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Rigidity of certain finite group actions on the complex projective plane

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In [HL], [W] the finite groups which act locally linearly on $P^2(\mathbb{C})$, inducing the identity on homology, were found to be just the subgroups of $PGL_3(\mathbb{C})$. Since any such subgroup acts on $P^2(\mathbb{C})$ as a group of collineations this raises the question of rigidity: namely, is every action topologically conjugate to a linear action? In this paper we prove that actions satisfying certain assumptions on the singular set are rigid, and give a construction for non-linear examples, based on the existence of knotted 2-spheres in S^4 invariant under cyclic group actions [G].

We will say that a locally-linear G-action has an *isolated fixed point* if G has a fixed point x_0 where the local tangential representation is free. This is a strong assumption: by [HL; 2.5], the local representation at an isolated fixed point identifies G with a subgroup of $U(2)$ acting freely on S^3. It follows that G has a unique non-trivial central element of order two, or G is cyclic of odd order (see [HL; §1]).

There are also two distinct possibilities for the singular set of an action with an isolated fixed point. When the action has an invariant 2-sphere which represents a generator of $H_2(P^2(\mathbb{C}); \mathbb{Z})$ we say the action has *type I*, and otherwise the action has *type II*. From [HL; 2.1] we see that in a type II action, G is a cyclic group of odd order acting semi-freely on $P^2(\mathbb{C})$ with three isolated fixed points $\{p_1, p_2, p_3\}$.

The linear G-actions on $P^2(\mathbb{C})$ are weakly complex in the sense that the tangent bundle has a G-$U(2)$ reduction. For a general locally linear action it turns out that the topological tangent bundle always has a G-vector bundle reduction, and if it further has a G-$U(2)$ reduction, the action is called *weakly complex*. Our main result is the following.

THEOREM A. *Let G be a finite group with a locally linear action on $P^2(\mathbb{C})$, inducing the identity on homology. If the action has an isolated fixed point and is*

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type I or weakly complex type II, then it is topologically conjugate to a linear action.

The proof can be outlined as follows: (i) we produce an isovariant homotopy equivalence to a linear action, which is a homeomorphism near the singular set, and (ii) we prove (in §3) that any action with this property is topologically conjugate to the linear model. We need to assume that type II actions are weakly complex in step (i), to show that the local representations at the three isolated fixed points agree with those in some linear action (see §2).

For a type II action, the local tangential representations $T_{p_i}P^2(\mathbb{C})$ are well-defined [MR], [HP]. They have complex structures, say

$$T_{p_i}P^2(\mathbb{C}) = V(a_i, b_i),$$

where the generator t of G acts on $V(a_i, b_i) = \mathbb{C}^2$ by

$$t \cdot (z_1, z_2) = (\zeta^{a_i} z_i, \zeta^{b_i} z_2), \qquad \zeta = \exp(2\pi\sqrt{-1}/|G|). \tag{0.1}$$

Of course, there is no canonical choice of (a_i, b_i) since the action does not distinguish between (a_i, b_i) and $(-a_i, -b_i)$, or prefer an ordering of the fixed points. We show in §1 that a type II action is weakly complex if and only if

$$\frac{a_1+b_1}{a_1b_1} + \frac{a_2+b_2}{a_2b_2} + \frac{a_3+b_3}{a_3b_3} \equiv 0 \pmod{|G|} \tag{0.2}$$

for some choice of rotation numbers (a_i, b_i).

If the topological tangent bundle has a G-$U(2)$ reduction, then (0.2) follows from (1.4) applied to the "determinant line bundle" $\Lambda^2(T_*P^2(\mathbb{C}))$.

For a linear type II action, the rotation numbers have the form

$$(a_1, b_1) = (a, b),\ (a_2, b_2) = (-a, b-a),\ (a_3, b_3) = (-b, a-b) \tag{0.3}$$

with a, b, and $a-b$ units in $\mathbb{Z}/|G|$. In this case, (0.2) follows by inspection.

Recently, A. Edmonds and J. Ewing [EE] have shown that the local representations in a type II action *always* agree with those in some linear action, using a more extensive analysis of the Atiyah–Singer formula. By combining their work with ours, we conclude that every type II action is weakly complex, and hence that Theorem A holds without this condition. It would be interesting to see a direct proof of (0.2), and thus provide a relatively elementary argument for the general result.

In the smooth category our methods do not prove rigidity, since smooth surgery does not work in dimension 4. In addition there is the intriguing possibility of the existence of a non-trivial smooth (inertial) 4-dimensional s-cobordism between spherical space forms. In fact, if there is such an example with the universal cover a product, then rigidity fails. In the topological category, the existence of non-trivial s-cobordisms does not prevent rigidity because the usual "infinite repetition" argument (see §3) overcomes this difficulty.

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Section 1: Equivariant Line Bundles

We begin with the classification of equivariant line bundles over a G-space X (the following convenient formulation is given in [LMS]):

$$[X, BU(1)]^G \cong [X \times_G EG, BU(1)]. \tag{1.1}$$

The right-hand side is just the Borel cohomology group $H^2_G(X; \mathbb{Z})$ and the left-hand side is the set of isomorphism classes of G-$U(1)$ bundles over X.

For type I actions on $X = P^2(\mathbb{C})$, there exists an invariant S^2. First we give a result essentially due to Freedman [F], concerning the existence of equivariant tubular neighbourhoods.

LEMMA 1.2. *Let G act locally-linearly on a closed oriented 4-manifold M. If the action is orientation-preserving and Σ is an invariant locally-flat surface in M, then there exist a closed equivariant neighborhood $(N, \partial N)$ for Σ which is G-homeomorphic as a pair to $(D(\nu), S(\nu))$ for some G-$U(1)$ vector bundle ν over (Σ^2, G).*

Proof. Let $\bar{G}$ be the quotient of G which acts effectively on the invariant surface. The singular set of the $\bar{G}$-action on Σ is a finite set S of points. After deleting a set $B(S)$ of balls around these points on which G acts linearly, the complement has a free G-action in a neighbourhood of Σ-$B(S)$. The result of Freedman [F, Thm. 10] applied to the orbit space gives a closed vector bundle neighbourhood for $(\Sigma\text{-}B(S))/G$. The covering space is then a G-vector bundle neighbourhood for this part of Σ. Inside $B(S)$ we can also find a G-tubular neighbourhood since the action is linear there. These two pieces fit together in $\partial B(S)$, by uniqueness of (smooth) G-tubular neighbourhoods in a 3-manifold. To justify the use of smoothness at the last stage, we need to know that any locally-linear action of a finite group on a closed 3-manifold is smoothable. But, since our action is orientation-preserving, the orbit space is also a topological 3-manifold and hence smoothable.

To understand the equivariant line bundles over S^2, we recall that any topological action of a finite group on S^2 is conjugate to a linear action [K, p. 229]. The relevant linear models are the Hopf bundles: let V be a complex 2-dimensional G-representation (where $G \subseteq SO(4)$, then

$$H = (\mathbb{C} \times_{S^1} S(V) \to (V - \{0\})/\mathbb{C}^* = S^2)$$

is a G-equivariant Hopf bundle.

LEMMA 1.3. *Let $v \searrow S^2$ be a G-$U(1)$ bundle over S^2 with Euler class ± 1. Then v is G-isomorphic to an equivariant Hopf bundle.*

Proof. From (1.1) the bundle v is classified by an element of

$$H^2_G(S^2; \mathbb{Z}) = H^2(G; \mathbb{Z}) \oplus \mathbb{Z}.$$

The group $H^2(G; \mathbb{Z})$ may be identified with the group of 1-dimensional (complex) representations L of G, or flat bundles over $(S^2, \bar{G})$. This group acts transitively on the G-$U(1)$ bundles with fixed Euler class ($v \to v \otimes L$) and hence the linear models represent all elements of $H^2_G(S^2; \mathbb{Z})$.

We assume now (and for the rest of the section) that our action on $X = P^2(\mathbb{C})$ is of type II. Let $G = C_n$ be a cyclic group of odd order n with a generator t. Fix an identification of the tangent planes $T_{p_i}X = V(a_i, b_i)$ as in (0.1). If L is a G-equivariant line bundle over X, we denote the isotropy representations at the fixed point p_i by $\{t^{\lambda_i}\}$ for $i = 1,2$ or 3. In the following result and its proof, we will establish certain relations among the $\{\lambda_i\}$ and the (a_i, b_i) expressed as congruences modulo n.

PROPOSITION 1.4. *There exists a G-equivariant line bundle L over X with $L|_{p_i} \cong t^{\lambda_i}$ if and only if*

$$\sum_i \frac{\lambda_i}{a_i b_i} \equiv 0 \pmod{n}.$$

Proof. Let B_i denote small open G-disks around the fixed points p_i, and let

$$Y=(X-B_1\cup B_2\cup B_3)/G. \tag{1.5}$$

Then Y is a compact 4-manifold with three boundary components Y_1, Y_2, Y_3 which are the lens spaces $L^3(V(a_i, b_i))$. The problem of finding the equivariant line bundle L over X is equivalent to showing that

$$L_i=S(V(a_i, b_i))\times_G t^{\lambda_i}$$

extends over Y.

Since S^1-bundles over Y are classified by an element in $H^2(Y;\mathbb{Z})$, we must study the image of the restriction map

$$H^2(Y;\mathbb{Z})\to\sum H^2(Y_i;\mathbb{Z}).$$

However, both $H^3(Y, \partial Y;\mathbb{Z})$ and $H^2(Y_i;\mathbb{Z})$ are isomorphic to $\mathbb{Z}/n$, and the natural map $H^2(Y_i;\mathbb{Z})\to H^3(Y, \partial Y;\mathbb{Z})$ between them can be identified (with the help of the linking pairing) with

$$\operatorname{Hom}(H_1(Y_i;\mathbb{Z}), \mathbb{Q}/\mathbb{Z})\to H_1(Y;\mathbb{Z})\cong\mathbb{Z}/n$$

$$\lambda_i\mapsto\frac{\lambda_i}{a_ib_i}. \tag{1.6}$$

Remark. The line bundle L is not specified uniquely in (1.4) since $H^2(Y;\mathbb{Z})\cong\mathbb{Z}\oplus\mathbb{Z}/n$. By considering intersection numbers in Y, we can see that $k^2=(c_1(L))^2$ is given by

$$k^2\equiv\sum\frac{\lambda_i^2}{a_ib_i}\ (\operatorname{mod} n). \tag{1.7}$$

This relation is derived in Section 2 by another method.

PROPOSITION 1.8. *For a type II action, the topological tangent bundle has a G-$U(2)$ reduction with $T_{p_i}X=V(a_i, b_i)$ if and only if* (0.2) *is satisfied.*

Proof. We first notice that the topological tangent bundle of X has a reduction to a G-vector bundle. Indeed, this follows from equivariant obstruction theory because $\pi_i(Top(4)/O(4))$ is 2-torsion for $i\le 3$ and $|G|$ is odd.

Given such a reduction, with $T_{p_i}X = V(a_i, b_i)$ we want to get a G-$U(2)$ reduction of TX. With the notation of (1.5), we want to lift $TX \mid \partial\tilde{Y}$ to a G-$U(2)$ bundle on $\tilde{Y}$. Since G acts freely on $\tilde{Y}$ (with orbit space Y), the obstructions lie in

$$H^i(Y, \partial Y; \pi_{i-1}(SO(4)/U(2)).$$

The only non-trivial group occurs for $i = 3$, where $\pi_2(SO(4)/U(2)) = \mathbb{Z}$. The mapping

$$SO(4)/U(2) \to BU(2) \xrightarrow{\text{Bdet}} BU(1)$$

is multiplication by 2 on π_2. Thus the obstruction to a G-$U(2)$ structure is the same as the obstruction to finding a line bundle L over X with

$$L \mid_{p_i} \cong \Lambda^2_{\mathbb{C}} V(a_i, b_i), \qquad i = 1, 2, 3.$$

This was analysed in (1.4).

Section 2: Type II Actions of Cyclic Groups of Odd Order

In this section we prove that the local representations for weakly complex type II actions on $P^2(\mathbb{C})$ agree with those in some linear action.

THEOREM 2.1. *Let G be a cyclic group of odd order n acting locally linearly and semi-freely on $P^2(\mathbb{C})$ with three isolated fixed points. If G is weakly complex and induces the identity on homology, then the local representations at the fixed points agree with those in a linear G-action.*

Before beginning the proof, we derive some more information about the rotation numbers (a_i, b_i) at the three isolated fixed points. We will see below that our method has difficulty with "small primes". However it is easy to verify Theorem 2.1 directly for $n = 3^r$ $(r \leq 3)$, 5, or 7, and from now on we will assume that these cases are known.

The linear G-actions on $P^2(\mathbb{C})$ are given by the formula

$$t \cdot (z_1, z_2, z_3) = (z_1, t^a z_2, t^b z_3) \tag{2.2}$$

in terms of homogeneous coordinates. This is a semi-free action with three isolated fixed points provided that $a \neq b$ are units $(\bmod n)$. If $p_1 = (1, 0, 0)$, $p_2 = (0, 1, 0)$ and $p_3 = (0, 0, 1)$, then the local rotation numbers (a_i, b_i) are

$$(a, b), (-a, b-a), \text{ and } (-b, a-b). \tag{2.3}$$

or

$$(a, b), (b-a, -a) \quad \text{and} \quad (a-b, -b). \tag{2.4}$$

In a possibly non-linear action (X, G) on $P^2(\mathbb{C})$, we have the relations arising from the G-signature Theorem [AS; Thm. 6.18]

$$\left(\frac{t^{a_1}+1}{t^{a_1}-1}\right)\left(\frac{t^{b_1}+1}{t^{b_1}-1}\right) + \left(\frac{t^{a_2}+1}{t^{a_2}-1}\right)\left(\frac{t^{b_2}+1}{t^{b_2}-1}\right) + \left(\frac{t^{a_3}+1}{t^{a_3}-1}\right)\left(\frac{t^{b_3}+1}{t^{b_3}-1}\right) = 1. \tag{2.5}$$

Note that we get an equation in the ring $R = \mathbb{Z}[t]/(1 + t + \cdots + t^{n-1})$ after multiplying both sides of (2.5) by $(t-1)^2$. Let I denote the ideal generated by $(t-1)$ in R. In order to compute the low order terms (≤ 6) in the I-adic expansion of the resulting left-hand side, we lift this equation to $\mathbb{Z}[t]$, expand in powers of $(t-1)$ and equate coefficients. From these relations we will obtain congruences modulo n involving the rotation numbers. Note that the indeterminacy in this procedure arises from the coefficients in the expansion of

$$g(t) = (1 + t + \cdots + t^{n-1}).$$

It is sufficient to do the cases $n = p^r$, for some prime p with $n \neq 3^r$ $(r \leq 3)$, 5, or 7 (i.e. $r(p-1) > 6$). The expression for $g(t)$ as a product of cyclotomic polynomials shows that

$$g(t) = h(t) \cdot (t-1)^{r(p-1)} + \cdots + p^r \cdot k(t)$$

for some $h(t), k(t) \in \mathbb{Z}[t]$. But in R, the prime p has I-adic valuation $(p-1)$, so the indeterminacy affects the terms of degree $\geq r(p-1)$. Since $r(p-1) > 6$, by considering the terms of degree up to six and reducing modulo n, one derives four congruences:

$$\sum \frac{1}{a_i b_i} = 0 \tag{2.6}$$

$$\sum \frac{a_i^2 + b_i^2}{a_i b_i} = 3 \tag{2.7}$$

$$\sum \frac{a_i^4 + b_i^4 - 5a_i^2 b_i^2}{a_i b_i} = 0 \tag{2.8}$$

$$\sum \frac{2a_i^6 - 7a_i^4 b_i^2 - 7a_i^2 b_i^4 + 2b_i^6}{a_i b_i} = 0. \tag{2.9}$$

Since the action is assumed to be semi-free with three isolated fixed points, then $a_i \neq b_i$ are units mod n (for $i = 1$, 2 or 3).

If L is a G-invariant line bundle over X, then the topological Index homomorphism applied to $[L] \in K_G(X)$ gives a formula for a certain character χ of G:

$$\begin{aligned} \chi(t) &= \sum_i \mathrm{ch}(L \mid p_i) e(v_{p_i})^{-1}[p_i] \\ \chi(1) &= ch(L)\mathscr{L}(X)[X]. \end{aligned} \tag{2.10}$$

By substituting the characters $\{t^{\lambda_i}\}$ for $L|_{p_i}$, and our local representations in (2.10), we get

$$\chi(t) = \sum t^{\lambda_i} \frac{(1+t^{a_i})}{(1-t^{a_i})} \frac{(1+t^{b_i})}{(1-t^{b_i})}.$$

Let $c_1(L)^2 = k^2$ and $x \in H^2(X; \mathbb{Z})$ denote a generator, then

$$\chi(1) = (1 + kx + k^2x^2/2)(4 + x^2)[X] = 2k^2 + 1.$$

If we substitute these expressions in (2.10) and expand, noting that $\chi(1) \equiv \chi(t) (\mathrm{mod}\, n)$, we get

$$\sum \frac{4}{a_i b_i} \left\{ \frac{1}{(t-1)^2} + \frac{(\lambda_i + 1)}{(t-1)} + \left(\frac{a_i^2 + b_i^2 + 1}{12} + \frac{\lambda_i(\lambda_i + 1)}{2} \right) + \cdots \right\} = 2k^2 + 1.$$

By comparing terms on both sides, we get (1.4) and (1.7).

When the action is weakly complex, we can supplement (2.6)–(2.9) with

$$\sum \frac{a_i + b_i}{a_i b_i} = 0 (\mathrm{mod}\, n). \tag{2.11}$$

From (2.6), (2.7) and (2.11) we get the equations

$$\begin{aligned} (a_1 + b_1 - a_2 - b_2)^2 &= 9(a_1 b_1 + a_2 b_2) \\ (a_1 + b_1 - a_3 - b_3)^2 &= 9(a_1 b_1 + a_3 b_3) \\ (a_2 + b_2 - a_3 - b_3)^2 &= 9(a_2 b_2 + a_3 b_3). \end{aligned} \tag{2.12}$$

Indeed, from (2.6)

$$a_3 b_3 = -\frac{a_1 b_1 a_2 b_2}{a_1 b_1 + a_2 b_2} \tag{2.13}$$

and then (2.11) gives

$$a_3 + b_3 = \frac{a_1 b_2 a_2 b_2}{a_1 b_1 + a_2 b_2} \left(\frac{a_1 + b_1}{a_1 b_1} + \frac{a_2 + b_2}{a_2 b_2} \right). \tag{2.14}$$

Substituting this in (2.7) leads to the equation

$$[(a_1 + b_1 - a_2 - b_2)^2 - 9(a_1 b_1 + a_2 b_2)] a_1 a_2 b_1 b_2 = 0,$$

and hence the first equation in (2.12). The others are similar. We remark that (2.6) implies that the quantities in brackets on the right-hand side of (2.12) are units (mod n).

Proof of (2.1) *for n an odd prime*: Let $n = |G|$ be an odd prime, and consider $G = C_n$ acting on $P^2(\mathbb{C})$. We will show that if $(a_1, b_1) = (a, b)$ are fixed, then the only rotation numbers satisfying the relations above are those from the linear models (2.3) or (2.4). The cases when $n = 3$, 5, or 7 are easy, so we leave them to the reader and suppose that $n > 7$.

The first step is to eliminate a_3, b_3 from (2.8) by using the relation

$$\frac{a_3^4 + b_3^4 - 5a_3^2 b_3^2}{a_3 b_3} = \frac{[(a_3 + b_3)^2 - 2a_3 b_3]^2}{a_3 b_3} - 7a_3 b_3,$$

and the above expressions for a_3b_3 and (a_3+b_3). This gives the equation

$$\frac{[(a_1+b_1)^2-2a_1b_1]^2}{a_1b_1}-7a_1b_1+\frac{[(a_2+b_2)^2-2a_2b_2]^2}{a_2b_2}$$
$$-7a_2b_2-\frac{\Delta^2}{(a_1b_1+a_2b_2)a_1b_1a_2b_2}+\frac{7a_1b_1a_2b_2}{(a_1b_1+a_2b_2)}=0$$

where

$$\Delta=[-(a_1+b_1)^2+9a_1b_1]a_1b_1+[(a_1+b_1)^2+2a_1b_1]a_2b_2$$
$$+2(a_1+b_1)(a_2+b_2)a_1b_1.$$

Finally we use the first equation in (2.12) to express a_2b_2 in terms of the sum a_2+b_2 (and a_1b_1). Putting

$$z=[a_2+b_2-a-b]/3,$$

we get (after some simplification)

$$z^6+2(a+b)z^5+(a^2+3ab+b^2)z^4-ab(a^2+3ab+b^2)z^2$$
$$-2ab(a+b)z-a^3b^3=0. \quad (2.15)$$

This equation factors as $(z^2-ab)(z+a)^2(z+b)^2=0$. Since $z^2-ab=a_2b_2$ by (2.12), and this is a unit, we get

$$(z+a)^2(z+b)^2=0. \quad (2.16)$$

Similarly, if $w=[a_3+b_3-a-b]3$,

$$(w+a)^2(w+b)^2=0. \quad (2.17)$$

These solutions lead immediately to the linear models, and this completes the proof for $|G|$ an odd prime.

Proof of (2.1), *general case.* Let $n=|G|$ be an odd integer. Notice first that $a_2+b_2=a_3+b_3(\text{mod } n)$ implies that $n=3$ or 9. Indeed the equations (2.12) show that $9a_2b_2=9a_3b_3$ and $9a_2b_2=-9a_3b_3$, hence $18a_2b_2=0$. Since n is odd and a_2b_2 is a unit this gives $9\equiv 0(\text{mod } n)$. The cases $n=3$, $n=9$ have already been dealt with, so assume below that $a_2+b_2\not\equiv a_3+b_3(\text{mod } n)$.

Write $(a, b) = (a_1, b_1)$ and suppose p is a prime divisor of n. The equations (2.16) and (2.17) give

$$(z + a) \equiv 0 (\mathrm{mod}\, p), \qquad (w + b) \equiv 0 (\mathrm{mod}\, p)$$

after possibly permuting a and b, so

$$a_2 + b_2 \equiv b - 2a, \ a_3 + b_3 \equiv a - 2b \quad (\mathrm{mod}\, p).$$

On the other hand, (2.12) gives

$$a_2 b_2 \equiv a(a - b), \ a_3 b_3 \equiv b(b - a) \qquad (\mathrm{mod}\, p).$$

Hence we can assume

$$(a_2, b_2) = (-a, b - a), \ (a_3, b_3) = (-b, a - b) \qquad (\mathrm{mod}\, p)$$

LEMMA 2.19. *Suppose* $n = p^r m$ *with* $(p, m) = 1$ *and* p *prime. If* (2.18) *is satisfied* $\mathrm{mod}\, p$ *then it is satisfied* $(\mathrm{mod}\, p^r)$.

Proof. Suppose (2.18) satisfied $(\mathrm{mod}\, p^t)$, $1 \le t < r$. Hence we can write $(\mathrm{mod}\, p^t)$

$$a_2 = -a + p^t x_2, \qquad b_2 = (b - a) + p^t y_2$$

$$a_3 = -b + p^t x_3, \qquad b_3 = (a - b) + p^t y_3.$$

When $p \neq 3$, substituting this back into (2.12) leads to the equations

$$y_2 = (3b/a - 1)x_2, \ x_3 = -(b^3/a^3)x_2, \ y_3 = -(3a/b - 1)(b^3/a^3)x_2 \qquad (\mathrm{mod}\, p).$$

For $p = 3$ we substitute into (2.6), (2.8) and (2.11) instead to obtain the same conclusion.

These relations can be substituted into (2.9), where the coefficient of x_2 becomes:

$$\frac{14(a - 2b)(a - b)(a + b)(2a - b)b}{a^2}.$$

This is non-zero $\mathrm{mod}\, p$ and hence $x_2 \equiv 0$ $(\mathrm{mod}\, p)$, unless $p = 7$ or we are in one of the degenerate cases $a = 2b$, $a = -b$, or $2a = b$. The latter are treated by a different method below. For $p=7$, we must substitute into the degree eight congruence derived from (2.5), and proceed in a similar way. Further details will be omitted.

The above lemma finishes the proof of Theorem 2.1 when G has prime power order and reduces the case of composite order to the square-free case. Write $n=pq$ with $(p, q)=1$, p prime and q square-free. Inductively we may assume

$$\begin{aligned} a_2 &= -a+qx_2, \quad b_2=(b-a)+qy_2 \\ a_3 &= -b+qx_3, \quad b_3=(a-b)+qy_3, \end{aligned} \tag{2.20}$$

and that the unordered pairs satisfy

$$\{a_2, b_2\} \equiv \{-a, b-a\}, \ \{a_3, b_3\} \equiv \{-b, a-b\} \qquad (\operatorname{mod} p),$$

or

$$\{a_2, b_2\} \equiv \{-b, a-b\}, \ \{a_3, b_3\} \equiv \{-a, b-a\} \qquad (\operatorname{mod} p).$$

This gives eight choices for the ordered pairs. We must rule out seven of them, namely the one corresponding to the cases (2.20) $(\operatorname{mod} p)$ where: $[x_2, y_2, x_3, y_3] \equiv$

$$\begin{aligned}
&1.\ [b/q, -b/q, 0, 0] \\
&2.\ [0, 0, a/q, -a/q] \\
&3.\ [b/q, -b/q, a/q, -a/q] \\
&4.\ [(a-b)/q, 2(a-b)/q, (b-a)/q, 2(b-a)/q] \\
&5.\ [(2a-b)/q, (a-2b)/q, (b-a)/q, 2(b-a)/q] \\
&6.\ [(a-b)/q, 2(a-b)/q, (2b-a)/q, (b-2a)/q] \\
&7.\ [(2a-b)/q, (a-2b)/q, (2b-a)/3, (b-2a)/q].
\end{aligned} \tag{2.21}$$

We first derive an equation in $\mathbb{F}_p \otimes \mathbb{Z}[\zeta_q]$ by substituting (2.18) back in the signature relation (2.5) and evaluating t at a generator $g \in G$. Let $t(g)=\zeta$ be a primitive pq'th root of 1. In $\mathbb{Z}[\zeta]$, $1-\zeta^q$ generates a p-adic ideal I and

$$\mathbb{Z}[\zeta]/I \cong \mathbb{F}_p \otimes \mathbb{Z}[\zeta_q], \qquad \zeta_q = \zeta^p$$

Notice that $\mathbb{F}_p \otimes \mathbb{Z}[\zeta_q]$ decomposes into $|(\mathbb{Z}/q)^{\times} : \langle p \rangle|$ factors, each isomorphic to the field $\mathbb{F}_p[\zeta_q]$. Write

$$1 \pm \zeta^{a_2} = (1+\zeta^{-a}) \pm \zeta^{-a}(\zeta^a - 1)\, \Delta(x_2)$$

with $\Delta(x_2) = ((\zeta^{qx_2} - 1)/(\zeta^q - 1))$. Then

$$(1-\zeta^{a_2})^{-1} \equiv (1-\zeta^{-a})^{-1}\left\{1 + \frac{\zeta^{-a}}{1-\zeta^{-a}}(\zeta^q - 1)\Delta(X_2)\right\} \quad (\mathrm{mod}\, I^2)$$

and

$$\frac{1+\zeta^{a_2}}{1-\zeta^{a_2}} = \frac{1+\zeta^{-a}}{1-\zeta^{-a}}\left\{1 + \frac{2\Delta(x_2)}{\zeta^a - \zeta^{-a}}(\zeta^q - 1)\right\} \quad (\mathrm{mod}\, I^2).$$

There is a similar calculation for $(1+\zeta^{b_2})/(1-\zeta^{b_2})$, and

$$\frac{1+\zeta^{a_2}}{1-\zeta^{a_2}} \cdot \frac{1+\zeta^{b_2}}{1-\zeta^{b_2}} \equiv \frac{1+\zeta^{-a}}{1-\zeta^{-a}} \cdot \frac{1+\zeta^{b-a}}{1-\zeta^{b-a}} \cdot \left\{1 + \left[\frac{2\Delta(x_2)}{\zeta^a - \zeta^{-a}} + \frac{2\Delta(y_2)}{\zeta^{a-b} - \zeta^{b-a}}\right](\zeta^q - 1)\right\} \quad (\mathrm{mod}\, I^2).$$

Substituting this expression and the similar one for (a_3, b_3) back in (2.5) and using that

$$\Delta(x) = x \quad (\mathrm{mod}\, I),$$

we get after clearing denominators the equation

$$(x_2 - 2y_2)\delta(a) + (x_3 - 2y_3)\delta(b) + (-2x_2 + 2x_3 + y_2 - y_3)\delta(a-b) + (y_2 + y_3)\delta(a+b) + x_2\delta(a-2b) + x_3\delta(b-2a) = 0 \quad (2.22)$$

where $\delta(u) = \zeta_q^u - \zeta_q^{-u}$, for $u \in (\mathbb{Z}/q)^\times$.

The Galois group $(\mathbb{Z}/q)^\times$ of the extension $\mathbb{Q}(\zeta)/\mathbb{Q}$ acts on the $\delta(u)$.

LEMMA 2.23. *Suppose the set* $\{u_1, \ldots, u_r\} \subseteq (\mathbb{Z}/q)^\times$ *injects into* $(\mathbb{Z}/q)^\times/\langle \pm 1\rangle$. *Then the units* $\zeta^{u_1} - \zeta^{-u_1}, \ldots, \zeta^{u_r} - \zeta^{-u_r}$ *are linearly independent in* $\mathbb{F}_p \otimes \mathbb{Z}[\zeta_q]$.

Proof. This follows easily from the fact that the elements $\{\zeta^u : u \in (\mathbb{Z}/q)^\times\}$ form a normal basis for the extension when q is square-free.

It follows from (2.23) that the elements $\delta(a)$, $\delta(b)$, $\delta(a-b)$, $\delta(a+b)$, $\delta(a-2b)$ and $\delta(b-2a)$ are linearly independent if there are six distinct elements in the set $\{1, b/a, 1-b/a, 1+b/a, 1-2b/a, -2+b/a\} \subseteq (\mathbb{Z}/q)^\times/\langle \pm 1\rangle$. This happens except in the special cases.

$$b/a=-1,\ b/a=\pm 2,\ b/a=3,\ b/a=1/3,\ b/a=1/2,\ b/a=3/2 \tag{2.24}$$

If we are not in the cases (2.24) we clearly get $x_2=x_3=y_2=y_3=0$ in $\mathbb{F}_p \otimes \mathbb{Z}[\zeta_q]$. For each of the special cases (2.24) one checks through the 7 cases of (2.21) separately, and derives a contradiction in each case. For example, if $a=-b$ then (2.22) reduces (upon replacing ζ_q^a by ζ_q) to

$$(x_2-x_3+2y_3-2y_2)\delta(1)+(-2x_2+2x_3+y_2-y_3)\delta(2)+(x_2-x_3)\delta(3)=0$$

so $x_2 \equiv x_3$ and $y_3 \equiv y_2 \pmod p$. Checking through (2.21) we see that this cannot be satisfied when $a=-b$. The other special cases are similar.

We also have some degenerate cases left over from the proof of (2.19). These too are handled by direct substitution of our relations for a_2, b_2, a_3 and b_3 $(\bmod\, p^t)$ into (2.5). Recall that in the argument of (2.19) we reduced the deviation of the (a_i, b_i) from a linear model to a single unknown x_2. It remains to show that $x_2 \equiv 0 \pmod p$. We write $n=pq$, with $q=p^{r-1}m$ divisible by p and expand (2.5) in powers of $I=(1-\zeta^q)$. This time the coefficients in the expansion lie in

$$\mathbb{Z}[\zeta]/I \cong \mathbb{F}_p(\zeta_m)[C_p].$$

After a straightforward calculation, we find that in each of the three cases $a=-b$, $a=2b$, and $2a=b$, the coefficient of I^2 is non-zero unless $x_2 \equiv 0 \pmod p$. Further details will be left to the reader. This ends the proof of (2.1).

Section 3: The Proof of Theorem A

From the results in Sections 1 and 2, we wish first to conclude that the actions are G-homotopy equivalent to some linear action. This leads via surgery theory to the proof of Theorem A.

LEMMA 3.1. *Suppose that a cyclic group G has a type I action on $P^2(\mathbb{C})$. Then the local representations of G at fixed points agree with those in a (complex) linear G-action.*

Proof. By (1.2) the invariant 2-sphere has a G-normal bundle. Suppose that the G action is non-trivial on the 2-sphere, so has two fixed points say p_2 and p_3. Then by (1.3), the rotation numbers at p_2 and p_3 are of the form (a_2, b_2) and $(a_2-b_2, -b_2)$. If we substitute these values into (2.5), the result follows easily.

PROPOSITION 3.2. *Let G be a finite group acting locally linearly on $P^2(\mathbb{C})$ inducing the identity on homology. Suppose that the action has an isolated fixed point and a disjoint invariant 2-sphere representing a generator of $H_2(P^2(\mathbb{C}); \mathbb{Z})$. Then there exists a G-homotopy equivalence to a linear action which is a homeomorphism on a closed equivariant neighbourhood of the singular set.*

Proof. Let $\rho: G \to SO(4)$ denote the local representative at the isolated fixed point. From [HL; 2.5] we may assume that the image of ρ lies in $U(2)$, so defines a complex representation V and the identification $P^2(V \oplus 1) \approx P^2(\mathbb{C})$ gives a linear action. We compare the given action to this linear model. Since the fixed point is isolated, the local representation is free and G must be either cyclic or contain a unique central element t of order 2. In either case, the linear model contains an invariant 2-sphere disjoint from the fixed point (this is just Fix t in the second case). Now (3.1) implies that a suitable closed neighborhood $(N, \partial N)$ of the invariant 2-sphere in the given action is G-homeomorphic to that in the linear model. In particular, ∂N is G-homeomorphic to $S(V)$. Since the invariant sphere represents a generator of $H_2(P^2(\mathbb{C}); \mathbb{Z})$ it follows that

$$W = (P^2(\mathbb{C}) - N \cup D(V))$$

is a free G-h-cobordism from ∂N to $S(V)$.

If our action satisfies the assumptions of (3.2), then the proof of Theorem A can be completed by the following argument. Let

$$f:(W, \partial W, \partial_+ W) \to (S(V)/G \times I, S(N)/G \times 0, S(V)/G \times 1)$$

be a homotopy equivalence. Then $f \mid \partial_+ W$ is a homotopy automorphism of the space form $S(V)/G$. By [R1], [R2] $f \mid \partial_+ W$ is homotopic to a homeomorphism. After a change of f we may therefore assume that $f \mid \partial_+ W$ is a homeomorphism such that f represents an element of the topological structure set,

$$[f] \in \mathcal{S}^h(S(V)/G \times I, \partial).$$

By results of Freedmann, the surgery exact sequence works in dimension 4, so

$$L_1^h(\mathbb{Z}G) \to \mathcal{S}^h(S(V)/G \times I, \partial) \to [S(V)/G \times I/\partial, F/TOP] \to L_0^h(\mathbb{Z}G)$$

is exact. The left-hand group is a finite 2-group [Wa2] and the normal invariant group is $Z \oplus H^2(G; \mathbb{Z}/2)$. The $\mathbb{Z}$ maps injectively forward to $L_0^h(\mathbb{Z}G)$, so we conclude that $\mathcal{S}^h(S(V)/G \times I, \partial)$ is finite 2-group, say of exponent 2^r.

COROLLARY 3.3. *In the situation above there is a homeomorphism*

$$F:(W\cup_{\partial W_+}\times[0,\infty),\partial_-W)\to(S(V)/G\times[0,\infty),S(V)/G\times 0)$$

which restricts to f on ∂_-W.

Proof. Replace the structure f by $2^{r+1}\cdot f$, f stacked on top of itself 2^{r+1} times. This map is homotopic to a homeomorphism (rel ∂) by the s-cobordism theorem [F], since 2^r was the exponent of the h-structure set, and one further doubling will eliminate the Whitehead torsion. Now use infinite repetition:

$$W\cup_{\partial W_+}\times[0,\infty)\cong 2^{r+1}\cdot W\cup 2^{r+1}\cdot W\cup\cdots.$$

It remains therefore to consider the case where G is cyclic and acts semi-freely with 3 isolated fixed points.

PROPOSITION 3.4. *Let G be a cyclic group acting as above on $P^2(\mathbb{C})$ semi-freely with 3 isolated fixed points. Suppose that the local representations at the fixed points agree with those in a linear G-action. Then there exists an isovariant G-homotopy equivalence to the linear action which is a local homeomorphism near the fixed point set.*

Proof. Write M for the G-manifold $(P^2(\mathbb{C}),G)$ and $p_i\in M$ for the 3 fixed points. We saw in (2.1) that there exists a linear model $P^2(V\oplus 1)$, the projective space of the representation $V=t^{\lambda_2-\lambda_1}\oplus t^{\lambda_3-\lambda_1}\oplus 1$. Let $q_i\in P^2(V\oplus 1)^G$ be the fixed point with $z_j=0$ for $j\neq i$, and the notation is arranged so that $T_{q_i}P^2(V\oplus 1)=T_{p_i}M$ as G-representations for $i=1,2,3$. The canonical line bundle $L(V)$ over $P^2(V\oplus 1)$ is G-equivariant and has fiber $L(V)_{q_i}\cong t^{\lambda_i}$ at q_i for $i=1,2,3$. By (1.4) there exists a complex G-bundle H over M with $H|_{p_i}=t^{\lambda_i}$, $i=1,2,3$.

$$T_{p_1}M=t^{\lambda_1-\lambda_2}\oplus t^{\lambda_1-\lambda_3},\qquad T_{p_2}M=t^{\lambda_2-\lambda_1}\oplus t^{\lambda_2-\lambda_3}$$
$$T_{p_3}M=t^{\lambda_3-\lambda_1}\oplus t^{\lambda_3-\lambda_2}. \tag{3.5}$$

The Chern class $(c_1(H))^2\equiv 1 \pmod n$ by (1.7), since this is true in the linear model. It is now easy to adjust H so that $(c_1(H))^2=1$.

Let U_i be a neighborhood of p_i in M, G-homeomorphic to $T_{p_i}M$. Then we can find an embedding

$$i:U=\coprod U_i\to P^2(V\oplus 1)$$

such that $L(V) \mid U \cong H \mid U$. The classifying G-map

$$f: P^2(V \oplus 1) \to BU(1)$$

for $L(V)$ maps q_i into the component of $BU(1)G$ which corresponds to t^{λ_i}. Hence we see that

$$U \xrightarrow{i} P^2(V \oplus 1) \xrightarrow{f} BU(1)$$

classifies $H \mid U$. The obstructions to extending this map to a G-map from M to $P^2(V \oplus 1)$ lie in the Bredon cohomology group

$$H^k_G(M, U; \pi_k(f))$$

where $\pi_k(f)$ is the coefficient system $\pi_k(f)(G/H) = \pi_k(f^H)$. By excision,

$$H^k_G(M, U; \pi_k(f)) = H^k_G(M, M^G; \pi_k(f))$$
$$= H^k(M/G, M^G; \pi_k(f)).$$

These groups vanish because $\pi_k(f) = 0$ for $k \leq 5$ and $(M/G, M^G)$ has relative CW-dimension 4. Then make $f: M \to P^2(V \oplus 1)$ transverse to $\{q_1, q_2, q_3\}$ without changing it in a neighbourhood of these points. This completes the proof of (3.4).

Now we finish the *Proof of Theorem A* for the G-actions of (3.4). Again we let M denote the given G-space and $P^2(V \oplus 1)$ the linear model (which exists by (2.1)). Since the G-map

$$f: M \to P^2(V \oplus 1)$$

given by (3.4) is a homeomorphism near the fixed set, we get a free G-homotopy equivalence

$$\bar{f}: (M - U, \partial) \to (P^2(V \oplus 1) - \coprod D^4_i, \partial) = (Y, \partial Y)$$

which is a homeomorphism on the boundary, hence an element of $\mathcal{S}^h(Y/G, \partial Y/G)$. But the Whitehead torsion of $\bar{f}/G$ vanishes by [Wal, 7.2] since

$SK_1(\mathbb{Z}G) = 0$ [Wa2, §5.4]. Finally

$$\mathcal{S}^s(Y/G, \partial Y/G) \cong \ker([Y/G, \partial Y/G; F/TOP] \to L_0^h(\mathbb{Z}G))$$
$$\cong H^2(Y/\partial y: \mathbb{Z}/2) \cong \mathbb{Z}/2,$$

and the normal invariant of our homotopy equivalence is zero. Indeed it is enough to notice that $\bar{f}$ is the restriction of a self-homotopy equivalence of $P^2(\mathbb{C})$, so homotopic to the identity or complex conjugation. Both of these have trivial normal invariant.

Section 4: Discussion

We remarked in the Introduction that the existence of knotted 2-spheres in S^4 fixed under a group action would prevent rigidity in general. Indeed, such knots can be constructed smoothly for cyclic groups of odd order [G, p.197]. Consider a linear action of $G = C_p \times C_p$ on $P^2(\mathbb{C})$ with an invariant projective triangle as the singular set. One may for example send the generators S and T of G to the matrices

$$\begin{pmatrix} \zeta & & \\ & 1 & \\ & & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & & \\ & \zeta & \\ & & 1 \end{pmatrix}$$

where ζ denotes a p'th root of unity. On one of the lines (say $x_1 = 0$) pick a free orbit of p points for the action of T, and replace the interiors of small linear balls around these points by the connected sum with p copies of the knotted C_p-invariant pair (D^4, K), where $K \approx D^2$ and $(\partial D^4, \partial K)$ is unknotted. It is easy to see that the fundamental group of the complement of the singular set is now different from any linear model. Indeed, in the linear models the complement of the singular set is just

$$Y \approx D^2 \times D^2 - (D^2 \times 0 \cup \times D^2) = S^1 \times S^1 \times D^2.$$

If Y' denotes the new complement and $\Gamma = \pi_1(D^4 - K)$, then

$$\pi_1(Y') \cong (\mathbb{Z} \oplus \mathbb{Z}) * \Gamma * \cdots * \Gamma/\langle z \rangle (p \text{ copies}),$$

is the free product amalgamated over the standard meridian for ∂K in ∂D^4. This contains a subgroup isomorphic to Γ, hence is different from $\pi_1(Y)$.

Finally it is worth observing that rigidity can hold in at least one action without an isolated fixed point. Consider $G = C_3 \times C_3$ with the linear action on $P^2(\mathbb{C})$ given by the representation

$$S(z_0, z_1, z_2) = (z_0, \omega z_1, \omega^2 z_2)$$

$$T(z_0, z_1, z_2) = (z_2, z_0, z_1)$$

in $PGL_3(\mathbb{C})$, where ω is a cube root of 1. Its singular set contains 12 points, the fixed points for each of the 4 subgroups of order three in G.

This G-action is also rigid (we are grateful to Stefan Bauer for help with the argument). Suppose that M denotes $P^2(\mathbb{C})$ with a locally-linear G-action and singular set M^S consisting of 12 points. Then M^G is empty and the localization theorem implies that the restriction map

$$H^2(EG \times_G M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$$

has cokernel $\mathbb{Z}/3$. Hence there exists a G-line bundle over M realizing the third power H^3 of the Hopf bundle. This is in agreement with the linear model, where the lens space $S^5/\langle [S, T] \rangle = L^5(\mathbb{Z}/3) \to P^2(\mathbb{C})$ is the total space of a G-bundle Λ restricting to H^3 but H itself does not lift to an equivariant line bundle.

To remedy this, we consider now the group Γ of upper triangular matrices in $GL_3(\mathbb{F}_3)$. It is the extension of $G = C_3 \times C_3$ by C_3

$$1 \to C_3 \to \Gamma \to G \to 1,$$

and $[S, T] = R$ generates C_3. We view M as a Γ-manifold and notice that

$$H^2(E\Gamma \times_\Gamma M, \mathbb{Z}) \to H^2(M, \mathbb{Z})$$

is onto, because $H^3(B\Gamma, \mathbb{Z}) = 0$. Thus we have a Γ-line bundle L over M realizing the Hopf bundle. Let V_+ and V_- be the $\mathbb{C}\Gamma$-modules induced up from the two irreducible faithful representations of C_3. Then $\mathbb{C}\Gamma \cong \mathbb{C}G \oplus \mathrm{End}(V_+) \oplus \mathrm{End}(V_-)$, so

$$BU(1)^{C_3} = \mathbb{P}(\mathbb{C}G^\infty) \coprod \mathbb{P}(V_+^\infty \coprod \mathbb{P}(V_-^\infty),$$

the disjoint union of the projective spaces of the indicated infinite direct sums of representations.

As before we look at the maps l and h which classify L and the standard

Γ-Hopf bundle

$$M \xrightarrow{l} BU(1)^{C_3} \xleftarrow{h} P^2(\mathbb{C}).$$

We may assume that both l and h map into the component $\mathbb{P}(V_+^\infty)$. On the singular set, the representations agree with the linear ones, so there exists a G-bijection f^s making the diagram

$$\begin{array}{ccc} M^s & \xrightarrow{l} & \mathbb{P}(V_+^\infty)^s \\ & \searrow^{f^s} & \uparrow h \\ & & P^2(\mathbb{C})^s \end{array}$$

commutative. This extends (by obstruction theory as before) to an isovariant G-homotopy equivalence $f: M \to P^2(\mathbb{C})$ which is a local homeomorphism near the singular set. Now the same argument used at the end of § 3 shows that f is G-homotopic to a homeomorphism.

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