C}_i(d, e, h^e(\mathcal{O}_X(-e)))$. $\square$

Making again use of the Kodaira vanishing theorem, we thus obtain:

(8.8) COROLLARY: *Let* $0 < i < e$. *Then*

$$h^i(\mathcal{O}_X(n)) \begin{cases} \leq \hat{B}_i(d, e, h^e(\mathcal{O}_X(-e)); n); & \textit{for } 0 \leq n < \hat{C}_i(d, e, h^e(\mathcal{O}_X(-e))). \\ = 0; & \textit{otherwise}. \end{cases}$$

Now, from (7.9) we get the following regularity bound:

(8.9) COROLLARY. $\operatorname{reg}(\mathcal{O}_X) \leq \hat{C}_1(d, e; h^e(\mathcal{O}_X(-e))) + 1$. $\square$

Next, we want to apply the previous results to bound the cohomological Hilbert functions of the vanishing ideal $\mathcal{I}_X \subseteq \mathcal{O}_{\mathbb{P}^d}$ of X. Thereby we clearly may restrict ourselves to the case $0 < e < d$.

Applying cohomology to the sequences

$$(8.10)\quad 0 \to \mathcal{I}_X(n) \to \mathcal{O}_{\mathbb{P}^d}(n) \to \iota_* \mathcal{O}_X(n) \to 0$$

and observing that $H^j(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(n)) \equiv 0$ for $j \neq 0, d$, we obtain

$$(8.11)\quad h^i(\mathcal{I}_X(n)) = h^{i-1}(\mathcal{O}_X(n)) \quad \textit{for } 1 < i \neq d.$$

As $H^d(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(n)) = 0$ for all $n \geq -d$, we get in addition

$$(8.12)\quad h^d(\mathcal{I}_X(n)) = h^{d-1}(\mathcal{O}_X(n)) \quad \textit{for all } n \geq -d.$$

So, it remains to give our upper bound for the first cohomological Hilbert function $n \mapsto h^1(\mathcal{I}_X(n))$. To do so, we introduce functions

$$B^* : \mathbb{N}^2 \times \mathbb{N}_0^2 \to \mathbb{N}_0, \quad C^* : \mathbb{N}^2 \times \mathbb{N}_0 \to \mathbb{N}_0$$

defined by

$$(8.13)\quad \text{(i) } B^*(a, b, c; n) := B^{(0)}_{a,1}(a, b; 0, \ldots, 0, \overset{b}{c}, 0, \ldots, 0; n)$$
$$\text{(ii) } C^*(a, b, c) := C^{(0)}_{a,1}(a, b; 0, \ldots, 0, \quad c, 0, \ldots, 0)$$

where $B^{(0)}_{a,1}$, $C^{(0)}_{a,1}$ are defined according to (6.8) and (6.9).

(8.14) PROPOSITION. *Let* $0 < e < d$. *Then*

(i) $h^1(\mathcal{I}_X(n)) \leq B^*(d, e+1, h^e(\mathcal{O}_X(-e)); n-1)$
for $0 < n \leq C^*(d, e+1, h^e(\mathcal{O}_X(-e)))$

(ii) $h^1(\mathcal{I}_X(n)) = 0$ *for all other* n.

Proof. Let $x \in \mathbb{P}^d$. Then (8.10) induces a short exact sequence $0 \to \mathcal{I}_{X,x} \to \mathcal{O}_{\mathbb{P}^d,x} \to \mathcal{O}_{X,x} \to 0$, which tells us that depth $(\mathcal{I}_X, x)$ equals $e+1$ or d, according to whether $x \in X$ or $x \notin X$. Therefore we have $\delta^{(0)}(\mathcal{I}_X) = \delta(\mathcal{I}_X) = e+1$, hence $\delta^{(0)}(\mathcal{I}_X(1)) = e+1$.

In view of (8.11) and (8.12)

$$h^i(\mathcal{I}_X(1)(-i)) = h^i(\mathcal{I}_X(1-i)) = h^{i-1}(\mathcal{O}_X(1-i)) = 0 \text{ or } h^e(\mathcal{O}_X(-e)),$$

according to whether $2 \leq i \neq e+1$ or $i = e+1$.

Applying cohomology to (8.10) (and observing that $H^0(\mathbb{P}^d, \mathcal{O}_{\mathbb{P}^d}(n)) = H^0(\mathbb{P}^d, \iota_* \mathcal{O}_X(n))$ for all $n \leq 0$) we see in addition that $H^1(\mathbb{P}^d, \mathcal{I}_X(n)) = 0$ for all $n \leq 0$. In particular $h^1(\mathcal{I}_X(1)(-1)) = h^1(\mathcal{I}_X) = 0$.

So, by (6.11) we get (cf. (8.4)(ii))

$$
\begin{aligned}
h^1(\mathscr{I}_X(n)) &= h^1(\mathscr{I}_X(1)(n-1)) \\
&\leq B^{(0)}_{d,1}(d, e+1; 0, \ldots, 0, \overbrace{h^e(\mathcal{O}_X(-e))}^{e+1}, 0, \ldots, 0; n-1) \\
&= B^*(d, e+1; h^e(\mathcal{O}_X(-e)); n-1)
\end{aligned}
$$

for $n - 1 \geq -1$. In view of (8.5)(ii) this proves our claim. □

(8.15) REMARK. Now, (8.7), (8.9) and (8.13) give the theorem (1.5). ○

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Mathematisches Institut der Universität Zürich
Rämistrasse 74
CH-8006 Zürich
Switzerland

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