Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 65 (1990)

Artikel: The maximum principle at infinity for minimal surfaces in flat three manifolds.

Autor: Meeks III, William H. / Rosenberg, H.

DOI: https://doi.org/10.5169/seals-49724

Comment. Math. Helvetici 65 (1990) 255–270 0010-2571/90/020255-16$1.50 + 0.20/0

The maximum principle at infinity for minimal surfaces in flat three manifolds

WILLIAM H. MEEKS III* AND HAROLD ROSENBERG

1. Introduction

Maximum principles are used as basic analytic tools for studying properties of functions defined on domains in $\mathbb{R}^n$ and satisfying certain equations (e.g. elliptic). In general these maximum principles play a fundamental role in analysis on complete Riemannian manifolds, especially in the study of variational problems. For example, the well-known maximum principle for harmonic functions has had both a simplifying and unifying effect on the fields of harmonic and complex analysis.

H. Hopf [18] gave an important general maximum principle for second order linear elliptic partial differential equations. The Hopf maximum principle easily yields a maximum principle for solutions of the minimal surface equation. In this context the principle states that if $D \subset \mathbb{R}^2$ is a smooth connected domain and f_1, f_2 are two smooth functions on D that satisfy the minimal surface equation, then the difference $f_1 - f_2$ cannot have an interior maximum or minimum unless the difference is constant.

The maximum principle for minimal graphs gives rise to the following geometric result for minimal surfaces in Riemannian three-manifolds: *If M_1 and M_2 are minimal surfaces in a Riemannian three-manifold that intersect at a common interior point p and M_1 is on one side of M_2 near p, then M_1 intersects M_2 in an open surface containing p.* In particular it follows that two differential minimal surfaces cannot intersect in their interiors at an isolated point. This geometric version of the maximum principle has many important applications to the general theory of minimal surfaces and, in its higher dimensional formulation, to the study of minimal hypersurfaces in n-dimensional Riemannian manifolds.

Recently Hoffman and Meeks [7] proved a theorem, called the Strong Halfspace Theorem, that is related to the maximum principle for minimal surfaces. Their

*The research described in this paper was supported by research grant DE-FG02-86ER250125 of the Applied Mathematical Science subprogram of Office of Energy Research, U.S. Department of Energy, and National Science Foundation grant DMS-8611574.

theorem can be interpreted as a kind of maximum principle at infinity for minimal surfaces. This theorem is based on the next fundamental result.

HALFSPACE THEOREM. *If $f: M \to \mathbb{R}^3$ is a proper connected minimal immersion that is contained in a halfspace, then $f(M)$ is a flat plane.*

The Halfspace Theorem fails for minimal hypersurfaces in $\mathbb{R}^n$, $n \geq 4$. In fact the n-dimensional $SO(n)$-invariant version of the catenoid, $C^n \subset \mathbb{R}^{n+1}$, is a properly embedded minimal surface with bounded x_{n+1}-coordinates.

Earlier using the work of do Carmo and Peng [2] and of Fischer-Colbrie and Schoen [4] on the geometry of stable minimal surfaces in $\mathbb{R}^3$, Meeks, Simon and Yau [14] showed that two properly immersed minimal surfaces in $\mathbb{R}^3$ either intersect at some point or each is contained in a halfspace. This result together with the Halfspace Theorem yielded the following [7].

STRONG HALFSPACE THEOREM. *Suppose M_1, M_2 are connected properly immersed minimal surfaces in $\mathbb{R}^3$. If M_1 and M_2 are disjoint, then M_1 and M_2 are parallel planes.*

It is the above generalized version of the Halfspace Theorem that has had many applications in recent years to global questions in the classical theory of minimal surfaces. However, for some applications of this type of result, there was a need to reformulate the Strong Halfspace Theorem to a more applicable form. Langevin and Rosenberg [9] gave a maximum principle at infinity for minimal surfaces of finite total curvature in $\mathbb{R}^3$. Their theorem stated that if M_1 and M_2 are disjoint, connected, properly embedded, minimal surfaces of finite total curvature and the boundaries of M_1 and of M_2 are compact (possibly empty), then $\operatorname{dist}(M_1, M_2) > 0$. They also found interesting applications of their maximum principle at infinity to the study of the uniqueness of solutions to the minimal surface equation on the exterior of the unit disk in $\mathbb{R}^2$. Choi, Meeks and White [1] gave a generalization that they needed in their study of the isometry group of a properly embedded minimal surface in $\mathbb{R}^3$. What they found is the following: *If M_1 and M_2 are two disjoint, connected, properly immersed, minimal surfaces that have compact boundary (possibly empty) and M_1 is asymptotic to a plane, then* $\operatorname{dist}(M_1, M_2) > 0$.

These maximum principles at infinity for minimal surfaces now play a fundamental role in virtually every aspect of the classical theory of minimal surfaces. In this paper we shall prove the following maximum principle at infinity for minimal surfaces in flat three-manifolds.

THEOREM 2 (Strong Maximum Principle at Infinity). *Suppose N is a complete flat three-dimensional manifold and M_1 and M_2 are disjoint, connected, properly* *immersed minimal surfaces in N with compact boundary (possibly empty). Then:*

1. *If ∂M_1 or ∂M_2 is nonempty, then, after possibly reindexing, there exists a point $x \in \partial M_1$ and a point $y \in M_2$, such that* $\text{dist}(x, y) = \text{dist}(M_1, M_2)$.
2. *If ∂M_1 and ∂M_2 are empty, then M_1 and M_2 are flat.*

When $N = \mathbb{R}^3$, the strong maximum principle at infinity is a simple consequence of the following weaker version (see the proof of Theorem 3 in Section 2.)

THEOREM 1 (Weak Maximum Principle at Infinity). *Suppose N is a complete flat three-dimensional manifold and M_1 and M_2 are connected properly immersed minimal surfaces in N with compact boundary (possibly empty). If M_1 and M_2 are disjoint, then* $\text{dist}(M_1, M_2) > 0$.

The proofs of the above maximum principles at infinity are informative and give some insight into the asymptotic behavior of minimal surfaces. Also their proofs introduce new constructions that are themselves useful in making nontrivial applications of the maximum principle at infinity. We refer the reader to [5], [8], [12], [13] for such applications.

The paper is arranged as follows. In Section 2 we prove the strong maximum principle at infinity for embedded minimal surfaces in $\mathbb{R}^3$. In Section 3 we reduce the weak maximum principle at infinity to the case where M_1 and M_2 are stable embedded minimal annuli of finite total curvature in $\mathbb{R}^3/S_\theta$ where S_θ is a screw-motion which is a nontrivial vertical translation composed with a rotation around the x_3-axis by θ, $0 \leq \theta < \pi$. In Section 4 we complete the proof of the weak maximum principle at infinity. Finally in Section 5 we show that the weak maximum principle at infinity implies the strong one.

2. The Strong Maximum Principle at Infinity for minimal surfaces in $\mathbb{R}^3$

LEMMA 1. *Suppose M_1 and M_2 are two disjoint connected minimally immersed hypersurfaces in a complete flat n-manifold. If the distance between the surfaces is realizable by a point in* $\text{Int}(M_1)$ *and a point in* $\text{Int}(M_2)$, *then M_1 and M_2 are totally geodesic.*

Proof. This proof appears in [11] but for completeness we repeat the proof here. Suppose $p \in M_1$ and $q \in M_2$ are points where the distance between M_1 and M_2 is realized. Let l be a line segment in N with end points p, q that realizes the distance. Note l is orthogonal to M_1 and M_2. Choose embedded disk neighborhoods $U_p \subset M_1$ and $V_q \subset M_2$ that are small enough so that $U_p \cup l \cup V_q$ is simply connected and lift this set to the universal cover $\mathbb{R}^n$. In $\mathbb{R}^n$ let $\tilde{U}_p$ denote the translate of U_p

along l so that p gets translated to q. Since l minimizes distance between p and q, $\tilde{U}_p$ lies on one side of V_q at q. The maximum principle implies that a smaller neighborhood $\hat{U}_p \subset \tilde{U}_p$ actually is contained in V_q. In particular any small parallel translate l' near p of l with one end point on U_p has its other end point on V_q. Since l' minimize the length between U_p and V_q, it is orthogonal to both surfaces. Hence the unit normal to U_p and V_q is parallel near p and q. This implies that U_p and V_q are totally geodesic and hence by analyticity M_1 and M_2 are also. □

COROLLARY 1. *Suppose M_1 and M_2 are disjoint proper minimally immersed hypersurfaces in a complete flat n-manifold. If M_1 is compact, then* $\operatorname{dist}(M_1, M_2) = \min\{\operatorname{dist}(\partial M_1, M_2), \operatorname{dist}(\partial M_2, M_1)\}$.

Proof. Since M_1 is compact and M_2 is proper, there exists points $p \in M_1$ and $q \in M_2$ such that $\operatorname{dist}(M_1, M_2) = \operatorname{dist}(p, q)$. If $p \in \partial M_1$ or $q \in \partial M_2$, then we are finished. If $p \in \operatorname{Int}(M_1)$ and $q \in \operatorname{Int}(M_2)$, then Lemma 1 states that M_1 and M_2 are totally geodesic. In this case the proof of the corrollary is immediate. □

LEMMA 2. *The weak maximum principle at infinity holds for properly embedded minimal surfaces of finite total curvature and compact boundary in $\mathbb{R}^3$. In other words, if M_1 and M_2 are two such disjoint surfaces, then* $\operatorname{dist}(M_1, M_2) > 0$.

Proof. Suppose M_1 and M_2 are two disjoint properly embedded minimal surfaces of finite total curvature in $\mathbb{R}^3$ with compact boundary. In this case $M_1 \cup M_2$ has a finite number of annular ends, each of which is asymptotic to a catenoid or to a plane [20]. Suppose $\operatorname{dist}(M_1, M_2) = 0$. This implies there exist annular ends E_1 of M_1 and E_2 of M_2, each asymptotic to a half-catenoid which we may assume is $C = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 = (\cosh x_3), x_3 \geq 0\}$, or to a plane, that we may assume is $\mathbb{R}^2$. Clearly we could choose E_1 and E_2 to be graphs over the exterior of a large disk D in $\mathbb{R}^2$. Since $E_1 \cap E_2 = \varnothing$, we may assume without loss of generality that E_1 lies above E_2. After a small vertical downward translation E_1' of E_1, $\partial E_1'$ still lies above E_2 but outside of a large ball, E_1' lies below E_2. It follows that $E_1' \cap E_2$ is a compact nonempty one-dimensional analytic subset of both E_1' and E_2.

We now show that $E_1' \cap E_2$ is a simple closed curve γ and E_1' is transverse to E along γ. Since E_1' is a graph over $\mathbb{R}^2 - D$, the projection $\Pi : \mathbb{R}^3 \to \mathbb{R}^2$ of $E_1' \cap E_2$ is a compact nonempty one-dimensional analytic variety in $\mathbb{R}^2$. If $\Pi(E_1' \cap E_2)$ is not a connected homotopically nontrivial simple closed curve in $\mathbb{R}^2 - D$, then $\mathbb{R}^2 - \Pi(E_1' \cap E_2)$ contains a compact component disjoint from D. This is impossible since the lifts of this component to E_2 and E_1' correspond to different solutions to the minimal surface equation with the same boundary values. Hence, E_1' intersects E_2 transversely in a single curve γ that is homotopically nontrivial on both E_1' and E_2. Let $\tilde{E}_1$ and $\tilde{E}_2$ denote the ends of E_1', E_2, respectively with boundary γ.

The surfaces $\tilde{E}_1$ and $\tilde{E}_2$ represents distinct solutions to the minimal surface equation over the unbounded region Δ of $\mathbb{R}^2$ with boundary curve $\Pi(\gamma)$ and they have the same boundary values along $\Pi(\gamma)$. Since $\tilde{E}_1$ and $\tilde{E}_2$ are asymptotic to translates of a fixed vertical catenoid, they have the same signed logarithmic growth in terms of $|\mathbf{x}|$, $\mathbf{x} \in \Delta$. (By the logarithmic growth rate of such an end $\tilde{E}$ we mean the following: $\tilde{E}$ is the graph of a function F on the exterior domain $|\mathbf{x}| > R$. The fact that F satisfies the minimal surface equation implies that F has an asymptotic expansion at infinity of the form: $F(\mathbf{x}) = a \log(|\mathbf{x}|) + \mathcal{O}(1)$, a is the logarithmic growth rate of $\tilde{E}$ [17]. Notice that the catenoid C to which we are assuming E_1 and E_2 are asymptotic, has logarithmic growth one.)

We will now give a simple geometric flux calculation to show that $\tilde{E}_1 = \tilde{E}_2$. (This proof easily generalizes to deal with similar uniqueness questions that arise in the proof of Theorem 1.)

First consider a simple closed homotopically nontrivial curve α on the half-catenoid C defined above. Suppose X is the gradient of the third coordinate function on C. Let η be the conormal of the unbounded component of $C - \alpha$. This means the unit vector field normal to α, tangent to C and pointing into the unbounded component of $C - \alpha$. The flux of X across α is

$$F(\alpha, X) = \int_\alpha X \cdot \eta = 2\pi.$$

This is clear if $\alpha = \partial C$ and follows for any α by the divergence theorem applied to the harmonic function x_3 on C. Similarly if $\tilde{C}$ is a minimal annulus that is a graph asymptotic to C, then the associated flux across the boundary of $\tilde{C}$ is also 2π. This follows from the Weierstrass Representation (see [20]).

Let X_1 and X_2 denote the gradient of the third coordinate functions of $\tilde{E}_1$ and $\tilde{E}_2$, respectively. From the above discussion we conclude that the flux of these vectors fields across their common boundary curve γ are equal. But since $\tilde{E}_1$ lies below $\tilde{E}_2$ along γ, $X_1 \cdot \eta_1 < X_2 \cdot \eta_2$ at every point of γ. Integrating this inequality along γ contradicts the fact that the flux of $\tilde{E}_1$ equals that of $\tilde{E}_2$. This contradiction proves Lemma 2. □

The following corollary to Lemma 2 was first proved by Langevin and Rosenberg [9] using a different method.

COROLLARY 2. *Suppose E_1 and E_2 are graphical solutions to the exterior Plateau problem for a compact domain in $\mathbb{R}^2$. If E_1 and E_2 each have the same limiting vertical normal vector, the same logarithmic growth and the same boundary, then $E_1 = E_2$.*

Proof. Suppose E_1 and E_2 satisfy the hypotheses of the corollary and $E_1 \neq E_2$. In this case E_i is asymptotic to an end-representative C_i of a catenoid or a horizontal plane. Note that C_1 and C_2 have the same logarithmic growth and limiting vertical normal vector. Hence, C_1 and C_2 can be chosen to be translates of each other.

By the flux argument in the proof of Lemma 2, E_1 does not lie above E_2 near their common boundary. Hence any small upward vertical translation of E_1 yields a E_1' such that E_1' intersects E_2 near ∂E_2. Since a large vertical upward translation of C_1 produces a surface that is a positive distance from C_2, a large upward translation of E_1 produces a surface that is disjoint from E_2. The maximum principle for minimal surfaces implies there exists a smallest $T > 0$ such that $(E_1 + (0, 0, T)) \cap E_2 = \varnothing$. Clearly $\operatorname{dist}(E_1 + (0, 0, T), E_2) = 0$, which contradicts Lemma 2. □

Recall that a noncompact surface in a Riemannian manifold is said to have *least-area* if compact subdomains have least-area with respect to their boundaries.

LEMMA 3. *The weak maximum principle at infinity holds for properly embedded minimal surfaces with compact boundary in* $\mathbb{R}^3$.

Proof. Suppose M_1 and M_2 are properly embedded disjoint minimal surfaces in $\mathbb{R}^3$ with compact boundary and suppose that $\operatorname{dist}(M_1, M_2) = 0$. Suppose B is a large ball that contains $\partial M_1 \cup \partial M_2$ in its interior and such that ∂B is transverse to $M_1 \cup M_2$. In this case $M_i - B$ consists of a finite number of components for $i = 1, 2$. Since $\operatorname{dist}(M_1, M_2) = 0$, it follows that a component of $M_1 - \operatorname{Int}(B)$ is a distance zero from a component of $M_2 - \operatorname{Int}(B)$. Hence, replacing M_1 and M_2 by these components we may assume that $\partial M_i = M_i \cap B \subset \partial B$ for $i = 1, 2$. By Corollary 1 we may assume that M_1 and M_2 are noncompact.

Our basic approach to proving the lemma will be to show that M_1 and M_2 can be separated by a pair of disjoint complete embedded minimal surfaces with compact boundary on ∂B and of finite total curvature. By Lemma 2 these finite total curvature surfaces are separated by a distance $\varepsilon > 0$, which gives a lower bound on the distance between M_1 and M_2. We now construct these finite total curvature surfaces.

The curves $\partial M_1 \cup \partial M_2$ bound a subdomain Δ of ∂B with at least one component having boundary in both ∂M_1 and ∂M_2. It follows that $M_1 \cup M_2 \cup \Delta$ is a connected properly embedded piecewise smooth surface in $\mathbb{R}^3$. This surface disconnects $\mathbb{R}^3$ into two components C, D where D is the closure of the component that contains $\operatorname{Int}(B)$. Note that $\partial M_1 \subset \partial D$ is homologous to zero in $B \subset D$. Since M_1 and M_2 are both noncompact and proper, there exists a proper arc $\delta : \mathbb{R} \to \partial C$ that intersects

∂M_1 transversely in a single point. If ∂M_1 bounded a compact surface E_1 in C, then since ∂M_1 bounds a compact surface E_2 in D, δ has odd intersection number with the cycle $E_1 \cup E_2$, which is impossible. Hence ∂M_1 is not homologous to zero mod 2 in C.

Notice that C has an analytic triangulation since it is an analytic manifold except along a finite number of compact transverse intersection curves. Change the metric in a compact neighborhood in C of $\Delta \subset \partial C$ in C so that the new metric satisfies (see the proof of Theorem 1 in [16]):

1. The 2-simplices of ∂C have nonnegative mean curvature and the edges of two adjacent simplices meet in an angle less than or equal to π.
2. If σ_1 is a 2-simplex in Δ and σ_2 is a 2-simplex in $M_1 \cup M_2$, σ_1 and σ_2 adjacent, then the angle between σ_1 and σ_2 is less than π along their common boundary.

We make this change of metric so that the least-area Plateau problem can be solved in C, i.e. any smooth 1-cycle in C that is null homologous in C is the boundary of a least-area surface $\Sigma \subset C$ and $\operatorname{Int}(\Sigma)$ is smooth and embedded. Moreover if Σ meets ∂C at a point x, then the maximum principle implies that the connected component of Σ containing x is contained in ∂C (see Theorem 2 in [16]). Let $\Sigma_1 \subset \Sigma_2 \subset \cdots$ be a compact exhaustion of M_1 by subdomains with smooth boundary and $\partial M_1 \subset \partial \Sigma_1$. Let $\tilde{\Sigma}_i$ be a least-area surface in C with $\partial \tilde{\Sigma}_i = \partial \Sigma_i$ and so that $\tilde{\Sigma}_i$ is $\mathbb{Z}_2$-homologous to Σ_i (rel $(\partial \Sigma_i)$). In this case $\tilde{\Sigma}_i \cup \Sigma_i$ is a boundary in C and hence $\tilde{\Sigma}_i$ is orientable.

We will now prove that a subsequence of the $\tilde{\Sigma}_i$ converge. This follows by showing that this family of surfaces satisfy uniform area and curvature estimates that we will now describe in detail.

Let B be a ball in C and W a least-area surface embedded in C, ∂W disjoint from B and W transverse to ∂B. (If $B \cap \partial C \neq \varnothing$, then assume $\partial B \cap \partial C$ is a disk.) Then $W \cap \partial B$ is the boundary of a region in ∂B of area at most half the area of ∂B. Consequently, there is a uniform local area bound for the $\tilde{\Sigma}_i$ (since $\tilde{\Sigma}_i$ minimizes in its $\mathbb{Z}_2$-homology class as a relative class.) Curvature estimates of Schoen [19] state that there exists a universal constant c such that for any stable orientable minimal surface T in a flat orientable three-manifold and $p \in T$ of distance d from ∂T, the Gaussian curvature is estimated by $|K(p)| \leq c/d^2$. This estimate leads to uniform curvature estimates for the family $\tilde{\Sigma}_i$ away from ∂M_1.

The above uniform area and curvature estimates for $\{\tilde{\Sigma}_i\}$ imply the family is compact, i.e., a subsequence of the surfaces $\tilde{\Sigma}_i$ converges to a proper least-area orientable minimal surface $\Gamma_1 \subset G$ with $\partial \Gamma_1 = \partial M_1$. (See the end of the proof of Theorem 3.1 in [15] for the proof that the smooth limit of least-area surfaces is again least-area.) This compactness property for $\{\tilde{\Sigma}_i\}$ is standard and for completeness we outline its proof.

Consider a small ball $B(r) \subset C - \partial M_1$ of radius r. By Schoen's curvature estimates, after choosing a possibly smaller r, every component of $\tilde{\Sigma}_i \cap B(r)$ that intersects $B(r/2)$ can be expressed as a graph of small gradient over a plane P_i in $B(r)$ passing through the center of the ball and P_i does not depend on the component. By the uniform area estimates, $B(r/2) \cap \tilde{\Sigma}_i$ contains a bounded number of components independent of i and hence there are a bounded number of associated graphs. Suppose for the moment that for every i, $\tilde{\Sigma}_i \cap B(r/2)$ contains one component and corresponding graph $G(i)$. Since a subsequence P_i, converge to a plane P in $B(r)$, the usual compactness theorems for minimal graphs imply that a subsequence $G(i_j)$ converges to a graph G over its projection to P. In the general case a subsequence of the corresponding graphs in $\tilde{\Sigma}_i \cap B(r)$ converge to a finite number of graphs. Note that $C - \partial M_1$ has a countable basis of balls $\{B_j\}$, where for each j and for every subsequence i_k the associated graphs $G(i_k, j)$ in $\tilde{\Sigma}_{i_k} \cap B_j$ have a convergent subsequence in B_j. Suppose that the subsequence $G(i_k, 1)$ converges in B_1. Then the associated subsequence of graphs in $\tilde{\Sigma}_{i_k} \cap B_2$ have a convergent subsequence in B_2 as well as B_1. Continuing in this manner *ad infinitum* from B_i to B_{i+1} and taking a diagonal sequence, yields a subsequence of the $\tilde{\Sigma}_i$ that converges in each B_j. The limit Γ_1 of this subsequence is a smooth properly embedded minimal surface in $C - \partial M_1$, has least area and has boundary ∂M_1. The boundary regularity theorem in [6] implies Γ_1 is smooth along ∂M_1. This completes our outline of the proof of compactness for the family $\{\tilde{\Sigma}_i\}$.

Suppose now that a subsequence of the $\tilde{\Sigma}_i$ converges to a properly embedded least-area surface Γ_1. Since Γ_1 is orientable and stable, it has finite total curvature (see [3] or Theorem 2.1 in [15]). Since $C - M_1$ is not smooth, the boundary maximum principle (see Theorem 2 in [16]) implies that either $\Gamma_1 = M_1$ or $\mathrm{Int}(\Gamma_1) \subset \mathrm{Int}(C)$. If $\Gamma_1 = M_1$, then M_1 has finite total curvature. If M_2 also has finite total curvature, then the lemma follows from Lemma 2. Thus, after possibly interchanging M_1 with M_2 we may assume that $\mathrm{Int}(\Gamma_1) \subset \mathrm{Int}(C)$.

The surface Γ_1 separates C into two components where one component contains M_1 and the other contains M_2. Let H denote the closure of the component containing M_2. Arguing as above for $M_2 \subset \partial H$ in place of $M_1 \subset \partial C$, we obtain a proper orientable smooth stable minimal surface $\Gamma_2 \subset H$ with $\Gamma_2 \cap \partial H = \partial M_2$. Note Γ_2 separates H into two components, one of which contains Γ_1 and the other that contains M_2.

It follows from Lemma 2 that $\mathrm{dist}(\Gamma_1, \Gamma_2) > 0$ since these surfaces have finite total curvature and are minimal in $\mathbb{R}^3$ outside of some compact neighborhood of their boundary curves. On the other hand, since $\mathrm{dist}(M_1, M_2) = 0$, there exist points $p \in M_1$, $q \in M_2$ far from the origin such that $\mathrm{dist}(p, q) < \mathrm{dist}(\Gamma_1, \Gamma_2)$. But

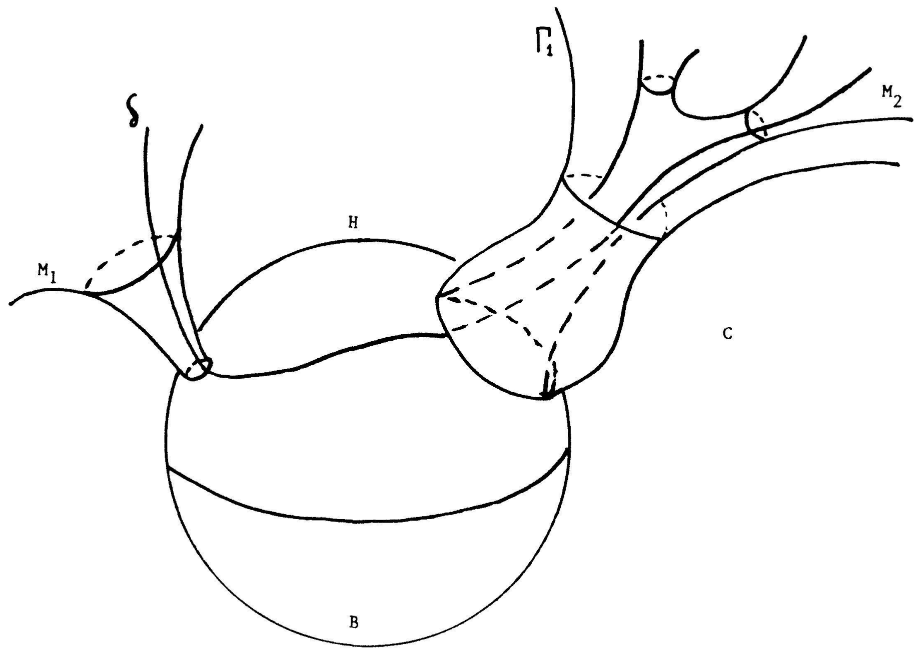

Fig. 1.

any arc joining p to q must contain a subarc in C joining a point of Γ_1 to a point of Γ_2. Hence dist $(p, q) >$ dist (Γ_1, Γ_2). This contradiction proves Lemma 3. $\square$

THEOREM 3. *Suppose M_1 and M_2 are disjoint properly embedded minimal surfaces in $\mathbb{R}^3$ with compact boundary and M_1 and M_2 are not parallel planes. Then*

$$\operatorname{dist}(M_1, M_2) = \min\{\operatorname{dist}(\partial M_1, M_2), \operatorname{dist}(\partial M_2, M_1)\}.$$

Proof. By the Strong Halfspace Theorem we may assume that ∂M_1 or ∂M_2 is nonempty. Let $(p_i, q_i) \in M_1 \times M_2$ be a sequence of points such that $\lim(\operatorname{dist}(p_i, q_i)) = \operatorname{dist}(M_1, M_2)$. Then a subsequence of the vectors $v_i = q_i - p_i$ converges to a point v on the sphere of radius dist (M_1, M_2). Let M_3 be the surface obtained by translating M_1 by the vector v. By Lemma 3 we know that $M_3 \cap M_2 \neq \varnothing$. There are two cases to consider:

1. $\partial M_3 \cap M_2 \neq \varnothing$ or $\partial M_2 \cap M_3 \neq \varnothing$.
2. $\operatorname{Int}(M_3) \cap \operatorname{Int}(M_2) \neq \varnothing$.

Lemma 1 shows possibility 2 occurs only when M_2 and M_3 are contained in a plane. Hence we are in case 1. But case 1 implies

$$\operatorname{dist}(M_1, M_2) = \min\{\operatorname{dist}(\partial M_1, M_2), \operatorname{dist}(\partial M_2, M_1)\},$$

which completes the proof of the theorem. $\square$

3. Reduction to the case of finite total curvature

In this section we reduce the proof of the weak maximum principle at infinity (Theorem 1 in the Introduction) to the case when the surfaces M_1 and M_2 are embedded stable minimal annuli with finite total curvature. We will call a noncompact surface an annulus if it is homeomorphic to $S^1 \times [0, 1)$.

LEMMA 4. *Suppose that the weak maximum principle at infinity holds in all flat manifolds of the form $\mathbb{R}^3/S_\theta$ for the special case of embedded stable minimal annuli of finite total curvature. Then the weak maximum principle at infinity holds in all complete flat three-manifolds.*

Proof. By Corollary 1 the weak maximum principle at infinity holds if M_1 or M_2 is compact. We will now assume they are both noncompact. Let N be an arbitrary flat three-manifold and suppose $M_1, M_2 \subset N$ are two properly immersed noncompact disjoint minimal surfaces with compact boundary (possibly empty). In particular, N is noncompact. By the classification of complete flat noncompact three-manifolds [22], we know that N is finitely covered by $\mathbb{R}^3$, by $\mathbb{R}^3/S_\theta$ or by $\mathbb{T} \times \mathbb{R}$ where $\mathbb{T}$ is a flat torus. After taking possibly a finite sheeted covering space of N and lifting the surfaces M_1, M_2, we may assume that N is $\mathbb{R}^3$, $\mathbb{R}^3/S_\theta$ or $\mathbb{T} \times \mathbb{R}$.

Choose a smooth compact analytic subdomain D of N such that ∂D intersects $M_1 \cup M_2$ transversely, $\partial M_1 \cup \partial M_2 \subset D$ and D has nonempty intersection with M_1 and with M_2. Without loss of generality we will replace M_1 and M_2 by their intersection with $N - \mathrm{Int}\,(D)$ and assume they are connected. Let C be a component of $N - (M_1 \cup M_2 \cup D)$ that contains points of M_1 and of M_2 in its boundary. We consider C with its induced metric (the distance between two points is the infimum of the lengths of paths in C joining the points). The metric completion of C, denoted $\tilde{\bar{C}}$, is a desingularization of $\bar{C}$ which is the closure of C.

Notice that $\tilde{\bar{C}}$ is an analytic manifold whose boundary is defined by analytic inequalities, hence by [10] the boundary of $\tilde{\bar{C}}$ has an analytic triangulation. We denote by $M_1(C)$, $M_2(C)$, $D(C)$ the points of $\tilde{\bar{C}}$ that project to M_1, M_2, D, respectively.

If $N = \mathbb{R}^3/S_\theta$ where S_θ is a screw motion, then choose the domain D to be a solid torus which is a regular neighborhood of the image of the axis of S_θ. If $N = \mathbb{T} \times \mathbb{R}$, then choose D to be of the form $\mathbb{T} \times [-t_0, t_0]$ for some t_0 and in the case $N = \mathbb{R}^3$ choose D to be a ball. In all cases, the fundamental group of each component Δ of $N - D$ is generated by the fundamental group of the boundary of the component. It then follows from separation theorems that a properly embedded surface Σ in Δ, separates Δ into two components. This separation property has the useful consequence in our constructions that if Σ is a properly embedded surface in $\tilde{\bar{C}}$ with

$\partial\Sigma = \partial M_1(C)$ or $\partial\Sigma = \partial M_2(C)$ or $\partial\Sigma = \varnothing$, then Σ separates $\tilde{C}$ into two components.

We next check that $\partial\tilde{C}$ is connected. If $\partial\tilde{C}$ is not connected, then $\tilde{C}$ contains a properly embedded connected surface $\Sigma \subset \mathrm{Int}(\tilde{C})$ that separates one component of $\partial\tilde{C}$ from another such component. The surface Σ can be considered to lie in N and Σ is disjoint from D. By our previous discussion Σ separates N into a component that contains D and another component that contains some point of M_1 or M_2. But since Σ is disjoint from $M_1 \cup M_2$ it is clear that either M_1 or M_2 is disjoint from D, which is contrary to our choice of D. Thus $\partial\tilde{C}$ is connected.

Change the metric in a compact neighborhood of $D(C)$ in $\tilde{C}$ so that the new metric satisfies:

1. The 2-simplices of $\partial\tilde{C}$ have nonnegative mean curvature and the edges of two adjacent simplices meet at an angle less than or equal to π.
2. If σ_1 is 2-simplex in $D(C)$ and σ_2 a 2-simplex in $M_1(C) \cup M_2(C)$ that are adjacent, then the angle between σ_1, σ_2 is less than or equal to π and different from π at some point.

We make this change of metric so that the least-area Plateau problem can be solved in $\tilde{C}$, i.e. if δ is a smooth cycle in $\tilde{C}$, that is null homologous in $\tilde{C}$ mod 2, then δ bounds a least-area surface Σ and $\mathrm{Int}(\Sigma)$ is smooth and embedded (see [16] and [21]). Moreover if $\mathrm{Int}(\Sigma)$ meets $\partial\tilde{C}$ at a point x, then the maximum principle implies that the connected component of Σ containing x is contained in $\partial\tilde{C}$.

Since $\mathrm{dist}(M_1, M_2) = 0$ and $\partial M_1 \cup \partial M_2$ is compact, we can choose the component C so that $\mathrm{dist}(M_1(C), M_2(C)) = 0$ in the metric induced by the Riemannian metric on $\tilde{C}$. Notice that the boundary of $M_1(C) \cup M_2(C)$ is contained in the boundary of $D(C)$. Let $\Sigma_1 \subset \Sigma_2 \subset \cdots$ be a compact exhaustion of $M_1(C)$ by piecewise smooth subdomains where $\partial M_1(C) \subset \partial\Sigma_1$. Let $\tilde{\Sigma}_i$ be a least-area surface in $\tilde{C}$ with $\partial\tilde{\Sigma}_i = \partial\Sigma_i$, and that is $\mathbb{Z}_2$-homologous to Σ_i (rel $(\partial\Sigma_i)$). The cycle $\tilde{\Sigma}_i \cup \Sigma_i$ bounds in $\tilde{C}$. In particular $\tilde{\Sigma}_i$ is orientable. (See Figure 1 where an analogous situation is described.) As in the proof of Lemma 3, a subsequence of the $\tilde{\Sigma}_i$ converge to a least-area surface Γ_1.

As in the construction of Γ_1 at the end of the proof of Lemma 3, we can assume that $\mathrm{Int}(\Gamma_1) \subset \mathrm{Int}(\tilde{C})$. The surface Γ_1 separates $\tilde{C}$ into two regions, one of which contains $M_1(C)$ and the other H that contains $M_2(C)$. Arguing as before with $M_2(C) \subset \partial H$ in place of $M_1(C) \subset \partial\tilde{C}$, we obtain a properly embedded stable minimal surface Γ_2 with $\partial\Gamma_2 = \partial M_2$. Furthermore, Γ_2 separates H into two components where one of the components has Γ_1, Γ_2 and part of $D(C)$ on its boundary.

Recall that C was chosen so that $\mathrm{dist}(M_1(C), M_2(C)) = 0$. Since Γ_1 and Γ_2 separate $M_1(C)$ and $M_2(C)$ in $\tilde{C}$, we conclude that $\mathrm{dist}(\Gamma_1, \Gamma_2) = 0$. However outside a compact subset of $\tilde{C}$, the metric on $\tilde{C}$ is flat. Theorem 2.1 in [15] states that a stable orientable properly immersed minimal surface with compact boundary in a flat orientable three-manifold has finite total curvature (also see [3]). Thus, Γ_1

and Γ_2 have a finite number of stable annular ends of finite total curvature. If $N = \mathbb{R}^3$, then Lemma 3 shows dist $(\Gamma_1, \Gamma_2) > 0$, a contradiction. If $N = \mathrm{T} \times \mathbb{R}$, it was shown in Theorem 3 in [13] that the ends of Γ_1 and Γ_2 stay a bounded distance from each other, which contradicts dist $(\Gamma_1, \Gamma_2) = 0$ (proved by a flux calculation similar to the calculation in the proof of Lemma 2). Thus, if the weak maximum principle at infinity holds in $\mathbb{R}^3/S_\theta$ for a pair of disjoint embedded stable minimal annuli, then the weak maximum principle at infinity holds in all flat three-manifolds. □

4. The Proof of the Weak Maximum Principle at Infinity

We now prove Theorem 1 (Weak Maximum Principle at Infinity) stated in the Introduction. By Lemma 4 we need only check the weak maximum principle at infinity for two properly embedded disjoint stable minimal annuli $A_1, A_2 \subset N = \mathbb{R}^3/S_\theta$ that have finite total curvature and such that dist $(A_1, A_2) = 0$. Let $\gamma \subset N$ denote the image of the x_3-axis. After removing compact subdomains from A_1, and A_2, we may assume that A_1 and A_2 are disjoint from γ. Let D_R denote the tubular neighborhood of γ of radius R.

Using the Weierstrass representation when $\theta = 0$ and a related analytic representation when $\theta \neq 0$, we derived analytic formulas for a minimal annulus A of finite total curvature in N [12]. When A is embedded, we proved that it is asymptotic to one of the following standard ends (see [12] for precise definitions):

1. A plane or catenoid in N;
2. A flat vertical annulus in N;
3. Helicoid-type ends.

We will now derive a contradiction if dist $(A_1, A_2) = 0$. It follows immediately from the description of standard ends in [12] that if A_1 is asymptotic to one of these standard ends S, then A_2 is also asymptotic to the same end S. In particular A_1 is asymptotic to A_2 and, after removing compact subdomains of A_1 and A_2, we may assume that A_1 is a small graph over A_2.

Suppose that the limiting unit normal vector to S is v. Note v is vertical when $N = \mathbb{R}^3/S_\theta$ and $\theta \neq 0$. It follows that N has a parallel Killing vector field V that is generated by translation in the direction v in $\mathbb{R}^3$. Without loss of generality, we may assume A_1 and A_2 are chosen so that the normals to A_1 and A_2 make a small angle with v. Thus, after a small translation of A_1 along the direction v, we obtain a new annulus A_3 whose boundary is above A_2 and that eventually lies below A_2. Standard ends do not intersect themselves after a small translation in the v or $-v$ directions. Thus, as in the proof of Lemma 2, A_3 intersects A_2 transversely in a simple closed curve α that is homotopically nontrivial on both A_2 and A_3.

Let E_2, E_3 denote the ends of A_2, A_3, respectively, with boundary curve α. Let V_2, V_3 denote the orthogonal projection of V onto E_2 and E_3. Let η_2, η_3 denote the conormals to E_2, E_3, respectively. Since V_2 and V_3 are divergence free, the fluxes

$$F_2 = \int_\alpha V_2 \cdot \eta_2, \qquad F_3 = \int_\alpha V_3 \cdot \eta_3$$

are geometric invariants of E_2 and E_3. However, as shown in [12], F_2 and F_3 only depend on the corresponding flux of S. We conclude that $F_2 = F_3$. However, since $\tilde{E}_3$ lies below $\tilde{E}_2$ along α, $V_2 \cdot \eta_2 < V_3 \cdot \eta_3$ along α and so $F_2 < F_3$. This contradiction completes the proof of the weak maximum principle at infinity. □

5. The Proof of the Strong Maximum Principle at Infinity

We are now in a position to prove Theorem 2 (Strong Maximum Princple at Infinity) stated in the Introduction. After possibly taking a finite sheeted cover of N and lifting the surfaces to this cover, we may assume that N is $\mathbb{R}^3$, $S^1 \times \mathbb{R}^2$, $\mathbb{T} \times \mathbb{R}$ or $\mathbb{R}^3/S_\theta$ where θ is not a rational multiple of π. First suppose that $N \neq \mathbb{R}^3/S_\theta$.

Suppose $\operatorname{dist}(\partial M_1, M_2) \leq \operatorname{dist}(\partial M_2, M_1)$ and that $\operatorname{dist}(\partial M_1, M_2) > \operatorname{dist}(M_1, M_2) > 0$. Consider a sequence of points $(p_i, q_i) \in M_1 \times M_2$ such that $\lim(\operatorname{dist}(p_i, q_i)) = \operatorname{dist}(M_1, M_2)$. Consider the isometry I_i of N taking p_i to q_i that lifts to a translation in $\mathbb{R}^3$. We may assume after picking a subsequence that I_i converges to an isometry $I: N \to N$. If $I(M_1) \cap M_2 \neq \varnothing$, then there exist interior points $p \in M_1$, and $q \in M_2$ with $\operatorname{dist}(p, q) = \operatorname{dist}(M_1, M_2)$, which is impossible by Lemma 1. On the other hand, $\operatorname{dist}(I(M_1), M_2) = 0$ so the weak maximum principle at infinity shows $I(M_1) \cap M_2 \neq \varnothing$. This proves the strong maximum principle at infinity in the case $N \neq \mathbb{R}^3/S_\theta$. Assume now that $N = \mathbb{R}^3/S_\theta$, θ an irrational multiple of π.

The proof of the strong maximum principle at infinity that we just gave for $N \neq \mathbb{R}^3/S_\theta$, θ an irrational multiple of π, fails to work when $N = \mathbb{R}^3/S_\theta$ because for $p \in M_1$ and $q \in M_2$ there does not always exist an isometry of N taking p to q. Let $(p_i, q_i) \in M_1 \times M_2$ with $\lim(\operatorname{dist}(p_i, q_i)) = \operatorname{dist}(M_1, M_2)$ and consider lifts $M_1(i)$ and $M_2(i)$ to $\mathbb{R}^3$ so the lifted points $\tilde{p}_i, \tilde{q}_i$ have the same distance in $\mathbb{R}^3$. If the vectors $(\tilde{q}_i - \tilde{p}_i)$ converge to a vertical vector v, then translation in $\mathbb{R}^3$ by v induces an isometry $I: N \to N$ that moves points a distance $\operatorname{dist}(M_1, M_2)$ and such that $\operatorname{dist}(I(M_1), M_2) = 0$. In this case the argument in the previous paragraph shows that the strong maximum principle at infinity holds for M_1 and M_2.

When M_1 and M_2 are embedded in N with finite total curvature, then the vector v is always vertical. To see this first note that the ends of M_1 and M_2 are asymptotic to standard ends and hence have vertical normal vectors at infinity (see Proposition 5.1 in [12]). Since the Gaussian curvature of M_1 and M_2 is asymptotic to zero, and the surfaces are a positive distance apart, it is clear that the sequence of points $(\tilde{q}_i - \tilde{p}_i)$ converges to a vertical vector. We will now reduce the proof of the general case to the case of embedded surfaces of finite total curvature (where the principle is true by the previous discussion).

For the moment assume that M_1 and M_2 are *embedded* in N. Also assume that $\operatorname{dist}(M_1, M_2) < \min\{\operatorname{dist}(\partial M_1, M_2), \operatorname{dist}(\partial M_2, M_1)\}$. Let γ denote the image of the x_3-axis in $\mathbb{R}^3$ and let D_R denote the tubular neighborhood of γ of radius R. The failure of the strong maximum principle at infinity to hold for M_1 and M_2 means that distance between M_1 and M_2 is never obtained by points on the surfaces. This property also holds if we remove a bounded subset from each of the surfaces. There exists a $T > 0$ such that after removing $M_i \cap D_T$ from M_i, $i = 1, 2$, the new surfaces (which we also call M_1 and M_2) have their boundary in ∂D_T. $M_1 \cup M_2$ separates $N - D_T$ into a finite number of components where 1 or 2 of these components have both M_1 and M_2 on their boundary. Let C be one of these components where the distance from M_1 to M_2 in C equals the distance from M_1 to M_2 in N.

Change the metric in a compact neighborhood of ∂C so that ∂C is a good barrier (see the proof of Lemma 4) for solving Plateau problems in C. Suppose M_1 does not have finite total curvature. By the argument in Lemma 4, ∂M_1 is the boundary of an embedded stable minimal surface Γ of finite total curvature and such that $\Gamma \subset \operatorname{Int}(C)$ and Γ separates C into a component containing M_1 and a component containing M_2. Furthermore, the ends of Γ consist of a finite number of annuli. These annuli are asymptotic to either a finite number of parallel flat planes or catenoids in N or they are asymptotic to a finite number of parallel helicoid-type ends in N.

In the case the ends of Γ are asymptotic to parallel planes or catenoids, then outside of some large D_R, Γ disconnects $N - D_R$ into regions in which $M_i \cap (N - D_R)$ lift with compact boundary to $\mathbb{R}^3$. Replace M_1 and M_2 by components of $M_i \cap (N - D_R)$, $i = 1, 2$, respectively, such that the new M_1 and M_2 are also closer at infinity than along their boundaries.

Let $\tilde{M}_1$ be a lift of M_1 to $\mathbb{R}^3$. First note that there are only a finite number of lifts $N_1, N_2, \ldots, N_k$ of M_2 to $\mathbb{R}^3$ such that the distance of the lift from $\tilde{M}_1$ is less than $2 \cdot \operatorname{dist}(M_1, M_2)$. This is because the lifts of M_2 to $\mathbb{R}^3$ are separated by parallel catenoid or planar type ends all essentially a constant distance apart. Clearly one of the surfaces N_i in $\{N_1, \ldots, N_k\}$ has distance $\operatorname{dist}(M_1, M_2)$ from $\tilde{M}_i$. However $\min\{\operatorname{dist}(\partial\tilde{M}_1, N_i), \operatorname{dist}(\partial N_i, \tilde{M}_1)\} > \operatorname{dist}(\tilde{M}_1, N_i)$. This contradicts the strong maximum principle at infinity in $\mathbb{R}^3$ (Theorem 3). We are left with the possibility that the ends of Γ are asymptotic to parallel helicoid-type ends.

Now choose R much larger than T. In particular we choose R large enough so that the ends of Γ intersect $\partial D_{R'}$ almost orthogonally in almost helices for $R' \geq R$. Let $\beta = M_1 \cap \partial D_R$ and note that β is homologous to ∂M_1 in the component H of $C - \Gamma$ that contains M_1. Applying the argument in the proof of Lemma 3 to β in H, we see that β is the boundary of a least-area orientable surface Γ_2 of finite total curvature and $\mathrm{Int}(\Gamma_2) \subset \mathrm{Int}(H)$.

Recall that the metric in H agrees with the induced metric as a subset of N except in some compact neighborhood Δ of $\partial D_T \cap C$. We claim that by choosing R sufficiently large, the surface Γ_2 will be disjoint from Δ and, hence, can be considered to be a minimal surface in N. First suppose that R is large enough so that $\Delta \subset D_{(1/10)R}$ and Γ intersects $\partial D_{R'}$ almost orthogonally in almost helices for $R' \geq \frac{1}{10}R$. In particular, the components of $\Gamma \cap (N - D_{\frac{1}{10}R})$ are very flat multisheeted graphs over their projection onto the (x_1, x_2)-plane. Consider a surface component E_R of $\Gamma_2 \cap (D_R - D_{\frac{1}{10}R})$. Since Γ_2 is a stable orientable minimal surface in a flat three-manifold, the curvature estimates of Schoen [19] imply that the Gaussian curvature of $x \in E_R$ is at most κ/d^2 where κ is a universal constant and d is the minimum of the distances of x to the boundary of D_R or $D_{\frac{1}{10}R}$. Hence, when R is large, the surface E_R is very flat near points in $\partial D_{\frac{1}{2}R} \cap E_R$. Since E_R is caught between the flat helicoid-type ends $\Gamma \cap (N - D_{\frac{1}{10}R})$, these curvature estimates imply the existence of an ε, $0 < \varepsilon < \frac{1}{10}R$, such that the projection of $E_R \cap (D_{\frac{1}{2}R} - D_{(\frac{1}{2}-\varepsilon)R})$ onto the (x_1, x_2)-plane is a submersion. It follows that if $E_R \cap D_{\frac{1}{10}R} \neq \varnothing$ for R large, then Area(E_R) grows quadratically in R. Assume that the translational part of S_θ is $(0, 0, 1)$. Since E_R is disjoint from D_T, ∂E_R bounds a surface in $H \cap \partial D_R$ of area less than πR. Since Γ_2 is a surface of least area, the area of E_R grows linearly in R, a contradiction. This proves that $\mathrm{Int}(\Gamma_2) \cap \Delta = \varnothing$ for R large, and hence, Γ_2 is a minimal surface in N.

Since Γ_2 separates $M_1 - D_R$ from M_2 and the surfaces M_1 and M_2 are asymptotically closer at infinity, $\mathrm{dist}(\Gamma_2, M_2) \leq \mathrm{dist}(M_1, M_2)$. Since M_1 and M_2 are asymptotically closer at infinity, it is clear that we can choose some large value R so that $\mathrm{dist}(\partial\Gamma_2, M_2) < 2 \cdot \mathrm{dist}(M_1, M_2)$ and $\mathrm{dist}(\partial M_2, \Gamma_2) > \frac{1}{2}R$. Since $\partial\Gamma_2 \subset M_1$ is compact, $\mathrm{dist}(\partial\Gamma_2, M_2) > \mathrm{dist}(M_1, M_2)$. Hence, Γ_2 and M_2 violate the strong maximum principle at infinity in N. If M_2 also has infinite total curvature, then repeating the above argument with Γ_2 and M_2, we can replace M_2 by a properly embedded minimal surface Γ_3 of finite total curvature such that Γ_2 and Γ_3 violate the strong maximum principle at infinity in N. As remarked earlier, the strong maximum principle at infinity holds for embedded surfaces of finite total curvature in N. This contradiction completes the proof in the case M_1 and M_2 are embedded.

If M_1 and M_2 are not embedded, the modification given in the proof of Lemma 4 by metrically completing components of $N - (M_1 \cup M_2 \cup D_T)$, reduces the argument to the embedded case. This completes the proof of Theorem 2.

REFERENCES

[1] T. CHOI, W. H. MEEKS III and B. WHITE, *A rigidity theorem for properly embedded minimal surfaces in* $\mathbb{R}^3$. Journal of Differential Geometry, March, 1990.

[2] M. DO CARMO and C. K. PENG, *Stable minimal surfaces in* $\mathbb{R}^3$ *are planes*. Bulletin of the AMS *1*, 903–906 (1979).

[3] D. FISCHER-COLBRIE, *On complete minimal surfaces with finite Morse index in 3-manifolds*. Inventiones Math. *82*, 121–132 (1985).

[4] D. FISCHER-COLBRIE and R. SCHOEN, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*. Comm. Pure Appl. Math. *33*, 199–211 (1980).

[5] C. FROHMAN and W. H. MEEKS III, *The topological uniqueness of complete one-ended minimal surfaces and Heegaard surfaces in* $\mathbb{R}^3$ (preprint).

[6] R. HARDT and L. SIMON, *Boundary regularity and embedded minimal solutions for the oriented Plateau problem*. Annals of Math. *110*, 439–486 (1979).

[7] D. HOFFMAN and W. H. MEEKS III, *The strong halfspace theorem for minimal surfaces*. Inventiones Math. (to appear).

[8] D. HOFFMAN and W. H. MEEKS III, *The asymptotic behavior of properly embedded minimal surfaces of finite topology*. Journal of the AMS *2*, 667–681 (1989).

[9] R. LANGEVIN and H. ROSENBERG, *A maximum principle at infinity for minimal surfaces and applications*. Duke Math. Journal *57*, 819–828, (1988).

[10] S. LOJASIEWICZ, *Triangulation of semianalytic sets*. Ann. Scuola Norm. Sup. Pisa *18*, 449–474 (1964).

[11] W. H. MEEKS III, *The topological uniqueness of minimal surfaces in three-dimensional Euclidean space*. Topology *20*, 389–410 (1981).

[12] W. H. MEEKS III and H. ROSENBERG, *The geometry of periodic minimal surfaces* (preprint).

[13] W. H. MEEKS III and H. ROSENBERG, *The global theory of doubly periodic minimal surfaces*. Inventiones Math. *97*, 351–379 (1989).

[14] W. H. MEEKS III, L. SIMON and S. T. YAU, *The existence of embedded minimal surfaces, exotic spheres and positive Ricci curvature*. Annals of Math. *116*, 221–259 (1982).

[15] W. H. MEEKS III and S. T. YAU, *The topological uniqueness theorem of complete minimal surfaces of finite topological type* (preprint).

[16] W. H. MEEKS III and S. T. YAU, *The existence of embedded minimal surfaces and the problem of uniqueness*. Math. Z. *179*, 151–168 (1982).

[17] R. OSSERMAN, *A Survey of Minimal Surfaces*, 2nd edition. Dover Publications, New York, 1986.

[18] M. PROTTER and H. WEINBERGER, *Maximum Principles in Differential Equations*. Prentice-Hall, Englewood 1967.

[19] R. SCHOEN, *Estimates for Stable Minimal Surfaces in Three Dimensional Manifolds*, volume 103 of *Annals of Math. Studies*. Princeton University Press, 1983.

[20] R. SCHOEN, *Uniqueness, symmetry, and embeddedness of minimal surfaces*. Journal of Differential Geometry *18*, 791–809 (1983).

[21] L. SIMON, *Lectures on geometric measure theory*. In *proceedings of the Center for Mathematical Analysis*, volume 3. Australian National University. Canberra 1983.

[22] J. A. WOLF, *Spaces of Constant Curvature*. McGraw-Hill, New York 1967.

Mathematics Department
University of Massachusetts
Amherst, MA 01003, USA

and

Department de Mathématique
Université de Paris 7
75251 Paris, France

Received May 9, 1989/December 13, 1989