

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 65 (1990)

Artikel: A relationship between volume, injectivity radius, and eigenvalues.
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DOI: <https://doi.org/10.5169/seals-49735>

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A relationship between volume, injectivity radius, and eigenvalues

BURTON RANDOL

Suppose M is a compact Riemannian manifold and C a measurable subset of M having measure A . Expand the indicator function χ of C in a Fourier series in orthonormal eigenfunctions of the Laplace operator to get (in L^2)

$$\chi(y) = \sum_{k=0}^{\infty} a_k \varphi_k(y).$$

By the Parseval theorem,

$$A = \int_M |\chi(y)|^2 dy = \sum_{k=0}^{\infty} |a_k|^2,$$

and since $a_0 = A/\sqrt{V}$, where $V = \text{vol}(M)$, this implies that

$$1 = \frac{A}{V} + \frac{1}{A} \sum' |a_k|^2, \tag{1}$$

where the prime on a summation sign means that the term corresponding to index 0 is omitted. This last identity is the core of Siegel's quantitative version of the Minkowski theorem for a convex symmetric body B in R^n , in which the role of C is played by $\frac{1}{2}B$ [7].

Equation (1) becomes more precise if we know something about the Fourier coefficients. We will illustrate this when M is hyperbolic and of dimension n , which we will henceforth assume to be the case. Take C to be a ball about a point x in M of radius equal to the injectivity radius R of M . It then follows from the Selberg pretrace formula (cf. [1], Chapter 11), that the Fourier coefficients are given by $a_k = h(r_k)\varphi_k(x)$, where r_k is either of the two roots of $\delta^2 + r^2 = \lambda_k$. Here $\delta = \frac{1}{2}(n - 1)$, λ_k is the k th eigenvalue of the Laplace operator, and the even function h is the Selberg transform of the point-pair invariant which is 1 if its two arguments are within R of each other, and 0 otherwise (cf. [1], Chapter 11).

Equation (1) thus becomes

$$1 = \frac{A}{V} + \frac{1}{A} \sum' |h(r_k)|^2 |\varphi_k(x)|^2,$$

where the summation is over one of the two r_k 's corresponding to each λ_k . For definiteness, we will suppose that the sum is taken over the r_k 's which lie on the union of the non-negative reals with the imaginary segment from 0 to δi . Note that the so-called small eigenvalues of M , i.e., those in $(0, \delta^2)$, correspond to r_k 's on the open imaginary segment. If $\lambda_k = \delta^2$ is an eigenvalue of multiplicity m , the corresponding $r_k = 0$ is counted m times.

Integrate now over x , to get

$$V = A + \frac{1}{A} \sum' |h(r_k)|^2,$$

from which we derive

THEOREM 1.

$$1 = \frac{A}{V} + \frac{1}{AV} \sum' |h(r_k)|^2.$$

In order to apply Theorem 1, we will need to calculate $h(r)$ for our particular point-pair invariant. Now by [1], equation (5), page 275,

$$h(r) = 2\omega_{n-2} \int_0^R \cos ru \, du \int_u^R (z(\rho) - z(u))^{\delta-1} \sinh \rho \, d\rho,$$

where

$$z(x) = \left(2 \sinh \frac{x}{2} \right)^2 = 2(\cosh x) - 2,$$

and ω_{n-2} is the area of the $(n-2)$ -sphere in R^{n-1} ($\omega_0 = 2$).

I.e.,

$$h(r) = 2^\delta \omega_{n-2} \int_0^R \cos ru \, du \int_u^R (\cosh \rho - \cosh u)^{\delta-1} \sinh \rho \, d\rho,$$

or

$$h(r) = \delta^{-1} 2^\delta \omega_{n-2} \int_0^R (\cosh R - \cosh u)^\delta \cos ru \, du. \quad (2)$$

Note that $h(r)$ is positive and decreasing along the segment from δi to 0, so that the values of $h(r)$ along this segment dominate $h(0)$, which is given by

$$\begin{aligned} h(0) &= \delta^{-1} 2^\delta \omega_{n-2} \int_0^R (\cosh R - \cosh u)^\delta \, du \\ &= \delta^{-1} 2^\delta \omega_{n-2} R \cosh^\delta R \int_0^1 \left(1 - \frac{\cosh Ru}{\cosh R}\right)^\delta \, du. \end{aligned}$$

Now A , the volume of the ball of radius R , is given by

$$\omega_{n-1} \int_0^R \sinh^{n-1} u \, du,$$

which is asymptotic to

$$\frac{\omega_{n-1}}{(n-1)2^{n-1}} e^{(n-1)R}$$

for large R . On the other hand, it follows easily from our last expression for $h(0)$, that $h(0)$ is positive for $R > 0$, and that $|h(0)|^2$ is asymptotic to

$$\delta^{-2} \omega_{n-2}^2 R^2 e^{(n-1)R}$$

for large R . It follows that $A^{-1}|h(0)|^2 \geq c_1(n, R)R^2$, where $c_1(n, R)$ is positive and asymptotic to

$$c_2(n) = \frac{2^{n+1} \omega_{n-2}^2}{(n-1) \omega_{n-1}}$$

for large R .

This has an interesting consequence, since it follows from Theorem 1 that

$$1 > \frac{A}{V} + \frac{1}{AV} \sum'' |h(r_k)|^2, \quad (3)$$

where the sum is over the small eigenvalues. On the other hand, we have seen that for such an eigenvalue, $|h(r_k)|^2 > |h(0)|^2$, so if we denote by $N(M)$ the number of small eigenvalues for M , we conclude that

$$1 > \frac{A}{V} + c_1(n, R) \frac{N(M)R^2}{V},$$

which implies the following theorem, which is of interest for large R :

THEOREM 2.

$$N(M) < \alpha(n, R) \frac{V - A}{R^2},$$

where $\alpha(n, R)$ is positive and asymptotic to $1/c_2(n)$ for large R .

We conclude with another application of Theorem 1. Recall that a_0 , the zeroth Fourier coefficient of χ , is equal to A/\sqrt{V} , and that $\varphi_0(y) \equiv 1/\sqrt{V}$. Since $a_0 = h(r_0)\varphi_0(y)$, it follows immediately that $h(r_0) = h(\delta i) = A$. Thus, if λ_k is close to 0, or equivalently, if r_k is close to δi , it will be the case that $h(r_k) \sim A$. In more detail, suppose $\epsilon \in (0, 1)$, and that $r_k = \delta' i$, where $|\delta - \delta'| < \epsilon/R$.

By (2),

$$A - h(\delta' i) = \delta^{-1} 2^\delta \omega_{n-2} \int_0^R (\cosh R - \cosh u)^\delta (\cosh \delta u - \cosh \delta' u) du,$$

and by the mean value theorem this last expression is equal to

$$\delta^{-1} 2^\delta \omega_{n-2} \int_0^R (\cosh R - \cosh u)^\delta (\sinh w(u)) (\delta - \delta') u du,$$

where $w(u)$ is between $\delta' u$ and δu .

This is dominated by

$$\epsilon \delta^{-1} 2^\delta \omega_{n-2} \int_0^R (\cosh R - \cosh u)^\delta (\cosh \delta u) du = \epsilon A.$$

I.e.,

$$|A - h(\delta' i)| < \epsilon A,$$

or

$$\left| 1 - \frac{h(\delta'i)}{A} \right| < \epsilon,$$

from which we easily deduce that

$$\frac{|h(\delta'i)|^2}{A} > A(1 - \epsilon)^2.$$

Suppose now that M has $s_\epsilon(M)$ very small eigenvalues in the above sense, i.e., eigenvalues for which $|r_k - \delta i| < \epsilon/R$.

By Theorem 1,

$$1 > \frac{A}{V} + \frac{1}{AV} \sum''' |h(r_k)|^2,$$

where the sum is taken over the $s_\epsilon(M)$ very small eigenvalues of M .

We conclude from this that

$$1 + (1 - \epsilon)^2 s_\epsilon(M) < \frac{V}{A},$$

which implies the following theorem which is of interest for large R :

THEOREM 3.

$$\frac{1}{(1 - \epsilon)^2} + s_\epsilon(M) < \frac{1}{(1 - \epsilon)^2} \frac{V}{A}.$$

COROLLARY.

$$1 + s_\epsilon(M) < \frac{1}{(1 - \epsilon)^2} \frac{V}{A}.$$

The corollary has an interesting consequence in two dimensions. Suppose M is of genus g , and $s_\epsilon(M) = 2g - 3$. (We remark that such examples can be produced, and that for a given genus, if ϵ is small enough this value of $s_\epsilon(M)$ is maximal, since it is known [2, 3, 6] that there exists $\epsilon(g) > 0$ such that for $\epsilon < \epsilon(g)$, $s_\epsilon(m) \leq 2g - 3$. Additionally, for a given genus, $s_\epsilon(M) = 2g - 3$ for ϵ sufficiently small implies that there are no other eigenvalues λ_k in $(0, \delta^2)$ [5].)

By the corollary, bearing in mind that $V = 4\pi(g - 1)$,

$$2g - 2 < \frac{1}{(1 - \epsilon)^2} \frac{4\pi(g - 1)}{A},$$

or

$$A < \frac{2\pi}{(1 - \epsilon)^2}.$$

Since $A = 2\pi(\cosh R - 1)$, we conclude that if $2g - 3$ very small eigenvalues are present, the injectivity radius R of M must be less than a quantity which for small ϵ is near $\cosh^{-1} 2 \approx 1.317$. Note that for fixed ϵ , this estimate on the injectivity radius is uniform in the genus. In general, in view of the corollary, in any dimension an inequality of the form $s_\epsilon(M) \geq cV$ imposes a computable upper bound on R . Similarly, in any dimension an inequality of the form $V/A \leq c$ imposes a computable upper bound on $s_\epsilon(M)$. Finally, since it is known that for fixed genus in two dimensions, λ_{2g-3} cannot tend to zero unless R tends to zero [2, 3, 6], results like the last one derive interest for small ϵ from the fact that they are uniform in the genus.

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Received June 29, 1989.