Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 65 (1990)

Artikel: On the genus of represenattion spheres.

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DOI: https://doi.org/10.5169/seals-49714

Comment. Math. Helvetici 65 (1990) 85–95
0010-2571/90/010085-11$1.50 + 0.20/0

On the genus of representation spheres

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0. Introduction

Let G be a topological group and X a G-space. The (G-) genus of X, $\gamma_G(X)$, is the minimal number n such that there exist subgroups $H_1, \ldots, H_n$ of G and a continuous equivariant map $X \to G/H_1 * \cdots * G/H_n$. We require that each H_i is the isotropy group of some element of X. $A * B$ denotes the join of the topological spaces A and B. In this note we provide lower bounds for the genus of spheres of (orthogonal) representation spaces of G when G is a cyclic group.

If $G = \mathbb{Z}/2$ acts via the antipodal map on S^{n-1} then the Borsuk–Ulam theorem tells us that $\gamma_G(S^{n-1}) = n$. More generally, in [Ba] it is has been proved for a compact Lie group G acting freely on a representation sphere SV that $\gamma_G(SV) \geq (\dim_{\mathbb{R}} V)/(1 + \dim G)$. The situation gets more complicated if the action of G on SV is not free (but has no fixed points). Already for $G = \mathbb{Z}/4$, the simplest nontrivial example, $\gamma_G(SV)$ will in general be smaller than $\dim_{\mathbb{R}} V$.

For $G = \mathbb{Z}/2$ the concept of genus is well known (cf. [K]) although sometimes under different names like B-index [Y], coindex [CF], level [DL]. In [PS] estimates for the $\mathbb{Z}/2$-genus (level) of projective spaces are given. As a corollary of our theorem we obtain lower bounds for the $\mathbb{Z}/2$-genus of lens spaces. This generalizes the result in [PS].

For $G = S^1$ a notion closely related to the genus (it is called index) has been used in [Be] to prove the existence of critical points of G-invariant functionals $SV \to \mathbb{R}$. This type of applications is one of the main reasons to study γ_G (aside from its intrinsic interest).

1. Statement of results

For a topological group G and a G-space X we use standard notations: X^G denotes the set of fixed points and X/G the orbit space. We write $I(X)$ for the set of conjugacy classes (H) of those subgroups H of G which are the isotropy group of some element of X. We now define two versions of the genus.

DEFINITION 1.1. If $X = \emptyset$ then $\gamma_G(X) := \tilde{\gamma}_G(X) := 0$. If $X \neq \emptyset$ then

$$\gamma_G(X) := \min \{n \in \mathbb{N} : \text{there exist } (H_1), \ldots, (H_n) \in I(X) \text{ and a continuous equivariant map } X \to G/H_1 * \cdots * G/H_n\},$$

$$\tilde{\gamma}_G(X) := \min \{n \in \mathbb{N} : \text{there exist closed subgroups } H_1, \ldots, H_n \text{ of } G, H_i \neq G, \text{ and a continuous equivariant map } X \to G/H_1 * \cdots * G/H_n\}.$$

We use the convention $\min \emptyset = \infty$.

Obviously, $\gamma_G(X) \geq \tilde{\gamma}_G(X)$ if $X^G = \emptyset$ and $\gamma_G(X) = 1$, $\tilde{\gamma}_G(X) = \infty$ if $X^G \neq \emptyset$. Furthermore, $\gamma_G(X) = \tilde{\gamma}_G(X)$ if $X^G = \emptyset$ and if every closed subgroup H of $G, H \neq G$, is contained in an isotropy group of some element of X.

As mentioned in the introduction for $G = \mathbb{Z}/2$, γ_G has long been known under a variety of names. We prefer to call it genus for several reasons. First for free G-spaces X $\gamma_G(X)$ has been introduced by Fadell in [F] as G-genus. Second, index theories abound. In addition it has become customary to speak of an index theory i (in the context considered here) if it has a number of properties (cf. [Be]), e.g. the monotonicity property: If there exists a G-map $f: X \to Y$ then $i(X) \leq i(Y)$. This is not true for γ_G. It is true, though, for $\tilde{\gamma}_G$. In fact, $\tilde{\gamma}_G$ is an index theory in the sense of [Be]. It is even equivalent to the one defined there for $G = S^1$.

Now let $G = \mathbb{Z}/n$ be cyclic and V a G-module, i.e. a finite-dimensional orthogonal representation space of G. Let SV denote the unit sphere of V and $d := \dim_{\mathbb{R}} V$. We assume $SV^G = \emptyset$.

THEOREM 1.2. (a) *If $n = p^k$ is a power of a prime p then $\tilde{\gamma}_G(SV) \geq (p \cdot d + n - p)/n$.*

(b) *If n is arbitrary suppose $t := \gcd\{|G/H| : (H) \in I(SV)\} \geq 2$ and let p be a prime dividing t. Let $n_p = p^k$ be the highest power of p dividing n. Then*

$$\gamma_G(SV) \geq (p \cdot d + n_p - p)/n_p.$$

REMARKS 1.3. (a) If $n_p = p$ then Theorem 1.2 gives $\gamma_G(SV) \geq \dim_{\mathbb{R}} V$. It is easy to see that equality holds.

(b) Let $G = \mathbb{Z}/4$ act on $S^3 \subset \mathbb{C}^2$ by scalar multiplication; $H := \mathbb{Z}/2 \subset G$. In [Ba] a G-map $S^3 \to G/H * G/H * G/H$ has been constructed (see also [BCP]). Thus $\tilde{\gamma}_G(S^3) \leq 3$ and the theorem yields equality. Since G acts freely on S^3 it is easy to see that $\gamma_G(S^3) = 4$. More generally, if G acts on $S^{2d-1} \subset \mathbb{C}^d$ with only one orbit type, G or G/H, then $\gamma_G(S^{2d-1}) = 2d$. I do not know the exact value of $\gamma_G(S^{2d-1})$ if both orbit types occur on S^{2d-1}.

(c) The following results are known for other groups G. If G is an elementary abelian p-group, $G = \mathbb{Z}/p \times \cdots \times \mathbb{Z}/p$, p a prime, which acts continuously and without fixed points on a sphere S^{n-1} then $\gamma_G(S^{n-1}) = \tilde{\gamma}_G(S^{n-1}) = n$.

If G is a torus, $G = S^1 \times \cdots \times S^1$, acting continuously and fixed point free on S^{2n-1} then $\gamma_G(SV) = \tilde{\gamma}_G(SV) = n$. A proof can be found in [CP]. In the case when the action of G is linear these two results can be easily obtained via a reduction to the case $\mathbb{Z}/p$.

If G is a compact Lie group and V a G-module such that the sphere SV is a free G-space then $\gamma_G(SV) \geq (\dim_{\mathbb{R}} V)/(1 + \dim G)$ (cf. [Ba]). See also [M] for related results.

A simple corollary of the theorem is the following result. Let $G = \mathbb{Z}/2n$ act on $V \cong \mathbb{C}^d$ such that all isotropy groups on SV are contained in $H := \mathbb{Z}/n$. The orbit space $LV := SV/H$ is called a lens space (in particular when the action of H on SV is free). There exists a free action of $\mathbb{Z}/2 \cong G/H$ on LV.

COROLLARY 1.4. (a) $\gamma_{\mathbb{Z}/2}(LV) \geq \dim_{\mathbb{R}} V$ *if n is odd.*
(b) $\gamma_{\mathbb{Z}/2}(LV) \geq 1 + (\dim_{\mathbb{R}} V)/2^r$ *if* $n = 2^r \cdot$ *odd*, $r > 0$.

If $n = 2$ and the action of G is free, $LV = \mathbb{R}P^{2d-1}$, we recover a result of [PS].

2. Reduction to an algebraic problem

We first prove part (a) of the theorem. Thus we consider the case $G = \mathbb{Z}/n \subset S^1$ has prime power order, $n = p^k, k \geq 1$.

Let V be a real G-module with $V^G = \{0\}$, SV its unit sphere, $d := \dim_{\mathbb{R}} V$. If $\gamma := \tilde{\gamma}_G(SV)$ there exist subgroups H_i of G, $H_i \neq G$, and a G-map $f: SV \to G/H_1 * \cdots * G/H_\gamma$. We have to show $\gamma \geq (p \cdot d + n - p)/n$.

The complex irreducible representations of G are denoted by $V_0, \ldots, V_{n-1}$, where all $V_i \cong \mathbb{C}$ and G acts on V_i via $\zeta \mapsto \zeta^i$.

LEMMA 2.1. *There exists a G-map* $\varphi : SV_1^d \to SV_m^\gamma$, $m = n/p$.

Proof. First consider the map

$$f * f: S(V \oplus V) \cong SV * SV \to G/H_1 * \cdots * G/H_\gamma * G/H_1 * \cdots * G/H_\gamma$$
$$\cong G/H_1 * G/H_1 * \cdots * G/H_\gamma * G/H_\gamma.$$

Next observe that $V \oplus V$ can be considered as a complex representation of G. Hence, it can be decomposed into $V_1, \ldots, V_{n-1}$. V_0 does not occur since $V^G = \{0\}$. The maps $V_1 \ni z \mapsto z^i \in V_i$ induce a G-map $SV_1^d \to S(V \oplus V)$. Moreover, there exists a G-map

$$G/H_i * G/H_i \to G/H * G/H \to SV_m.$$

Here $H = \mathbb{Z}/m$, all $H_i \subset H$. This gives the left map. The right one is induced by the map

$$G/H * G/H \ni [t_1, g_1 H, t_2, g_2 H] \mapsto t_1 g_1^m + t_2 g_2^m e^{\pi i/2} \in V_m \backslash \{0\}.$$

It is easy to check that this map is well defined, continuous and equivariant ($t_i \geq 0$, $t_1 + t_2 = 1$). Putting all these maps together yields $\varphi : SV_1^d \to SV_m^\gamma$. □

We now apply equivariant K-theory K_G to the map φ in order to get the required inequality. All facts about K_G which we need can be found in [A] and [S]. For a complex G-module W one can compute $K_G(SW)$ as follows. The Gysin sequence of W yields the exact sequence

$$K_G(pt) \xrightarrow{e_W} K_G(pt) \to K_G(SW) \to 0.$$

The map denoted by e_W is simply multiplication with the Euler class of W. Since the Euler class is multiplicative, $e_{W_1 \oplus W_2} = e_{W_1} \cdot e_{W_2}$, we only have to compute e_{V_i}. Now $K_G(pt) \cong RG \cong \mathbb{Z}[x]/(1 - x^n)$ and the representation V_i corresponds to the monomial x^i. The Euler class e_W corresponds to the element $\Sigma_{j=0}^{\dim W} \Lambda^j W$ where $\Lambda^j W$ is the j-th exterior power of W. In particualr, e_{V_i} corresponds to $V_0 - V_i$, i.e. to $1 - x^i$. Thus $K_G(SV_i^\alpha) \cong \mathbb{Z}[x]/(1 - x^n, (1 - x^i)^\alpha)$. Next the homomorphism $\varphi^* : K_G(SV_m^\gamma) \to K_G(SV_1^d)$ is simply given by $\varphi^*(x) = x$. To see this observe that φ induces the identity

$$K_G(pt) \cong K_G(DV_m^\gamma) \xrightarrow{\varphi^*} K_G(DV_1^d) \cong K_G(pt).$$

Here DW denotes the unit disc of W. So we have a homomorphism

$$\varphi^* : \mathbb{Z}[x]/(1 - x^n, (1 - x^m)^\gamma) \to \mathbb{Z}[x]/(1 - x^n, (1 - x)^d)$$

with $\varphi^*(x) = x$. This implies of course that

$$(1 - x^m)^\gamma \in (1 - x^n, (1 - x)^d).$$

In the next section (Proposition 3.1) we shall show that this implies

$$(m - 1) \cdot (\gamma - 1) \geq d - \gamma \quad \text{or} \quad \gamma \geq \frac{d + m - 1}{m}.$$

Using $m = n/p$ this is the desired inequality needed to prove (a).

To prove (b) let $G = \mathbb{Z}/n$ with $n \in \mathbb{N}$ arbitrary. If the prime p divides $gcd\{|G/H| : (H) \in I(SV)\}$ it also divides n. Let $G_p := \mathbb{Z}/n_p \subset G$, $n_p = p^k$ the highest power of p dividing n (G_p is the p-Sylow subgroup of G). It is easy to see that $\gamma_G(SV) \geq \gamma_{G_p}(SV)$. This is true for all abelian groups G and all closed subgroups of finite index. It is false for $\tilde{\gamma}$. G_p acts without fixed points on SV since p divides $|G/H|$ for all $(H) \in I(SV)$. Thus we can apply (a) to get (b). □

Finally we prove the Corollary. Here $G = \mathbb{Z}/2n \supset \mathbb{Z}/n = H$, $n = 2^r \cdot$ odd. Writing $\gamma := \gamma_{\mathbb{Z}/2}(LV)$ there exists a $\mathbb{Z}/2$-map $LV \to S^{\gamma-1}$. Here we consider the antipodal action of $\mathbb{Z}/2$ on $S^{\gamma-1}$. This induces a G-map $SV \to S^{\gamma-1}$ where G acts on $S^{\gamma-1}$ via $G \to G/H$. We need only consider the 2-Sylow subgroup G_2 of G since G_2 acts without fixed points on SV; $|G_2| = 2^{r+1}$. So we know

$$\begin{aligned}\gamma \geq \tilde{\gamma}_{G_2}(SV) &\geq (2 \cdot \dim_{\mathbb{R}} V + 2^{r+1} - 2)/2^{r+1} \\ &= (\dim_{\mathbb{R}} V + 2^r - 1)/2^r.\end{aligned}$$

□

3. Computations in $\mathbb{Z}[x]$

The goal of this section is to prove the following proposition which we needed in Section 2. Fix a prime p and let n be a power of p, $m := n/p$, and γ, d be natural numbers.

PROPOSITION 3.1. *If* $(1 - x^m)^\gamma$ *is an element of the ideal generated by* $1 - x^n$ *and* $(1 - x)^d$ *in* $\mathbb{Z}[x]$ then $(m - 1) \cdot (\gamma - 1) \geq d - \gamma$.

Proof. First observe that $1 - x^m$ divides $1 - x^n$. Hence, under the assumptions of the Proposition

$$(1 - x^m)^{\gamma-1} \in (1 + x^m + \cdots + x^{(p-1)m}, (1 - x)^{d-1}).$$

Now assume $\gamma \leq d$. (If not the proposition is true). Hence,

$$(1 + x + \cdots + x^{m-1})^{\gamma-1} \in (1 + x^m + \cdots + x^{(p-1)m}, (1 - x)^{d-\gamma}).$$

We now set $a := \gamma - 1$, $b := d - \gamma$ and have to show $b \leq (m - 1)a$. Substituting $y = 1 - x$ denote

$$\varphi := \sum_{i=0}^{m-1} (1 - y)^i \in \mathbb{Z}[y] \quad \text{and} \quad \psi := \sum_{i=0}^{p-1} (1 - y)^{mi} \in \mathbb{Z}[y].$$

Now

$$\varphi^a \in (\psi, y^b) \quad \text{iff} \quad \varphi^a/\psi = \sum_{i=0}^{\infty} s_i y^i \in \mathbb{Q}[[y]]$$

is such that $s_0, \ldots, s_{b-1} \in \mathbb{Z}$. Set

$$1/\psi = \sum_{i=0}^{\infty} r_i y^i \in \mathbb{Q}[[y]].$$

LEMMA 3.2. (a) *For all* $i \geq 1$ *and all* $0 \leq j < (p-1)mi$: $p^i r_j \in \mathbb{Z}$.
(b) *For all* $i \geq 0$: $p^{i+1} r_{(p-1)mi} \in \pm 1 + p\mathbb{Z}$.

Proof. We use induction on i to prove both statements simultaneously. If $i = 0$ then $p \cdot r_0 = 1$ by definition of r_i. The first statement is trivially true.

Suppose the lemma is true for $i \geq 0$. We have to show:

(i) For all $0 \leq j < (p-1)m(i+1) =: J$: $p^{i+1} r_j \in \mathbb{Z}$,
(ii) $p^{i+2} r_J \in \pm 1 + p\mathbb{Z}$,

(i) is true by induction for all $j \leq (p-1)mi$. Take $j > (p-1)mi$ and suppose (i) is true up to $j-1$. Then by definition

$$p \cdot r_j = \sum_{\nu=1}^{j} (-1)^{\nu+1} r_{j-\nu} \sum_{\mu=1}^{p-1} \binom{\mu m}{\nu}.$$

Here and in the sequel we use the convention that $\binom{a}{b}$ is zero for $b > a$. We now compute mod 1.

$$\begin{aligned} p^{i+1} r_j &= p^i \sum_{\nu=1}^{j} (-1)^{\nu+1} r_{j-\nu} \sum_{\mu=1}^{p-1} \binom{\mu m}{\nu} \\ &\equiv p^i \sum_{\nu=1}^{j-(p-1)mi} (-1)^{\nu+1} r_{j-\nu} \sum_{\mu=1}^{p-1} \binom{\mu m}{\nu} \equiv 0. \end{aligned}$$

The first congruence holds by induction. The second is true since for

$$\nu \leq j - (p-1)mi < (p-1)m: \sum_{\mu=1}^{p-1} \binom{\mu m}{\nu} \in p\mathbb{Z},$$

(see Lemma 3.4). And $p^{i+1}r_{j-\nu} \in \mathbb{Z}$ by induction on j. To prove (ii) we compute mod p.

$$\begin{aligned} p^{i+2}r_J &= p^{i+1}\sum_{\nu=1}^{J}(-1)^{\nu+1}r_{J-\nu}\sum_{\mu=1}^{p-1}\binom{\mu m}{\nu} \\ &= p^{i+1}\sum_{\nu=1}^{(p-1)m}(-1)^{\nu+1}r_{J-\nu}\sum_{\mu=1}^{p-1}\binom{\mu m}{\nu} \\ &\equiv p^{i+1}(-1)^{(p-1)m+1}r_{(p-1)mi}\sum_{\mu=1}^{p-1}\binom{\mu m}{(p-1)m} \equiv \pm 1. \end{aligned}$$

All congruences hold by induction (and Lemma 3.4). □

LEMMA 3.3. *Set* $J_a := (m-1)\cdot a$. *Remember*: $\varphi^a/\psi = \Sigma_{i=0}^{\infty} s_i y^i$.
(a) *For all* $0 \le j < J_a : s_j \in \mathbb{Z}$.
(b) *For all* $i \ge 1$ *and all* $0 \le j < (p-1)mi : p^i s_{J_a+j} \in \mathbb{Z}$.
(c) *For all* $i \ge 0 : p^{i+1}s_{J_a+(p-1)mi} \in \pm 1 + p\mathbb{Z}$.

Proof. We use induction on a.
If $a = 0$ then $J_a = 0$ and $s_j = r_j$. Hence, (a) is trivial and (b) and (c) correspond to (a) and (b) from Lemma 3.2.

Now suppose the lemma is true for $a \ge 0$. We write $\varphi^{a+1}/\psi = \Sigma_{i=0}^{\infty} t_i y^i$ and have to show (a), (b) and (c) with $a+1$ instead of a and t_i instead of s_i. The t_i and s_i are related by the equation

$$\sum_{i=0}^{\infty} t_i y^i = \left(\sum_{i=0}^{\infty} s_i y^i\right)\cdot\left(\sum_{i=0}^{m-1}(1-y)^i\right).$$

We first prove (a). We only treat the case $p \mid m$. If $m = 1$ (i.e. $n = p$) then $J_a = J_{a+1} = 0$. This case is easier (and Theorem 1.2 is known for $n = p$).
If $j < J_a$ then $t_j \in \mathbb{Z}$ since it is true for all $s_j, j < J_a$. Now take $J_a \le j < J_{a+1}$ and suppose $t_{j-1} \in \mathbb{Z}$. Then

$$\begin{aligned} t_j - t_{j-1} &= \sum_{\nu=0}^{j}\sum_{\mu=\nu}^{m-1} s_{j-\nu}(-1)^\nu\binom{\mu}{\nu} - \sum_{\nu=0}^{j-1}\sum_{\mu=\nu}^{m-1} s_{j-1-\nu}(-1)^\nu\binom{\mu}{\nu} \\ &= m\cdot s_j + \sum_{\nu=1}^{j}(-1)^\nu\cdot s_{j-\nu}\left[1+\sum_{\mu=\nu}^{m-1}\binom{\mu+1}{\nu}\right] \\ &= m\cdot s_j + \sum_{\nu=1}^{j}(-1)^\nu\left[\binom{m}{\nu}+\binom{m}{\nu+1}\right]\cdot s_{j-\nu}. \end{aligned}$$

If $\nu < m-1$ then

$$\binom{m}{\nu}+\binom{m}{\nu+1}\in p\mathbb{Z},$$

and $p \cdot s_{j-\nu} \in \mathbb{Z}$ since $j-\nu < J_{a+1} < J_a + (p-1)m$.

If $\nu \geq m-1$ then $j-\nu < J_{a+1}-(m-1) = J_a$, hence $s_{j-\nu} \in \mathbb{Z}$. This yields $t_j \in \mathbb{Z}$ as required.

We next prove (b). Take $i \geq 1$ and $0 \leq j < (p-1)mi$. Set $k := J_{a+1}+j$. Then

$$p \cdot t_k = p^i \sum_{\nu=0}^{k} s_{k-\nu} \sum_{\mu=\nu}^{m-1} (-1)^\nu \binom{\mu}{\nu}.$$

If $\nu \geq m-1$ then $k-\nu \leq J_a + j$, hence $p^i s_{k-\nu} \in \mathbb{Z}$.

If $\nu < m-1$ then

$$\sum_{\mu=\nu}^{m-1} \binom{\mu}{\nu} = \binom{m}{\nu+1} \in p\mathbb{Z}.$$

Since $k-\nu \leq k < J_a + (p-1)m(i+1)$ we have $p^{i+1} \cdot s_{k-\nu} \in \mathbb{Z}$.

Finally, we prove (c). Take $i \geq 0$ and set $j := J_{a+1} + (p-1)mi$. Then

$$p^{i+1} t_j = p^{i+1} \sum_{\nu=0}^{j} s_{j-\nu} \sum_{\mu=\nu}^{m-1} (-1)^\nu \binom{\mu}{\nu}.$$

If $\nu < m-1$ then

$$\sum_{\mu=\nu}^{m-1} \binom{\mu}{\nu} \in p\mathbb{Z} \quad \text{and} \quad p^{i+2} s_{j-\nu} \in p\mathbb{Z},$$

since $p^{i+1} s_{j-\nu} \in \mathbb{Z}$.

If $\nu = m-1$ we have $j-\nu = J_a + (p-1)mi$, hence $p^{i+1} s_{j-\nu} \in \pm 1 + p\mathbb{Z}$. This yields $p^{i+1} t_j \in \pm 1 + p\mathbb{Z}$. □

The proposition is now a consequence of Lemma 3.3(c). Namely, if $i = 0$ then $p \cdot s_{J_a} \in \pm 1 + p\mathbb{Z}$ which implies $s_{J_a} \notin \mathbb{Z}$. As mentioned before Lemma 3.2 $\varphi^a \in (\psi, y^b)$ iff $s_0, \ldots, s_{b-1} \in \mathbb{Z}$. We obtain $b \leq J_a = (m-1) \cdot a$ as required. □

In the proof of Lemma (3.2) we used a property of the binomial coefficients which we now prove.

LEMMA 3.4. *For all* $0 \le \nu < (p-1)m : \sum_{\mu=1}^{p-1} \binom{\mu m}{\nu} \in p\mathbb{Z}$.

Proof. Remember that m is a power of p, $m = p^l$.

Claim 1: For all $l \ge 1$ and all $\nu \notin p\mathbb{Z}$: $\binom{p^l}{\nu} \in p\mathbb{Z}$.

This is clear for $l = 1$. We compute mod p.
Using

$$\binom{a+b}{c} = \sum_{i=0}^{b} \binom{b}{i}\binom{a}{c-i}$$

we get

$$\binom{p^{l+1}}{\nu} \equiv \sum_{i=0}^{p} \binom{p}{i}\binom{p^l}{\nu - pi} \equiv \binom{p^l}{\nu} + \binom{p^l}{\nu - p^2} \equiv 0.$$

Claim 2: For all $\mu \ge 1$ and all $\nu \notin p\mathbb{Z}$: $\binom{\mu m}{\nu} \in p\mathbb{Z}$.

For $\mu = 1$ this is just Claim 1. Mod p we have

$$\binom{\mu m + m}{\nu} = \sum_{j=0}^{m} \binom{m}{j}\binom{\mu m}{\nu - j} \equiv \sum_{\substack{j=0 \\ p \mid j}}^{m} \binom{m}{j}\binom{\mu m}{\nu - j} \equiv 0.$$

Claim 2 proves Lemma 3.4 if $\nu \notin p\mathbb{Z}$.

Claim 3: For all $a, b \in \mathbb{N}$: $\binom{a}{b} \equiv \binom{pa}{pb} \bmod p$.

This is trivial for $a = 0$. Computing mod p we get

$$\binom{pa+p}{pb} = \sum_{i=0}^{p} \binom{p}{i}\binom{pa}{pb-i} \equiv \binom{pa}{pb} + \binom{pa}{pb-p}$$
$$\equiv \binom{a}{b} + \binom{a}{b-1} = \binom{a+1}{b}.$$

Now, if $\nu \in p\mathbb{Z} \backslash m\mathbb{Z}$, i.e. $\nu = \lambda p^r$ with $\lambda \notin p\mathbb{Z}$ and $r < l$, then

$$\binom{\mu m}{\nu} \equiv \binom{up^{l-r}}{\lambda} \equiv 0.$$

The first congruence is true because of Claim 3, the second because of Claim 2. Finally, if $\nu = \lambda m$ with $0 \leq \lambda < p - 1$ we have mod p

$$\sum_{\mu=1}^{p-1} \binom{\mu m}{\nu} \equiv \sum_{\mu=1}^{p-1} \binom{\mu}{\lambda} = \binom{p}{\lambda + 1} \equiv 0. \qquad \square$$

4. Problems and remarks

(4.1) The assumption $t := gcd\{|G/H| : (H) \in I(SV)\} \geq 2$ in Theorem 1.2 allowed us to reduce the problem to the case of a cyclic group of prime power order. What can be said if $t = 1$? Of course, one still has to exclude fixed points ($SV^G = \emptyset$).

(4.2) As mentioned in Remark 1.3(c) $\gamma_G(SV)$ has been computed for elementary abelian subgroups and tori. This can be done with elementary methods. It is natural to consider more general groups. The most promising to attack are p-groups. From the point of view of applications, e.g. the dihedral groups or other subgroups of $O(3)$ are important.

(4.3) Related to the genus $\gamma_G(X)$ is the equivariant Lusternik–Schnirelman category $\mathrm{cat}_G(X)$. This is the smallest integer n (or ∞) such that there exists a numerable covering of X consisting of n invariant subsets of X which can be equivariantly deformed inside X to an orbit. It is easy to see that $\mathrm{cat}_G(X) \geq \gamma_G(X)$; cf. [CP] or [Ba] for properties of cat_G, some computations and applications (in particular for $G \subset O(3)$). In [BCP] equivariant stable cohomotopy is used to show that $\mathrm{cat}_G(SV) = \infty$ for all p-groups G and infinite-dimensional G-modules V with $V^G = \{0\}$. Unfortunately, the argument there does not give any estimates of $\mathrm{cat}_G(SV)$ for finite-dimensional V. In the very special case $G = \mathbb{Z}/p^k$ Theorem 1.2 implies $\gamma_G(SV) = \infty$ if $\dim V = \infty$.

(4.4) For applications lower estimates of $\gamma_G(SV)$ are more important than upper estimates. Still it would be very interesting to give upper estimates or even to compute $\gamma_G(SV)$.

(4.5) Another problem is to compute $\gamma_G(X)$ for other G-spaces X, e.g. G-manifolds. If $G = \mathbb{Z}/2$ and $X = \mathbb{R}P^{2n-1}, \mathbb{C}P^{2n-1}$ see [PS]. What if G acts nonlinearly on a sphere?

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Received February 20, 1989.