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# On the genus of representation spheres

THOMAS BARTSCH

## 0. Introduction

Let  $G$  be a topological group and  $X$  a  $G$ -space. The ( $G$ -) genus of  $X$ ,  $\gamma_G(X)$ , is the minimal number  $n$  such that there exist subgroups  $H_1, \dots, H_n$  of  $G$  and a continuous equivariant map  $X \rightarrow G/H_1 * \dots * G/H_n$ . We require that each  $H_i$  is the isotropy group of some element of  $X$ .  $A * B$  denotes the join of the topological spaces  $A$  and  $B$ . In this note we provide lower bounds for the genus of spheres of (orthogonal) representation spaces of  $G$  when  $G$  is a cyclic group.

If  $G = \mathbb{Z}/2$  acts via the antipodal map on  $S^{n-1}$  then the Borsuk–Ulam theorem tells us that  $\gamma_G(S^{n-1}) = n$ . More generally, in [Ba] it is has been proved for a compact Lie group  $G$  acting freely on a representation sphere  $SV$  that  $\gamma_G(SV) \geq (\dim_{\mathbb{R}} V)/(1 + \dim G)$ . The situation gets more complicated if the action of  $G$  on  $SV$  is not free (but has no fixed points). Already for  $G = \mathbb{Z}/4$ , the simplest nontrivial example,  $\gamma_G(SV)$  will in general be smaller than  $\dim_{\mathbb{R}} V$ .

For  $G = \mathbb{Z}/2$  the concept of genus is well known (cf. [K]) although sometimes under different names like  $B$ -index [Y], coindex [CF], level [DL]. In [PS] estimates for the  $\mathbb{Z}/2$ -genus (level) of projective spaces are given. As a corollary of our theorem we obtain lower bounds for the  $\mathbb{Z}/2$ -genus of lens spaces. This generalizes the result in [PS].

For  $G = S^1$  a notion closely related to the genus (it is called index) has been used in [Be] to prove the existence of critical points of  $G$ -invariant functionals  $SV \rightarrow \mathbb{R}$ . This type of applications is one of the main reasons to study  $\gamma_G$  (aside from its intrinsic interest).

## 1. Statement of results

For a topological group  $G$  and a  $G$ -space  $X$  we use standard notations:  $X^G$  denotes the set of fixed points and  $X/G$  the orbit space. We write  $I(X)$  for the set of conjugacy classes ( $H$ ) of those subgroups  $H$  of  $G$  which are the isotropy group of some element of  $X$ . We now define two versions of the genus.

DEFINITION 1.1. If  $X = \emptyset$  then  $\gamma_G(X) := \tilde{\gamma}_G(X) := 0$ . If  $X \neq \emptyset$  then

$\gamma_G(X) := \min \{n \in \mathbb{N} : \text{there exist } (H_1), \dots, (H_n) \in I(X) \text{ and a continuous equivariant map } X \rightarrow G/H_1 * \dots * G/H_n\}$ ,

$\tilde{\gamma}_G(X) := \min \{n \in \mathbb{N} : \text{there exist closed subgroups } H_1, \dots, H_n \text{ of } G, H_i \neq G, \text{ and a continuous equivariant map } X \rightarrow G/H_1 * \dots * G/H_n\}$ .

We use the convention  $\min \emptyset = \infty$ .

Obviously,  $\gamma_G(X) \geq \tilde{\gamma}_G(X)$  if  $X^G = \emptyset$  and  $\gamma_G(X) = 1$ ,  $\tilde{\gamma}_G(X) = \infty$  if  $X^G \neq \emptyset$ . Furthermore,  $\gamma_G(X) = \tilde{\gamma}_G(X)$  if  $X^G = \emptyset$  and if every closed subgroup  $H$  of  $G$ ,  $H \neq G$ , is contained in an isotropy group of some element of  $X$ .

As mentioned in the introduction for  $G = \mathbb{Z}/2$ ,  $\gamma_G$  has long been known under a variety of names. We prefer to call it genus for several reasons. First for free  $G$ -spaces  $X$   $\gamma_G(X)$  has been introduced by Fadell in [F] as  $G$ -genus. Second, index theories abound. In addition it has become customary to speak of an index theory  $i$  (in the context considered here) if it has a number of properties (cf. [Be]), e.g. the monotonicity property: If there exists a  $G$ -map  $f: X \rightarrow Y$  then  $i(X) \leq i(Y)$ . This is not true for  $\gamma_G$ . It is true, though, for  $\tilde{\gamma}_G$ . In fact,  $\tilde{\gamma}_G$  is an index theory in the sense of [Be]. It is even equivalent to the one defined there for  $G = S^1$ .

Now let  $G = \mathbb{Z}/n$  be cyclic and  $V$  a  $G$ -module, i.e. a finite-dimensional orthogonal representation space of  $G$ . Let  $SV$  denote the unit sphere of  $V$  and  $d := \dim_{\mathbb{R}} V$ . We assume  $SV^G = \emptyset$ .

THEOREM 1.2. (a) If  $n = p^k$  is a power of a prime  $p$  then  $\tilde{\gamma}_G(SV) \geq (p \cdot d + n - p)/n$ .

(b) If  $n$  is arbitrary suppose  $t := \gcd\{|G/H| : (H) \in I(SV)\} \geq 2$  and let  $p$  be a prime dividing  $t$ . Let  $n_p = p^k$  be the highest power of  $p$  dividing  $n$ . Then

$$\gamma_G(SV) \geq (p \cdot d + n_p - p)/n_p.$$

REMARKS 1.3. (a) If  $n_p = p$  then Theorem 1.2 gives  $\gamma_G(SV) \geq \dim_{\mathbb{R}} V$ . It is easy to see that equality holds.

(b) Let  $G = \mathbb{Z}/4$  act on  $S^3 \subset \mathbb{C}^2$  by scalar multiplication;  $H := \mathbb{Z}/2 \subset G$ . In [Ba] a  $G$ -map  $S^3 \rightarrow G/H * G/H * G/H$  has been constructed (see also [BCP]). Thus  $\tilde{\gamma}_G(S^3) \leq 3$  and the theorem yields equality. Since  $G$  acts freely on  $S^3$  it is easy to see that  $\gamma_G(S^3) = 4$ . More generally, if  $G$  acts on  $S^{2d-1} \subset \mathbb{C}^d$  with only one orbit type,  $G$  or  $G/H$ , then  $\gamma_G(S^{2d-1}) = 2d$ . I do not know the exact value of  $\gamma_G(S^{2d-1})$  if both orbit types occur on  $S^{2d-1}$ .

(c) The following results are known for other groups  $G$ . If  $G$  is an elementary abelian  $p$ -group,  $G = \mathbb{Z}/p \times \dots \times \mathbb{Z}/p$ ,  $p$  a prime, which acts continuously and without fixed points on a sphere  $S^{n-1}$  then  $\gamma_G(S^{n-1}) = \tilde{\gamma}_G(S^{n-1}) = n$ .

If  $G$  is a torus,  $G = S^1 \times \cdots \times S^1$ , acting continuously and fixed point free on  $S^{2n-1}$  then  $\gamma_G(SV) = \tilde{\gamma}_G(SV) = n$ . A proof can be found in [CP]. In the case when the action of  $G$  is linear these two results can be easily obtained via a reduction to the case  $\mathbb{Z}/p$ .

If  $G$  is a compact Lie group and  $V$  a  $G$ -module such that the sphere  $SV$  is a free  $G$ -space then  $\gamma_G(SV) \geq (\dim_{\mathbb{R}} V)/(1 + \dim G)$  (cf. [Ba]). See also [M] for related results.

A simple corollary of the theorem is the following result. Let  $G = \mathbb{Z}/2n$  act on  $V \cong \mathbb{C}^d$  such that all isotropy groups on  $SV$  are contained in  $H := \mathbb{Z}/n$ . The orbit space  $LV := SV/H$  is called a lens space (in particular when the action of  $H$  on  $SV$  is free). There exists a free action of  $\mathbb{Z}/2 \cong G/H$  on  $LV$ .

**COROLLARY 1.4.** (a)  $\gamma_{\mathbb{Z}/2}(LV) \geq \dim_{\mathbb{R}} V$  if  $n$  is odd.  
 (b)  $\gamma_{\mathbb{Z}/2}(LV) \geq 1 + (\dim_{\mathbb{R}} V)/2^r$  if  $n = 2^r \cdot \text{odd}$ ,  $r > 0$ .

If  $n = 2$  and the action of  $G$  is free,  $LV = \mathbb{R}P^{2d-1}$ , we recover a result of [PS].

## 2. Reduction to an algebraic problem

We first prove part (a) of the theorem. Thus we consider the case  $G = \mathbb{Z}/n \subset S^1$  has prime power order,  $n = p^k$ ,  $k \geq 1$ .

Let  $V$  be a real  $G$ -module with  $V^G = \{0\}$ ,  $SV$  its unit sphere,  $d := \dim_{\mathbb{R}} V$ . If  $\gamma := \tilde{\gamma}_G(SV)$  there exist subgroups  $H_i$  of  $G$ ,  $H_i \neq G$ , and a  $G$ -map  $f: SV \rightarrow G/H_1 * \cdots * G/H_\gamma$ . We have to show  $\gamma \geq (p \cdot d + n - p)/n$ .

The complex irreducible representations of  $G$  are denoted by  $V_0, \dots, V_{n-1}$ , where all  $V_i \cong \mathbb{C}$  and  $G$  acts on  $V_i$  via  $\zeta \mapsto \zeta^i$ .

**LEMMA 2.1.** *There exists a  $G$ -map  $\varphi: SV_1^d \rightarrow SV_m^\gamma$ ,  $m = n/p$ .*

*Proof.* First consider the map

$$\begin{aligned} f * f: S(V \oplus V) &\cong SV * SV \rightarrow G/H_1 * \cdots * G/H_\gamma * G/H_1 * \cdots * G/H_\gamma \\ &\cong G/H_1 * G/H_1 * \cdots * G/H_\gamma * G/H_\gamma. \end{aligned}$$

Next observe that  $V \oplus V$  can be considered as a complex representation of  $G$ . Hence, it can be decomposed into  $V_1, \dots, V_{n-1}$ .  $V_0$  does not occur since  $V^G = \{0\}$ . The maps  $V_1 \ni z \mapsto z^i \in V_i$  induce a  $G$ -map  $SV_1^d \rightarrow S(V \oplus V)$ . Moreover, there exists a  $G$ -map

$$G/H_i * G/H_i \rightarrow G/H * G/H \rightarrow SV_m.$$

Here  $H = \mathbb{Z}/m$ , all  $H_i \subset H$ . This gives the left map. The right one is induced by the map

$$G/H * G/H \ni [t_1, g_1 H, t_2, g_2 H] \mapsto t_1 g_1^m + t_2 g_2^m e^{\pi i/2} \in V_m \setminus \{0\}.$$

It is easy to check that this map is well defined, continuous and equivariant ( $t_i \geq 0$ ,  $t_1 + t_2 = 1$ ). Putting all these maps together yields  $\varphi : SV_1^d \rightarrow SV_m^\gamma$ .  $\square$

We now apply equivariant  $K$ -theory  $K_G$  to the map  $\varphi$  in order to get the required inequality. All facts about  $K_G$  which we need can be found in [A] and [S]. For a complex  $G$ -module  $W$  one can compute  $K_G(SW)$  as follows. The Gysin sequence of  $W$  yields the exact sequence

$$K_G(pt) \xrightarrow{e_W} K_G(pt) \rightarrow K_G(SW) \rightarrow 0.$$

The map denoted by  $e_W$  is simply multiplication with the Euler class of  $W$ . Since the Euler class is multiplicative,  $e_{W_1 \oplus W_2} = e_{W_1} \cdot e_{W_2}$ , we only have to compute  $e_{V_i}$ . Now  $K_G(pt) \cong RG \cong \mathbb{Z}[x]/(1 - x^n)$  and the representation  $V_i$  corresponds to the monomial  $x^i$ . The Euler class  $e_W$  corresponds to the element  $\sum_{j=0}^{\dim W} \Lambda^j W$  where  $\Lambda^j W$  is the  $j$ -th exterior power of  $W$ . In particular,  $e_{V_i}$  corresponds to  $V_0 - V_i$ , i.e. to  $1 - x^i$ . Thus  $K_G(SV_i^\alpha) \cong \mathbb{Z}[x]/(1 - x^n, (1 - x^i)^\alpha)$ . Next the homomorphism  $\varphi^* : K_G(SV_m^\gamma) \rightarrow K_G(SV_1^d)$  is simply given by  $\varphi^*(x) = x$ . To see this observe that  $\varphi$  induces the identity

$$K_G(pt) \cong K_G(DV_m^\gamma) \xrightarrow{\varphi^*} K_G(DV_1^d) \cong K_G(pt).$$

Here  $DW$  denotes the unit disc of  $W$ . So we have a homomorphism

$$\varphi^* : \mathbb{Z}[x]/(1 - x^n, (1 - x^m)^\gamma) \rightarrow \mathbb{Z}[x]/(1 - x^n, (1 - x)^d)$$

with  $\varphi^*(x) = x$ . This implies of course that

$$(1 - x^m)^\gamma \in (1 - x^n, (1 - x)^d).$$

In the next section (Proposition 3.1) we shall show that this implies

$$(m-1) \cdot (\gamma-1) \geq d-\gamma \quad \text{or} \quad \gamma \geq \frac{d+m-1}{m}.$$

Using  $m = n/p$  this is the desired inequality needed to prove (a).

To prove (b) let  $G = \mathbb{Z}/n$  with  $n \in \mathbb{N}$  arbitrary. If the prime  $p$  divides  $\gcd\{|G/H| : (H) \in I(SV)\}$  it also divides  $n$ . Let  $G_p := \mathbb{Z}/n_p \subset G$ ,  $n_p = p^k$  the highest power of  $p$  dividing  $n$  ( $G_p$  is the  $p$ -Sylow subgroup of  $G$ ). It is easy to see that  $\gamma_G(SV) \geq \gamma_{G_p}(SV)$ . This is true for all abelian groups  $G$  and all closed subgroups of finite index. It is false for  $\tilde{\gamma}$ .  $G_p$  acts without fixed points on  $SV$  since  $p$  divides  $|G/H|$  for all  $(H) \in I(SV)$ . Thus we can apply (a) to get (b).  $\square$

Finally we prove the Corollary. Here  $G = \mathbb{Z}/2n \supset \mathbb{Z}/n = H$ ,  $n = 2^r \cdot \text{odd}$ . Writing  $\gamma := \gamma_{\mathbb{Z}/2}(LV)$  there exists a  $\mathbb{Z}/2$ -map  $LV \rightarrow S^{\gamma-1}$ . Here we consider the antipodal action of  $\mathbb{Z}/2$  on  $S^{\gamma-1}$ . This induces a  $G$ -map  $SV \rightarrow S^{\gamma-1}$  where  $G$  acts on  $S^{\gamma-1}$  via  $G \rightarrow G/H$ . We need only consider the 2-Sylow subgroup  $G_2$  of  $G$  since  $G_2$  acts without fixed points on  $SV$ ;  $|G_2| = 2^{r+1}$ . So we know

$$\begin{aligned} \gamma &\geq \tilde{\gamma}_{G_2}(SV) \geq (2 \cdot \dim_{\mathbb{R}} V + 2^{r+1} - 2)/2^{r+1} \\ &= (\dim_{\mathbb{R}} V + 2^r - 1)/2^r. \end{aligned}$$

$\square$

### 3. Computations in $\mathbb{Z}[x]$

The goal of this section is to prove the following proposition which we needed in Section 2. Fix a prime  $p$  and let  $n$  be a power of  $p$ ,  $m := n/p$ , and  $\gamma, d$  be natural numbers.

**PROPOSITION 3.1.** *If  $(1 - x^m)^\gamma$  is an element of the ideal generated by  $1 - x^n$  and  $(1 - x)^d$  in  $\mathbb{Z}[x]$  then  $(m-1) \cdot (\gamma-1) \geq d-\gamma$ .*

*Proof.* First observe that  $1 - x^m$  divides  $1 - x^n$ . Hence, under the assumptions of the Proposition

$$(1 - x^m)^{\gamma-1} \in (1 + x^m + \cdots + x^{(p-1)m}, (1 - x)^{d-1}).$$

Now assume  $\gamma \leq d$ . (If not the proposition is true). Hence,

$$(1 + x + \cdots + x^{m-1})^{\gamma-1} \in (1 + x^m + \cdots + x^{(p-1)m}, (1 - x)^{d-\gamma}).$$

We now set  $a := \gamma - 1$ ,  $b := d - \gamma$  and have to show  $b \leq (m-1)a$ . Substituting  $y = 1 - x$  denote

$$\varphi := \sum_{i=0}^{m-1} (1-y)^i \in \mathbb{Z}[y] \quad \text{and} \quad \psi := \sum_{i=0}^{p-1} (1-y)^{mi} \in \mathbb{Z}[y].$$

Now

$$\varphi^a \in (\psi, y^b) \quad \text{iff} \quad \varphi^a/\psi = \sum_{i=0}^{\infty} s_i y^i \in \mathbb{Q}[[y]]$$

is such that  $s_0, \dots, s_{b-1} \in \mathbb{Z}$ . Set

$$1/\psi = \sum_{i=0}^{\infty} r_i y^i \in \mathbb{Q}[[y]].$$

LEMMA 3.2. (a) *For all  $i \geq 1$  and all  $0 \leq j < (p-1)mi$ :  $p^i r_j \in \mathbb{Z}$ .*

(b) *For all  $i \geq 0$ :  $p^{i+1} r_{(p-1)mi} \in \pm 1 + p\mathbb{Z}$ .*

*Proof.* We use induction on  $i$  to prove both statements simultaneously. If  $i = 0$  then  $p \cdot r_0 = 1$  by definition of  $r_i$ . The first statement is trivially true.

Suppose the lemma is true for  $i \geq 0$ . We have to show:

- (i) For all  $0 \leq j < (p-1)m(i+1) =: J$ :  $p^{i+1} r_j \in \mathbb{Z}$ ,
- (ii)  $p^{i+2} r_J \in \pm 1 + p\mathbb{Z}$ ,

(i) is true by induction for all  $j \leq (p-1)mi$ . Take  $j > (p-1)mi$  and suppose (i) is true up to  $j-1$ . Then by definition

$$p \cdot r_j = \sum_{v=1}^j (-1)^{v+1} r_{j-v} \sum_{\mu=1}^{p-1} \binom{\mu m}{v}.$$

Here and in the sequel we use the convention that  $\binom{a}{b}$  is zero for  $b > a$ . We now compute mod 1.

$$\begin{aligned} p^{i+1} r_j &= p^i \sum_{v=1}^j (-1)^{v+1} r_{j-v} \sum_{\mu=1}^{p-1} \binom{\mu m}{v} \\ &\equiv p^i \sum_{v=1}^{j-(p-1)mi} (-1)^{v+1} r_{j-v} \sum_{\mu=1}^{p-1} \binom{\mu m}{v} \equiv 0. \end{aligned}$$

The first congruence holds by induction. The second is true since for

$$v \leq j - (p-1)mi < (p-1)m: \sum_{\mu=1}^{p-1} \binom{\mu m}{v} \in p\mathbb{Z},$$

(see Lemma 3.4). And  $p^{i+1}r_{j-v} \in \mathbb{Z}$  by induction on  $j$ . To prove (ii) we compute mod  $p$ .

$$\begin{aligned} p^{i+2}r_J &= p^{i+1} \sum_{v=1}^J (-1)^{v+1} r_{J-v} \sum_{\mu=1}^{p-1} \binom{\mu m}{v} \\ &= p^{i+1} \sum_{v=1}^{(p-1)m} (-1)^{v+1} r_{J-v} \sum_{\mu=1}^{p-1} \binom{\mu m}{v} \\ &\equiv p^{i+1} (-1)^{(p-1)m+1} r_{(p-1)m} \sum_{\mu=1}^{p-1} \binom{\mu m}{(p-1)m} \equiv \pm 1. \end{aligned}$$

All congruences hold by induction (and Lemma 3.4).  $\square$

LEMMA 3.3. Set  $J_a := (m-1) \cdot a$ . Remember:  $\varphi^a/\psi = \sum_{i=0}^{\infty} s_i y^i$ .

- (a) For all  $0 \leq j < J_a$ :  $s_j \in \mathbb{Z}$ .
- (b) For all  $i \geq 1$  and all  $0 \leq j < (p-1)mi$ :  $p^i s_{J_a+j} \in \mathbb{Z}$ .
- (c) For all  $i \geq 0$ :  $p^{i+1} s_{J_a+(p-1)mi} \in \pm 1 + p\mathbb{Z}$ .

*Proof.* We use induction on  $a$ .

If  $a = 0$  then  $J_a = 0$  and  $s_j = r_j$ . Hence, (a) is trivial and (b) and (c) correspond to (a) and (b) from Lemma 3.2.

Now suppose the lemma is true for  $a \geq 0$ . We write  $\varphi^{a+1}/\psi = \sum_{i=0}^{\infty} t_i y^i$  and have to show (a), (b) and (c) with  $a+1$  instead of  $a$  and  $t_i$  instead of  $s_i$ . The  $t_i$  and  $s_i$  are related by the equation

$$\sum_{i=0}^{\infty} t_i y^i = \left( \sum_{i=0}^{\infty} s_i y^i \right) \cdot \left( \sum_{i=0}^{m-1} (1-y)^i \right).$$

We first prove (a). We only treat the case  $p \mid m$ . If  $m = 1$  (i.e.  $n = p$ ) then  $J_a = J_{a+1} = 0$ . This case is easier (and Theorem 1.2 is known for  $n = p$ ).

If  $j < J_a$  then  $t_j \in \mathbb{Z}$  since it is true for all  $s_j, j < J_a$ . Now take  $J_a \leq j < J_{a+1}$  and suppose  $t_{j-1} \in \mathbb{Z}$ . Then

$$\begin{aligned} t_j - t_{j-1} &= \sum_{v=0}^j \sum_{\mu=v}^{m-1} s_{j-v} (-1)^v \binom{\mu}{v} - \sum_{v=0}^{j-1} \sum_{\mu=v}^{m-1} s_{j-1-v} (-1)^v \binom{\mu}{v} \\ &= m \cdot s_j + \sum_{v=1}^j (-1)^v \cdot s_{j-v} \left[ 1 + \sum_{\mu=v}^{m-1} \binom{\mu+1}{v} \right] \\ &= m \cdot s_j + \sum_{v=1}^j (-1)^v \left[ \binom{m}{v} + \binom{m}{v+1} \right] \cdot s_{j-v}. \end{aligned}$$

If  $v < m - 1$  then

$$\binom{m}{v} + \binom{m}{v+1} \in p\mathbb{Z},$$

and  $p \cdot s_{j-v} \in \mathbb{Z}$  since  $j - v < J_{a+1} < J_a + (p-1)m$ .

If  $v \geq m - 1$  then  $j - v < J_{a+1} - (m-1) = J_a$ , hence  $s_{j-v} \in \mathbb{Z}$ . This yields  $t_j \in \mathbb{Z}$  as required.

We next prove (b). Take  $i \geq 1$  and  $0 \leq j < (p-1)mi$ . Set  $k := J_{a+1} + j$ . Then

$$p \cdot t_k = p^i \sum_{v=0}^k s_{k-v} \sum_{\mu=v}^{m-1} (-1)^v \binom{\mu}{v}.$$

If  $v \geq m - 1$  then  $k - v \leq J_a + j$ , hence  $p^i s_{k-v} \in \mathbb{Z}$ .

If  $v < m - 1$  then

$$\sum_{\mu=v}^{m-1} \binom{\mu}{v} = \binom{m}{v+1} \in p\mathbb{Z}.$$

Since  $k - v \leq k < J_a + (p-1)m(i+1)$  we have  $p^{i+1} \cdot s_{k-v} \in \mathbb{Z}$ .

Finally, we prove (c). Take  $i \geq 0$  and set  $j := J_{a+1} + (p-1)mi$ . Then

$$p^{i+1} t_j = p^{i+1} \sum_{v=0}^j s_{j-v} \sum_{\mu=v}^{m-1} (-1)^v \binom{\mu}{v}.$$

If  $v < m - 1$  then

$$\sum_{\mu=v}^{m-1} \binom{\mu}{v} \in p\mathbb{Z} \quad \text{and} \quad p^{i+2} s_{j-v} \in p\mathbb{Z},$$

since  $p^{i+1} s_{j-v} \in \mathbb{Z}$ .

If  $v = m - 1$  we have  $j - v = J_a + (p-1)mi$ , hence  $p^{i+1} s_{j-v} \in \pm 1 + p\mathbb{Z}$ . This yields  $p^{i+1} t_j \in \pm 1 + p\mathbb{Z}$ .  $\square$

The proposition is now a consequence of Lemma 3.3(c). Namely, if  $i = 0$  then  $p \cdot s_{J_a} \in \pm 1 + p\mathbb{Z}$  which implies  $s_{J_a} \notin \mathbb{Z}$ . As mentioned before Lemma 3.2  $\varphi^a \in (\psi, y^b)$  iff  $s_0, \dots, s_{b-1} \in \mathbb{Z}$ . We obtain  $b \leq J_a = (m-1) \cdot a$  as required.  $\square$

In the proof of Lemma (3.2) we used a property of the binomial coefficients which we now prove.

**LEMMA 3.4.** For all  $0 \leq v < (p-1)m$ :  $\sum_{\mu=1}^{p-1} \binom{\mu m}{v} \in p\mathbb{Z}$ .

*Proof.* Remember that  $m$  is a power of  $p$ ,  $m = p^l$ .

*Claim 1:* For all  $l \geq 1$  and all  $v \notin p\mathbb{Z}$ :  $\binom{p^l}{v} \in p\mathbb{Z}$ .

This is clear for  $l = 1$ . We compute mod  $p$ .

Using

$$\binom{a+b}{c} = \sum_{i=0}^b \binom{b}{i} \binom{a}{c-i}$$

we get

$$\binom{p^{l+1}}{v} \equiv \sum_{i=0}^p \binom{p}{i} \binom{p^l}{v-pi} \equiv \binom{p^l}{v} + \binom{p^l}{v-p^2} \equiv 0.$$

*Claim 2:* For all  $\mu \geq 1$  and all  $v \notin p\mathbb{Z}$ :  $\binom{\mu m}{v} \in p\mathbb{Z}$ .

For  $\mu = 1$  this is just Claim 1. Mod  $p$  we have

$$\binom{\mu m + m}{v} = \sum_{\substack{j=0 \\ p|j}}^m \binom{m}{j} \binom{\mu m}{v-j} \equiv \sum_{\substack{j=0 \\ p|j}}^m \binom{m}{j} \binom{\mu m}{v-j} \equiv 0.$$

Claim 2 proves Lemma 3.4 if  $v \notin p\mathbb{Z}$ .

*Claim 3:* For all  $a, b \in \mathbb{N}$ :  $\binom{a}{b} \equiv \binom{pa}{pb} \pmod{p}$ .

This is trivial for  $a = 0$ . Computing mod  $p$  we get

$$\begin{aligned} \binom{pa+p}{pb} &= \sum_{i=0}^p \binom{p}{i} \binom{pa}{pb-i} \equiv \binom{pa}{pb} + \binom{pa}{pb-p} \\ &\equiv \binom{a}{b} + \binom{a}{b-1} = \binom{a+1}{b}. \end{aligned}$$

Now, if  $v \in p\mathbb{Z} \setminus m\mathbb{Z}$ , i.e.  $v = \lambda p^r$  with  $\lambda \notin p\mathbb{Z}$  and  $r < l$ , then

$$\binom{\mu m}{v} \equiv \binom{up^{l-r}}{\lambda} \equiv 0.$$

The first congruence is true because of Claim 3, the second because of Claim 2. Finally, if  $v = \lambda m$  with  $0 \leq \lambda < p - 1$  we have mod  $p$

$$\sum_{\mu=1}^{p-1} \binom{\mu m}{v} \equiv \sum_{\mu=1}^{p-1} \binom{\mu}{\lambda} = \binom{p}{\lambda+1} \equiv 0.$$

□

#### 4. Problems and remarks

(4.1) The assumption  $t := \gcd\{|G/H| : (H) \in I(SV)\} \geq 2$  in Theorem 1.2 allowed us to reduce the problem to the case of a cyclic group of prime power order. What can be said if  $t = 1$ ? Of course, one still has to exclude fixed points  $(SV^G = \emptyset)$ .

(4.2) As mentioned in Remark 1.3(c)  $\gamma_G(SV)$  has been computed for elementary abelian subgroups and tori. This can be done with elementary methods. It is natural to consider more general groups. The most promising to attack are  $p$ -groups. From the point of view of applications, e.g. the dihedral groups or other subgroups of  $O(3)$  are important.

(4.3) Related to the genus  $\gamma_G(X)$  is the equivariant Lusternik–Schnirelman category  $\text{cat}_G(X)$ . This is the smallest integer  $n$  (or  $\infty$ ) such that there exists a numerable covering of  $X$  consisting of  $n$  invariant subsets of  $X$  which can be equivariantly deformed inside  $X$  to an orbit. It is easy to see that  $\text{cat}_G(X) \geq \gamma_G(X)$ ; cf. [CP] or [Ba] for properties of  $\text{cat}_G$ , some computations and applications (in particular for  $G \subset O(3)$ ). In [BCP] equivariant stable cohomotopy is used to show that  $\text{cat}_G(SV) = \infty$  for all  $p$ -groups  $G$  and infinite-dimensional  $G$ -modules  $V$  with  $V^G = \{0\}$ . Unfortunately, the argument there does not give any estimates of  $\text{cat}_G(SV)$  for finite-dimensional  $V$ . In the very special case  $G = \mathbb{Z}/p^k$  Theorem 1.2 implies  $\gamma_G(SV) = \infty$  if  $\dim V = \infty$ .

(4.4) For applications lower estimates of  $\gamma_G(SV)$  are more important than upper estimates. Still it would be very interesting to give upper estimates or even to compute  $\gamma_G(SV)$ .

(4.5) Another problem is to compute  $\gamma_G(X)$  for other  $G$ -spaces  $X$ , e.g.  $G$ -manifolds. If  $G = \mathbb{Z}/2$  and  $X = \mathbb{R}P^{2n-1}, \mathbb{C}P^{2n-1}$  see [PS]. What if  $G$  acts nonlinearly on a sphere?

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