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Cohomological dimension and symmetric automorphisms of a free group*

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0. Introduction

Among a number of recent results concerning the cohomology of groups one of the most interesting is that obtained by Gersten [10] and Culler and Vogtmann [7] to the effect that if F is a free group of rank n then its outer automorphism group Out F has virtual cohomological dimension $2n - 3$. In this paper we shall apply the method of Culler and Vogtmann to the subgroup of Out F consisting of "symmetric" automorphisms and shall show that this group has virtual cohomological dimension $n - 2$.

Let F be free with finite basis S. An automorphism α of F is *symmetric* if, for every generator s in S, the image $\alpha(s)$ is a conjugate of an element of $S \cup S^{-1}$. Clearly the symmetric automorphisms form a group which we shall denote by $\Sigma A(F)$ and we shall write $\Sigma O(F)$ for the corresponding image in Out F.

THEOREM. *If the free group F has rank n then the group $\Sigma O(F)$ of symmetric outer automorphisms has virtual cohomological dimension $n - 2$.*

COROLLARY. $vcd(\Sigma A(F)) = n - 1$. □

Our interest in $\Sigma A(F)$ and $\Sigma O(F)$ came originally from our interest in the automorphism groups of free products. There are reasonably close parallels between $\Sigma A(F)$ and Aut G, where $G = *G_i$ is a non-trivial free product of indecomposable groups G_i, none of which is infinite cyclic. In particular, if α is any automorphism of G, then the image $\alpha(G_i)$ of any factor G_i is a conjugate of some factor G_j isomorphic to G_i. In [5] we show that if $G = *G_i$ is a free product of n finite groups then

(i) Aut G is virtually torsion-free;
(ii) $vcd(\text{Aut } G) \geq n - 1$.

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It seems reasonable to conjecture that in fact $vcd(\mathrm{Aut}\, G) = n - 1$.

Another reason for studying $\Sigma A(F)$ stems from the fact that the braid group B_n (see Birman [3]) is the subgroup of $\Sigma A(F)$ consisting of all automorphisms fixing the product $s_1 s_2 \cdots s_n$, where F has basis $S = \{s_1, s_2, \ldots, s_n\}$. It is shown in [3] that B_n has a subgroup P_n (the group of pure braids) of finite index which can be expressed as a semidirect product $P_n = U_n \rtimes P_{n-1}$, with U_n a finitely generated free group. By a theorem of Feldman [8] (see p. 70 of Bieri [2]) it follows that

$$cd(P_n) = cd(U_n) + cd(P_{n-1}) = cd(P_{n-1}) + 1.$$

Inductively $cd(P_{n-1}) = n - 2$ and thus $cd(B_n) = n - 1$. So certainly, if F has rank n, $vcd(\Sigma A(F)) \geqslant n - 1$.

The starting point of our belief that the method of [7] could be applied to $\Sigma A(F)$ and $\Sigma O(F)$ was the observation that the well-known theorem of Whitehead [14] about equivalence of elements under automorphisms of F remains valid when the domain of discussion is restricted to symmetric automorphisms. (The proof of this is entirely straightforward since all that has to be done is to follow the proof of Whitehead's theorem as in, for example, [11] and add the word symmetric at appropriate places. Indeed the argument is very much simpler since many cases do not occur. Now the main technical step in the proof of Whitehead's theorem — what we have called Peak Reduction in [5] and is called the Higgins–Lyndon Lemma in [7] – is also the basic ingredient of Culler and Vogtmann's argument in [7]. They obtain $vcd(\mathrm{Out}\, F)$ by constructing a connected contractible simplicial complex K of dimension $2n - 3$ on which $\mathrm{Out}\, F$ acts. By introducing a condition of symmetry on vertices of K, we pick out a subcomplex K^{Σ} of dimension $n - 2$ on which $\Sigma O(F)$ acts.

The hardest part of the argument is to show that K^{Σ} is contractible. We are grateful to Marc Culler and Karen Vogtmann for discussions from which it emerged that it might be easier to apply the results obtained in [7] rather then slavishly copy the proof. We are also grateful to Martin Lustig for explaining to us how to set about proving the "Poset Lemma" of [7].

1. Symmetric automorphisms

Let F be free with finite basis S. Then $\alpha \in \mathrm{Aut}\, F$ is *symmetric* if, for every $s \in S$, $\alpha(s) = w(s)^{-1}\pi(s)w(s)$, where $\pi(s) \in S \cup S^{-1}$. Clearly the symmetric automorphisms form a group ΣA(F). If $\pi(s) = s$, for every $s \in S$, then we call *α pure symmetric*. We say α is a permutation automorphism if $\alpha(s) = \pi(s)$, for every

$s \in S$. We call α a *symmetric Nielsen* automorphism if there exist $x \in S$ and $y \in S \cup S^{-1}$ such that

$$\alpha(s) = \begin{cases} y^{-1}xy & \text{if } s = x \\ s & \text{otherwise} \end{cases}$$

1.1. PROPOSITION

(i) *The permutation automorphisms form a finite subgroup $\Omega(F)$ of $\Sigma A(F)$.*

(ii) *The pure symmetric automorphisms form a torsion-free normal subgroup $P\Sigma A(F)$ of $\Sigma A(F)$ which is generated by the symmetric Nielsen automorphisms.*

(iii) *$\Sigma A(F)$ is the semidirect product $P\Sigma A(F) \rtimes \Omega(F)$.*

Proof. (i) This is trivial.

(ii) Obviously $P\Sigma A(F)$ is a group and is torsion-free by the theorem of Baumslag–Taylor [1], since it lies in the kernel of the natural map from Aut F to $GL(n, \mathbb{Z})$. If $\alpha(s) = w(s)^{-1}\pi(s)w(s)$, then

$$(\pi^{-1}\alpha\pi)(s) = \pi^{-1}(w(\pi(s)))^{-1}s\pi^{-1}(w(\pi(s)))$$

which yields normality. The fact that $P\Sigma A(F)$ is generated by symmetric Nielsen automorphisms follows from a standard cancellation argument (see Humphries [12] for an exhaustive account). □

A subset $A \subseteq S \cup S^{-1}$ is *symmetric* if there is a unique *distinguished* element $x \in S \cup S^{-1}$ such that $x \in A$ and $x^{-1} \notin A$. Thus if $y \in S \cup S^{-1}$ and $y \neq x^{\pm 1}$, then either y, $y^{-1} \in A$ or $y, y^{-1} \notin A$. Clearly A is symmetric if and only if its complement $\bar{A}$ in $S \cup S^{-1}$ is symmetric. Given any symmetric set A with distinguished element x there is defined a corresponding Whitehead automorphism, denoted by (A, x), and defined by

$$(A, x) : s \mapsto \begin{cases} x^{-1}sx & \text{if } s, s^{-1} \in A \\ s & \text{otherwise.} \end{cases}$$

1.2. PROPOSITION (Peak Reduction Lemma). *Let u, v and w be n-tuples of cyclic words of F and let σ and τ be symmetric Whitehead automorphisms such that $\sigma(w) = u$ and $\tau(w) = v$. Suppose that*

(i) $|u| \leq |w| \geq |v|$;

(ii) $|u| < |w|$ or $|w| > |v|$.

Then there exist symmetric Whitehead automorphisms $\theta_1, \theta_2, \ldots, \theta_r$ *such that* $\sigma^{-1}\tau = \theta_r\theta_{r-1} \cdots \theta_1$ *and* $|\theta_i \cdots \theta_1(u)| < |w|$, $i = 1, 2, \ldots, r-1$. $\square$

We shall omit the proof since the lemma is not necessary for the subsequent argument and, as noted earlier, is quite easily derived from, say, [11]. We do, however, record the fact that the lemma implies, by the same kind of argument as in the general case, that there is an algorithm to determine of any two n-tuples of cyclic words (or indeed linear words) whether or not they are equivalent under a symmetric automorphism.

2. Culler–Vogtmann revisited

We review here the main ideas of Culler–Vogtmann [7], and have endeavoured to make our account as self-contained as possible. Our basic viewpoint is combinatorial but, so as not to diverge too for from [7], we provide a topological gloss. A graph, therefore, is a connected one-dimensional CW-complex with vertices (0-cells) and edges (1-cells). Combinatorially, edges come in oriented pairs with $\bar{e}$ (or e^{-1}) the reverse of e. If e is an edge it runs from its *source* vertex $s(e)$ to its *target* vertex $t(e)$. We write $V(X)$ for the vertex set and $E(X)$ for the set of (oriented) edges of the graph X and $\deg(v)$ for the *degree* (or *valency*) of the vertex v. All graphs considered will be assumed to be *reduced* i.e. will be assumed

(i) not homotopy equivalent to a proper subgraph;
(ii) to have no vertices of degree less than three;
(iii) to have no separating edges.

We fix the graph Y consisting of a single vertex and n loops, and identify the free group F with $\pi_1 Y$, regarding the set $S \cup S^{-1}$ of oriented edges of Y as an "oriented basis" for F. A *marking* on a graph X is a homotopy equivalence $g: Y \to X$ (combinatorially g assigns to the edges of Y closed paths at a basepoint so that the images generate $\pi_1 X$) and two markings $g: Y \to X$ and $g': Y \to X'$ are equivalent if there exists a cellular homeomorphism $h: X \to X'$ (combinatorially an automorphism) such that the diagram commutes up to free homotopy. We have an equivalence relation and the class of $g: Y \to X$ is denoted by (g, X).

$$\begin{tikzcd} & X \arrow[dd, "h"] \\ Y \arrow[ur, "g"] \arrow[dr, "g'"'] & \\ & X' \end{tikzcd}$$

The vertices of the simplicial complex K are the equivalence classes (g, X) of markings. A *collapsing map* $d: X \to X'$ is a cellular homotopy equivalence which collapses one or more edges of X. Then a k-simplex of K is a $(k+1)$-tuple $(\xi_0, \xi_1, \ldots, \xi_k)$ of vertices such that there is a representative $g_i: Y \to X_i$ of ξ_i and a collapsing map $d_j: X_j \to X_{j-1}$, $0 \leq i < k$, $1 \leq j < k$, such that the diagram below is homotopy commutative. An Euler characteristic argument shows that $\dim K = 2n - 3$.

$$\begin{tikzcd}
X_k \arrow[r, "d_k"] & X_{k-1} \arrow[r] & \cdots \arrow[r] & X_1 \arrow[r, "d_1"] & X_0 \\
 & & Y \arrow[ull, "g_k"] \arrow[ul, "g_{k-1}"'] \arrow[ur, "g_1"] \arrow[urr, "g_0"'] & &
\end{tikzcd}$$

It is convenient here to stress a point only briefly mentioned in [7]. Suppose ξ_0, ξ_1, ξ_2 are vertices of K such that (ξ_0, ξ_1) and (ξ_1, ξ_2) are 1-simplices. Then we can form the diagram below where $\xi_2 = (g_2, X_2)$, $\xi_1 = (g_1, X_1) = (g_1', X_1')$, $\xi_0 = (g_0, X_0)$, d_2 and d_1 are collapsing maps and h is an isomorphism. The

$$\begin{tikzcd}
X_2 \arrow[r, "d_2"] & X_1 \arrow[r, "h"] & X_1' \arrow[r, "d_1"] & X_0 \\
 & Y \arrow[ul, "g_2"] \arrow[u, "g_1"] \arrow[ur, "g_1'"] \arrow[urr, "g_0"'] & &
\end{tikzcd}$$

composite $d_1 h d_2$ must also be a collapsing map, and so $\{\xi_0, \xi_2\}$ is a 1-simplex. Thus, as noted in [7], K defines a category, where an arrow is defined by a collapsing map, and clearly the vertices of K in fact form a partially ordered set (*poset*) with respect to the relation:

$$\xi_1 < \xi_2 \quad \text{if} \quad \{\xi_1, \xi_2\} \text{ is a 1-simplex.}$$

We record this formally.

2.1. LEMMA. *The vertices of K form a poset of finite height with* $\xi_1 < \xi_2$ *if and only if* ξ_2 *can be "collapsed" to* ξ_1. □

There is a natural right action of Aut F on K given as follows. Any $\alpha \in \operatorname{Aut} F$ can be regarded as a cellular homotopy equivalence $\alpha: Y \to Y$ and so given $g: Y \to X$ we obtain

$$\begin{tikzcd}
Y \arrow[r, "g"] & X \\
Y \arrow[u, "\alpha"] \arrow[ur, "g\alpha"'] &
\end{tikzcd}$$

Clearly inner automorphisms act trivially, by the definition of equivalence, and so Out F acts on K.

A *rose* ρ is an equivalence class (α, Y), with $\alpha \in \operatorname{Aut} F$. Given any tuple W of cyclic words of F (i.e. conjugacy classes of F) there is defined on the set of all roses a norm $\|-\|_W$ given by $\|\rho\|_W = \sum_{w \in W} |\alpha(w)|$, where $\rho = (\alpha, Y)$ and $|\alpha(w)|$ is the length of the cyclic word $\alpha(w)$. (This is just Definition 1.3.2 of [4].) Given W, write $K_{\min(W)} = \bigcup st(\rho)$, where $st(\rho)$ denotes the star of ρ and the union is over all roses of minimal norm. The main result of [7] is the following.

2.2. THEOREM. [7] (i) *For any W, K is contractible to $K_{\min(W)}$.*

(ii) *There exists W such that $K_{\min(W)}$ is contractible and hence K is contractible.* □

We note that according to [7], Gersten [10] also proves that K is contractible but by somewhat different methods with which we are not familiar.

3. Symmetric graphs and the complex K^{Σ}

We call a graph X *symmetric* if every edge of X lies in a unique circuit (here we identify cyclic rearrangements of a closed path with one another.)

3.1. LEMMA. *If X is a symmetric reduced graph, then* $\deg(v) \geq 4$, *for every vertex v of X.*

Proof. Suppose that $\deg(v) = 3$. If some loop is incident to v then the remaining edge incident to v will be a separating edge, contradicting reducedness. So suppose no edge incident to v is a loop. Then we have the situation of Fig. 1. The unique circuit γ containing e must have the form, say, $\gamma = (e, \ldots, \bar{e}_1)$. Now e_2 does not lie in γ or $\bar{\gamma}$ since γ is a circuit and so the unique circuit δ containing e_2 is distinct from γ and $\bar{\gamma}$. But clearly any circuit containing e_2 must contain $\bar{e}$ or $\bar{e}_1$ which is a contradiction. □

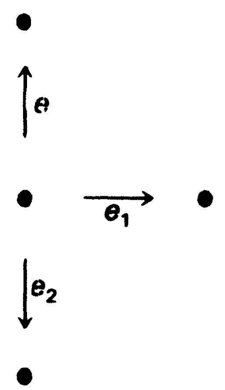

Fig. 1

3.2. LEMMA. *If X is a symmetric reduced graph with fundamental group of rank n, then $|V(X)| \leq n-1$.*

Proof. We have $\sum_{v \in V(X)} \deg(v) = 2(n + |V(X)| - 1)$ from the "handshaking lemma" for graphs. Hence

$$4|V(X)| \leq 2n + 2|V(X)| - 2. \qquad \square$$

We are now ready to define K^{Σ} and establish some of its easier properties. Let W be the n-tuple $(s_1, \ldots, s_n)$ of *cyclic* words. Then a rose $\rho = (\alpha, Y)$ is of minimal norm with respect to $\| - \|_W$ if and only if α is a symmetric automorphism since $\sum_{i=1}^{n} |\alpha(s_i)| = n$ if and only if $|\alpha(s_i)| = 1$, $1 \leq i \leq n$. Then $K_{\min(W)} = \bigcup_{\rho \in \mathcal{R}} st(\rho)$ where $\mathcal{R}$ is the set of all roses (α, Y) such that α is symmetric. (Since Aut Y just consists of suitable permutations of $E(Y)$, every α in a given such rose is symmetric.) A vertex ξ lies in $K_{\min(W)}$ if and only if $\xi = (g, X)$ and there exists a collapsing map $d: X \to Y$ such that $dg: \pi_1 Y \to \pi_1 Y$ is symmetric. Clearly $K_{\min(W)}$ is invariant under the action of $\Sigma A(F)$.

Now the dimension of $K_{\min(W)}$ is still $2n-3$ and we need to replace $K_{\min(W)}$ by a subcomplex of smaller dimension. This is achieved by imposing the condition of symmetry defined above. Namely, we define K^{Σ} to be the subcomplex of $K_{\min(W)}$ generated by all vertices $\xi = (g, X)$ with X symmetric. A discussion of the motivation for the definition of K^{Σ} is given at the end of 4.

3.3. PROPOSITION. $\dim K^{\Sigma} = n-2$.

Proof. This is immediate from Lemma 3.2, since it is easy to construct a symmetric reduced graph with fundamental group of rank n and having $(n-1)$ vertices.

Certain automorphisms introduced by Gersten [9] in looking at fixed-point subgroups play a role in determining the virtual cohomological dimension of Out F. These are the "change of maximal tree" or CMT automorphisms which may be described as follows.

Let $d: X \to Y$ be a collapsing map that collapses the maximal tree T and let X have a given basepoint v. For each edge x of $E(Y)$ there is a unique edge e_x of X mapped to x by d. Further there are unique paths in T from v to $s(e_x)$ and $t(e_x)$ which may be written, respectively, in the form $a_x b_x$ and $a_x c_x$ with a_x of maximal length. The maximality implies that $b_x e_x \bar{c}_x$ is a circuit. We define $d^{-1}: Y \to X$ by $d^{-1}: x \mapsto a_x b_x e_x \bar{c}_x \bar{a}_x$; then d^{-1} is a canonical homotopy inverse for d.

Now let $d: X \to Y$ and $d': X \to Y$ be collapsing maps with corresponding maximal trees T and T' respectively. The induced automorphism $d'_* d_*^{-1}$ is a CMT

Fig. 2

automorphism and is calculated by evaluating d' on the closed paths $d^{-1}(x)$, $x \in S$.

3.4. PROPOSITION. *K^{Σ} is connected.*

Proof. It suffices, by Proposition 1.1, to show that if α and α' are symmetric automorphisms and $\alpha' = \sigma\alpha$, where σ is a symmetric Nielsen automorphism, then there is a path in K^{Σ} from the rose $\rho = (\alpha, Y)$ to the rose $\rho' = (\alpha', Y)$. So suppose $\sigma(x) = y^{-1}xy$ and $\sigma(s) = s$ for $s \in S$, $s \neq x$. Then with $d: X \to Y$ and $d': X \to Y$ defined by the diagrams in Fig. 2 we obtain $\sigma = d'_* d_*^{-1}$ and hence the vertex $(d^{-1}\alpha, X) \in K^{\Sigma}$ is adjacent to ρ and ρ'. □

3.5. PROPOSITION. *$\Sigma A(F)$ acts on K^{Σ} with finite stabilisers and finite quotient.*

Proof. This is proved in exactly the same way as the corresponding statement in [7]. If α stabilises (g, X) then the diagram below yields an injection $\alpha \mapsto h$ from Stab (g, X) to Aut X.

$$\begin{tikzcd} Y \arrow[r, "g"] & X \arrow[d, "h"] \\ Y \arrow[u, "\alpha"] \arrow[r, "g"'] & X \end{tikzcd}$$

The quotient is finite since every vertex is equivalent under $\Sigma A(F)$ to a vertex of the star of the trivial rose $(1, Y)$. □

4. Contractibility of K^{Σ}

We shall show K^{Σ} is contractible by contracting $K_{\min(W)}$, with $W = (s_1, s_2, \ldots, s_n)$, onto K^{Σ}. The homotopy theory result we need is:

4.1. LEMMA. *Let P be a poset of finite height and $f: P \to P$ a poset morphism satisfying*

(a) $f(\xi) \leq \xi$,

(b) $f(f(\xi)) = f(\xi)$.

for all $\xi \in P$. Then f induces a deformation retraction of the corresponding simplicial complex $K(P)$ to the simplicial complex $K(f(P))$.

Proof. This is a special case of the Poset Lemma (Lemma 6.2.1 of [7]) taken from the paper [13] of Quillen. In an appendix we sketch an elementary direct proof of this special case. □

We want, then, to define a retraction $r: K_{\min(W)} \mapsto K^{\Sigma}$ by making r a poset morphism on the vertices of $K_{\min(W)}$. So let $\xi = (g, X) \in K_{\min(W)}$; we shall define a collapsing map $d^{\Sigma}: X \to X^{\Sigma}$ so that $(d^{\Sigma}g, X^{\Sigma}) \in K^{\Sigma}$ and then set $r(\xi) = (d^{\Sigma}g, X^{\Sigma})$. In order that we know which edges of X to collapse so that X^{Σ} is symmetric, we need an alternative characterisation of symmetric graphs.

Let X be a (reduced) graph, with T a maximal tree of X and $e_0 \in E(T)$. Then we can decompose X relative to T and e_0 as follows. Deleting $\{e_0, \bar{e}_0\}$ from T gives two components T_1 and T_2. We define, for $i = 1, 2$, $V_i = V(T_i)$ and $E_i = \{e \in E(X); s(e), t(e) \in V_i\}$. Then V_i, E_i constitute the vertex and edge set of a subgraph X_i of X, with T_i as a maximal tree. Every edge of X not in X_1 or X_2 has one endpoint in X_1 and the other in X_2. We frequently use this or similar notation, not always with further explanation, and depict x as in Fig. 3, noting that $m \geq 1$ since X is reduced. We call $e_1, \ldots, e_m$ the *companion edges* of e_0 and say that e_0 is *symmetric* (*relative to* T) if $m = 1$, i.e. e_0 has a unique companion edge. Clearly e_0 is symmetric relative to T if and only if $\bar{e}_0$ is as well.

4.2. LEMMA. *The reduced graph X is symmetric if and only if for every maximal tree T and every edge $e \in E(T)$, e is symmetric relative to T.*

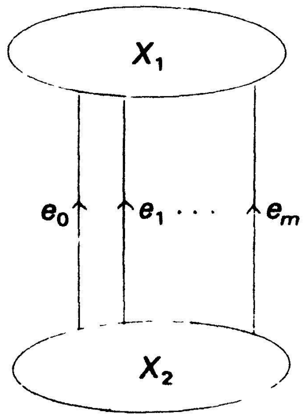

Fig. 3

Proof. Suppose X is symmetric and let T and $e_0 \in E(T)$ be given. We decompose X relative to T and e_0 and suppose e_0 has m companion edges, $m > 1$.

For $1 \leq j \leq m$, let γ_{1j} be the reduced path in $T_1 = T \cap X_1$ from $t(e_0)$ to $t(e_j)$ and let γ_{2j} be the reduced path in $T_2 = T \cap X_2$ from $s(e_j)$ to $s(e_0)$. Then it follows that $\delta_j = (e_0, \gamma_{1j}, \bar{e}_j, \gamma_{2j})$ is a circuit and obviously $\delta_j \neq \delta_k$ when $j \neq k$, giving a contradiction.

Conversely suppose for every maximal tree T and $e \in E(T)$, e is symmetric relative to T. Let $e_0 \in E(X)$; if e_0 is a loop there is nothing to prove. So suppose e_0 is not a loop; take a maximal tree T containing e_0 and let e_1 be the unique companion edge for e_0 in the decomposition of X relative to T and e_0 (see Fig. 4).

Let γ_1 be the reduced path in T from $t(e)$ to $t(e_1)$ and γ_2 the reduced path in T from $s(e_1)$ to $s(e)$. Then $\gamma = (e_0, \gamma_1, \bar{e}_1, \gamma_2)$ is a circuit containing e_0. If e lies in another circuit δ, then $\delta = (e_0, \bar{\delta}_1, \bar{e}_1, \delta_2)$, where δ_1 is a reduced path in X_1 and δ_2 a reduced path in X_2. (Some of γ_1, γ_2, δ_1, δ_2 may be trivial.) Without loss of generality suppose $\gamma_1 \neq \delta_1$. Then γ_1 must be non-trivial (otherwise δ is not a circuit) and hence there is an edge e in γ_1 that does not appear in δ_1.

We consider the decomposition of X relative to T and e. By first decomposing X_1 relative to $T_1 = T \cap X$ and e, we see that the decomposition of X must have the form given in Fig. 5 with $\gamma_1 = (\gamma_{11}, e, \gamma_{12})$, e_1 the unique companion edge for e, and $X_{11} \cup X_2 \cup \{e_0, \bar{e}_0\}$ and X_{12} the corresponding two subgraphs. However the path δ_1 begins in X_{11} and ends in X_{12} without ever leaving X_1. Consequently it must involve e which is a contradiction. □

It actually suffices, in order that X be symmetric, that there exist at least one maximal tree T whose edges are symmetric (relative to T). This is the content of the next lemma.

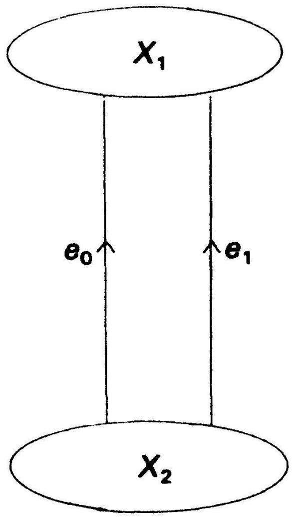

Fig. 4

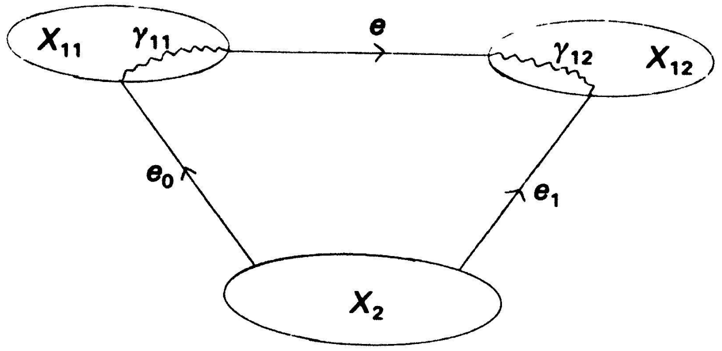

Fig. 5

4.3. LEMMA.

(i) *Let T be a maximal tree in the reduced graph X and let e_0 be symmetric relative to T with companion edge e_0'. Let $\hat{T} = (T - \{e_0, \bar{e}_0\}) \cup \{e_0', \bar{e}_0'\}$. Then for any $e \in E(T)$, $e \neq e_0$, $\bar{e}_0$, e is symmetric relative to T if and only if e is symmetric relative to $\hat{T}$.*

(ii) *Let T and T' be maximal trees of the reduced graph X. Then every edge of T is symmetric relative to T if and only if every edge of T' is symmetric relative to T'.*

Proof. Let $X = X_1 \cup X_2 \cup \{e_0, \bar{e}_0, e_0', \bar{e}_0'\}$ be the decomposition of X relative to T and e_0, and let $e \in E(T)$, $e \neq e_0$, $\bar{e}_0$. We may assume $e \in E(X_1)$ and, possibly replacing e by $\bar{e}$, it follows that the partion of X relative to T and e has the form (a) or (b) given in Fig. 6, depending on whether or not $t(e_0)$ and $t(e_0')$ lies in the

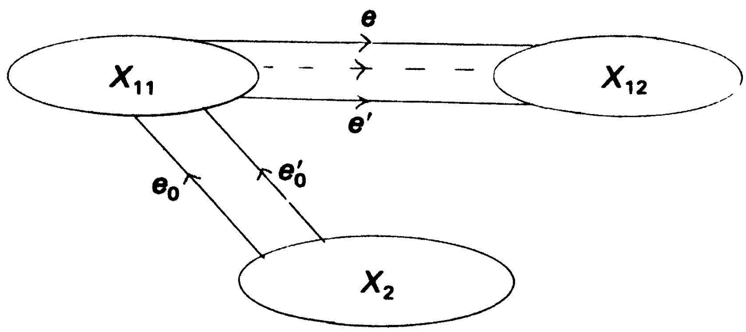

Fig. 6(a)

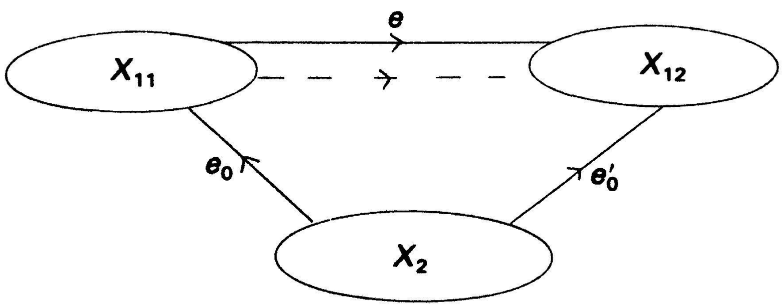

Fig. 6(b)

same part of X_1 determined by decomposing X_1 relative to $T \cap X_1$ and e. In case (a), e has precisely the same companion edges relative to T and $\hat{T}$ while in case (b), e_0' is exchanged for $\bar{e}_0$.

In both cases e is symmetric relative to T if and only if e is symmetric relative to $\hat{T}$.

(ii) Suppose all edges of T are symmetric relative to T and let $e_0 \in E(T) - E(T')$. Let $\hat{T}$ be defined as in (i). Then, clearly all edges of $\hat{T}$ are symmetric relative to $\hat{T}$. Since e_0' must lie in $E(T')$, it follows that $|E(\hat{T}) - E(T')| < |E(T) - E(T')|$ and the result follows by induction. □

We are now in a position to describe our retraction $r: K_{\min(W)} \to K^{\Sigma}$. Let $\xi = (g, X) \in K_{\min(W)}$ and suppose $d: X \to Y$ is a collapsing map, with maximal tree T, such that $(gd)_*: \pi_1 Y \to \pi_1 Y$ is symmetric. We shall define $r(\xi) = (gd^{\Sigma}, X^{\Sigma})$ where $d^{\Sigma}: X \to X^{\Sigma}$ is the collapsing map obtained by collapsing the unsymmetric edges of T. Some work, though, is needed to ensure that r is well-defined.

4.4. LEMMA. *Let $d: X \to Y$ and $d': X \to Y$ be collapsing maps with corresponding maximal trees T and T' which induce a symmetric* CMT *automorphism. If $e_0 \in E(T) - E(T')$ then e_0 is symmetric relative to T.*

Proof. Suppose that e_0 has companion edges $e_1, e_2, \ldots, e_m$, $m \geq 2$. Let $d(e_1) = x$ and $d(e_2) = y$, $x, y \in E(Y)$. Then, in the notation for CMT automorphisms introduced in 3., $e_1 = e_x$ and $e_2 = e_y$. We consider the corresponding circuits $b_x e_x \bar{c}_x$ and $b_y e_y \bar{c}_y$. It follows that $\bar{e}_0$ lies in both circuits (see Fig. 7). Moreover, since d and d' induce a symmetric CMT automorphism the circuits $b_x e_x \bar{e}_x$ and $b_y e_y \bar{c}_y$ contain only a single edge not in $E(T')$ which must be $\bar{e}_0$. But then there exists a path in $T' \cap X_1$ from $t(e_x)$ to $t(e_y)$ and a path in $T' \cap X_2$ from $s(e_y)$ to $s(e_x)$ from which it follows that T' contains a circuit contradicting the fact that T' is a tree. □

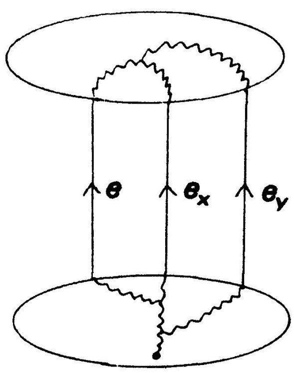

Fig. 7

Let T be a maximal tree of X. We write $E_{-\Sigma}(T)=\{e\in E(T);\ e$ is unsymmetric relative to $T\}$.

4.5. PROPOSITION. *Let $d:X\to Y$ and $d':X\to Y$ be collapsing maps, with corresponding maximal trees T and T', which induce a symmetric* CMT *automorphism. Then $E_{-\Sigma}(T)=E_{-\Sigma}(T')$.*

Proof. We use induction on $|E(T)-E(T')|$. Obviously if $|E(T)-E(T')|=0$ there is nothing to prove. So let $e_0\in E(T)-E(T')$. By Lemma 4.3, e_0 has a unique companion edge e_0' in the decomposition of X relative to T and e_0, and clearly $e_0'\in E(T')-E(T)$. If we define $\hat{T}=(T-\{e_0,\bar{e}_0\})\cup\{e_0',\bar{e}_0'\}$ and $\hat{d}$ by $\hat{d}(e_0)=d(\bar{e}_0')$, $\hat{d}(e_0')=d(\bar{e}_0)$ and $\hat{d}(e)=d(e)$, for $e\neq e_0,\ \bar{e}_0,\ e_0',\ \bar{e}_0'$, then $\hat{d}$ is a collapsing map with maximal tree $\hat{T}$. Furthermore, taking a basepoint in the subgraph X_2, which contains $s(e_0)$, of the decomposition of X relative to T and e_0, d and $\hat{d}$ induce the symmetric Whitehead automorphism given by

$$y\mapsto\begin{cases}x^{-1}yx & y=d(e),\quad e\in E(X_1)-E(T)\\ y & y=d(e),\quad e\in E(X_2)-E(T)\quad\text{or}\quad y=x\end{cases}$$

where $x=d(e_0')$. Then $\hat{d}$ and d' induce a symmetric CMT automorphism and so we are done by induction if we can show that $E_{-\Sigma}(T)=E_{-\Sigma}(\hat{T})$. Since e_0' is clearly symmetric relative to $\hat{T}$, this follows immediately from Lemma 4.3. □

4.6 LEMMA

(i) *Let T be a maximal tree in the reduced graph and let $\hat{d}:X\to\hat{X}$ be a map collapsing a single edge e_0 of T. Let $e\in E(T)$, $e\neq e_0,\ \bar{e}_0$. Then e is symmetric relative to T if and only if $\hat{d}(e)$ is symmetric relative to $\hat{T}=\hat{d}(T)$.*

(ii) *Let T be a maximal tree in the reduced graph X and $d^{\Sigma}:X\to X^{\Sigma}$ the map defined by collapsing all the edges in $E_{-\Sigma}(T)$. Then X^{Σ} is symmetric.*

Proof. (i) This is essentially obvious. Let e_0 have companion edges $e_1,\ldots,e_m$ relative to T. After contraction of an edge $e\in E(T)$ different from e_0, $\bar{e}_0$, clearly e_0 still has companion edges $e_1,e_2,\ldots,e_m$ (see for example Fig. 8).

(ii) By Lemmas 4.3 and 4.2 it suffices to show that every edge of $T^{\Sigma}=d^{\Sigma}(T)$ is symmetric relative to T^{Σ}. This follows immediately from (i) by induction on $|E_{-\Sigma}(T)|$. □

4.7. THEOREM. *K^{Σ} is contractible.*

Proof. We recall that with $W=(s_1,s_2,\ldots,s_m)$, $K_{\min(W)}$ is contractible. We

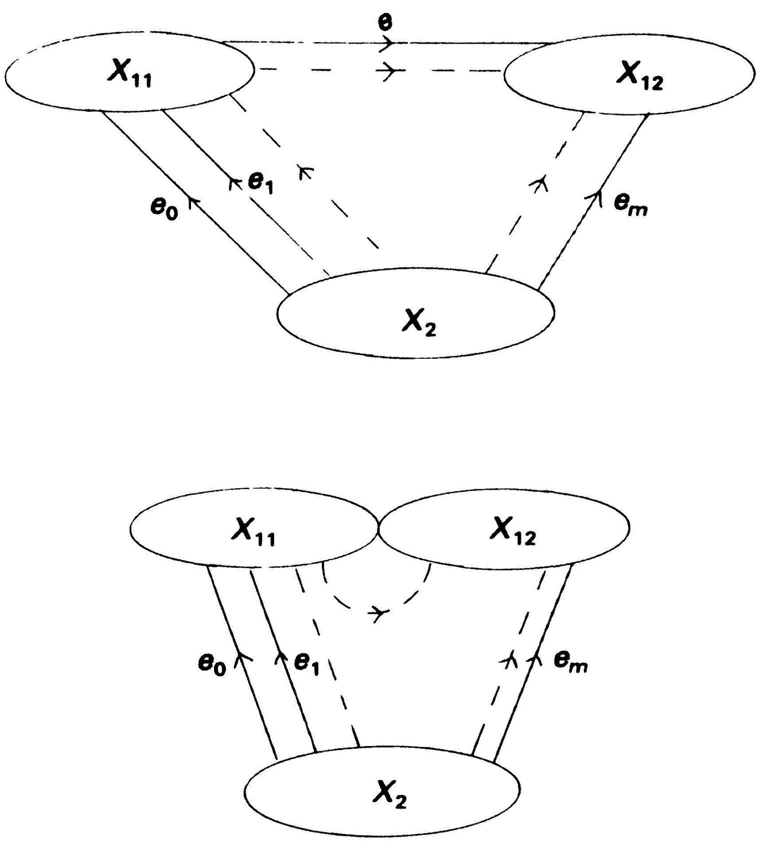

Fig. 8

define $r: K_{\min(W)} \to K^{\Sigma}$ as follows. Given $\xi = (g, X) \in K_{\min(W)}$, choose $d: X \to Y$, with maximal tree T, such that $(dg)_*$ is symmetric. Let $d^{\Sigma}: X \to X^{\Sigma}$ collapse the edges of $E_{-\Sigma}(T)$ and set

$$r(\xi) = (d^{\Sigma}g, X^{\Sigma}).$$

By Lemma 4.6, x^{Σ} is symmetric and it follows that $(d^{\Sigma}g, X^{\Sigma}) \in K^{\Sigma}$. We have to check that $r(\xi)$ is independent of the choice of g, X and d.

So let $\xi = (g', X')$ with $d': X' \to Y$ be such that $(d'g)_*$ is symmetric. Then we have an isomorphism $h: X \to X'$ such that

$$\begin{tikzcd} & X \arrow[dd, "h"] \\ Y \arrow[ur, "g"] \arrow[dr, "g'"'] & \\ & X' \end{tikzcd}$$

commutes. We claim that if T' is the maximal tree for d', then $E_{-\Sigma}(T') = h(E_{-\Sigma}(T))$ and hence that h induces an isomorphism $h^{\Sigma}: X^{\Sigma} \to X'^{\Sigma}$ such that

$$\begin{tikzcd} & X^{\Sigma} \arrow[dd, "h^{\Sigma}"] \\ Y \arrow[ur, "d^{\Sigma}g"] \arrow[dr, "d'g^{\Sigma}"'] & \\ & X'^{\Sigma} \end{tikzcd}$$

commutes. Now the CMT automorphism $(d'hd^{-1})_*$ is symmetric and so, by Proposition 4.5 applied to dh^{-1} and d', $E_{-\Sigma}(T') = E_{-\Sigma}(h(T)) = h(E_{-\Sigma}(T))$. Thus r is well-defined.

Clearly $r(\xi) \leqslant \xi$ and $r(r(\xi)) = r(\xi)$ so that, to apply Lemma 4.1, it remains only to verify that $\xi_1 \geqslant \xi_2$ implies $r(\xi_1) \geqslant r(\xi_2)$. Suppose we have

$$\begin{tikzcd} & X_1 \arrow[dd, "d_{12}"] \\ Y \arrow[ur, "g_1"] \arrow[dr, "g_2"'] & \\ & X_2 \end{tikzcd}$$

Choose a collapsing map $d_2: X_2 \to Y$ so that $(d_2g_2)_*$ is symmetric. Then if we put $d_1 = d_2d_{12}$ it follows that $(d_1g_1)_*$ is symmetric, and $r(\xi_1) = (d_1^{\Sigma}g_1, X_1^{\Sigma})$, $r(\xi_2) = (d_2^{\Sigma}g_2, X_2^{\Sigma})$. Moreover if T_2 is the maximal tree for d_2, then the maximal tree T_1 for d_1 is just the inverse image of T_2 with respect to d_{12}. We are looking for $d_{12}^{\Sigma}: X_1^{\Sigma} \to X_2^{\Sigma}$ such that

$$\begin{tikzcd} & X_1^{\Sigma} \arrow[dd, "d_{12}^{\Sigma}"] \\ Y \arrow[ur, "g_1^{\Sigma}g_1"] \arrow[dr, "d_2^{\Sigma}g_2"'] & \\ & X_2^{\Sigma} \end{tikzcd}$$

If w is a vertex of X_1^{Σ} then its inverse image under d_1^{Σ} is a subtree T_w of T_1, all of whose edges are unsymmetric. By Lemma 4.6 the edges of $d_{12}(T_w)$ are unsymmetric relative to T_2 and so $d_2^{\Sigma}(d_{12}(T_w))$ is just a vertex v of X_2^{Σ}. We define $d_{12}^{\Sigma}(w) = v$. If e is an edge of X_1^{Σ} then there is a unique edge e_1 of X_1 such that $d_1^{\Sigma}(e_1) = e$ and we can define $d_{12}^{\Sigma}(e) = d_2^{\Sigma}(d_{12}(e_1))$. It is easy to check that d_{12}^{Σ} has the required properties. This completes the proof:

Discussion: We describe here the ideas underlying the definition of K^{Σ}. Let $d:X\to Y$ be a collapsing map with maximal tree T. Given any edge e_0 of $E(T)$ let $X_1(e)$ and $X_2(e)$ be the two subgraphs of X occurring in the decomposition of X relative to T and e_0. Further define

$$A_d(e_0)=\{d(e):e\in E(X)-E(T),\ t(e)\in X_1(e_0)\}$$

and let $\mathcal{F}=\mathcal{F}(X,d)=\{A_d(e_0);\ e_0\in E(T)\}$. Then it is easy to verify that $\mathcal{F}$ is a *complete compatible* family of *ideal edges* of $E(Y)$, i.e. $\mathcal{F}$ is a family of subsets of $E(Y)$ such that:

(i) (ideal edges) for any $A\in\mathcal{F}$, (a) there exists $x\in E(Y)$ such that $x\in A$ and $\bar{x}\notin A$, and (b) $2\leq|A|\leq n-2$;

(ii) (complete) for any $A\subseteq E(Y)$, $A\in\mathcal{F}$ if and only if its complement $\bar{A}\in\mathcal{F}$;

(iii) (compatible) for any A, $B\in\mathcal{F}$, one of $A\cap B$, $A\cap\bar{B}$, $\bar{A}\cap B$ and $\bar{A}\cap\bar{B}$ is empty.

All this is to be found in [7] where an elegant converse is provided. Given a complete compatible family $\mathcal{F}$ of ideal edges there is defined a graph $X_{\mathcal{F}}$ and a collapsing map $d_{\mathcal{F}}:X_{\mathcal{F}}\to Y$ from which the original family $\mathcal{F}$ is recovered by applying the process described above. It follows (Proposition 2.2.2 of [7]) that for any rose ρ of K, $st(\rho)$ is homeomorphic to the poset complex of ideal edges of Y. In particular collapsing an edge pair $\{e,\bar{e}\}$ of the graph X corresponds to discarding the ideal edges $A_d(e)$ and $A_d(\bar{e})$.

Now given $d:X\to Y$ with maximal tree T, it is clear that $e\in E(T)$ is symmetric relative to T if and only if $A_d(e)$ is symmetric in the sense defined in 1. and then, if e' is the companion edge of e and $x=d(e')$, $(A_d(e),x)$ is a symmetric automorphism. The graph X is symmetric if and only if the family $\mathcal{F}(X,d)$ consists entirely of symmetric sets and, for arbitrary X, the map $d^{\Sigma}:X\to X^{\Sigma}$ corresponds to casting out unsymmetric sets from $\mathcal{F}(X,d)$. With this interpretation the fact that the retraction $r:K_{\min(W)}\to K^{\Sigma}$ is a poset morphism becomes transparent.

5. Conclusion

5.1. THEOREM. *Let F be free of rank n and let $\Sigma O(F)$ be the group of symmetric outer automorphisms of F. Then $vcd(\Sigma O(F))=n-2$.*

Proof. We have shown that $\Sigma O(F)$ acts on a connected contractible complex of dimension $n-2$, with finite stabilisers and quotient. Hence, see [2], $vcd(\Sigma O(F))\leq n-2$. The equality is obtained by observing that if F is free on

$\{s_1, s_2, \ldots, s_n\}$ then the automorphisms

$$\sigma_i : \begin{cases} s_i \mapsto s_n^{-1} s_i s_n \\ s_j \mapsto s_j, \qquad j \neq i, \end{cases}$$

$i = 1, 2, \ldots, n-1$ generate a free abelian subgroup of $P\Sigma A(F)$ of rank $n-1$ whose image in $\Sigma O(F)$ has rank $n-2$. □

5.2. COROLLARY. $vcd(\Sigma A(F)) = n - 1$.

Proof. $\Sigma A(F)$ is an extension of the free group Inn F by $\Sigma O(F)$ and hence

$$vcd(\Sigma A(F)) \leq 1 + vcd(\Sigma O(F)) n - 1.$$

Equality follows since $cd(\langle \sigma_1, \ldots, \sigma_{n-1} \rangle) = n - 1$. □

6. Appendix

We sketch here an elementary proof of Lemma 4.1 suggested to us by Martin Lustig. We have a poset P of finite height and a poset morphism $f: P \to P$ satisfying $f(\xi) \leq \xi$ and $f(f(\xi)) = f(\xi)$ for all $\xi \in P$. It suffices to show that f, regarded as a continuous map of the complex $K(P)$, is homotopic to the identity map.

Step 1. Let $P_0 = \{\xi \in P; f(\xi) \neq \xi$ and $\zeta < \xi$ implies $f(\zeta) = \zeta\}$. So the elements of P_0 are the "minimal" elements moved by f. Define $f_0: P \to P$ by

$$f_0(\xi) = \begin{cases} f(\xi) & \text{if } \xi \in P_0 \\ \xi & \text{if } \xi \notin P_0 \end{cases}$$

The minimality condition ensures that f_0 is a poset morphism. Moreover f_0 is homotopic to id_P.

To see this note that a simplex σ can meet P_0 in a most one point, since the elements of P_0 are incomparable. If $\xi_0 < \xi_1 < \cdots < \xi_k$ is a simplex and $\xi_i \in P_0$ then, possibly adjoining $f(\xi_i)$, we may assume $\xi_{i-1} = f(\xi_i)$ and f_0 restricted to σ is just the contraction of the 1-simplex $\xi_{i-1} < \xi_i$.

Step 2. Put $P' = P - P_0$ and define $f': P' \to P$ by $f'(\xi) = f(\xi)$. Note that $f(P') \subseteq P'$ since $P_0 \cap f(P) = \varnothing$. Then f' is a poset morphism and satisfies the same conditions as f. Furthermore if we define $\mu(P,f)$ to be the length of the longest chain $\xi_0<\xi_1<\cdots<\xi_k$ such that $f(\xi_0)\neq\xi_0$ then $\mu(P',f')<\mu(P,f)$ and so by induction f' is homotopic to $id_{P'}$. However for any $\xi\in P$, $f(\xi)=f'(f_0(\xi))$ whence it follows that if $h_0:K(P)\times I\to K(P)$ is the homotopy from id_P to f_0 and $h':K(P')\times I\to K(P')$ is the homotopy from $id_{p'}$ to f', then $h:K(P)\times I\to K(P)$ defined by

$$h(z,t)=\begin{cases} h_0(z,2t) & 0\leq t\leq\frac{1}{2} \\ h'(f_0(z),2t-1), & \frac{1}{2}\leq t\leq 1\end{cases}$$

is a homotopy from id_P to f, as required.

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