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Amalgamated products and finitely presented groups

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§1. Introduction

§1.1. It has long been recognized that a decomposition of a group G as a free product of two groups with one amalgamated subgroup can often be used to obtain important properties of G. The objective of this paper is to show that if a finitely presented group has a presentation of the "right kind" then the group can be decomposed into an amalgamated product in an interesting way.

It is customary to term an amalgamated product

$$G = A *_C B \tag{$*$}$$

non-trivial if the amalgamated subgroup C is a proper subgroup of each of the factors A and B. In the event that C is of index 2 in both A and B then we say that the decomposition $(*)$ is of *dihedral type*; if C is proper in both A and B, and has index greater than 2 in one of them then the decomposition $(*)$ is called *proper*. In general, a group has a decomposition of dihedral type if and only if it admits a homomorphism onto the infinite dihedral group (see §3). It is not hard to show that many solvable groups admit decompositions of dihedral type. By way of contrast, as is well-known, the groups with proper decompositions always contain free subgroups of rank two (see §2). It is with such proper decompositions that we are mainly concerned. Indeed our goal here is to obtain conditions on a finite presentation of a group G that ensure that G has a proper amalgamated product decomposition. The following theorem exemplifies what we have in mind.

THEOREM 2. *Let G be a group given by a finite presentation of deficiency at least two (i.e. one with at least two more generators than relators). Then G admits a proper decomposition where the factors (and amalgamated subgroup) are all finitely generated.*

It follows immediately from Theorem 2 that a finitely presented group with a presentation of deficiency at least two contains a free subgroup of rank two. This observation should be compared with the theorem of B. Baumslag & S. J. Pride [1] which states that a finitely presented group G with a presentation of deficiency at least two contains a normal subgroup K such that G/K is a finite cyclic group and K maps onto a free group of rank two.

It also follows immediately from Theorem 2 that

COROLLARY 2.1. *A one-relator group given by a presentation with at least* 3 *generators and one defining relator has a proper decomposition in which the factors (and amalgamated subgroup) are finitely generated.*

C. T. C. Wall posed the following question in [13]:

Which one-relator groups are (non-trivial) amalgamated products?

Corollary 2.1 answers this question for one-relator groups with at least three generators in a most emphatic way.

The key to Theorem 2 is the following

THEOREM 1. *Let G be a finitely presented group with a given non-trivial amalgamated product decomposition $G = A *_C B$. Then G also has a non-trivial amalgamated product decomposition $G = A' *_{C'} B'$ where A', B' and C' are finitely generated and $A' \leq A$, $B' \leq B$, and $C' \leq C$.*

It has been pointed out to us by Martin Dunwoody, in a letter, after this paper was submitted for publication, that Theorem 1 has also been proved, independently, by M. Bestvina and M. Feighn in some as yet unpublished work of theirs. Their work depends on earlier work of Dunwoody.

It is worth noting that if a finitely presented group G has an amalgamated product decomposition $(*)$ in which the factors A and B are finitely generated, then the amalgamated subgroup C is necessarily finitely generated (this is essentially the content of G. Baumslag [2]). It is also worth noting that it is possible for a finitely presented group to have an amalgamated product decomposition in which the factors and the amalgamated subgroup are finitely generated despite the fact that none of them are finitely related. We give such an example in §6.

Theorem 1 will be proved in §2.

C. F. Miller has pointed out to us that the proof of Theorem 1 applies, essentially without change, to HNN extensions. Dunwoody has informed us that this result has also been obtained by Bestvina and Feighn.

Theorem 2 will be deduced from Theorem 1 on appealing to four lemmas, which will be proved in §3.

There is a simple variation of the argument used to prove Theorem 2 which can be applied to finitely presented groups of a rather different kind:

THEOREM 3. *Let the group G be given by the finite presentation*

$$G = \langle x_1, \ldots, x_m; r_1, \ldots, r_n \rangle$$

where $m \geq 2$ and each of the relators r_i lie in the third derived group of the free group on $x_1, \ldots, x_m$. Then G admits a proper amalgamated product decomposition in which the factors (and amalgamated subgroup) are all finitely generated.

We will give the proof of Theorem 3 in §5.

So Theorem 3 implies in particular that the non-cyclic free solvable groups of derived length at least three are infinitely presented – in fact A. I. Mal'cev [8] has proved that *all* non-cyclic free solvable groups of derived length at least *two* are infinitely presented (cf. also R. Bieri & R. Strebel [6]). We refer the interested reader to R. Strebel [12] for the last word on this subject.

In conclusion we would like to point out that much of the work in this paper was stimulated by an earlier unpublished theorem of ours that we proved in 1982, namely

THEOREM O. *Let G be a group given by a finite presentation on m generators and n relations. If $d = m - n$ and*

$$3d - 3 > \dim (H_1(G, Z/2Z)),$$

(where here $H_1(G, Z/2Z)$ is the first homology group of G with coefficients in the integers mod 2 *and* $\dim (H_1(G, Z/2Z))$ *is the dimension of $H_1(G, Z/2Z)$ thought of as a vector space over $Z/2Z$) then G is an amalgamated product in which the amalgamated subgroup is of infinite index in one factor and of index at least two in the other.*

The proof of Theorem O relies on the representation-theoretic ideas that were introduced by Culler and Shalen in [7]. Now the hypothesis of Theorem O implies that $d > 1$. Consequently Theorem 2 can also be applied here, yielding a proper amalgamated product decomposition, a conclusion only slightly weaker than that reached in Theorem O.

§2. The proof of Theorem 1

§2.1. Let G be a group generated by its subgroups A and B. Suppose that $A \cap B = C$. We term G an *amalgamated product* of A and B amalgamating C if every strictly alternating product

$$a_1b_1 \cdots a_nb_n(a_i \in A, a_i \notin C, b_i \in B, b_i \notin C)$$

is different from 1 (see B. H. Neumann [11]) and we express this fact by writing

$$G = A *_C B.$$

We observe next, as promised in the introduction, the well-known

LEMMA 1. *Let $G = A *_C B$ be a proper decomposition. Then G contains a free subgroup of rank two.*

We give the proof here for completeness. Thus suppose that C is of index at least 3 in A and of index at least 2 in B. Let a_1 and a_2 be two elements of A which lie outside C and such that $a_1a_2^{-1} \notin C$. Then $a_1^{-1}b^{-1}a_1b$ and $a_2^{-1}b^{-1}a_2b$ freely generate a free group of rank two.

§2.2. Our objective now is to prove Theorem 1. Thus suppose G is a finitely presented group given as a non-trivial amalgamated product:

$$G = A *_C B.$$

Since G is finitely presented, it is finitely generated. Hence we can find a finite set

$$a_1, \ldots, a_m, b_1, \ldots, b_n$$

of generators of G where the $a_i \in A$ and the $b_j \in B$. By a theorem of B. H. Neumann [10], G can be finitely presented on this set of generators:

$$G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n; r_1, \ldots, r_p \rangle. \tag{1}$$

Let $A_1 = gp(a_1, \ldots, a_m)$ and let $B_1 = gp(b_1, \ldots, b_n)$. Suppose that we present A_1 and B_1 on the generators given above:

$$A_1 = \langle a_1, \ldots, a_m; R_1 \rangle \quad \text{and} \quad B_1 = \langle b_1, \ldots, b_n; S_1 \rangle.$$

Of course R_1 and S_1 are consequences of the defining relations for G. So we can add both sets to the presentation (1) for G:

$$G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n; R_1, S_1, r_1, \ldots, r_p \rangle. \tag{2}$$

Now each of the relators r_k can be written as alternating products which take the form

$$r_k = u_1(\vec{a})v_1(\vec{b}) \cdots u_q(\vec{a})v_q(\vec{b}).$$

Since $A_1 \leq A$ and $B_1 \leq B$ it follows that either one of $u_i(\vec{a}) \in C$ or $v_j(\vec{b}) \in C$. Suppose, for example, that $u_i(\vec{a}) \in C$. Then put $y_1 = u_i(\vec{a})$ and add y_1 to the set of generators for G together with the extra relation $u_i(\vec{a})y_1^{-1}$:

$$G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n, y_1; R_1, S_1, r_1, \ldots, r_p, u_i y_1^{-1} \rangle. \tag{3}$$

We now set $B_2 = gp(B_1, y_1)$, $A_2 = A_1$ and present A_2 and B_2

$$A_2 = \langle a_1, \ldots, a_m; R_2 \rangle \quad \text{and} \quad B_1 = \langle b_1, \ldots, b_n, y_1; S_2 \rangle;$$

here $R_2 = R_1$, but S_2 may well be different from S_1. R_2 and S_2 are consequences of the defining relators in the presentation (3) of G. So

$$G = \langle a_1, \ldots, a_m, b_1, \ldots, b_n, y_1; R_2, S_2, r_1, \ldots, r_p, u_i y_1^{-1} \rangle. \tag{4}$$

Notice that we can think of y_1 as an element of B. The relator r_k can then be replaced in (4) by the relator

$$r'_k = u_1(\vec{a})v_1(\vec{b}) \cdots u_{i-1}(\vec{a})(v_{i-1}(\vec{b})y_1 v_i(\vec{b}))u_{i+1}(\vec{a})v_{i+1}(\vec{b}) \cdots u_q(\vec{a})v_1(\vec{b}),$$

which has smaller "syllable" length. This procedure can be repeated, leading to a presentation of G which takes the form we shall now describe. First of all the generators of G are partitioned into two finite sets:

$$\{a_1, \ldots, a_m, x_1, \ldots, x_h\}, \qquad \{b_1, \ldots, b_n, y_1, \ldots, y_j\},$$

where $a_1, \ldots, a_m, x_1, \ldots, x_h \in A$, $b_1, \ldots, b_n, y_1, \ldots, y_j \in B$. The defining relators in this presentation are arrived at as follows. First we put

$$A' = gp(a_1, \ldots, a_m, x_1, \ldots, x_h), \qquad B' = gp(b_1, \ldots, b_n, y_1, \ldots, y_j).$$

Then A' and B' can be defined on these generators:

$$A' = \langle a_1, \ldots, a_m, x_1, \ldots, x_h; R' \rangle, \qquad B' = \langle b_1, \ldots, b_n, y_1, \ldots, y_j; S' \rangle.$$

The defining relators of G then consist of

$$R' \cup S' \cup \{r_1^\#, \ldots, r_q^\#\} \cup \{u_1^\# y_1^{-1}, \ldots, u_j^\# y_j^{-1}, x_1^{-1} v_1^\#, \ldots, x_h^{-1} v_h^\#\},$$

where each of the $r_i^\#$ are words in the generators of A' and B' of syllable length 1, $u_1^\#, \ldots, u_j^\# \in A' \cap C, v_1^\#, \ldots, v_h^\# \in B' \cap C$ and

$$u_1^\# = y_1, \ldots, u_j^\# = y_j, \qquad x_1 = v_1^\#, \qquad x_h = v_h^\#$$

in G. Thus

$$G = \langle a_1, \ldots, a_m, x_1, \ldots, x_h, b_1, \ldots, b_n, y_1, \ldots, y_j;$$
$$R', S', r_1^\#, \ldots, r_q^\#, u_1^\# y_1^{-1}, \ldots, u_j^\# y_j^{-1}, x_1^{-1} v_1^\#, \ldots, x_h^{-1} v_h \rangle. \qquad (5)$$

Since we have included R' and S' among the relators, each of the $r_i^\#$ can be omitted from the presentation (5). Moreover if we put

$$C' = gp(u_1^\#, \ldots, u_j^\#, x_1, \ldots, x_h)$$

then

$$C' = gp(y_1, \ldots, y_j, v_1^\#, \ldots, v_h^\#).$$

It follows from the presentation (5) that

$$G = A' *_{C'} B'.$$

Notice that this decomposition is non-trivial, because if, e.g. $C' = A'$, then $G = B$, contradicting the hypothesis. This completes the proof of Theorem 1.

§3. Preparation for the proof of Theorem 2

§3.1. We begin with the proof of the following simple

LEMMA 2. *A group G has a decomposition of dihedral type if and only if it admits a homomorphism onto the infinite dihedral group.*

The proof of Lemma 2 is straightforward. Thus suppose that G has a decomposition of dihedral type $G = A *_C B$ – so here C is of index two in both A and B.

Hence C is a normal subgroup of G and G/C is the infinite dihedral group. Conversely, suppose that G maps onto the infinite dihedral group $D = \langle a, b; a^2 = b^2 = 1\rangle$. Let A be the preimage of $gp(a)$, B the preimage of $gp(b)$ and C the preimage of 1. Then $G = A *_C B$ is a decomposition of G of dihedral type.

§3.2. The next lemma turns out to be crucial in the proof of Theorem 2 (cf. G. Baumslag [3])

LEMMA 3. *Let G be a group given by the finite presentation*

$$G = \langle a_1, \ldots, a_m; r_1, \ldots, r_n\rangle$$

where $m - n \geq 2$. *Let K be any normal subgroup of G with G/K infinite cyclic. Then the tensor product*

$$K_{ab} \otimes \mathbb{Q}$$

of the abelianisation K_{ab} *of K with the additive group* $\mathbb{Q}$ *of rational numbers (over* $\mathbb{Z}$*) is infinite dimensional. So, in particular, K is not finitely generated.*

Proof. Suppose that $G/K = gp(tK)$. We add t to the given generators of G and one extra relation expressing t as a word w in the given generators of G. This yields the presentation

$$G = \langle a_1, \ldots, a_m, t; r_1, \ldots, r_n, tw^{-1}\rangle$$

of deficiency $m - n$. Since t generates G modulo K we can find integers $e_1, \ldots, e_m$ such that $a_1t^{e_1}, \ldots, a_mt^{e_m}$ all lie in K. Put $b_i = a_it^{e_i}$ $(i = 1, \ldots, m)$. Then the elements $t, b_1, \ldots, b_m$ again generate G and we can present G on these $m + 1$ generators subject to $n + 1$ relators $s_1, \ldots, s_{n+1}$, where now t occurs with exponent sum zero in each of these relators:

$$G = \langle t, b_1, \ldots, b_m; s_1, \ldots, s_{n+1}\rangle.$$

It follows then that the normal closure in G of $b_1, \ldots, b_m$ (i.e. the least normal subgroup of G containing $b_1, \ldots, b_m$) is K. We now view K_{ab} as a right module over the integral group ring $\mathbb{Z}[t, t^{-1}]$ of the infinite cyclic group generated by t, where t acts on K_{ab} via conjugation. This allows us to think of K_{ab} as an m-generator module subject to $n + 1$ defining relators. So we can view

$$M = K_{ab} \otimes \mathbb{Q}$$

as an m-generator module over $A = \mathbb{Q}[t, t^{-1}]$ subject to $n+1$ defining relators. Since A is a principal ideal domain, M is a direct sum of cyclic modules. Moreover, since $m > n+1$ at least one of these cyclic submodules is free and so is infinite dimensional over $\mathbb{Q}$. This completes the proof of Lemma 3.

§3.3. We are now in a position to prove Lemma 4, another key step in the proof of Theorem 2. We adopt throughout the notation and hypothesis of Lemma 3.

LEMMA 4. *The Klein bottle group*

$$H = \langle u, v; u^2 = v^2 \rangle$$

is a quotient of G.

In order to prove Lemma 4 we first prove that $\langle a, t; tat^{-1} = a^{-1} \rangle$ is a homomorphic image of G. To this end let N be a rational vector space with basis $\{\alpha\}$. We turn N into an A-module by setting

$$\alpha \cdot t = -\alpha.$$

There is an obvious module homomorphism γ of M onto N obtained by first mapping M onto a free cyclic summand of M and then mapping this summand onto N. Now the composition of the canonical homomorphism of K into M with γ is a homomorphism, say μ, of K into N such that $N/K\mu$ is a torsion abelian group, which means that $K\mu$ is non-zero. Now $K\mu$, thought of as a $\mathbb{Z}[t, t^{-1}]$-module, is finitely generated. It follows that $K\mu$ is infinite cyclic. The kernel L of μ is invariant under conjugation by t and is therefore normal in G. The quotient group G/L can be presented in the form $G/L = \langle a, t; tat^{-1} = a^{-1} \rangle$. It is easy then to see that G/L is simply H with a different presentation – we need only put $u = t$, $v = at$.

§3.3. Finally we will need the following observation

LEMMA 5. *Let G be a group given by a presentation of deficiency at least two with m generators and n relators. Then every subgroup of G of finite index has a presentation of deficiency at least two.*

Let J be a subgroup of G of finite index j. It follows immediately from the method of Reidemeister and Schreier that J can be presented on $1 + j(m-1)$ generators and defined in terms of these generators by jn relators (see e.g. M. Hall [8]). The desired conclusion follows immediately.

§4. The proof of Theorem 2

§4.1. Let G be a group given by a finite presentation of deficiency $d > 1$. Then, by Lemma 4, G maps onto the Klein bottle group H. So, by Lemma 2 G has a decomposition of dihedral type, i.e.

$$G = A *_C B$$

where C is of index two in both A and B. We now apply Theorem 1 to obtain a non-trivial decomposition

$$G = A' *_{C'} B'$$

where A', B' and C' are finitely generated. This decomposition is necessarily proper. In order to see this suppose that C' is of index two in both A' and B'. Then, by Lemma 2, G contains a subgroup S of index two containing C' with S/C' is infinite cyclic. By Lemma 5, S has a presentation of deficiency at least two. Hence, by Lemma 3, C' is not finitely generated. This contradiction completes the proof of Theorem 2.

§5. The proof of Theorem 3

§5.1. Let G be a finitely presented group satisfying the hypothesis of Theorem 3. Then G maps onto the infinite dihedral group and so has a decomposition

$$G = A *_C B$$

of dihedral type. So, by Theorem 1, G also has a non-trivial decomposition

$$G = A' *_{C'} B'$$

in which both A' and B' are finitely generated. Our objective is to prove that this decomposition is proper. It suffices then, as in the proof of Theorem 2 in §4, to prove that a normal subgroup K of G with G/K isomorphic to the infinite dihedral group is not finitely generated. This follows from the following

LEMMA 6. *Let S be a non-cyclic free solvable group of derived length three. If T is a normal subgroup of S such that S/T is isomorphic to the infinite dihedral group, then T_{ab} is not finitely generated.*

Proof. By definition

$$S \cong F/\delta_3(F)$$

where F is an absolutely free group of rank at least two and $\delta_3(F)$ is the third derived group of F. In terms of this isomorphism, the normal subgroup T of S corresponds to a normal subgroup D of F containing $\delta_3(F)$ with F/D isomorphic to the infinite dihedral group. Hence there is a subgroup E of F containing D of index 2 in F with E/D infinite cyclic. It follows from the subgroup theorem for free groups that D is a free group of infinite rank. Hence $D_{ab} = D/[D, D]$ is free abelian of infinite rank. Now $F/[D, D]$ is a solvable group of derived length at most 3. Hence

$$\delta_3(F) \leq [D, D].$$

This implies that

$$T/[T, T] \cong D/[D, D]$$

and therefore completes the proof of the lemma.

This analysis does not work if one replaces the free solvable group of derived length three by the free solvable group of derived length two, i.e. by the free metabelian group. Indeed if S is the free metabelian group of rank two on x and y and if T is the normal closure in S of x^2 and y^2, then S/T is the infinite dihedral group but T is generated by 5 elements.

§6. An example

§6.1. The content of Theorem 1 is that if a finitely presented group has a non-trivial amalgamated product decomposition then it also has a non-trivial decomposition in which the factors are finitely generated. It is not hard to show that in this instance the amalgamated subgroup is necessarily finitely generated. Our objective here is to give an example of a finitely presented amalgamated product in which the factors are both finitely generated but not finitely presented, and the amalgamated subgroup is also finitely generated but not finitely presented. To this end, let

$$M = \langle a, s, t; [s, t] = 1, [a, a^s] = 1, a^t = aa^s \rangle.$$

Then M is a finitely presented group in which $gp(a, s) = \langle a \rangle \backslash \langle s \rangle$, the wreath product of two infinite cyclic groups, is not finitely presented (G. Baumslag [4]). Put

$$G = M * \langle v \rangle,$$

the free product of M and the infinite cyclic group on v and let $u = tv$. Then it is not hard to see that if

$$A = gp(a, s, u), \qquad B = gp(a, s, v),$$

then

$$A = gp(a, s) * \langle u \rangle \quad \text{and} \quad B = gp(a, s) * \langle v \rangle.$$

So neither A nor B is finitely presented. Next observe that

$$U = gp(a, s, a^u, s^u) = gp(a, s) * gp(a^u, s^u)$$

and

$$V = gp(a, s, (aa^s)^v, s^v) = gp(a, s) * gp((aa^s)^v, s^v).$$

So U and V are isomorphic. Let $\phi : U \to V$ be the isomorphism defined by

$$\phi : a \mapsto a, \qquad s \mapsto s, \qquad a^u \mapsto (aa^s)^v, \qquad s^u \mapsto s^v.$$

Then it follows without difficulty that G is an amalgamated product of A and B where U and V are identified according to the isomorphism ϕ:

$$G = \{A * B; U = V\}.$$

Now U is the free product of two copies of $\langle a \rangle \backslash \langle s \rangle$. So U is also finitely generated but not finitely presented. Since G is patently finitely presented we have concocted the desired example.

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