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On the genus of representation spheres

THOMAS BARTSCH

0. Introduction

Let G be a topological group and X a G -space. The (G -) genus of X , $\gamma_G(X)$, is the minimal number n such that there exist subgroups H_1, \dots, H_n of G and a continuous equivariant map $X \rightarrow G/H_1 * \dots * G/H_n$. We require that each H_i is the isotropy group of some element of X . $A * B$ denotes the join of the topological spaces A and B . In this note we provide lower bounds for the genus of spheres of (orthogonal) representation spaces of G when G is a cyclic group.

If $G = \mathbb{Z}/2$ acts via the antipodal map on S^{n-1} then the Borsuk–Ulam theorem tells us that $\gamma_G(S^{n-1}) = n$. More generally, in [Ba] it has been proved for a compact Lie group G acting freely on a representation sphere SV that $\gamma_G(SV) \geq (\dim_{\mathbb{R}} V)/(1 + \dim G)$. The situation gets more complicated if the action of G on SV is not free (but has no fixed points). Already for $G = \mathbb{Z}/4$, the simplest nontrivial example, $\gamma_G(SV)$ will in general be smaller than $\dim_{\mathbb{R}} V$.

For $G = \mathbb{Z}/2$ the concept of genus is well known (cf. [K]) although sometimes under different names like B -index [Y], coindex [CF], level [DL]. In [PS] estimates for the $\mathbb{Z}/2$ -genus (level) of projective spaces are given. As a corollary of our theorem we obtain lower bounds for the $\mathbb{Z}/2$ -genus of lens spaces. This generalizes the result in [PS].

For $G = S^1$ a notion closely related to the genus (it is called index) has been used in [Be] to prove the existence of critical points of G -invariant functionals $SV \rightarrow \mathbb{R}$. This type of applications is one of the main reasons to study γ_G (aside from its intrinsic interest).

1. Statement of results

For a topological group G and a G -space X we use standard notations: X^G denotes the set of fixed points and X/G the orbit space. We write $I(X)$ for the set of conjugacy classes (H) of those subgroups H of G which are the isotropy group of some element of X . We now define two versions of the genus.

DEFINITION 1.1. If $X = \emptyset$ then $\gamma_G(X) := \tilde{\gamma}_G(X) := 0$. If $X \neq \emptyset$ then

$$\gamma_G(X) := \min \{n \in \mathbb{N} : \text{there exist } (H_1), \dots, (H_n) \in I(X) \text{ and a continuous equivariant map } X \rightarrow G/H_1 * \dots * G/H_n\},$$

$$\tilde{\gamma}_G(X) := \min \{n \in \mathbb{N} : \text{there exist closed subgroups } H_1, \dots, H_n \text{ of } G, H_i \neq G, \text{ and a continuous equivariant map } X \rightarrow G/H_1 * \dots * G/H_n\}.$$

We use the convention $\min \emptyset = \infty$.

Obviously, $\gamma_G(X) \geq \tilde{\gamma}_G(X)$ if $X^G = \emptyset$ and $\gamma_G(X) = 1$, $\tilde{\gamma}_G(X) = \infty$ if $X^G \neq \emptyset$. Furthermore, $\gamma_G(X) = \tilde{\gamma}_G(X)$ if $X^G = \emptyset$ and if every closed subgroup H of G , $H \neq G$, is contained in an isotropy group of some element of X .

As mentioned in the introduction for $G = \mathbb{Z}/2$, γ_G has long been known under a variety of names. We prefer to call it genus for several reasons. First for free G -spaces X $\gamma_G(X)$ has been introduced by Fadell in [F] as G -genus. Second, index theories abound. In addition it has become customary to speak of an index theory i (in the context considered here) if it has a number of properties (cf. [Be]), e.g. the monotonicity property: If there exists a G -map $f: X \rightarrow Y$ then $i(X) \leq i(Y)$. This is not true for γ_G . It is true, though, for $\tilde{\gamma}_G$. In fact, $\tilde{\gamma}_G$ is an index theory in the sense of [Be]. It is even equivalent to the one defined there for $G = S^1$.

Now let $G = \mathbb{Z}/n$ be cyclic and V a G -module, i.e. a finite-dimensional orthogonal representation space of G . Let SV denote the unit sphere of V and $d := \dim_{\mathbb{R}} V$. We assume $SV^G = \emptyset$.

THEOREM 1.2. (a) If $n = p^k$ is a power of a prime p then $\tilde{\gamma}_G(SV) \geq (p \cdot d + n - p)/n$.

(b) If n is arbitrary suppose $t := \gcd\{|G/H| : (H) \in I(SV)\} \geq 2$ and let p be a prime dividing t . Let $n_p = p^k$ be the highest power of p dividing n . Then

$$\gamma_G(SV) \geq (p \cdot d + n_p - p)/n_p.$$

REMARKS 1.3. (a) If $n_p = p$ then Theorem 1.2 gives $\gamma_G(SV) \geq \dim_{\mathbb{R}} V$. It is easy to see that equality holds.

(b) Let $G = \mathbb{Z}/4$ act on $S^3 \subset \mathbb{C}^2$ by scalar multiplication; $H := \mathbb{Z}/2 \subset G$. In [Ba] a G -map $S^3 \rightarrow G/H * G/H * G/H$ has been constructed (see also [BCP]). Thus $\tilde{\gamma}_G(S^3) \leq 3$ and the theorem yields equality. Since G acts freely on S^3 it is easy to see that $\gamma_G(S^3) = 4$. More generally, if G acts on $S^{2d-1} \subset \mathbb{C}^d$ with only one orbit type, G or G/H , then $\gamma_G(S^{2d-1}) = 2d$. I do not know the exact value of $\gamma_G(S^{2d-1})$ if both orbit types occur on S^{2d-1} .

(c) The following results are known for other groups G . If G is an elementary abelian p -group, $G = \mathbb{Z}/p \times \dots \times \mathbb{Z}/p$, p a prime, which acts continuously and without fixed points on a sphere S^{n-1} then $\gamma_G(S^{n-1}) = \tilde{\gamma}_G(S^{n-1}) = n$.

If G is a torus, $G = S^1 \times \cdots \times S^1$, acting continuously and fixed point free on S^{2n-1} then $\gamma_G(SV) = \tilde{\gamma}_G(SV) = n$. A proof can be found in [CP]. In the case when the action of G is linear these two results can be easily obtained via a reduction to the case \mathbb{Z}/p .

If G is a compact Lie group and V a G -module such that the sphere SV is a free G -space then $\gamma_G(SV) \geq (\dim_{\mathbb{R}} V)/(1 + \dim G)$ (cf. [Ba]). See also [M] for related results.

A simple corollary of the theorem is the following result. Let $G = \mathbb{Z}/2n$ act on $V \cong \mathbb{C}^d$ such that all isotropy groups on SV are contained in $H := \mathbb{Z}/n$. The orbit space $LV := SV/H$ is called a lens space (in particular when the action of H on SV is free). There exists a free action of $\mathbb{Z}/2 \cong G/H$ on LV .

COROLLARY 1.4. (a) $\gamma_{\mathbb{Z}/2}(LV) \geq \dim_{\mathbb{R}} V$ if n is odd.

(b) $\gamma_{\mathbb{Z}/2}(LV) \geq 1 + (\dim_{\mathbb{R}} V)/2^r$ if $n = 2^r \cdot \text{odd}$, $r > 0$.

If $n = 2$ and the action of G is free, $LV = \mathbb{R}P^{2d-1}$, we recover a result of [PS].

2. Reduction to an algebraic problem

We first prove part (a) of the theorem. Thus we consider the case $G = \mathbb{Z}/n \subset S^1$ has prime power order, $n = p^k$, $k \geq 1$.

Let V be a real G -module with $V^G = \{0\}$, SV its unit sphere, $d := \dim_{\mathbb{R}} V$. If $\gamma := \tilde{\gamma}_G(SV)$ there exist subgroups H_i of G , $H_i \neq G$, and a G -map $f: SV \rightarrow G/H_1 * \cdots * G/H_\gamma$. We have to show $\gamma \geq (p \cdot d + n - p)/n$.

The complex irreducible representations of G are denoted by V_0, \dots, V_{n-1} , where all $V_i \cong \mathbb{C}$ and G acts on V_i via $\zeta \mapsto \zeta^i$.

LEMMA 2.1. *There exists a G -map $\varphi: SV_1^d \rightarrow SV_m^\gamma$, $m = n/p$.*

Proof. First consider the map

$$\begin{aligned} f * f: S(V \oplus V) &\cong SV * SV \rightarrow G/H_1 * \cdots * G/H_\gamma * G/H_1 * \cdots * G/H_\gamma \\ &\cong G/H_1 * G/H_1 * \cdots * G/H_\gamma * G/H_\gamma. \end{aligned}$$

Next observe that $V \oplus V$ can be considered as a complex representation of G . Hence, it can be decomposed into V_1, \dots, V_{n-1} . V_0 does not occur since $V^G = \{0\}$. The maps $V_1 \ni z \mapsto z^i \in V_i$ induce a G -map $SV_1^d \rightarrow S(V \oplus V)$. Moreover, there exists a G -map

$$G/H_i * G/H_i \rightarrow G/H * G/H \rightarrow SV_m.$$

Here $H = \mathbb{Z}/m$, all $H_i \subset H$. This gives the left map. The right one is induced by the map

$$G/H * G/H \ni [t_1, g_1 H, t_2, g_2 H] \mapsto t_1 g_1^m + t_2 g_2^m e^{\pi i/2} \in V_m \setminus \{0\}.$$

It is easy to check that this map is well defined, continuous and equivariant ($t_i \geq 0, t_1 + t_2 = 1$). Putting all these maps together yields $\varphi : SV_1^d \rightarrow SV_m^\gamma$. \square

We now apply equivariant K -theory K_G to the map φ in order to get the required inequality. All facts about K_G which we need can be found in [A] and [S]. For a complex G -module W one can compute $K_G(SW)$ as follows. The Gysin sequence of W yields the exact sequence

$$K_G(pt) \xrightarrow{e_W} K_G(pt) \rightarrow K_G(SW) \rightarrow 0.$$

The map denoted by e_W is simply multiplication with the Euler class of W . Since the Euler class is multiplicative, $e_{W_1 \oplus W_2} = e_{W_1} \cdot e_{W_2}$, we only have to compute e_{V_i} . Now $K_G(pt) \cong RG \cong \mathbb{Z}[x]/(1-x^n)$ and the representation V_i corresponds to the monomial x^i . The Euler class e_W corresponds to the element $\sum_{j=0}^{\dim W} \Lambda^j W$ where $\Lambda^j W$ is the j -th exterior power of W . In particular, e_{V_i} corresponds to $V_0 - V_i$, i.e. to $1 - x^i$. Thus $K_G(SV_i^\alpha) \cong \mathbb{Z}[x]/(1-x^n, (1-x^i)^\alpha)$. Next the homomorphism $\varphi^* : K_G(SV_m^\gamma) \rightarrow K_G(SV_1^d)$ is simply given by $\varphi^*(x) = x$. To see this observe that φ induces the identity

$$K_G(pt) \cong K_G(DV_m^\gamma) \xrightarrow{\varphi^*} K_G(DV_1^d) \cong K_G(pt).$$

Here DW denotes the unit disc of W . So we have a homomorphism

$$\varphi^* : \mathbb{Z}[x]/(1-x^n, (1-x^m)^\gamma) \rightarrow \mathbb{Z}[x]/(1-x^n, (1-x)^d)$$

with $\varphi^*(x) = x$. This implies of course that

$$(1-x^m)^\gamma \in (1-x^n, (1-x)^d).$$

In the next section (Proposition 3.1) we shall show that this implies

$$(m-1) \cdot (\gamma-1) \geq d-\gamma \quad \text{or} \quad \gamma \geq \frac{d+m-1}{m}.$$

Using $m = n/p$ this is the desired inequality needed to prove (a).

To prove (b) let $G = \mathbb{Z}/n$ with $n \in \mathbb{N}$ arbitrary. If the prime p divides $\gcd\{|G/H| : (H) \in I(SV)\}$ it also divides n . Let $G_p := \mathbb{Z}/n_p \subset G$, $n_p = p^k$ the highest power of p dividing n (G_p is the p -Sylow subgroup of G). It is easy to see that $\gamma_G(SV) \geq \gamma_{G_p}(SV)$. This is true for all abelian groups G and all closed subgroups of finite index. It is false for $\tilde{\gamma}$. G_p acts without fixed points on SV since p divides $|G/H|$ for all $(H) \in I(SV)$. Thus we can apply (a) to get (b). \square

Finally we prove the Corollary. Here $G = \mathbb{Z}/2n \supset \mathbb{Z}/n = H$, $n = 2^r \cdot \text{odd}$. Writing $\gamma := \gamma_{\mathbb{Z}/2}(LV)$ there exists a $\mathbb{Z}/2$ -map $LV \rightarrow S^{\gamma-1}$. Here we consider the antipodal action of $\mathbb{Z}/2$ on $S^{\gamma-1}$. This induces a G -map $SV \rightarrow S^{\gamma-1}$ where G acts on $S^{\gamma-1}$ via $G \rightarrow G/H$. We need only consider the 2-Sylow subgroup G_2 of G since G_2 acts without fixed points on SV ; $|G_2| = 2^{r+1}$. So we know

$$\begin{aligned} \gamma &\geq \tilde{\gamma}_{G_2}(SV) \geq (2 \cdot \dim_{\mathbb{R}} V + 2^{r+1} - 2)/2^{r+1} \\ &= (\dim_{\mathbb{R}} V + 2^r - 1)/2^r. \end{aligned}$$

 \square

3. Computations in $\mathbb{Z}[x]$

The goal of this section is to prove the following proposition which we needed in Section 2. Fix a prime p and let n be a power of p , $m := n/p$, and γ, d be natural numbers.

PROPOSITION 3.1. *If $(1 - x^m)^\gamma$ is an element of the ideal generated by $1 - x^n$ and $(1 - x)^d$ in $\mathbb{Z}[x]$ then $(m - 1) \cdot (\gamma - 1) \geq d - \gamma$.*

Proof. First observe that $1 - x^m$ divides $1 - x^n$. Hence, under the assumptions of the Proposition

$$(1 - x^m)^{\gamma-1} \in (1 + x^m + \cdots + x^{(p-1)m}, (1 - x)^{d-1}).$$

Now assume $\gamma \leq d$. (If not the proposition is true). Hence,

$$(1 + x + \cdots + x^{m-1})^{\gamma-1} \in (1 + x^m + \cdots + x^{(p-1)m}, (1 - x)^{d-\gamma}).$$

We now set $a := \gamma - 1$, $b := d - \gamma$ and have to show $b \leq (m - 1)a$. Substituting $y = 1 - x$ denote

$$\varphi := \sum_{i=0}^{m-1} (1 - y)^i \in \mathbb{Z}[y] \quad \text{and} \quad \psi := \sum_{i=0}^{p-1} (1 - y)^{mi} \in \mathbb{Z}[y].$$

Now

$$\varphi^a \in (\psi, y^b) \quad \text{iff} \quad \varphi^a/\psi = \sum_{i=0}^{\infty} s_i y^i \in \mathbb{Q}[[y]]$$

is such that $s_0, \dots, s_{b-1} \in \mathbb{Z}$. Set

$$1/\psi = \sum_{i=0}^{\infty} r_i y^i \in \mathbb{Q}[[y]].$$

LEMMA 3.2. (a) For all $i \geq 1$ and all $0 \leq j < (p-1)mi$: $p^i r_j \in \mathbb{Z}$.

(b) For all $i \geq 0$: $p^{i+1} r_{(p-1)mi} \in \pm 1 + p\mathbb{Z}$.

Proof. We use induction on i to prove both statements simultaneously. If $i = 0$ then $p \cdot r_0 = 1$ by definition of r_i . The first statement is trivially true.

Suppose the lemma is true for $i \geq 0$. We have to show:

- (i) For all $0 \leq j < (p-1)m(i+1) =: J$: $p^{i+1} r_j \in \mathbb{Z}$,
- (ii) $p^{i+2} r_J \in \pm 1 + p\mathbb{Z}$,

(i) is true by induction for all $j \leq (p-1)mi$. Take $j > (p-1)mi$ and suppose (i) is true up to $j-1$. Then by definition

$$p \cdot r_j = \sum_{v=1}^j (-1)^{v+1} r_{j-v} \sum_{\mu=1}^{p-1} \binom{\mu m}{v}.$$

Here and in the sequel we use the convention that $\binom{a}{b}$ is zero for $b > a$. We now compute mod 1.

$$\begin{aligned} p^{i+1} r_j &= p^i \sum_{v=1}^j (-1)^{v+1} r_{j-v} \sum_{\mu=1}^{p-1} \binom{\mu m}{v} \\ &\equiv p^i \sum_{v=1}^{j-(p-1)mi} (-1)^{v+1} r_{j-v} \sum_{\mu=1}^{p-1} \binom{\mu m}{v} \equiv 0. \end{aligned}$$

The first congruence holds by induction. The second is true since for

$$v \leq j - (p-1)mi < (p-1)m: \sum_{\mu=1}^{p-1} \binom{\mu m}{v} \in p\mathbb{Z},$$

(see Lemma 3.4). And $p^{i+1}r_{j-v} \in \mathbb{Z}$ by induction on j . To prove (ii) we compute mod p .

$$\begin{aligned} p^{i+2}r_j &= p^{i+1} \sum_{v=1}^j (-1)^{v+1} r_{j-v} \sum_{\mu=1}^{p-1} \binom{\mu m}{v} \\ &= p^{i+1} \sum_{v=1}^{(p-1)m} (-1)^{v+1} r_{j-v} \sum_{\mu=1}^{p-1} \binom{\mu m}{v} \\ &\equiv p^{i+1} (-1)^{(p-1)m+1} r_{(p-1)m} \sum_{\mu=1}^{p-1} \binom{\mu m}{(p-1)m} \equiv \pm 1. \end{aligned}$$

All congruences hold by induction (and Lemma 3.4). □

LEMMA 3.3. Set $J_a := (m-1) \cdot a$. Remember: $\varphi^a/\psi = \sum_{i=0}^{\infty} s_i y^i$.

- (a) For all $0 \leq j < J_a : s_j \in \mathbb{Z}$.
- (b) For all $i \geq 1$ and all $0 \leq j < (p-1)mi : p^i s_{J_a+j} \in \mathbb{Z}$.
- (c) For all $i \geq 0 : p^{i+1} s_{J_a+(p-1)mi} \in \pm 1 + p\mathbb{Z}$.

Proof. We use induction on a .

If $a = 0$ then $J_a = 0$ and $s_j = r_j$. Hence, (a) is trivial and (b) and (c) correspond to (a) and (b) from Lemma 3.2.

Now suppose the lemma is true for $a \geq 0$. We write $\varphi^{a+1}/\psi = \sum_{i=0}^{\infty} t_i y^i$ and have to show (a), (b) and (c) with $a+1$ instead of a and t_i instead of s_i . The t_i and s_i are related by the equation

$$\sum_{i=0}^{\infty} t_i y^i = \left(\sum_{i=0}^{\infty} s_i y^i \right) \cdot \left(\sum_{i=0}^{m-1} (1-y)^i \right).$$

We first prove (a). We only treat the case $p \mid m$. If $m = 1$ (i.e. $n = p$) then $J_a = J_{a+1} = 0$. This case is easier (and Theorem 1.2 is known for $n = p$).

If $j < J_a$ then $t_j \in \mathbb{Z}$ since it is true for all $s_j, j < J_a$. Now take $J_a \leq j < J_{a+1}$ and suppose $t_{j-1} \in \mathbb{Z}$. Then

$$\begin{aligned} t_j - t_{j-1} &= \sum_{v=0}^j \sum_{\mu=v}^{m-1} s_{j-v} (-1)^v \binom{\mu}{v} - \sum_{v=0}^{j-1} \sum_{\mu=v}^{m-1} s_{j-1-v} (-1)^v \binom{\mu}{v} \\ &= m \cdot s_j + \sum_{v=1}^j (-1)^v \cdot s_{j-v} \left[1 + \sum_{\mu=v}^{m-1} \binom{\mu+1}{v} \right] \\ &= m \cdot s_j + \sum_{v=1}^j (-1)^v \left[\binom{m}{v} + \binom{m}{v+1} \right] \cdot s_{j-v}. \end{aligned}$$

If $v < m - 1$ then

$$\binom{m}{v} + \binom{m}{v+1} \in p\mathbb{Z},$$

and $p \cdot s_{j-v} \in \mathbb{Z}$ since $j - v < J_{a+1} < J_a + (p-1)m$.

If $v \geq m - 1$ then $j - v < J_{a+1} - (m-1) = J_a$, hence $s_{j-v} \in \mathbb{Z}$. This yields $t_j \in \mathbb{Z}$ as required.

We next prove (b). Take $i \geq 1$ and $0 \leq j < (p-1)mi$. Set $k := J_{a+1} + j$. Then

$$p \cdot t_k = p^i \sum_{v=0}^k s_{k-v} \sum_{\mu=v}^{m-1} (-1)^v \binom{\mu}{v}.$$

If $v \geq m - 1$ then $k - v \leq J_a + j$, hence $p^i s_{k-v} \in \mathbb{Z}$.

If $v < m - 1$ then

$$\sum_{\mu=v}^{m-1} \binom{\mu}{v} = \binom{m}{v+1} \in p\mathbb{Z}.$$

Since $k - v \leq k < J_a + (p-1)m(i+1)$ we have $p^{i+1} \cdot s_{k-v} \in \mathbb{Z}$.

Finally, we prove (c). Take $i \geq 0$ and set $j := J_{a+1} + (p-1)mi$. Then

$$p^{i+1} t_j = p^{i+1} \sum_{v=0}^j s_{j-v} \sum_{\mu=v}^{m-1} (-1)^v \binom{\mu}{v}.$$

If $v < m - 1$ then

$$\sum_{\mu=v}^{m-1} \binom{\mu}{v} \in p\mathbb{Z} \quad \text{and} \quad p^{i+2} s_{j-v} \in p\mathbb{Z},$$

since $p^{i+1} s_{j-v} \in \mathbb{Z}$.

If $v = m - 1$ we have $j - v = J_a + (p-1)mi$, hence $p^{i+1} s_{j-v} \in \pm 1 + p\mathbb{Z}$. This yields $p^{i+1} t_j \in \pm 1 + p\mathbb{Z}$. \square

The proposition is now a consequence of Lemma 3.3(c). Namely, if $i = 0$ then $p \cdot s_{J_a} \in \pm 1 + p\mathbb{Z}$ which implies $s_{J_a} \notin \mathbb{Z}$. As mentioned before Lemma 3.2 $\varphi^a \in (\psi, y^b)$ iff $s_0, \dots, s_{b-1} \in \mathbb{Z}$. We obtain $b \leq J_a = (m-1) \cdot a$ as required. \square

In the proof of Lemma (3.2) we used a property of the binomial coefficients which we now prove.

LEMMA 3.4. For all $0 \leq v < (p-1)m : \sum_{\mu=1}^{p-1} \binom{\mu m}{v} \in p\mathbb{Z}$.

Proof. Remember that m is a power of p , $m = p^l$.

Claim 1: For all $l \geq 1$ and all $v \notin p\mathbb{Z}$: $\binom{p^l}{v} \in p\mathbb{Z}$.

This is clear for $l = 1$. We compute mod p .

Using

$$\binom{a+b}{c} = \sum_{i=0}^b \binom{b}{i} \binom{a}{c-i}$$

we get

$$\binom{p^{l+1}}{v} \equiv \sum_{i=0}^p \binom{p}{i} \binom{p^l}{v-pi} \equiv \binom{p^l}{v} + \binom{p^l}{v-p^2} \equiv 0.$$

Claim 2: For all $\mu \geq 1$ and all $v \notin p\mathbb{Z}$: $\binom{\mu m}{v} \in p\mathbb{Z}$.

For $\mu = 1$ this is just Claim 1. Mod p we have

$$\binom{\mu m + m}{v} = \sum_{j=0}^m \binom{m}{j} \binom{\mu m}{v-j} \equiv \sum_{\substack{j=0 \\ p|j}}^m \binom{m}{j} \binom{\mu m}{v-j} \equiv 0.$$

Claim 2 proves Lemma 3.4 if $v \notin p\mathbb{Z}$.

Claim 3: For all $a, b \in \mathbb{N}$: $\binom{a}{b} \equiv \binom{pa}{pb} \pmod{p}$.

This is trivial for $a = 0$. Computing mod p we get

$$\begin{aligned} \binom{pa+p}{pb} &= \sum_{i=0}^p \binom{p}{i} \binom{pa}{pb-i} \equiv \binom{pa}{pb} + \binom{pa}{pb-p} \\ &\equiv \binom{a}{b} + \binom{a}{b-1} = \binom{a+1}{b}. \end{aligned}$$

Now, if $v \in p\mathbb{Z} \setminus m\mathbb{Z}$, i.e. $v = \lambda p^r$ with $\lambda \notin p\mathbb{Z}$ and $r < l$, then

$$\binom{\mu m}{v} \equiv \binom{up^{l-r}}{\lambda} \equiv 0.$$

The first congruence is true because of Claim 3, the second because of Claim 2. Finally, if $v = \lambda m$ with $0 \leq \lambda < p - 1$ we have mod p

$$\sum_{\mu=1}^{p-1} \binom{\mu m}{v} \equiv \sum_{\mu=1}^{p-1} \binom{\mu}{\lambda} = \binom{p}{\lambda+1} \equiv 0. \quad \square$$

4. Problems and remarks

(4.1) The assumption $t := \gcd\{|G/H| : (H) \in I(SV)\} \geq 2$ in Theorem 1.2 allowed us to reduce the problem to the case of a cyclic group of prime power order. What can be said if $t = 1$? Of course, one still has to exclude fixed points ($SV^G = \emptyset$).

(4.2) As mentioned in Remark 1.3(c) $\gamma_G(SV)$ has been computed for elementary abelian subgroups and tori. This can be done with elementary methods. It is natural to consider more general groups. The most promising to attack are p -groups. From the point of view of applications, e.g. the dihedral groups or other subgroups of $O(3)$ are important.

(4.3) Related to the genus $\gamma_G(X)$ is the equivariant Lusternik–Schnirelman category $\text{cat}_G(X)$. This is the smallest integer n (or ∞) such that there exists a numerable covering of X consisting of n invariant subsets of X which can be equivariantly deformed inside X to an orbit. It is easy to see that $\text{cat}_G(X) \geq \gamma_G(X)$; cf. [CP] or [Ba] for properties of cat_G , some computations and applications (in particular for $G \subset O(3)$). In [BCP] equivariant stable cohomotopy is used to show that $\text{cat}_G(SV) = \infty$ for all p -groups G and infinite-dimensional G -modules V with $V^G = \{0\}$. Unfortunately, the argument there does not give any estimates of $\text{cat}_G(SV)$ for finite-dimensional V . In the very special case $G = \mathbb{Z}/p^k$ Theorem 1.2 implies $\gamma_G(SV) = \infty$ if $\dim V = \infty$.

(4.4) For applications lower estimates of $\gamma_G(SV)$ are more important than upper estimates. Still it would be very interesting to give upper estimates or even to compute $\gamma_G(SV)$.

(4.5) Another problem is to compute $\gamma_G(X)$ for other G -spaces X , e.g. G -manifolds. If $G = \mathbb{Z}/2$ and $X = \mathbb{R}P^{2n-1}$, $\mathbb{C}P^{2n-1}$ see [PS]. What if G acts nonlinearly on a sphere?

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