**Zeitschrift:** Commentarii Mathematici Helvetici
**Herausgeber:** Schweizerische Mathematische Gesellschaft
**Band:** 65 (1990)

**Artikel:** Rigidity for surfaces of non-positive curvature.
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**DOI:** https://doi.org/10.5169/seals-49718

Comment. Math. Helvetici 65 (1990) 150–169 0010-2571/90/010150-20\$1.50 + 0.20/0


# Rigidity for surfaces of non-positive curvature

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## Introduction

In this paper we consider the question of boundary rigidity for surfaces of nonpositive curvature. Given a compact manifold, $M$, with smooth boundary $N$, a riemannian metric $g_0$ on $M$ induces a nonnegative real valued function, $d_0$, on $N \times N$ where $d_0(p, q)$ is the distance in $(M, g_0)$ between $p$ and $q$. A riemannian manifold $(M, g_0)$ is called boundary rigid if for any riemannian manifold $(M_1, g_1)$ with the same boundary, $N$, if $d_1 = d_0$ then $g_1$ is isometric to $g_0$. This question was recently considered by the author in [C] where one was led to the quesiton: "Are all SGM manifolds boundary rigid?". The condition SGM is a condition on the boundary distance function $d_0$ which roughly speaking is equivalent to the condition that all geodesic segments in $M$ are the unique minimizing paths between the endpoints (see [C] for a precise definition.) By geodesic segments we mean geodesics that intersect the boundary at most at the boundary points (i.e. they do not "graze" the boundary at interior points of the segment.) Any compact subdomain in the interior of a convex manifold with possibly empty boundary (i.e. between any two points there is a unique geodesic) will be SGM. Hence, in particular, any subdomain of a complete simply connected manifold of nonpositive curvature will be SGM. Also any disk of nonpositive curvature will be SGM. In this paper we show:

THEOREM A. *If $(M_0^2, g_0)$ is a compact, nonpositively curved, SGM, surface with boundary then it is boundary rigid.*

It should be emphasized that no assumptions are made a-priori about the curvature (or even the topology) of the possible $(M_1, g_1)$. Other manifolds are known to be boundary rigid. It has been shown by Gromov and Michel (see [G] sec. 5.5B and [M]) that $(M_0, g_0)$ is boundary rigid in any of the following three cases: (1) $M_0^n$ admits an isometric immersion into $\mathbb{R}^n$, (2) $M_0^n$ admits a 1-1 immersion into a convex subset of the round $n$-sphere, (3) $M_0^2$ admits a 1-1 immersion into the hyperbolic plane. All of the above three cases are for manifolds of constant curvature.

The reader is referred to [C] for a more extensive history of this problem as well as its relationship to other problems such as the uniqueness of "geodesic lenses". Also in [C] the reader will find a case made for the condition SGM through examples that are not boundary rigid.

A problem related to the above boundary rigidity problem is the question of compact manifolds without boundary whose geodesic flows are conjugate. We will say that $M_0$ and $M_1$ have conjugate geodesic flows via $F$ if $F$ is a $C^1$ diffeomorphism, $F : UM_1 \to UM_0$, between the unit tangent bundles which commutes with the geodesic flows i.e. $\zeta_0^t \circ F = F \circ \zeta_1^t$ for all $t$ where $\zeta_i^t$ is the goedesic flow (for time $t$) on $UM_i$.

THEOREM B. *If $M_0$ is a compact surface (without boundary) of genus $\geq 2$ with non-positive sectional curvature and $M_1$ is a compact surface whose geodesic flow is conjugate via $F$ to $M_0$ then $F = \zeta_0^t \circ dI$ (or $dI \circ \zeta_1^t$), where $I$ is an isometry from $M_1$ to $M_0$ and $t$ is a fixed number.*

We emphasize that in this theorem as well there are no a-priori assumptions about the compact surface $M_1$.

The question of geodesic conjugacy has come up in many contexts recently. In particular the recent work of Feres and Katok [F–K] extending the results of Kanai [K] shows that if $M$ is a compact manifold of negative quarter pinched curvature such that at least one of the horospheric foliations is smooth then the geodesic flow on $M$ is smoothly conjugate to the geodesic flow on a manifold of constant negative curvature. Hence a higher dimensional version of theorem B would answer part of a long standing conjecture.

In the case that both $M_0$ and $M_1$ are surfaces of negative curvature they will have conjugate geodesic flows if and only if they have the same marked length spectrum (see [B–K] sec. 10 and [F–O]) and hence by the above they will be isometric if and only if they have the same marked length spectrum. The marked length spectrum for a surface of negative curvature is the function that takes elements of $\pi_1$ (or conjugacy classes) to the length of the shortest closed geodesic in the free homotopy class. The length spectrum (the image of the above function) is not enough to determine a surface of negative curvature up to isometry as was shown by Vignéras [V] (also see [Su]) who gave examples of two nonisometric surfaces of constant negative curvature $-1$ with the same eigenvalue spectrum and hence (by our curvature conditions) the same length spectrum (see [D–G] or [CV]).

The fact that two surfaces of negative curvature are isometric if they have the same marked length spectrum was proved independently (and apparently some months earlier than the author) by Otal [O1] and had been conjectured in [B–K]. A result similar to Theorem A was also proved independently by Otal [O2]. The methods used in Otal's papers are different from the ones used here and the results stated here are more general. In particular Otal makes additional assumptions about the metric of $M_1$ in both his theorems (some of these assumptions are not hard to drop) and he needs to assume negative rather than nonpositive curvature. An earlier version of theorem A (with additional assumptions) can be found in [G–N].

In the final section of this paper we discuss the case where $M_0$ has genus 1 (i.e. is a flat torus.) We show that $M_1$ must be isometric to $M_0$ but $F$ need not be of the form $\zeta_0^t \circ dI$.

It should be pointed out that for general surfaces there is no theorem like Theorem B. In particular Zoll surfaces have geodesic flows that are conjugate to the geodesic flow on the round sphere (see [W]).

The author would like to thank P. Eberlein and K. Burns for helpful conversations. In particular much of the section about flat tori grew out of a discussion with K. Burns.

## I. Preliminaries

We begin with an analytic lemma that will be used in the proof of both Theorem A and Theorem B.

LEMMA 1.1. *Let $j$ and $\bar{j}$ be positive real valued continuous functions defined on intervals of $\mathbb{R}^1$. For constants $C_1$ and $C_2$ with $C_2 > 0$ define $f: [a, b] \to [\bar{a}, \bar{b}]$ by:*

$$C_2 \cdot \int_a^{f(t)} \frac{ds}{\bar{j}^2(s)} + C_1 = \int_a^t \frac{ds}{j^2(s)} \tag{i}$$

*where $j$ is assumed to be defined at least on $[a, b]$ and $\bar{j}$ on $[a, \bar{b}] \cup [\bar{a}, \bar{b}]$. Then we have:*

$$\int_a^b \frac{C_2 \cdot j(t)}{\bar{j}(f(t))}\, dt \geq \left[\frac{(b-a)^3 \cdot C_2}{(\bar{b}-\bar{a})}\right]^{1/2}$$

*with equality if and only if*

$$f(t) = \frac{\bar{b}-\bar{a}}{b-a}(t-a) + \bar{a} \quad \textit{and} \quad \frac{j(t)}{\bar{j}(f(t))} = \left[\frac{(b-a)}{C_2(\bar{b}-\bar{a})}\right]^{1/2}.$$

*Proof.* Differentiating (i) with respect to $t$ we see that

$$f'(t) = \frac{\bar{j}^2(f(t))}{C_2 \cdot j^2(t)}.$$

Hence using the substitution $u = f(t)$ gives:

$$\int_a^b \frac{C_2 \cdot j(t)}{\bar{j}(f(t))}\,dt = \int_{\bar a}^{\bar b} \frac{C_2^2 \cdot j^3(f^{-1}(u))}{\bar{j}^3(u)}\,du$$

(Note that $C_2 > 0$ implies $f'(t) > 0$ and hence that $f^{-1}(u)$ is well defined.) A Hölder inequality applied to the right hand side, RHS, of the above yields:

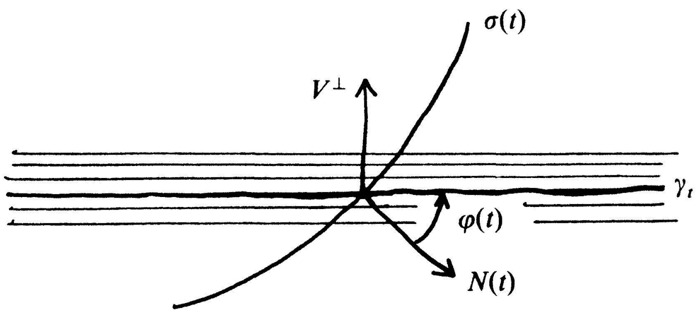
$$[\mathrm{RHS}]^{2/3} \cdot [\bar b - \bar a]^{1/3} \geq \int_{\bar a}^{\bar b} \frac{C_2^{4/3} \cdot j^2(f^{-1}(u))}{\bar{j}^2(u)}\,du = C_2^{1/3} \cdot (b - a). \tag{ii}$$

The equality above comes from the substitution $t = f^{-1}(u)$. The inequality in (ii) will be equality if and only if $j(f^{-1}(u))/(\bar{j}(u))$ is a constant, say $F$. Rearranging (ii) yields the inequality in the lemma. If equality holds then we see that $C_2 \cdot F \cdot (b - a) = [C_2(b - a)^3/(\bar b - \bar a)]^{1/2}$ and hence $F = [(b - a)/\{C_2(\bar b - \bar a)\}]^{1/2}$. Further our computation of $f'(t)$ yields in the equality case $f'(t) = 1/(C_2 \cdot F^2) = (\bar b - \bar a)/(b - a)$. These results plus the fact that $f(a) = \bar a$ yield the equality case in the lemma. ■

In both applications of the lemma $C_2$ will be 1.

The next lemma will help in interpreting the boundary term in the Gauss-Bonnet theorem in two dimensions.

Let $H : (-\varepsilon, \varepsilon) \times (a, b) \to M^2$ be a $C^2$-differentiable variation of unit speed geodesics in $M$. That is, for each fixed $t$, $\gamma_t(s) = H(t, s)$ is a unit speed geodesic in $M$. Let $h : (-\varepsilon, \varepsilon) \to (a, b)$ be a function such that $\sigma(t) = H(t, h(t))$ is an embedded $C^2$-differentiable curve transverse to $\gamma_t$ for all $t$ near 0. Let $N(t)$ be the continuous unit normal to $\sigma$ at $t$ near 0 with $\langle \gamma_t'(h(t)), N(t) \rangle > 0$. We now let $Kg(t)$ be the geodesic curvature of $\sigma(t)$ with respect to $N(t)$ and $\varphi(t) \in (-\pi/2, \pi/2)$ be the angle between $N(t)$ and $\gamma_t'(h(t))$. Let $J_t(s)$ be the variation field (Jacobi field) along $\gamma_t$ of this variation. Let $V^\perp = -\sin(\varphi(t)) \cdot N(t) + \cos(\varphi(t)) \cdot \sigma'(t)/|\sigma'(t)|$ be a unit vector perpendicular to $\gamma_t'(h(t))$ (see figure.)



LEMMA 1.2. *In the situation described above we have*:

$$\nabla_{\gamma_t'(h(t))} J_t = \left(\frac{d\varphi}{dt} - Kg(t) \cdot |\sigma'(t)|\right) \cdot V^\perp$$

*Proof.* We first claim that we may assume that $h(t) = 0$ for all $t$ near 0. Consider the new variation $H^N(t, s) = H(t, s + h(t))$, then $\sigma(t) = H^N(t, 0)$ and the new variation field $J_t^N$ satisfies $J_t^N(s) = J_t(s - h(t)) + h'(t)\gamma_t'(s - h(t))$ and hence its covariant derivative in $\gamma_t'$ direction agrees with that of $J_t$. Hence we may assume that $h(t) = 0$.

We let $T(t) = \sigma'(t)/|\sigma'(t)|$ then $\gamma_t'(0) = \cos(\varphi(t)) \cdot N(t) + \sin(\varphi(t)) \cdot T(t)$. Thus

$$\begin{aligned}\nabla_{\gamma_t'(0)} J_t(s)|_{s=0} = \nabla_{J_t(0)}\gamma_t'(0) = \nabla_{\sigma'(t)}\gamma_t'(0) &= -\sin(\varphi(t)) \cdot \frac{d\varphi}{dt} \cdot N(t) \\ &+ |\sigma'(t)| \cdot \cos(\varphi(t)) \cdot \nabla_{T(t)} N(t) + \cos(\varphi(t)) \cdot \frac{d\varphi}{dt} \cdot T(t) \\ &+ |\sigma'(t)| \cdot \sin(\varphi(t)) \cdot \nabla_{T(t)} T(t).\end{aligned}$$

Now from $\nabla_T T = Kg \cdot N$ and $\nabla_T N = -Kg \cdot T$ we find that $\nabla_{\gamma_t'(0)} J_t(s)|_{s=0}$ is

$$\begin{aligned}&\sin(\varphi(t)) \cdot \left\{\frac{-d\varphi}{dt} + Kg(t) \cdot |\sigma'(t)|\right\} \cdot N(t) + \cos(\varphi(t)) \cdot \left\{\frac{d\varphi}{dt} - Kg(t)|\sigma'(t)|\right\} \cdot T(t) \\ &= \left\{\frac{d\varphi}{dt} - Kg(t)|\sigma'(t)|\right\} \cdot V^\perp\end{aligned}$$

and the lemma follows. ■

For $\theta \in S^1$ let $a(\theta) < b(\theta)$ be bounded functions such that $a$ is continuous and $b$ is $C^1$-differentiable for all but a finite set $\{\theta_1, \theta_2, \ldots, \theta_k\}$ where the derivatives have left and right limits ($b$ need not be continuous at $\theta_i$.) Let

$$Q = \text{closure}\{(\theta, s)|a(\theta) \le s \le b(\theta)\} \subset S^1 \times \mathbb{R}^1.$$

$Q$ has two boundary components

$\partial_0 = \{(\theta, a(\theta))\}$ and $\partial_1 = \{(\theta, b(\theta))\} \cup_{i=1}^k \{(\theta_i, s)|s$ is in the interval between the two half limits of $b$ at $\theta_i\}$.

Let $H : Q \to M^2$ be a map into a two dimensional riemannian manifold with the following properties:

(i) Each curve $\gamma_\theta(s) = H(\theta, s)$ is a unit speed geodesic in $M$.

(ii) On the interior of $Q$, $H$ is a $C^1$ immersion.

(iii) The image $H(Q)$ is a manifold whose boundary is the 1-1 image of $\partial_1$.

(iv) The image of $\partial_0$ lies in the interior of $H(Q)$.

We will let $J(\theta, s)$ be the variation field $H_*(d/d\theta)$. Hence for fixed $\theta$, $J(\theta, s)$ is a Jacobi field along $\gamma_\theta$. We also choose a unit normal field $\gamma_\theta^\perp$ along each geodesic $\gamma_\theta$ which we assume has $\langle J(\theta, s), \gamma_\theta^\perp \rangle > 0$ for $a(\theta) < \theta < b(\theta)$ (we can do this since $H$ is an immersion on the interior of $Q$).

LEMMA 1.3. *If in the above $M$ has nonpositive curvature then we have*:

$$2\pi \geq \int_{S^1} \langle \nabla_{\dot\gamma_\theta(a(\theta))} J(\theta, s), \gamma_\theta^\perp \rangle \, d\theta$$

*If $M$ has negative curvature then equality will hold if and only if $H$ is one to one on the interior of $Q$ and $H(Q)$ is a disk.*

*Proof.* For $m$ in $M$ let $k(m)$ represent the curvature of $M$ at $m$. Since $H$ may be more than 1 to 1 and since $k(m) < 0$ we have

$$\int_{H(Q)} k(m) \, dm \geq \int_0^{2\pi} \int_{a(\theta)}^{b(\theta)} k(H(\theta, s)) \langle J(\theta, s), \gamma_\theta^\perp \rangle \, ds \, d\theta$$

with equality holding when $M$ has negative curvature if and only if $H$ is one to one on the interior of $Q$. Using the jacobi equation along $\gamma_\theta^\perp$ we find that the integrand of the right hand side is

$$-\langle \nabla_{\dot\gamma_\theta(s)} \nabla_{\dot\gamma_\theta(s)} J(\theta, s), \gamma_\theta^\perp \rangle = -\frac{d}{ds} \langle \nabla_{\dot\gamma_\theta(s)} J(\theta, s), \gamma_\theta^\perp \rangle .$$

Hence the right hand side becomes

$$\int_0^{2\pi} \langle \nabla_{\dot\gamma_\theta(a(\theta))} J(\theta, s), \gamma_\theta^\perp \rangle \, d\theta - \int_0^{2\pi} \langle \nabla_{\dot\gamma_\theta(b(\theta))} J(\theta, s), \gamma_\theta^\perp \rangle \, d\theta.$$

Since the boundary component of $H(Q)$ is a single circle the euler characteristic is $\leq 1$ (in our applicaitons $H(Q)$ will in fact always be a disk) and hence the left hand side is less than or equal to $2\pi$—boundary term, $B\partial$, of Gauss–Bonnet. Hence the lemma follows when we see that

$$B\partial = \int_0^{2\pi} \langle \nabla_{\dot\gamma_\theta(b(\theta))} J(\theta, s), \gamma_\theta^\perp \rangle \, d\theta.$$

In the case that the boundary $H(\partial_1)$ and the variation is piecewise $C^2$ integrating the result of Lemma 1.2 will show that this is true. In general we can approximate by such and see the above result in each of the approximating cases and hence in the limit. ∎

## II. The proof of Theorem A

We now consider the case of Theorem A, that is $(M_0, g_0)$ is an SGM riemannian manifold of nonpositive curvature with boundary $N$ and $(M_1, g_1)$ is another riemannian manifold with the same boundary $N$ and the same boundary distance function (and hence is also SGM.) Since all geodesics, $\gamma$, hit the boundary we will always parameterize them by arclength with parameter $\geq 0$ and $\gamma(0) \in N$.

$M_0$ and $M_1$ are equivalent as lenses (see [C]). This means that for every geodesic segment $\gamma$ in $M_1$ the geodesic segment $\bar{\gamma}$ in $M_0$ which begins at the same boundary point with the same angle as $\gamma$ intersects $N$ again at the same point with the same angle and at the same parameter value as $\gamma$. This allows us to define a natural map $F$ from $UM_1$ (the unit tangent bundle of $M_1$) to $UM_0$ as follows. Given $u$ in $UM_1$ let $v$ be the unique unit vector of $M_1$ at the boundary $N$ such that $u = \gamma_v'(t)$ where $\gamma_v$ is the geodesic of $M_1$ with $\gamma_v'(0) = v$ and $0 \leq t \leq 1(\gamma)$ where $1(\gamma)$ is the first parameter value greater than zero with $\gamma(1(\gamma)) \in N$. (In particular if $\gamma$ "grazes" $N$ then it is considered to stop there.) We then let $F(u) = \bar{\gamma}_v(t)$ where $\bar{\gamma}_v$ is the geodesic in $M_0$ with the corresponding initial condition as above. It is not hard to see that the map $F$ above is continuous, it commutes with the geodesic flow, and $F(-u) = -F(u)$. In particular $F$ is measure preserving and hence $\mathrm{Vol}\,(UM_1) = \mathrm{Vol}\,(UM_0)$ so $M_1$ and $M_2$ have the same volume. All of the above holds in all dimensions. For further details see [C].

Although it is not clear that the map $F$ is smooth for all $u$ in $UM_1$ it is clear for those $u$'s such that $\gamma_v$ is not tangent to the boundary at 0. Further since $F(-u) = -F(u)$, $F$ will be smooth at all $u$ except those where $\gamma_v$ is tangent to $N$ at 0 and $1(\gamma_v)$. We will call a point $x$ in $M_1$ (or $M_0$) "generic" if $x$ does not lie on any geodesic that grazes $N$ at both endpoints. Non-generic $x$ form a set of measure zero and $F$ is smooth at all $u$ that are tangent to a geodesic that passes through a generic $x$.

We saw above that to each geodesic $\gamma$ of $M_1$ there corresponds a geodesic $F(\gamma)$ in $M_0$ namely the one having the same end points (and hence the same initial conditions). Hence for every jacobi field $J$ along $\gamma$ we will associate the Jacobi field $\Phi(J)$ coming from the corresponding variation of geodesics. This means that $\Phi(J)$ is the jacobi field along $F(\gamma)$ having $J(0)$ correspond to $\Phi(J)(0)$ and $J(1(\gamma))$ correspond to $\Phi(J)(1(\gamma))$. $\Phi$ is thus linear. Note that the SGM condition guarantees that $\gamma$ and $F(\gamma)$ have no conjugate points so the above correspondence can always be made. We used the fact that for $q \in N$ there is a natural isometry between $T_qM_1$ and $T_qM_0$ which is the identity on $T_qN$ and takes inward normal to inward normal. In the above it may not be the case that $J'(0)$ corresponds to $\Phi(J)'(0)$ since the second fundamental forms of $N$ in the two manifolds need not be the same. The relationship between $J'(0)$ and $\Phi(J)'(0)$ in all dimensions is studied in the appendix to [C]. It was shown in [M] that if $M_0$ (and hence $M_1$) is assumed to be convex then the boundaries agree up to second order hence in particular $J'(0)$ and $\Phi(j)'(0)$ do correspond.

We now restrict our attention to jacobi fields along a fixed geodesic $\gamma$ and its corresponding geodesic $F(\gamma)$. We choose parallel unit vector fields $X_\gamma$ and $X_{F(\gamma)}$ along $\gamma$ and $F(\gamma)$ which are perpendicular to the geodesics and correspond at 0. We will show that they also agree at $1(\gamma)$. From now on we will only consider jacobi fields perpendicular to the geodesics which can and will be thought of as functions (since they are functional multiples of $X_\gamma$ and $X_{F(\gamma)}$.) We can tell if a jacobi field vanishes on the interior of a geodesic simply by looking at its values at the endpoints. If it has the same sign at both endpoints then it does not vanish since it cannot vanish twice (no conjugate points) and it cannot vanish along with its derivative. On the other hand if it changes sign it clearly vanishes. We will let $J_\gamma(t)$ be the jacobi field such that $J_\gamma(0) = J_\gamma(1(\gamma)) = 1$. Since $J_\gamma$ never vanishes we can define a jacobi field

$$J_\gamma^0(t) = J_\gamma(t) \cdot \int_0^t \frac{ds}{J_\gamma(s)^2}.$$

Similarly we can define $\Phi(J_\gamma)^0$. $J_\gamma^0$ is the jacobi field with $J_\gamma^0(0) = 0$ and $J_\gamma^{0\prime}(0) = 1$. Thus $J_\gamma^0$ comes from a standard variation of geodesics all starting at $\gamma(0)$. It is clear that the corresponding variation in $M_0$ gives rise to a jacobi field with initial value 0 and initial derivative 1, i.e. $\Phi(J_\gamma)^0 = \Phi(J_\gamma^0)$. Since $\Phi(J_\gamma^0)$ cannot vanish for $t > 0$ we see that $\Phi(J_\gamma^0)(1(\gamma)) > 0$, and hence that $X_\gamma$ corresponds to $X_{F(\gamma)}$ at $1(\gamma)$. Further since $J_\gamma^0$ and $\Phi(J_\gamma^0)$ correspond at $1(\gamma)$

$$\int_0^{1(\gamma)} \frac{ds}{J_\gamma(s)^2} = \int_0^{1(\gamma)} \frac{ds}{\Phi(J_\gamma)(s)^2}.$$

(The $n$ dimensional version of this appears in [C].)

Now fix $a \in [0, 1(\gamma)]$. The jacobi field that vanishes at $a$ and has derivative 1 at $a$ is

$$J_\gamma^a(t) = J_\gamma(a) \cdot J_\gamma(t) \cdot \int_a^t \frac{ds}{J_\gamma(s)^2} = J_\gamma(a) \cdot J_\gamma^0(t) + J_\gamma(a) \cdot \int_a^0 \frac{ds}{J_\gamma(s)^2} \cdot J_\gamma(t).$$

Thus by the linearity of $\Phi$, $\Phi(J_\gamma^a)(t)$ must be

$$J_\gamma(a) \cdot \Phi(J_\gamma^0)(t) + J_\gamma(a) \cdot \int_a^0 \frac{ds}{J_\gamma(s)^2} \cdot \Phi(J_\gamma)(t).$$

It is important to note that $\Phi(J_\gamma^a)(t)$ is not necessarily the jacobi field along $F(\gamma)$ that vanishes at $a$ (i.e. $\Phi(J_\gamma^a) \neq \Phi(J_\gamma)^a$). We will let $f_\gamma(a) \in [0, 1(t)]$ be the place where it does vanish. We will later think of $f$ as a function on the unit sphere bundle and write $f(\gamma'(a))$ for $f_\gamma(a)$. Similarly for $u = \gamma'(a)$ we will write $J^u$ and $\Phi(J^u)$ for $J_\gamma^a$ and $\Phi(J_\gamma^a)$. The above formulas give:

LEMMA 2.1

$$\int_0^{f_\gamma(a)} \frac{ds}{\Phi(J_\gamma)(s)^2} = \int_0^a \frac{ds}{J_\gamma(s)^2} \tag{1}$$

$$\Phi(J_\gamma^a)'(f_\gamma(a)) = \frac{J_\gamma(a)}{\Phi(J_\gamma)(f_\gamma(a))}. \tag{2}$$

*Proof.* Putting the formula for $\Phi(J_\gamma^0)$ into the formula for $\Phi(J_\gamma^a)$ gives:

$$\Phi(J_\gamma^a)(t) = J_\gamma(a) \cdot \Phi(J_\gamma)(t) \cdot \left\{ \int_0^t \frac{ds}{\Phi(J_\gamma)(s)^2} - \int_0^a \frac{ds}{J_\gamma(s)^2} \right\}.$$

(1) follows from the fact that $J_\gamma$ is positive and $\Phi(J_\gamma^a)(f_\gamma(a)) = 0$.
(2) comes from differentiating the above and using 1). ■

If $x$ is a generic point of $M_1$ it will be called regular if only a finite number of the geodesics through $x$ graze the boundary. For a regular $x$ let $D_x$ = closure of $\{p \in M_1 | \text{There is a geodesic segment from } p \text{ to } x\}$. The boundary $\partial D_x$ is a circle consisting of two parts: $B$, which is the union of intervals $B_i$ of $N$, and a union of geodesic segments $\tau_i$. We will let $F(\partial D_x)$ be the corresponding circle in $M_0$, i.e. it consists of $B$ and the $F(\tau_i)$'s. We will let $F(D_x)$ be the closure of $\{q \in M_0 | q \text{ lies on a } F(\gamma) \text{ where } \gamma \text{ is a geodesic segment passing through } x\}$.

LEMMA 2.2. *If $x$ is a regular point of $M_1$, then $F(D_x)$ is domain of $M_0$ with boundary $F(\partial D_x)$.*

*Proof.* We first show that $F(\partial D_x)$ is an imbedded circle, i.e. that if $i \neq j$ then $F(\tau_i) \cap F(\tau_j)$ is empty. Let $p_i$ and $q_i$ be the points on $N$ such that $\tau_i$ is the geodesic segment from $p_i$ to $q_i$ (note that $p_i$ is closer to $x$ than $q_i$ and that $\tau_i$ is tangent to $N$ at $p_i$). If $F(\tau_i)$ intersects $F(\tau_j)$ then triangle inequalities show $d_0(p_i, q_j) + d_0(q_i, p_j) < d_0(p_i, q_i) + d_0(p_j, q_j)$. On the other hand, $d_1(q_i, p_j) \geq d_1(q_i, x) - d_1(p_j, x)$ and $d_1(q_j, p_i) \geq d_1(q_j, x) - d_1(p_i, x)$. Adding these two inequalities and using $d_1(p_i, q_i) = d_1(q_i, x) - d_1(p_i, x)$ yields $d_1(q_i, p_j) + d_1(q_j, p_i) \geq d_1(p_i, q_i) + d_1(p_j, q_j)$. However the $p$'s and $q$'s lie on $N$ hence $d_0 = d_1$ and we get a contradiction. Thus $F(\partial D_x)$ is imbedded.

We can also use the above to see that $F(\gamma_u) \cap F(\tau_i) \subset B$. Assume that for some $v$ and $i$ we had $F(\gamma_v)$ intersect $F(\tau_i)$ in the interior. Let $I \subset U_x$ be $\{u | F(\gamma_u)$ intersects $F(\tau_i)$ in the interior$\}$. $F(\tau_i)$ is the extension of some $F(\gamma_w)$. Variations near $w$ make it clear that for $u$ in a neighborhood of $w$, $u$ is not in $I$. Thus there is a $u \neq w$ on the boundary of $I$. But it is clear that this can only happen if $F(\gamma_u)$ does not intersect $F(\tau_i)$ but an extension $F(\tau_j)$ of it does. This contradicts the previous paragraph and gives $F(\gamma_u)$ never intersects $F(\tau_i)$ in the interior.

A similar argument says $F(\gamma_u)$ does intersect $F(\gamma_v)$ for all $u$ and $v$. Let $I = \{u | F(\gamma_u)$ intersects $F(\gamma_v)\}$. Variations near $v$ show that $I$ includes a neighborhood of $v$. If $I \neq U_x$ then there is a boundary point $w$. But as before $F(\gamma_w)$ must extend to $F(\tau_i)$ which intersects $F(\gamma_v)$ which contradicts the previous paragraph.

We can use this to show that $F(\partial D_x)$ is contractible in $M_0$ and hence separates $M_0$. Fix $v \in U_x$. Since each $F(\gamma_u)$ intersects $F(\gamma_v)$ we can homotop in the obvious way $F(\partial D_x)$ along the $F(\gamma_u)$ so that the image lies in a small neighborhood of $F(\gamma_v)$ and hence can be contracted. Since the homotopy was made through points of $F(D_x)$ we see that $F(D_x)$ contains one of the components of $M_0 - F(\partial D_x)$. On the other hand since the $F(\gamma_u)$ do not intersect $F(\partial D_x)$ except at endpoints we see that $F(D_x)$ is the closure of this component and the lemma follows. ∎

*Remark.* The above proof shows that $F(D_x)$ is in fact a disk.

LEMMA 2.3. *If $x$ is a generic point in the interior of $M_1$ then*

$$2\pi \geq \int_{U_x} \Phi(J^u)'(f(u))\, du$$

*where $U_x$ is the circle of unit tangent vectors at $x$ with the usual measure $du$.*

*Proof.* This is an applicaiton of Lemma 1.3. For $u \in U_x$ there is a number $t(u)$ and a geodesic $\gamma_u$ such that $\gamma_u'(t(u)) = u$. The functions $a$ and $b$ in the definition of $Q$ are given by $a(u) = f(u) - t(u)$, $b(u) = 1(\gamma_u) - t(u)$. The reason for introducing $t(u)$ is that for our choice of parameter for geodesics (i.e. $\gamma(0) \in N$) $f$ is not a smooth (or even continuous) function of $u$, however $f(u) - t(u)$ will be smooth when $x$ is generic. The fact that $x$ is generic also guarantees that the map $H : Q \to M_0$ defined by $H(u, s) = F(\gamma_u)(s + t(u))$ is smooth. It is an immersion on the interior since the variation field vanishes only for $s = a(u)$. We may assume that there are finitely many places where $b$ is not smooth since if not we need only look at nearby regular $x$ where there are finitely many, prove that lemma for this $x$ and then take limits. Let $D_x$, $\partial D_x$, $F(D_x)$, and $F(\partial D_x)$ be as in Lemma 2.2. It is clear that $F(\partial D_x)$ is the image under $H$ of $\partial_1$. It may appear that only part of $F(\gamma_u)$ is in the image $H(Q)$ but the other part shows up as part of $F(\gamma_{-u})$ since $F(\gamma_u)(f(u)) = F(\gamma_{-u})(f(-u))$.

Hence $H(Q)$ is $F(D_x)$, the only thing left to show in order to apply Lemma 1.3 is that $H(\partial_0) \cap \partial F(D_x)$ is empty. But every point of $H(\partial_0)$ is an interior point of a $F(\gamma_u)$ and hence cannot intersect the boundary. ■

*Proof of Theorem A.* Let $(M_1, g_1)$ be a surface with boundary with the same boundary distance function as $(M_0, g_0)$ (i.e. $d_1 = d_0$.) We have seen in Lemma 2.3 that for all but a set of measure 0 points $x$ in $M_1$

$$\int_{U_x} \Phi(J^u)'(f(u))\, du \leq 2\pi.$$

Integrating this over all $x$ in $M_1$ leads to

$$\int_{UM_1} \Phi(J^u)'(f(u))\, du \leq 2\pi \cdot \mathrm{Vol}\,(M_1)$$

where here $du$ represents the standard measure on $UM_1$. Let $\Gamma$ represent the space of geodesic segments on $M_1$ with standard measure $d\gamma$. Then using Santaló's formula (see [Sa] pp. 336–338 or [C] sec. III) the above says

$$\int_\Gamma \int_0^{1(\gamma)} \Phi(J_{\gamma'(t)})'(f(\gamma'(t)))\, dt\, d\gamma \leq \mathrm{Vol}\,(UM_1).$$

Lemma 2.1 tells us that

$$\int_\Gamma \int_0^{1(\gamma)} \frac{J_\gamma(t)}{\Phi(J_\gamma)(f(\gamma'(t)))}\, dt\, d\gamma \leq \mathrm{Vol}(UM_1).$$

Now for each fixed $\gamma$ use Lemma 1.1 with $j = J_\gamma$, $\bar{j} = \Phi(J_\gamma)$, $f(t) = f(\gamma'(t))$, $C_1 = 0$, $C_2 = 1$, $a = \bar{a} = 0$, and $b = \bar{b} = 1(\gamma)$ (note that Lemma 2.1 says that $i$ holds.) This yields $\int_\Gamma 1(\gamma)\, d\gamma \leq \mathrm{Vol}\,(UM_1)$ and hence equality must hold in all the inequalities. In particular by Lemma 1.1 we have $f(\gamma'(t)) = t$ and $J_\gamma(t) = \Phi(J_\gamma)(t)$ for all $\gamma$ and $t$ and hence the spaces are isometric. ■

## III. The proof of Theorem B

In this section we will assume that $M_0$ is a compact surface of genus $\geq 2$ with a Riemannian metric of nonpositive curvature. $F: UM_1 \to UM_0$ will be a $C^1$ diffeomorphism which induces a conjugacy of geodesic flows where $M_1$ is a Riemannian surface. All geodesics will be parameterized by arclength unless otherwise stated. If $\gamma$ is an oriented geodesic in $M_1$ then $F$ will take its tangent vector field, $T\gamma$, to the tangent vector field of a geodesic in $M_0$ which we will denote by $F(\gamma)$. For $i = 0, 1$ we will let $Z_i$ be the vector field on $UM_i$ generating the geodesic flows $\zeta_i^t$. So $F_*(Z_1) = Z_0$.

It was pointed out in [B–K] that for orientable surfaces of genus $\geq 2$ if $K_i$ is the subgroup of $\pi_1(UM_i)$ generated by the fiber (i.e. the kernel of the projection map) then $K_i$ is the center of $\pi_1(UM_i)$. In the nonorientable case $K_i = \{a \in \pi_1(UM_i) \mid bab^{-1} = a \text{ or } a^{-1} \text{ for all } b \in \pi_1(UM_i)\}$. In either case $F_*$ must take $K_1$ to $K_0$. In particular we see:

LEMMA 3.1. *$F$ lifts to a map from $U\tilde{M}_1$ to $U\tilde{M}_0$ where $\tilde{M}_i$ is the universal covering space of $M_i$.*

By abuse of notation we will also refer to this lifted map as $F$.

*Proof.* The only requirement for the existence of such a lift is that $(\pi_0 \circ F)_*(K_1)$ is trivial but this follows from the above remarks. ■

*Remark.* In Section IV we will see that this is false for flat tori.

The fact that $F_*K_1 = K_0$ also implies that the map $F$ induces an isomorphism from $\pi_1(M_1)$ to $\pi_1(M_0)$ and hence the map $F$ on closed geodesics induces an isomorphism of free homotopy classes. In particular since two freely homotopic closed geodesics in $M_0$ have the same length the same is true for $M_1$.

LEMMA 3.2. *$M_1$ has no conjugate points.*

*Proof.* By the above, every closed geodesic $\gamma$ is the shortest curve in its free homotopy class. Since this applies as well to all iterates of $\gamma$ we see that the lift $\tilde{\gamma}$ of $\gamma$ to $\tilde{M}_1$ is minimizing and hence has no conjugate pairs. But by [B] the set of closed geodesics is dense in $UM_0$ and hence via $F^{-1}$ in $UM_1$. Thus there are no conjugate points in $M_1$. ■

The space of jacobi fields $\Psi$ along a geodesic $\gamma$ splits naturally as $\Psi = \Psi^\perp + \Psi^t + \Psi^b$ where $\Psi^\perp$ consists of those jacobi fields that are perpendicular to $\gamma$, $\Psi^t$ is spanned by $\gamma'$, and $\Psi^b$ is spanned by $t\gamma'$. Although all jacobi fields arise from variations of geodesics only those in $\Psi^\perp + \Psi^t$ come from variations of geodesics $\gamma_s$ which are all parameterized by arclength.

Let $J$ be a jacobi field along a geodesic $\gamma$ in $\Psi^\perp + \Psi^t$. We define a vector field $TJ$ along $T\gamma$ in the unit tangent bundle as the variation field of the variation $T\gamma_s$ where $\gamma_s$ is a variation of geodesics whose variation field is $J$. $TJ$ is determined by the fact that $\pi_*(TJ) = J$ and that the vertical (with respect to the usual connection) component $v(TJ)$ of $TJ$ is equal to $J'$, the covariant derivative of $J$ with respect to $\gamma'$ where $v(TJ)$ and $J'$ are thought of as tangent vectors perpendicular to $\gamma'$. (Note that $J'$ is perpendicular to $\gamma'$ since $J \in \Psi^{\perp} + \Psi^{t}$.) In particular $|TJ(t)|^2 = |J(t)|^2 + |J'(t)|^2$. $\Psi^{\perp}$ is thus the subspace of $J$ where $TJ$ is perpendicular to $Z$.

The subspace of $\Psi$ consisting of jacobi fields $J$ such that $|TJ(t)|$ goes to 0 as $t$ goes to $\infty$ (resp. $-\infty$) will be denoted $\Psi^s$ (resp. $\Psi^u$.) It is easy to see that $\Psi^s$ (resp. $\Psi^u$) $\subset \Psi^{\perp}$ since if it had any component in $\Psi^t + \Psi^b$ it could not vanish at $\infty$. We will let $\Psi^{ws}$ (resp. $\Psi^{wu}$) be those $J \in \Psi^{\perp}$ such that $|TJ(t)|$ stays bounded as $t$ goes to $\infty$ (resp. $-\infty$.) By definition $\Psi^s \subset \Psi^{ws} \subset \Psi^{\perp}$.

Since $F$ takes geodesics to geodesics (actually tangent fields to tangent fields) then $F$ induces a map, $\Phi$, from the jacobi fields along a geodesic $\gamma$ in $\Psi_1^{\perp} + \Psi_1^t$ to the jacobi fields along $F(\gamma)$ in $\Psi_0^{\perp} + \Psi_0^t$ by taking variations to variations. We thus see that $F_*(TJ) = T\Phi(J)$ and hence $\pi_{0*}(F_*(TJ)) = \Phi(J)$. In particular $\Phi$ is a linear isomorphism.

LEMMA. 3.3. *Along every geodesic $\gamma$ of $M_1$ $\Phi$ takes the sets $\Psi_1^{\perp}$, $\Psi_1^s$, $\Psi_1^{ws}$, $\Psi_1^u$ and $\Psi_1^{wu}$ to the corresponding sets $\Psi_0^{\perp}$, $\Psi_0^s$, $\Psi_0^{ws}$, $\Psi_0^u$, and $\Psi_0^{wu}$ along $F(\gamma)$.*

*Proof.* Since $F$ is a $C^1$ map between compact manifolds there is a number $a > 1$ such that $1/a|V| < |F_*(V)| < a|V|$ for all $V \in TUM_1$ where all norms are with respect to the usual metric. In particular for a jacobi field $J$, $|TJ(t)|$ goes to zero at $\infty$ if and only if $|T\Phi(J)(t)|$ goes to zero at $\infty$. Thus we see $\Phi(\Psi_1^s) = \Psi_0^s$ and similarly $\Phi(\Psi_1^u) = \Psi_0^u$. Along a dense set of geodesics in $M_0$ (for example those closed geodesics that pass through a region of negative curvature — see [B]) $\Psi_0^s$ and $\Psi_0^u$ span $\Psi_0^{\perp}$ and hence $\Phi^{-1}\Psi_0^{\perp} \subset \Psi_1^{\perp}$. For dimension reasons $\Phi\Psi_1^{\perp} = \Psi_0^{\perp}$. By continuity this holds for all geodesics. The fact that $\Phi\Psi_1^{ws} = \Psi_0^{ws}$ (resp. $\Psi^{wu}$) follows from the same argument as for $\Psi^s$ along with the fact that $\Phi\Psi_1^{\perp} = \Psi_0^{\perp}$. ■

In particular the lemma says that $dF$ takes $Z_1^{\perp}$ to $Z_0^{\perp}$ at each point of $UM_1$ and hence preserves the cannonical contact form $\theta$ and thus the canonical volume form $\theta \wedge d\theta$ (and thus as well the orientation.) This yields:

LEMMA 3.4. *$F$ is orientation and volume preserving.*

Along a geodesic $\gamma$ where no pair of points on $\gamma$ are conjugate along $\gamma$ it is natural to look at $\Psi^{\perp} = \Psi^n \cup \Psi^z$ where $\Psi^n$ consists of those jacobi fields that never vanish and $\Psi^z$ those that do.

By [Gre] or [E] along a geodesic without conjugate points a jacobi field that vanishes must be unbounded at $\infty$ and $-\infty$ and hence $\Psi^{ws}$ and $\Psi^{wu}$ are contained in $\Psi^n$. Along a geodesic where $K \leq 0$ it is easy to find nontrivial elements of $\Psi^{ws}$

and $\Psi^{wu}$ (these elements may coincide if the curvature is identically 0.) Via $\Phi^{-1}$ we thus see there are nontrivial elements of $\Psi_1^{ws}$ and $\Psi_1^{wu}$.

For a geodesic on a surface we can choose a parallel unit field $X$ normal to $\gamma'$ along $\gamma$. Every jacobi field $J(t)$ in $\Psi^{\perp}$ can (and will) be written as $J(t) = j(t) \cdot X(t)$ where $j(t)$ is a function. We will sometimes confuse the jacobi field with the function $j$.

For a fixed geodesic $\gamma$ of $M_1$ we will from now on denote by $J_1^s$ the element of $\Psi^{ws}$ with $J_1^s(0) = 1$. Similarly define $J_1^u$ (which may coincide with $J_1^s$). By the above $J_1^s$ never vanishes and so we can define a new jacobi field $J_1^z$ by

$$j_1^z(t) = j_1^s(t) \cdot \int_0^t \frac{ds}{j_1^s(s)^2}. \tag{1}$$

Any jacobi field $J$ in $\Psi^{\perp}$ is a linear combination of $J_1^s$ and $J_1^z$. For $v = \gamma'(x)$ we let $J_1^v$ be the jacobi field along $\gamma$ such that $j_1^v(x) = 0$ and $j_1^{v\prime}(x) = 1$. We see that

$$J_1^v(t) = j_1^s(x) \cdot J_1^z(t) + j_1^s(x) \cdot \int_x^0 \frac{ds}{j_1^s(s)^2} \cdot J_1^s(t). \tag{2}$$

Along the geodesic $\Phi(\gamma)$ we will let $J_0^s = \Phi(J_1^s)$. By Lemma 3.3 we know that $J_0^s \in \Psi_0^{ws} \subset \Psi^n$. We define $J_0^z$ from $J_0^s$ in the same way that $J_1^z$ was defined from $J_1^s$. We know there are constants $c_1$ and $c_2$ such that

$$\Phi(J_1^z) = c_1 \cdot J_0^s + c_2 \cdot J_0^z. \tag{3}$$

LEMMA 3.5. *In the above* $c_2 = 1$.

*Proof.* The fact that $F$ is measure preserving, and takes $Z_1$ to $Z_0$ and $Z_1^{\perp}$ to $Z_0^{\perp}$, implies that is it measure preserving on $Z^{\perp}$. This translates to the fact that $j_1^s(t) \cdot j_1^{z\prime}(t) - j_1^{s\prime}(t) \cdot j_1^z(t) = \Phi(j_1^s)(t) \cdot \Phi(j_1^z)'(t) - \Phi(j_1^s)'(t) \cdot \Phi(j_1^z)(t)$ for all $t$. Using the formula for $j_1^{z\prime}$ in terms of $j_1^s$ at $t = 0$ yields the left hand side to be 1. For the right hand side we get: $j_0^s(0) \cdot \{c_1 j_0^{s\prime}(0) + c_2 j_0^{z\prime}(0)\} - j_0^{s\prime}(0) \cdot \{c_1 j_0^s(0) + c_2 j_0^z(0)\} = c_2\{j_0^s(0) \cdot j_0^{z\prime}(0) - j_0^{s\prime}(0) \cdot j_0^z(0)\}$. Using the definition of $j_0^z$ in terms of $j_0^s$ we find that the left hand side is $c_2$ at $t = 0$ and the lemma follows. ■

If (as we shall show next) $\Phi(\Psi_1^z) \subset \Psi_0^z$ for all geodesics in $M_1$ then we define a function $g : UM_1 \to \mathbb{R}$ as follows: for $v \in UM_1$ let $\gamma_v$ be the geodesic determined by $v$. Then $\Phi(J_1^v)$ will vanish once along $\Phi(\gamma_v)$ say at $\Phi(\gamma_v)(t_0)$. We let $g(v) = t_0$. (In the above we thought of $v = \gamma'(0)$ if instead $v = \gamma'(t_1)$ then take $g(v) = t_0 - t_1$ to be consistent with different choices of parameter for $\gamma_v$.) We also define $G : UM_1 \to UM_0$ by $G(v) = \Phi(\gamma_v)'(g(v))$.

LEMMA 3.6. *We have* $\Phi(\Psi_1^n) = \Psi_0^n$ *and* $\Phi(\Psi_1^z) = \Psi_0^z$.
*Further the maps G and g are continuous and hence g is bounded* (*say* $|g(v)| \le g_0$).

*Proof.* We first show $\Phi(\Psi_1^z) \subset \Psi_0^z$ and hence that we can define $G$ and $g$. We need to show that $\Phi(J_1^v)$ vanishes for all $v = UM_1$. Let $\gamma$ be a geodesic in $M_1$. Using equations 1 (with 0 subscripts), 2, 3 and Lemma 3.5 we see that

$$\Phi(J_1^v)(t) = j_1^s(x) \cdot \left\{ c_1 j_0^s(t) + j_0^s(t) \int_0^t \frac{ds}{j_0^s(s)^2} \right\} + j_1^s(x) \int_x^0 \frac{ds}{j_1^s(s)^2} j_0^s(t).$$

Hence

$$\Phi(J_1^v)(t) = j_1^s(x) j_0^s(t) \cdot \left\{ c_1 + \int_0^t \frac{ds}{j_0^s(s)^2} - \int_0^x \frac{ds}{j_1^s(s)^2} \right\}. \tag{4}$$

Thus $\Phi(J_1^v)$ will vanish somewhere if and only if there is a $t_x$ such that

$$c_1 + \int_0^{t_x} \frac{ds}{j_0^s(s)^2} = \int_0^x \frac{ds}{j_1^s(s)^2} \tag{5}$$

If such a $t_{x_0}$ exists for some $x_0$ then it must exist for all $x > x_0$ since both sides of the equation are monotone increasing to $\infty$ (since $j_0^s$ and $j_1^s$ are bounded at $\infty$). Since we can also pick $x$ so that the right hand side is $> c_1$ we see that such $t_x$ exists for all large $x$. Now we could have gone through the whole process above (starting just before equation (1) starting with $j_1^u$ in place of $j_1^s$ to derive the equations corresponding to 4 and 5 only with $j_1^s$ and $j_0^s$ replaced with $j_i^u$ and $j_0^u$ (where $c_1$ may be different) since the only property of $j_1^s$ that we used was that it never vanished. In this case since $j_1^u$ and $j_0^u$ are bounded at $-\infty$ we see that $\Phi(J^v)$ must vanish somewhere for all small (near $-\infty$) $x$ and hence for all $x$ by our previous discussion. Thus we see that there is a $t_x$ for all $x$ so that equation 5 is satisfied and $\Phi(\Psi_1^z) \subset \Psi_0^z$.

As $v$ varies continuously the jacobi equations (thought of as an equation on the reals) $j''(t) + K_v(t) \cdot j(t) = 0$, where $K_v(t)$ represents the curvature of the surface $M_0$ at $\Phi(\gamma)(t)$, will vary continuously. Also $T\Phi(J^v)(0) = F_*(TJ^v(0))$ varies continuously with $v$ and hence so do the initial conditions $\Phi(J^v)(0)$ and $\Phi(J^v)'(0)$. Thus by the theory of ordinary differential equations $\Phi(J^v)(t)$ varies continuously with $v$. On a surface without conjugate points jacobi fields $\Phi(J^v)(t)$ that vanish are 0 at exactly one point and they cross the $t$ axis transversely and hence the 0 varies continuously with $v$. Thus $G(v)$ and $g(v)$ are continuous and in particular $g(v)$ is bounded.

The boundedness and continuity of $g$ imply that for any $\gamma$ as $t$ varies from $-\infty$ to $\infty$ so does $g(\gamma'(t)) + t$ and hence $\Phi(\Psi_1^z) = \Psi_0^z$. ∎

LEMMA 3.7. *There is a number $R > 0$ such that if $\gamma$ and $\sigma$ are geodesics in $\tilde{M}_1$ such that $\gamma(0) = \sigma(0)$ and $\gamma \neq \sigma$ then $F(\sigma)(R) \notin F(\gamma)[-g_0, \infty)$.*

*Proof.* Fix $p \in M_0$ and $v \in U_p$. For any $\tilde{p} \in \tilde{M}_0$ which projects to $p$ and any $w \in U_p$ we let $\tilde{\gamma}_v$ and $\tilde{\gamma}_w$ be the geodesics in $\tilde{M}_0$ starting at $\tilde{p}$ with initial tangents that project to $v$ and $w$ respectively. Lemma 3.6 guarantees the existence of a $\theta_v > 0$ such that if $w$ makes an angle less than $\theta_v$ with $v$ then the geodesics $F^{-1}(\tilde{\gamma}_v)$ and $F^{-1}(\tilde{\gamma}_w)$ intersect at some $F^{-1}(\tilde{\gamma}_v)(t)$ for $t < g_0 + 1$ and hence never intersect for $t > g_0 + 1$ as long as they do not coincide. Easy continuity arguments along with the compactness of $UM_0$ allow us to choose a $\theta$ with $\theta_v > \theta > 0$ for all $v$ in $UM_0$.

Now let $\gamma$ and $\sigma$ be as in the statement of the lemma and let $R$ be greater than $\max\{g_0 + 1, \pi a/\sin(\theta), g_0 + \pi a\}$ where $a$ is as in the proof of Lemma 3.3. Assume $F(\gamma)(t_0) = F(\sigma)(R)$ then the first paragraph says that $F(\sigma)'(R)$ and $F(\gamma)'(t_0)$ make an angle greater than $\theta$ (since $R > g_0 + 1$.) If $t_0 \geq 0$ then $d(F(\sigma)(0), F(\gamma)(0)) \geq R\sin(\theta) > \pi a$ since $\tilde{M}_0$ has nonpositive curvature. If $t_0 < 0$ (here we need to worry about the angle close to $\pi$) the triangle inequality gives again $d(F(\sigma)(0), F(\gamma)(0)) \geq R - g_0 > \pi a$. On the other hand there is a path in $U\tilde{M}_1$ from $\gamma'(0)$ to $\sigma'(0)$ of length $\leq \pi$. By the definition of $a$ its image in $U\tilde{M}_0$ is a curve of length $\leq \pi a$ which when projected to $\tilde{M}_0$ becomes a curve of length $\leq \pi a$ from $F(\gamma)(0)$ to $F(\sigma)(0)$. This contradiction yields the lemma.

PROPOSITION 3.8. *In the situation of Theorem B we have for every $p \in M_1$ (we parameterize geodesics $\gamma_v$ so that $\gamma_v'(0) = v$ for all $v \in U_p$)*

$$2\pi \geq \int_{U_p} \Phi(J^v)'(g(v))\, dv$$

*Proof.* The inequality is an application of Lemma 1.3. We can parameterize $U_p$ as usual by $\theta$ in $[0, 2\pi]$ then $a(\theta)$ will be $g(\theta)$ and $b(\theta) = R$ where $R$ comes from Lemma 3.7. We define the map $H(\theta, s) = F(\gamma_\theta)(s)$ into $\tilde{M}_0$. We need only show that $H$ has all the right properties from the fact that the jacobi fields $\Phi(J^\theta)$ vanish only at $g(\theta)$. By Lemma 3.7 $H$ maps $\partial_1$ in a 1-1 fashion to an imbedded circle $\partial$ in $\tilde{M}_0$ which will bound a disk $D$.

Since $\Phi(J^\theta)$ is perpendicular to $F(\gamma_\theta)$ and $\Phi(J^\theta)(R)$ is tangent to $\partial$ we see that $F(\gamma_\theta)$ is the geodesic perpendicular to $\partial$ at $\partial(\theta)$.

As $s$ goes to $\infty$ $F(\gamma_\theta)(s)$ goes to $\infty$ and hence eventually lies outside $D$. By Lemma 3.7 $F(\gamma_\theta)(R, \infty) \cap \partial = \emptyset$ and hence $F(\gamma_\theta)(R, \infty)$ lies outside $D$ and since $F(\gamma_\theta)[-g_0, R) \cap \partial = \emptyset$ we have $F(\gamma_\theta)[-g_0, R)$ lies in $D$. In particular $H(\partial_0)$ lies in the interior of $D$ and property iv is satisfied.

For any $p \in D$ let $\tau$ be a minimizing geodesic from $p$ to $\partial$. Then $\tau$ is perpendicular to $\partial$ so $p = \Phi(\gamma_\theta)(t)$ for some $\theta$ and $t$. We need to show that $g(\theta) \le t \le R$. By the previous paragraph $t \le R$. Since $\Phi(J^\theta)(g(\theta)) = 0$ and $\Phi(J^\theta)$ is the variation field of the variation of normal geodesics the usual variation argument will say that, since $\tau$ is the shortest path from $p$ to $\partial$, $t$ cannot be $< g(\theta)$. Hence $D$ is the image of $H$ and property iii is satisfied.

We can thus apply Lemma 1.3 to yield the inequality. ■

*Proof of Theorem B.* Integrating the inequality of Lemma 3.8 over $M_1$ we get:

$$2\pi \cdot \mathrm{Vol}\,(M_1) \ge \int_{UM_1} \Phi(J^v)'(g(v))\, dv.$$

From the invariance of the canonical measure under the geodesic flow we get for each $L > 0$:

$$2\pi L \cdot \mathrm{Vol}\,(M_1) \ge \int_{UM_1} \int_0^L \Phi(J^{\zeta^t v})'(g(\zeta^t v))\, dt\, dv.$$

For fixed $v$ let $\gamma(t)$ be the geodesic with $\gamma'(0) = v$ so that $\zeta^t(v) = \gamma'(t)$. Differentiating Equation 4 with respect to $t$, plugging in $g(\gamma'(t))$, and using Equation 5 yields:

$$\Phi(J^{\zeta^t v})'(g(\zeta^t v)) = \frac{j_1^s(t)}{j_0^s(g(\gamma'(t)) + t)}.$$

(In the above one must be careful with parameters since $\Phi(j^{\gamma'(t)})$ is a jacobi field along the geodesic $F(\gamma)$ with the parameter shifted by $t$.)

Apply Lemma 1.1 with $f(t) = g(\gamma'(t)) + t$, $j = j_1^s$, and $\bar{j} = j_0^s$ we find that

$$2\pi L \cdot \mathrm{Vol}\,(M_1) \ge \int_{UM_1} \frac{L^{3/2}}{(L + g(\zeta^L v) - g(v))^{1/2}}\, dv.$$

Rearranging terms we see:

$$1 \ge \frac{1}{\mathrm{Vol}\,(UM_1)} \cdot \int_{UM_1} \frac{1}{\left(1 + \dfrac{g(\zeta^L v) - g(v)}{L}\right)^{1/2}}\, dv.$$

Using a Jensen inequality for the function $x^{-1/2}$ we see that

$$1 \geq \left[\frac{1}{\operatorname{Vol}(UM_1)} \cdot \int_{UM_1} \left(1 + \frac{g(\zeta^L v) - g(v)}{L}\right) dv\right]^{-1/2}$$

with equality holding only if $g(\zeta^L(v)) = g(v) + c(L)$ where $c(L)$ is a constant depending at most on $L$. On the other hand the invariance of $dv$ under $\zeta^t$ says

$$\int_{UM_1} g(\zeta^t v)\, dv = \int_{UM_1} g(v)\, dv$$

and hence we get equality everywhere and further $c(L) = 0$ and hence $g(v) = g(\zeta^L v)$ for all $v$ and $L$. Since there are dense geodesics (see [B–B–E]) we see that $g(v)$ is a constant $K$. By composing with $\zeta^K$ we can assume that $g(v) = 0$ (i.e. we consider $\zeta^K \circ F$ instead of $F$ and will show it is $dI$.)

We claim that $F$ covers a map $f: M_1 \to M_0$. To see this let $x \in M_1$ and let $c(\theta)$ in $UM_1$ be the curve of unit vectors at $x$. Then $c'(\theta)$ corresponds to the jacobi field $J^0$ along $\gamma_\theta$ and hence $(\pi_0 \circ F)_*(c'(\theta)) = \Phi(J^0)(0) = 0$. Thus $(\pi_0 \circ F)(c(\theta)) = f(x)$ is independent of $\theta$.

To finish the proof we need only note that $f$ is an isometry and $df = F$. But this follows since $f$ takes unit speed geodesics $\gamma_v$ to unit speed geodesics $\gamma_{F(v)}$. In particular if $\gamma$ is a minimizing geodesic from $p$ to $q$ then $f(\gamma)$ is a minimizing geodesic of the same length from $f(p)$ to $f(q)$.

## IV. The genus one case

In this section we take up the one case of non-positive curvature not covered by Theorem B. This is the case where $M_0$ is a flat torus. (In the Klein bottle case for algebraic reasons any diffeomorphism of unit tangent bundles will lift to a diffeomorphism of the unit tangent bundles of the oriented double covers.)

EXAMPLE 4.1. If $M_0$ is a flat two torus, say $M_0 = \mathbb{R}^2/\Gamma$ for a lattice $\Gamma$. Let $(x, y)$ be standard parameters for $\mathbb{R}^2$ and $\theta$ the angle from the $x$-axis. Then $UM_0 = \{(x, y, \theta) \in \mathbb{R}^2/\Gamma \times \mathbb{R}^1/2\pi\}$. Note that the geodesic flow vector field at $(x, y, \theta)$ is $\cos(\theta) \cdot d/dx + \sin(\theta) \cdot d/dy$. Hence diffeomorphisms $F: (x, y, \theta) \to (x + a(\theta), y + b(\theta), \theta)$ induce a conjugacy of the geodesic flows when $a$ and $b$ are functions of $\theta$ such that $a(0) = b(0) = 0$ and $(a(2\pi), b(2\pi)) \in \Gamma$.

It is easy to see that if $(a(2\pi), b(2\pi)) \in \Gamma - (0, 0)$ then $F$ is not homotopic to a fiber preserving map so cannot be of the form $dI \circ \zeta^t$. Even if $(a(2\pi), b(2\pi)) = (0, 0)$ as long as $a$ or $b$ is not identically 0, $F$ is not fiber preserving and (except for special choices $a(\theta) = t(1 - \cos(\theta))$, $b(\theta) = -t \sin(\theta)$) cannot be made so by following by a fixed amount. Hence again $F$ is not $dI \circ \zeta^t$.

Although the above shows that Theorem B does not hold in its strongest form we do have:

THEOREM C. *If the geodesic flow of a closed surface $M_1$ is conjugate to that of a flat torus $M_0$ then $M_1$ is isometric to $M_0$.*

*Proof.* We first show that the map on geodesics induced by the conjugacy $F$ induces a 1-1 correspondence between $\pi_1(M_1)$ and $\pi_1(M_0)$. $UM_0$ is homeomorphic to $S^1 \times S^1 \times S^1$ and $\pi_1(UM_0)$ is isomorphic to $\mathbb{Z}^3$ with generators $\alpha, \beta, \gamma$. We can assume that $\alpha$ and $\beta$ come from tangent vector fields to closed geodesics on $M_0$ while $\gamma$ comes from the fiber. In particular, there is a natural identification between the $\mathbb{Z}^2$ spanned by $\alpha$ and $\beta$ and $\pi_1(M_0)$ given by lifting a closed geodesic to its tangent vector field in $UM_0$. Let $P_i : UM_i \to M_i$ be the projection. Then $(P_1 \circ F^{-1})_* : \mathrm{span}\{\alpha, \beta\} \to \pi_1(M_1)$ induces a homomorphism from $\pi_1(M_0)$ to $\pi_1(M_1)$. This homomorphism is onto since each element of $\pi_1(M_1)$ can be represented by a closed geodesic $\gamma_1$ and $F^{-1}(T\gamma_1)$ is $T\gamma_0$ for some geodesic $\gamma_0$ hence is in the span of $\alpha$ and $\beta$. This homomorphism must thus be injective.

We now claim that every closed geodesic $\gamma_1$ in $M_1$ is the shortest in its homotopy class. To see this let $\tau_1$ be a closed geodesic homotopic to $\gamma_1$. The corresponding geodesics $\gamma_0$ and $\tau_0$ in $M_0$ must be homotopic by the previous paragraph and hence have the same length (since $M_0$ is a flat torus.) Thus $\gamma_1$ and $\tau_1$ have the same length.

Since closed geodesics are dense in $UM_0$ they are also in $UM_1$ via $F^{-1}$. Proceeding now as in the proof of Lemma 3.2 we see that $M_1$ has no conjugate points. By E. Hopf's theorem [H] $M_1$ is flat. It is easy to check that two flat two tori with the same length spectrum are isometric. ■

REFERENCES


[B] W. BALLMANN, *Axial isometries of manifolds of nonpositive curvature.* J. Diff. Geom. *18* (1983), 701–721.

[B–B–E] W. BALLMANN, M. BRIN and P. EBERLEIN, *Structure of manifolds of non-positive curvature I.* Univ. of Maryland, Technical Report, TR 84-8 (1984).

[B–K] K. BURNS and A. KATOK, *Manifolds with non-positive curvature.* Ergod. Th. of Dynam. Sys. *5* (1985), 307–317.

[CV] Y. COLIN DE VERDIERÉ, *Spectre du laplacian et longueurs des géodésiques periodiques II.* Compos. Math. *27* (1973), 159–184.

[C] C. CROKE, *Rigidity and the distance between boundary points.* (to appear in J. Diff. Geom.).

[D–G] J. DUISTERMAAT and V. GUILLEMIN, *The spectrum of positive elliptic operators and periodic bicharacteristics.* Invent. Math. *29* (1975), 30–79.

[E–O] J. ESCHENBERG and J. O'SULLIVAN, *Growth of Jacobi fields and divergence of geodesics.* Math Z. *150* (1976) 221–237.

[F–O] J. FELDMAN and D. ORNSTEIN, *Semi-rigidity of horocycle flows over surfaces of variable negative curvature.* Ergod. Th. & Dynam. Syst. *7* (1987), 49–72.

[F–K] R. FERES and A. KATOK, *Invariant tensor fields of dynamical systems with pinched Lyapunov exponents and rigity of geodesic flows.* (preprint).

[G–N] M. Gerver and N. Nadirashvili, *An isometricity condition for Riemannian metris on a disk.* Soviet Math. Dokl. *29* (1984) no. 2 199–203.

[Gre] L. Green, *Surfaces without conjugate points.* Trans. Amer. Math. Soc. *76* (1954) 529–546.

[G] M. Gromov, *Filling Riemannian manifolds.* J. Diff. Geom. *18* (1983), 1–147.

[H] E. Hopf, *Closed surfaces without conjugate points.* Proc. Nat. Ac. Sc. USA *34* (1948), 47–51.

[K] M. Kanai, *Geodesic flows of negatively curved manifolds with smooth stable and unstable foliations.* (to appear in Ergod. Th. Dynam. Systems).

[M] R. Michel, *Sur la rigidité imposée par la longueur des géodésiques.* Inv. Math. *65* (1981), 71–83.

[O1] J.-P. Otal, *Le spectre marqué des longueurs des surfaces à courbure négative.* (preprint).

[O2] J.-P. Otal, *Sur les longueurs des géodésiques d'une métrique à courbure négative dans le disque.* (preprint).

[Sa] L. Santaló, *Integral Geometry and Geometric Probability,* (*Encyclopedia of Mathematics and its Applications*). Addison–Wesley, London–Amsterdam–Don Mills, Ontario–Sydney–Tokyo 1976.

[Su] T. Sunada, *Riemannian covering and isospectral manifolds.* Ann. of Math. *121* (1985), 169–186.

[V] M. Vignéras, *Variétés riemanniennes isospectrales et non isométriques.* Ann. of Math. *110* (1980), 21–32.

[W] A. Weinstein, *Fourier integral operators, quantization and the spectra of Riemannian manifolds.* Colloques Internationaux C.N.R.S. no. 273 – *Géométrie symplectique et physique mathematique* (1975) 289–298.

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Received May 2, 1989

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