

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 65 (1990)

Artikel: On group homomorphisms including mod-p cohomology isomorphisms.
Autor: Mislin, Guido
DOI: <https://doi.org/10.5169/seals-49736>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 13.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

On group homomorphisms inducing mod- p cohomology isomorphisms

GUIDO MISLIN

Let $\rho : F_1 \rightarrow F_2$ be a homomorphism of finite groups F_1 and F_2 inducing an isomorphism $H^*(F_2; \mathbb{Z}/p) \rightarrow H^*(F_1; \mathbb{Z}/p)$, p a fixed prime. By a result of S. Jackowski [5] it is known that then

- (i) $\ker(\rho)$ is of order prime to p ,
- (ii) $\text{im}(\rho)$ has index prime to p .

Simple examples show that in general (i) and (ii) alone do not suffice for ρ to induce a \mathbb{Z}/p -cohomology isomorphism. The purpose of this note is to describe necessary and sufficient conditions on ρ in group theoretic terms for ρ to induce an $H^*\mathbb{Z}/p$ -isomorphism. It turns out to be natural to work in the more general setting of compact Lie groups. The following notations and terminology will be used throughout this note.

For $\rho : G \rightarrow H$ a morphism of compact Lie groups we write

$$C(\rho) = \{h \in H \mid h\rho(g) = \rho(g)h \text{ for all } g \in G\}$$

for the centralizer of ρ ,

$$N(\rho) = \{h \in H \mid h\rho(G) = \rho(G)h\}$$

for the normalizer of ρ , and

$$W(\rho) = N(\rho)/C(\rho)$$

for the Weyl group of ρ . Note that $W(\rho)$ is a compact Lie group. It is a finite group, if for instance $\rho(G)$ is a finite subgroup of H . In case $\rho : T \rightarrow G$ stands for the inclusion of a maximal torus into a compact connected Lie group, $W(\rho) = W(G)$, the classical Weyl group of G . As usual

$$\text{Rep}(G, H)$$

stands for the representations of G in H , that is, the set of H -conjugacy classes of continuous homomorphisms $G \rightarrow H$. For p a prime we write

$$Q_p(G)$$

for the Quillen-category of finite p -subgroups of G ; its objects are the finite p -subgroups of G , and morphisms $P_1 \rightarrow P_2$ are homomorphisms of the form $c^g : x \rightarrow g^{-1}xg$ for some $g \in G$.

Our theorem then takes the following form.

THEOREM. *Let $\rho : G \rightarrow H$ be a morphism of compact Lie groups and let p be a prime. Then the following are equivalent:*

- (A) $H^*B\rho : H^*(BH; \mathbb{Z}/p) \rightarrow H^*(BG; \mathbb{Z}/p)$ is an isomorphism.
- (B) $\text{Rep}(\rho) : \text{Rep}(\pi, G) \rightarrow \text{Rep}(\pi, H)$ is a bijection for every finite p -group π .
- (C) $Q_p(\rho) : Q_p(G) \rightarrow Q_p(H)$ is an equivalence of categories.

REMARK. The reader verifies easily that (B) implies

(Bi): $\ker(\rho)$ contains no element of order p .

(Bii): every finite p -subgroup in H is conjugate to a subgroup in $\rho(G) \subset H$.

These statements generalize (i) and (ii) above to the case of compact Lie groups.

Before proving the Theorem, we want to recall some basic facts on homotopy fixed-points. All spaces considered are supposed to be of the homotopy type of CW-complexes. If X denotes a π -space, π a group, one writes $X^{h\pi}$ for the homotopy fixed-point space of the π action on X . It is by definition equal to $\text{map}_\pi(E\pi, X)$, the space of π -maps from the universal π -space $E\pi$ to X . The space of fixed-points X^π maps naturally to $X^{h\pi}$ and the induced map

$$(Z/p)_\infty(X^\pi) \rightarrow ((Z/p)_\infty X)^{h\pi}, \tag{1}$$

is known to be an equivalence, if π is a finite p -group and X a finite dimensional π -space (Theorem of Carlsson, Miller and Lannes, cf. [2]). The functor $(Z/p)_\infty(-)$ denotes the Bousfield–Kan Z/p -completion functor [1]. It has the basic property that it turns $H_*(; Z/p)$ -isomorphisms into homotopy equivalences. The following lemma is implicit in Carlsson’s paper [2].

LEMMA 1. *Let X be a finite dimensional π -space, π a finite p -group. Then the natural map*

$$X^{h\pi} \rightarrow ((Z/p)_\infty X)^{h\pi}$$

induces a bijection of connected components.

Proof. From the equivalence (1) we see that

$$\pi_0((Z/p)_\infty X)^{h\pi} \cong \pi_0(X^\pi).$$

By [2, VI.12],

$$\pi_0(X^{h\pi}) \cong \pi_0(((Z/p)_\infty^{\text{tot}} X)^\pi),$$

where the space $((Z/p)_\infty^{\text{tot}} X)^\pi$ consists of a disjoint union of certain partial completions of the components of X^π (cf. [2, IV.3]). Therefore,

$$\pi_0(((Z/p)_\infty^{\text{tot}} X)^\pi) \cong \pi_0(X^\pi)$$

and the lemma follows.

We will also need the following result which applies to arbitrary (not necessarily finite dimensional) spaces X .

LEMMA 2. *Suppose X is an i -connected π -space, π a finite p -group and $i \geq 2$. Then the canonical map*

$$\Theta : X^{h\pi} \rightarrow ((Z/p)_\infty X)^{h\pi}$$

induces a π_0 -bijection, and isomorphisms

$$\pi_j(X^{h\pi}, x) \rightarrow \pi_j(((Z/p)_\infty X)^{h\pi}, \Theta x)$$

for $j < i$ and all $x \in X^{h\pi}$.

Proof. Since X is i -connected, $(Z/p)_\infty X$ is i -connected too and it follows that the fibre F of $X \rightarrow (Z/p)_\infty X$ is $(i-1)$ -connected, with uniquely p -divisible homotopy groups. The (homotopy) fibre F_y of Θ over a point $y \in ((Z/p)_\infty X)^{h\pi}$ may be identified with $F^{h\pi}$ for some action of π on F . Since F is 1-connected mod- p acyclic, $F^{h\pi}$ is p -acyclic too [3, 2.3]. Thus F_y is non-empty and connected, which implies that Θ is a π_0 -bijection. The obstruction theory spectral sequence

$$H^*(\pi, \pi_* F) \Rightarrow \pi_*(F^{h\pi})$$

then collapses, because the groups $\pi_k F$ are all uniquely p -divisible, and it follows that

$$\pi_k(F^{h\pi}) \cong (\pi_k F)^\pi$$

for all k . In particular, $\pi_k(F^{h\pi}) = \pi_k(F_y) = 0$ for $k < i$ since F is $(i - 1)$ -connected. It follows then that Θ is a π_j -isomorphism for $j < i$.

Proof of the Theorem. (A) \Rightarrow (B). By Dwyer–Zabrodsky [3] one has a natural bijection

$$\text{Rep}(\pi, G) \rightarrow \pi_0 \text{map}(B\pi, BG), \tag{2}$$

associating with a homomorphism $\varphi : \pi \rightarrow G$ the component of $\text{map}(B\pi, BG)$ containing $B\varphi$; we denote that component by $\text{map}(B\pi, BG)_\varphi$. As $B\rho : BG \rightarrow BH$ is an $H_*(; \mathbb{Z}/p)$ -isomorphism, the induced map

$$\text{map}(B\pi, (\mathbb{Z}/p)_\infty BG) \rightarrow \text{map}(B\pi, (\mathbb{Z}/p)_\infty BH), \tag{3}$$

is an equivalence. Thus, to prove (B) it suffices to show that for a general compact Lie group G

$$\pi_0(\text{map}(B\pi, BG)) \rightarrow \pi_0(\text{map}(B\pi, (\mathbb{Z}/p)_\infty BG)), \tag{4}$$

is a bijection. This is certainly so for $G = SU(n)$ as we see from Lemma 2 (trivial π -action on $BSU(n)$). In the general case we choose an embedding $\epsilon : G \rightarrow SU(n)$ for some n , and we look at the fibration

$$SU(n)/G \rightarrow BG \rightarrow BSU(n), \tag{5}$$

If we fix a map $\sigma : \pi \rightarrow SU(n)$ which factors through $G \subset SU(n)$, then we obtain a fibration sequence

$$Z \rightarrow \coprod_\alpha \text{map}(B\pi, BG)_{\sigma_\alpha} \rightarrow \text{map}(B\pi, BSU(n))_\sigma, \tag{6}$$

where $\sigma_\alpha : \pi \rightarrow G$ runs over all G -conjugacy classes for which $\epsilon\sigma_\alpha$ is $SU(n)$ -conjugate to σ . We can identify Z with the space of sections of the fibration

$$SU(n)/G \rightarrow E\pi \times_\pi (SU(n)/G) \rightarrow B\pi$$

which is obtained by pulling back (5) along $B\sigma : B\pi \rightarrow BSU(n)$. As a result

$$Z \cong \text{map}_\pi(E\pi, SU(n)/G) = (SU(n)/G)^{h\pi}$$

where π acts on $SU(n)/G$ via σ . Since $BSU(n)$ is simply connected, $(\mathbb{Z}/p)_\infty(-)$ turns

(5) into a fibration sequence

$$(Z/p)_\infty(SU(n)/G) \rightarrow (Z/p)_\infty BG \rightarrow (Z/p)_\infty BSU(n), \tag{7}$$

which will give rise, as before, to a fibration

$$(((Z/p)_\infty(SU(n)/G))^{h\pi} \rightarrow \text{map}(B\pi, (Z/p)_\infty BG)_{R(\sigma)} \rightarrow \text{map}(B\pi, (Z/p)_\infty BSU(n))_\sigma, \tag{8}$$

where $\text{map}(B\pi, (Z/p)_\infty BG)_{R(\sigma)}$ denotes the disjoint union of those connected components of $\text{map}(B\pi, (Z/p)_\infty BG)$ which map to $\text{map}(B\pi, (Z/p)_\infty BSU(n))_\sigma$, the component of $(Z/p)_\infty B\sigma$ of $\text{map}(B\pi, (Z/p)_\infty BSU(n))$. To ensure that the map in (4) is bijective it obviously suffices to check that

$$\pi_0\left(\coprod_x \text{map}(B\pi, (Z/p)_\infty BG)_{\sigma_x}\right) \rightarrow \pi_0(\text{map}(B\pi, (Z/p)_\infty BG)_{R(\sigma)}), \tag{9}$$

is bijective for every $\sigma : \pi \rightarrow SU(n)$ which factors through $G \subset SU(n)$. For this, consider the natural map of the fibration (6) to that of (8). Because

$$\pi_1 \text{map}(B\pi, BSU(n))_\sigma \rightarrow \pi_1 \text{map}(B\pi, (Z/p)_\infty BSU(n))_\sigma$$

is an isomorphism (Lemma 2) we see that (9) is a bijection, if the map on fibres

$$\pi_0(SU(n)/G)^{h\pi} \rightarrow \pi_0((Z/p)_\infty(SU(n)/G))^{h\pi}$$

is a bijection. But this is the case by Lemma 1.

(B) \Rightarrow (C). We first check that $Q_p(\rho)$ induces a bijection on isomorphism classes of objects. Let A, B be finite p -subgroups of G with $\rho(A)$ and $\rho(B)$ isomorphic as objects of $Q_p(H)$ so that there exists an $h \in H$ with $c^h : \rho(A) \rightarrow \rho(B)$ a group isomorphism. Note that $A \rightarrow \rho(A)$ is injective in view of (B). Thus, there is a group isomorphism $\Theta : A \rightarrow B$ rendering the diagram

$$\begin{array}{ccc} A & \xrightarrow{\rho} & \rho(A) \\ \Theta \downarrow & & \downarrow c^h \\ B & \xrightarrow{\rho} & \rho(B) \end{array}$$

commutative; we will show that $\Theta = c^g$ for some $g \in G$, proving that A is isomor-

phic to B in $Q_p(G)$. Namely, because the bijection

$$\text{Rep}(\rho) : \text{Rep}(A, G) \rightarrow \text{Rep}(A, H)$$

maps the class of $\tilde{\Theta} : A \rightarrow G, (x \rightarrow \Theta x)$, to $\rho\tilde{\Theta} = c^h\rho : A \rightarrow H$, which is the same as the image under $\text{Rep}(\rho)$ of the inclusion $A \subset G$, we infer that $\tilde{\Theta}$ is G -conjugate to this inclusion; thus $\tilde{\Theta} = c^g : A \rightarrow B$ for some $g \in G$. This shows that $Q_p(G) \rightarrow Q_p(H)$ is one-one on isomorphism classes of objects. Actually, the same argument shows that $Q_p(\rho)$ is full: for any objects $A, B \in Q_p(G)$, the induced map of Q_p -morphisms

$$\text{Mor}(A, B) \rightarrow \text{Mor}(\rho(A), \rho(B))$$

is surjective.

If P is any finite p -subgroup of H , we apply (B) with $\pi = P$ to infer a commutative diagram

$$\begin{array}{ccc} & & G \\ & \nearrow f & \downarrow \rho \\ P & & H \\ & \searrow c^h & \end{array}$$

Thus $P \in Q_p(H)$ is isomorphic, as object of $Q_p(H)$, to $\rho(fP)$, showing that $Q_p(G) \rightarrow Q_p(H)$ is onto on isomorphism classes of objects.

It remains to check that $Q_p(\rho)$ is faithful, i.e., that for any $A, B \in Q_p(G)$

$$\text{Mor}(A, B) \rightarrow \text{Mor}(\rho A, \rho B)$$

is injective. But this is obvious because $\text{Mor}(A, B) \subset \text{Hom}(A, B)$, $\text{Mor}(\rho A, \rho B) \subset \text{Hom}(\rho A, \rho B)$ and $\rho : B \rightarrow \rho(B)$ is a group isomorphism as observed earlier.

(C) \Rightarrow (A). Define a cofunctor $F : Q_p(G) \rightarrow \text{Ab}$ by mapping P to $H^*(BP; Z/p)$. The natural map

$$\text{Res} : H^*(BG; Z/p) \rightarrow \lim_{\leftarrow} F$$

is then an isomorphism. In the case of a finite group G this follows from the classical result describing $H^*(BG; Z/p)$ in terms of the stable elements in the cohomology of a p -Sylow subgroup of G ; the general case was dealt with in [4, Theorem 2.3]. The implication (C) \Rightarrow (A) is then plain.

The next result is an immediate consequence of the Theorem. It relates Weyl-groups of maps with group cohomology.

COROLLARY 1. *Let $\rho : G \rightarrow H$ be a map of compact Lie groups inducing an isomorphism $H^*(BH; \mathbb{Z}/p) \rightarrow H^*(BG, \mathbb{Z}/p)$. Then for every homomorphism $\varphi : \pi \rightarrow G$ with π a finite p -group, the induced map of Weyl-groups*

$$\rho_* : W(\varphi) \rightarrow W(\rho\varphi)$$

is a group isomorphism.

Proof. Note that $W(\varphi)$ is the automorphism group of the object $\varphi(\pi) \in Q_p(G)$; similarly for $W(\rho\varphi)$. Thus part (C) of the theorem shows that the natural map $W(\varphi) \rightarrow W(\rho\varphi)$ is an isomorphism.

It seems surprising that \mathbb{Z}/p -cohomology information can contain such precise information on Weyl-groups, which are in general not p -groups. The following application shall illustrate this; as a variation of the theme we use rational cohomology information as input.

COROLLARY 2. *Let $\rho : G \rightarrow H$ be a map of connected compact Lie groups inducing an isomorphism*

$$H^*(BH; \mathbb{Q}) \rightarrow H^*(BG; \mathbb{Q}).$$

Then ρ induces an isomorphism of Weyl-groups $W(G) \rightarrow W(H)$.

Proof. Choose a prime p large enough such that $H^*B\rho : H^*(BH; \mathbb{Z}/p) \rightarrow H^*(BG; \mathbb{Z}/p)$ is an isomorphism (any prime which does not divide the order of the kernel and cokernel of the map $H_*(G; \mathbb{Z}) \rightarrow H_*(H; \mathbb{Z})$ will do). Clearly, G and H have the same rank and, because in addition there is no element of order p in the kernel of ρ , ρ maps a maximal torus $T(G) \subset G$ onto a maximal torus $\rho T(G) = T(H) \subset H$. The union of the finite p -subgroups is dense in $T(G)$ and $T(H)$. As a result, we can find a finite p -subgroup $\pi \subset T(G)$ with centralizer $C(\pi) = C(T(G)) = T(G)$, and $C(\rho\pi) = C(T(H)) = T(H)$; here we used the fact that in a compact Lie group closed subgroups satisfy the descending chain condition and that in a connected compact Lie group, a maximal torus is its own centralizer. Similarly, we may assume that the normalizer of π satisfies $N(\pi) = N(T(G))$, and $N(\rho\pi) = N(T(H))$. Then it follows that the induced map of Weyl-groups $W(G) \rightarrow W(H)$ is an isomorphism as one sees by applying the previous Corollary to the given map $\rho : G \rightarrow H$ and the inclusion map $\varphi : \pi \rightarrow G$.

Of course, this corollary could also be proved in a more conventional way by observing that the hypothesis implies that $\rho : G \rightarrow H$ induces an isomorphism of associated Lie algebras.

REFERENCES

- [1] A. K. BOUSFIELD and D. M. KAN, *Homotopy limits, completions, and localizations*. Lecture Notes in Math. 304 (1972).
- [2] G. CARLSSON, *Equivariant stable homotopy and Sullivan's conjecture* (to appear).
- [3] W. DWYER and A. ZABRODSKY, *Maps between classifying spaces*. Lecture Notes in Math. 1298, 106–119 (1987).
- [4] E. FRIEDLANDER and G. MISLIN, *Locally finite approximation of Lie groups II*. Math. Proc. Camb. Phil. Soc. 100, 505–517 (1986).
- [5] S. JACKOWSKI, *Group homomorphisms inducing isomorphisms of cohomology*. Top. 17, 303–307 (1978).
- [6] J. LANNES, *Cohomology of groups and function spaces* (preprint 1986).
- [7] H. MILLER, *The fixed-point conjecture* (preprint).

Mathematik
ETH-Zentrum
CH-8092 Zürich
July 1989

Received August 25, 1989