Zeitschrift: Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

Band: 64 (1989)

Artikel: You can not hear the mass of a homology class.

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DOI: https://doi.org/10.5169/seals-48965

Comment. Math. Helvetici 64 (1989) 589-617 0010-2571/89/040589-29$01.50+0.20/0

You can not hear the mass of a homology class

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Two Riemannian metrics on a compact Riemannian manifold M are said to be *isospectral* if their associated Laplacians have the same eigenvalues. During the last quarter-century, since the discovery of the first pair of isospectral (but not isometric) metrics by Milnor [Mi] on the 16-dimensional torus, the spectrum of the Laplacian has been the object of intense study by analysts and geometers. (See Berard's monograph [Be] for background and an extensive bibliography.) On the one hand, numerous examples of isospectral manifolds have been discovered. On the other, various geometric and topological properties of manifolds have been found to be determined by the Laplace spectrum. Following the classic article of Mark Kac [Ka], and thinking of the eigenvalues as the frequencies of the normal modes of vibration of an idealized elastic medium, the "drum", we say that a geometric property can be "heard" if it is determined by the Laplace spectrum. While a great deal is known about properties that are determined by the Laplace spectrum, the proofs that the examples of isospectral manifolds are in fact not isometric frequently rely on quite abstract arguments.

Our purpose here is to exhibit specific geometric invariants that can not be "heard". They in turn help to answer the question: "How can a drum change shape, while sounding the same?"

We will focus entirely on a particular 6-dimensional manifold M and a one-parameter isospectral family of metrics g_t on it. This family was discovered by C. S. Gordon and E. N. Wilson [Go–Wi] (see also [DeT–Go]), along with many other examples of isospectral deformations of metrics.

By the *mass* of a homology class in a compact Riemannian manifold, let us mean roughly the minimum volume of any cycle in that class. (The precise definition is given in §5 in the language of currents.) By the *shape* of the manifold, we mean the function which assigns to each homology class its mass.

We will apply the method of calibrated geometries in §7 to prove

THEOREM A. *The shape of the manifold (M, g_t) varies with t.*

The manifold M is the compact quotient of a nilpotent Lie group G by a discrete subgroup. The family of metrics g_t on M is constructed with the aid of a family of almost-inner automorphisms of G. The arithmetic character of M lends an arithmetic character to the search for the appropriate calibrating forms.

Our ongoing research indicates that Theorem A is true for many, perhaps all, of the isospectral deformations constructed using the methods of [Go–Wi] and [DeT–Go]. The results of these investigations will be reported in a subsequent paper.

To prove Theorem A, it is natural to look first in dimension one at closed geodesics on (M, g_t). Many authors have explored relationships between the Laplace spectrum and the length spectrum (i.e., the collection of lengths of closed geodesics) of a Riemannian manifold. The metrics in our family can not be distinguished by their length spectra [Go]; indeed, the mass of each 1-dimensional homology class is independent of t.

Analogous to the length spectrum, we define an *area spectrum* of (M, g_t) by collecting the masses of all the integral 2-dimensional homology classes of M, measured in the metric g_t, together with multiplicities. In contrast to the length spectrum, we prove in §7

THEOREM B. *The area spectrum of (M, g_t) varies with t.*

The change in the area spectrum is suggested by the behavior of the closed geodesics. Although the masses of the 1-dimensional homology classes are independent of t, the location of their minimizing cycles depends on t, as follows. The shortest closed geodesics in a certain homology class foliate a 5-dimensional closed submanifold P of M, independent of t. Those in a second homology class foliate a 4-dimensional closed submanifold Q_t of M, which does depend on t. At time O, we have Q_0 contained in P. But as t increases, Q_t separates from P. Indeed, their distance apart parametrizes the isometry classes of metrics in the deformation.

This change of location within the 1-dimensional classes causes a change of mass for related 2-dimensional classes. Two of these classes are especially interesting.

In one of the classes, there is a moving family of tori T_t, located half way between the submanifolds P and Q_t mentioned above. Each torus T_t in the family minimizes area in the given homology class for the metric g_t, and this minimum area changes with t. A similar phenomenon happens in the second class, but there we are only able to exhibit a mass minimizing 2-dimensional current, and not an ordinary area-minimizing surface.

Theorem B of course implies Theorem A.

The idea of looking at the volumes of higher-than-one-dimensional minimizing cycles to show that isospectral metrics are not isometric has some precedent in the work done on isospectral flat tori. For J. Milnor's now classic example of sixteen-dimensional tori, E. Witt [Wt] has already shown that there is a correspondence between 2-dimensional homology classes of the two isospectral tori which preserves the area of minimizing cycles, but that no such correspondence is possible for 4-dimensional homology. Later, M. Kneser [Kn] showed that there is also a volume-preserving correspondence between the 3-dimensional homology groups. We thank Professor Kneser for pointing this out to us.

This paper is organized into the following sections:

1. AN ISOSPECTRAL FAMILY OF METRICS
2. REAL HOMOLOGY AND COHOMOLOGY VIA INVARIANT FORMS AND CURRENTS
3. INTEGRAL HOMOLOGY VIA CLASSICAL CYCLES
4. INTEGRAL COHOMOLOGY VIA GYSIN SEQUENCES
5. HOW TO FIND THE SMALLEST CYCLES IN A HOMOLOGY CLASS
6. CLOSED GEODESICS
7. AREA-MINIMIZING SURFACES.

Sections 2 through 4 describe the topology of the underlying manifold M, while §§5 through 7 describe the change in its geometry as t varies.

We thank Chris Croke for his help with §7. We also thank the National Science Foundation, the North Atlantic Treaty Organization and the Alfred P. Sloan Foundation for their support.

1. An isospectral family of metrics

Let G be the matrix group consisting of all real matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & z_1 & 0 & 0 & 0 \\ 0 & 1 & 0 & y_1 & 0 & 0 & 0 \\ 0 & 0 & 1 & y_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x_1 & z_2 \\ 0 & 0 & 0 & 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

For simplicity, we denote the above matrix by

$$h = (x_1, x_2, y_1, y_2, z_1, z_2).$$

The first four components of the product hh' are

$$x_1 + x_1', \ x_2 + x_2', \ y_1 + y_1', \ y_2 + y_2'.$$

The fifth component of hh' is

$$z_1 + z_1' + x_1 y_1' + x_2 y_2',$$

and the sixth is

$$z_2 + z_2' + x_1 y_2'.$$

These last two components reflect the non-commutativity of the multiplication. The inverse of h is

$$(-x_1, \ldots, -y_2, -z_1 + x_1 y_1 + x_2 y_2, -z_2 + x_1 y_2).$$

G is a two-step nilpotent Lie group.

Let Γ be the discrete subgroup of G consisting of matrices with integer entries. The set $M = \Gamma \backslash G$ of right cosets Γh of Γ is a compact smooth 6-dimensional manifold.

We will define a family of left-invariant metrics g_t on G, which will descend to metrics of the same name on M.

First look at the Lie algebra $\mathscr{G}$ of G. It has a basis

$$B = \{X_1, X_2, Y_1, Y_2, Z_1, Z_2\},$$

with brackets

$$[X_1, Y_1] = Z_1 = [X_2, Y_2] \quad \text{and} \quad [X_1, Y_2] = Z_2,$$

and all other brackets zero.

A left-invariant metric on G can be specified by an inner product on $\mathscr{G}$. Define g_t to be the left-invariant metric for which

$$B_t = \{X_1, X_2, Y_1, Y_2(t) = Y_2 - tZ_2, Z_1, Z_2\}$$

is an orthonormal basis. We will denote g_0 by g.

PROPOSITION. *The metrics g_t form an isospectral family of metrics on M. Two such metrics g_t and g_r are isometric if and only if the distance from t to its nearest integer equals the distance from r to its nearest integer.*

The isospectrality of the metrics is a special case of a general theorem of [Go–Wi]. In fact, the particular family of manifolds (M, g_t) appears as Example 2.4(i) of [Go–Wi], and is also discussed in [DeT–Go]. The isospectrality comes from the fact that the linear map of $\mathscr{G}$, which carries the ordered basis B_t back to the ordered basis B_0, is the differential of an automorphism Φ_t of G given by

$$\Phi_t(x_1, \ldots, z_2) = (x_1, x_2, y_1, y_2, z_1, z_2 + ty_2).$$

This automorphism of G is "almost-inner", that is, for each $h \in G$,

$$\Phi_t(h) = h'hh'^{-1},$$

but h' depends on h. When t is nonzero, Φ_t is *not* an inner automorphism.

As metrics on G, we have $g_t = \Phi_t^* g$. (In particular, g_t and g are isometric metrics on G, but the isometry does not descend to $\Gamma\backslash G$.) The main theorem of [Go–Wi] states that if a left-invariant metric on a compact nilmanifold M (i.e., a metric whose lift to the nilpotent Lie group covering M is left-invariant) is deformed by a family of almost-inner automorphisms, then the deformation is isospectral.

To obtain the last statement of the proposition, let

$K = \{\sigma \in \mathrm{Aut}(G): \sigma^* g = g\}$, and

$D = \{\delta \in \mathrm{Aut}(G): \delta(\Gamma) = \Gamma\}$,

where Aut (G) denotes the group of automorphisms of G. By Corollary 5.3 of [Go–Wi], $\Phi_t^* g = \Phi_r^* g$ as metrics on $\Gamma\backslash G$ if and only if there exists a $\sigma \in K$ such that $\Phi_r^{-1}\sigma\Phi_t \in D\,\mathrm{Inn}(G)$. By normality of the subgroup Inn (G) of inner automorphisms of G, the product $D\,\mathrm{Inn}(G) = \mathrm{Inn}(G)\,D$ is itself a subgroup of Aut (G). If $t \equiv r \bmod Z$, then $\Phi_r^{-1}\Phi_t \in D$, and we can take $\sigma = Id$. If $t + r \in Z$, we may take

$$\sigma(x_1, \ldots, z_2) = (-x_1, x_2, y_1, -y_2, -z_1, z_2)$$

and check that $\Phi_r^{-1}\sigma\Phi_t \in D$. Finally, by explicitly computing K, we see that no other pairs are isometric.

In Figure 1, we display M as a bundle over a flat 4-torus T^4 with fibre a flat 2-torus T^2.

The 6-dimensional nilmanifold M is a non-commutative version of the 6-dimensional flat torus. We will see that the non-commutativity robs us of homology: the 1-dimensional homology of M has rank 4, while for T^6 it has rank 6; the 2-dimensional homology of M has rank 8, while for T^6 it has rank 15. Most, but not all, of the homology of M in these dimensions is carried by one or the other of the two 4-dimensional subtori shown in Figure 1.

In the next three sections, we will describe the topology of M.

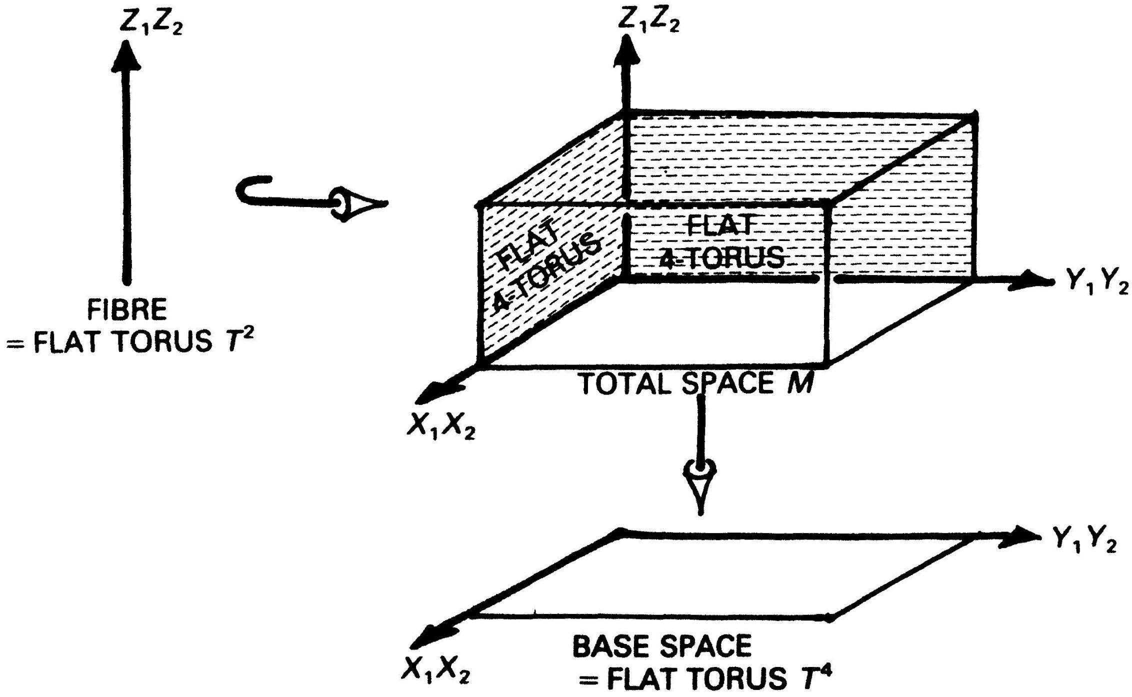

FIGURE 1

2. Real homology and cohomology via invariant forms and currents

The arithmetic character of M makes this easy to compute.

We begin by setting notation.

In the previous section, we introduced, on the Lie group G, the left-invariant vector fields X_1, X_2, Y_1, Y_2, Z_1, Z_2, with Lie bracket relations

$$[X_1, Y_1] = Z_1 = [X_2, Y_2] \quad \text{and} \quad [X_1, Y_2] = Z_2.$$

They agree with the coordinate vector fields $\partial/\partial x_1, \ldots, \partial/\partial z_2$ at the identity of G, but one quickly computes that in general:

$$\begin{aligned}
&X_1 = \partial/\partial x_1 \qquad X_2 = \partial/\partial x_2\\
&Y_1 = \partial/\partial y_1 + x_1\,\partial/\partial z_1\\
&Y_2 = \partial/\partial y_2 + x_2\,\partial/\partial z_1 + x_1\,\partial/\partial z_2\\
&Z_1 = \partial/\partial z_1 \qquad Z_2 = \partial/\partial z_2.
\end{aligned}$$

These left-invariant vector fields on G descend to well-defined vector fields of the same name on the right coset space $M = \Gamma\backslash G$. By abuse of language, we refer to these as *left-invariant vector fields on* M, even though G does not have a left action on M.

We denote the dual basis of left-invariant 1-forms on G by

$$\alpha_1,\ \alpha_2,\ \beta_1,\ \beta_2,\ \gamma_1 \text{ and } \gamma_2.$$

In local coordinates, we have:

$$\begin{aligned}
&\alpha_1 = dx_1 \qquad \alpha_2 = dx_2\\
&\beta_1 = dy_1 \qquad \beta_2 = dy_2\\
&\gamma_1 = dz_1 - x_1\,dy_1 - x_2\,dy_2\\
&\gamma_2 = dz_2 - x_1\,dy_2.
\end{aligned}$$

These left-invariant 1-forms on G likewise descend to "left-invariant" 1-forms on M.

On either G or M, the exterior derivatives of these 1-forms can be read off from the Lie brackets of the vector fields via the formula

$$d\varphi(X, Y) = -\varphi([X, Y]),$$

in which φ is any left-invariant 1-form and X and Y are any left-invariant vector fields. Alternatively, one differentiates directly in local coordinates. Either way:

$$d\alpha_1 = 0 \qquad d\alpha_2 = 0$$

$$d\beta_1 = 0 \qquad d\beta_2 = 0$$

$$d\gamma_1 = -dx_1\, dy_1 - dx_2\, dy_2 = -\alpha_1\beta_1 - \alpha_2\beta_2$$

$$d\gamma_2 = -dx_1\, dy_2 = -\alpha_1\beta_2.$$

The left-invariant 1-forms may be combined via exterior multiplication to yield the left-invariant k-forms. The exterior derivative on k-forms is already determined, via the Leibniz rule, by its values on the 1-forms. So it will be easy to calculate which of the k-forms are closed.

We will use "k-current" in the sense of deRham to denote a continuous linear functional on smooth k-forms. Exterior products of vector fields define currents by evaluation:

$$X_1 \wedge \cdots \wedge X_k(\varphi) = \int_M \varphi(X_1 \wedge \cdots \wedge X_k)\, d\,\mathrm{vol}.$$

We will call a current "left-invariant" if it is a linear combination of exterior products of left-invariant vector fields. The boundary map ∂ on the space of k-currents is the adjoint of the exterior derivative on $k-1$ forms. In particular, the boundary of a left-invariant k-current is a left-invariant $k-1$ current.

By a theorem of Nomizu [No], the cohomology of left-invariant forms on any nilpotent Lie group G is isomorphic in the obvious way to the real cohomology of the coset space $M = \Gamma\backslash G$. By duality, the homology of left-invariant currents on G is isomorphic to the real homology of M. This provides an effective scheme, which we now carry out, for calculating the real homology and cohomology of M.

We begin with cohomology, concentrating on dimensions 1 and 2. From the above table of exterior derivatives of 1-forms, we see immediately that $H^1(M; R) \cong R^4$, generated by the classes of the closed 1-forms α_1, α_2, β_1 and β_2.

Using the table together with the Leibniz rule, we compute the exterior derivatives of left-invariant 2-forms:

$$d(\alpha_1\alpha_2)=0 \qquad d(\alpha_1\beta_1)=0 \qquad d(\alpha_1\beta_2)=0$$
$$d(\alpha_1\gamma_1)=\alpha_1\alpha_2\beta_2$$
$$d(\alpha_1\gamma_2)=0 \qquad d(\alpha_2\beta_1)=0 \qquad d(\alpha_2\beta_2)=0$$
$$d(\alpha_2\gamma_1)=-\alpha_1\alpha_2\beta_1$$
$$d(\alpha_2\gamma_2)=-\alpha_1\alpha_2\beta_2 \qquad d(\beta_1\beta_2)=0$$
$$d(\beta_1\gamma_1)=-\alpha_2\beta_1\beta_2$$
$$d(\beta_1\gamma_2)=-\alpha_1\beta_1\beta_2$$
$$d(\beta_2\gamma_1)=\alpha_1\beta_1\beta_2 \qquad d(\beta_2\gamma_2)=0$$
$$d(\gamma_1\gamma_2)=-\alpha_1\beta_1\gamma_2-\alpha_2\beta_2\gamma_2+\alpha_1\beta_2\gamma_1.$$

From this table, we find ten generators for the 2-dimensional cocycles, and two generators for the 2-dimensional coboundaries. Hence $H^2(M;R)\cong R^8$, generated by the classes of the closed 2-forms:

$$\alpha_1\alpha_2, \qquad \alpha_1\beta_1, \qquad \alpha_2\beta_1, \qquad \beta_1\beta_2,$$
$$\alpha_1\gamma_2, \qquad \alpha_1\gamma_1+\alpha_2\gamma_2,$$
$$\beta_1\gamma_2+\beta_2\gamma_1 \quad \text{and} \quad \beta_2\gamma_2.$$

We turn to homology, again looking just at dimensions 1 and 2.

The 1-dimensional left-invariant currents

$$X_1, X_2, Y_1, Y_2, Z_1 \text{ and } Z_2$$

are all closed, hence represent homology classes in $H_1(M;R)$, which is isomorphic to R^4 by duality. Of course, these homology classes can not all be independent.

There are fifteen generators for the 2-dimensional left-invariant currents:

$$X_1X_2, X_1Y_1, \ldots, Z_1Z_2.$$

Twelve of these are closed, three are not:

$$\partial(X_1Y_1)=-Z_1=\partial(X_2Y_2)$$
$$\partial(X_1X_2)=-Z_2.$$

We see that the 1-cycles Z_1 and Z_2 are boundaries, leaving X_1, X_2, Y_1 and Y_2 to provide a 1-dimensional homology base.

In addition, we get a thirteenth 2-cycle:

$$X_1Y_1 - X_2Y_2.$$

The boundaries of the 3-dimensional currents provide five independent homologies among the thirteen 2-cycles:

$$[X_2Z_1] = 0, \qquad [Y_2Z_1] = [Y_1Z_2],$$
$$[X_1Z_1] = [X_2Z_2], \qquad [Y_1Z_1] = 0 \quad \text{and} \quad [Z_1Z_2] = 0.$$

Thus $H_2(M; R) \cong R^8$, with a basis provided by the following 2-cycles:

X_1X_2, Y_1Y_2, X_2Y_1, $X_1Y_1 - X_2Y_2$,
X_1Z_1 (which is homologous to X_2Z_2), X_1Z_2,
Y_1Z_2 (which is homologous to Y_2Z_1) and Y_2Z_2.

This basis turns out to be dual to the one given earlier for the 2-forms.

3. Integral homology via classical cycles

By a "classical cycle" we mean a singular Lipschitz chain, that is, a chain built from finitely many Lipschitz maps of individual simplexes.

It is easy to find classical cycles in the homology classes of the closed 1-dimensional currents X_1, X_2, Y_1 and Y_2. For example, the one-parameter subgroup $\{(t, 0, 0, 0, 0, 0)\}$ of G descends to a circle in M which is homologous to the current X_1. And likewise for X_2, Y_1 and Y_2.

It is also easy to find classical cycles in most of the homology classes represented by our chosen basis of 2-dimensional currents. Consider the 4-dimensional subtori $\{y_1 = y_2 = 0\}$ and $\{x_1 = x_2 = 0\}$ of M, included earlier in Figure 1.

Each of the 2-cycles

$$X_1X_2,\ Y_1Y_2,\ X_1Z_1,\ X_1Z_2,\ Y_1Z_2 \text{ and } Y_2Z_2$$

is easily seen to be homologous to an appropriate 2-torus inside one or the other of the above 4-dimensional subtori of M.

The subgroup $\{(0, x_2, y_1, 0, 0, 0)\}$ of G covers a 2-dimensional torus in M which is homologous to the closed 2-dimensional current X_2Y_1.

This leaves us yet to represent the closed 2-current $X_1Y_1 - X_2Y_2$, which turns out to be interesting for two reasons:

1) It is the only "indecomposable" 2-current in our basis, and hence the only one which can *not* be visualized as a foliation, and then represented by a compact toral leaf.
2) The homology class of this closed 2-current turns out *not* to be integral, though twice it is.

To help understand the homology class of $X_1Y_1 - X_2Y_2$, we construct an orientable surface of genus 2 (a double torus) in M as follows. The subgroup $G_1 = \{(x_1, 0, y_1, 0, z_1, 0)\}$ of G covers a 3-dimensional Heisenberg submanifold H_1 of M. H_1 is a quotient of the unit cube in $x_1y_1z_1$-space: the front face $y_1 = 0$ is identified with the back face $y_1 = 1$ by translation in the y_1 direction, and the bottom face $z_1 = 0$ is identified with the top face $z_1 = 1$ by translation in the z_1 direction. However, the left face $x_1 = 0$ is identified with the right face $x_1 = 1$ by the "shear"

$$(0, 0, y_1, 0, z_1, 0) \to (1, 0, y_1, 0, y_1 + z_1, 0),$$

as shown in Figure 2.

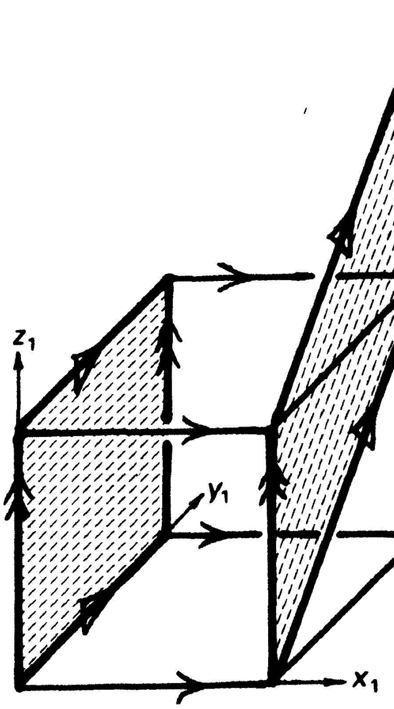

FIGURE 2

Consider the surface S shaded in Figure 3; it is a disk whose boundary is the loop

$$X_1Y_1X_1^{-1}Y_1^{-1}Z_1^{-1}.$$

The image $\underline{S}$ of S in M is obtained by performing the indicated identifications, so is a punctured torus whose boundary is the Z_1-circle. $\underline{S}$ can be parametrized by the charts

$$(s, t) \to (s, 0, t, 0, 1-s+st, 0), \text{ for } 0 \le s, t \le 1, \text{ and}$$
$$(u, v) \to (u, 0, 0, 0, v, 0), \text{ for } 0 \le u, v, u+v \le 1.$$

Similarly, the subgroup $G_2 = \{(0, x_2, 0, y_2, z_1, 0)\}$ of G covers a 3-dimensional Heisenberg submanifold H_2 of M, and inside it is a punctured torus parametrized by

$$(s, t) \to (0, s, 0, t, 1-s+st, 0), \text{ for } 0 \le s, t \le 1, \text{ and}$$
$$(u, v) \to (0, u, 0, 0, v, 0), \text{ for } 0 \le u, v, u+v \le 1.$$

Both punctured tori have the same boundary circle, parametrized by

$$v \to (0, 0, 0, 0, v, 0), \text{ for } 0 \le v \le 1,$$

so they join up to form a double torus DT^2 in M.

We can compute the homology class of this double torus by integrating over it

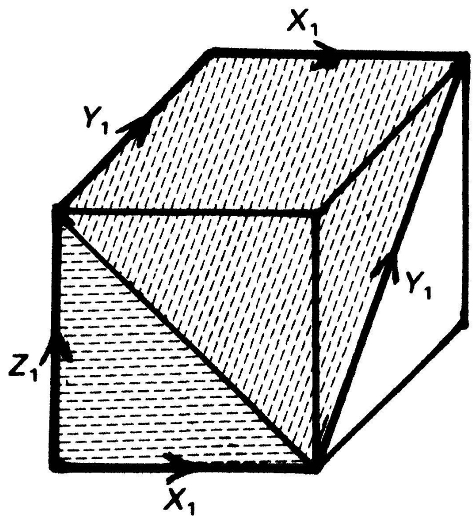

FIGURE 3

each of the eight basis two-forms, and find:

$$[DT^2] = [X_1Y_1 - X_2Y_2] + (1/2)[X_1Z_1] + (1/2)[Y_1Z_2].$$

In summary, we have seen that the closed left-invariant 2-currents

$$X_1X_2,\ Y_1Y_2,\ X_1Z_2,\ Y_1Z_2,\ Y_2Z_2,\ X_2Y_1$$
$$\text{and}\quad X_1Y_1 - X_2Y_2 + (1/2)X_1Z_1 + (1/2)Y_1Z_2$$

represent *integral* homology classes which constitute a basis for the *real* homology $H_2(M; R)$. That they are also a basis for the integral homology will be seen in the next section.

4. Integral cohomology via Gysin sequences

Earlier, we described M as a bundle over a flat 4-torus with fibre a flat 2-torus. In this section we will view M as an iterated circle bundle, and then calculate its integral cohomology by two applications of the Gysin sequence. It will turn out that this integral cohomology has no torsion, and hence injects into the real cohomology. In particular, integral cohomology classes can be represented by differential forms.

To this end, let G for the moment be the 5-dimensional Heisenberg group, that is, the matrix group consisting of all real matrices of the form

$$\begin{pmatrix} 1 & x_1 & x_2 & z_1 \\ 0 & 1 & 0 & y_1 \\ 0 & 0 & 1 & y_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let Γ be the discrete subgroup of G consisting of matrices with integer entries. The set $L = \Gamma\backslash G$ of right cosets is a compact smooth 5-dimensional *Heisenberg manifold.*

We view M as a circle bundle over L by dropping the z_2 coordinate, and L as a circle bundle over the 4-torus T^4 by dropping the z_1 coordinate:

$$\begin{array}{l} S^1 \hookrightarrow M^6 \\ \qquad\quad \downarrow \\ S^1 \hookrightarrow L^5 \\ \qquad\quad \downarrow \\ \qquad\quad T^4. \end{array}$$

If we use real coefficients, we can quickly compute the cohomology of L just as we did for M in §2.

To get the integral cohomology of L, we consider the Gysin sequence of the circle bundle with total space $E = L$ and base space $B = T^4$:

$$\cdots \longrightarrow H^{k-2}B \xrightarrow{\cup e} H^kB \xrightarrow{\pi^*} H^kE \xrightarrow{\Delta} H^{k-1}B \longrightarrow \cdots$$

where $e \in H^2B$ is the Euler class, $\pi: E \to B$ is the projection map, and Δ the "boundary map" given by integration along the fibre. We may read this sequence with either integral or real coefficients.

We let

$$X_1, X_2, Y_1, Y_2 \text{ and } Z_1$$

denote the obvious "left-invariant" vector fields on L, and

$$\alpha_1, \alpha_2, \beta_1, \beta_2 \text{ and } \gamma_1$$

the dual "left-invariant" 1-forms.

We have the relation in $E = L$:

$$d\gamma_1 = -\alpha_1\beta_1 - \alpha_2\beta_2,$$

which reveals the bundle's Euler class

$$e = -\underline{\alpha}_1\underline{\beta}_1 - \underline{\alpha}_2\underline{\beta}_2$$

in $B = T^4$. We underline Greek letters to indicate forms on the base. To pull back to the total space, simply delete the underline.

Because the Euler class is nonzero, the map $H^0B \xrightarrow{\cup e} H^2B$ is injective. Hence from the Gysin sequence,

$$H^1L = H^1E \cong H^1B \cong Z^4,$$

generated by the classes of the closed 1-forms

$$\alpha_1, \alpha_2, \beta_1 \text{ and } \beta_2.$$

Next, one quickly checks that the map $H^1B \xrightarrow{\cup e} H^3B$ is an isomorphism. Hence

$$H^2L = H^2E \cong H^2B/(\text{image of } H^0B \text{ under } \cup e) \cong Z^5,$$

generated by the classes of the closed 2-forms

$$\alpha_1\alpha_2,\ \alpha_1\beta_1,\ \alpha_1\beta_2,\ \alpha_2\beta_1 \text{ and } \beta_1\beta_2.$$

To compute the 1- and 2-dimensional integral cohomology of M in terms of that of L, we view M as the total space of a circle bundle over the base space L, and appeal to the corresponding Gysin sequence.

Notationally, forms which live on L will be underlined, since L is now our base space.

The relation in the total space $E = M$:

$$d\gamma_2 = -\alpha_1\beta_2$$

reveals the bundle's Euler class

$$e = -\underline{\alpha}_1\underline{\beta}_2$$

in the base space $B = L$.

Because this Euler class is nonzero, the map $H^0B \xrightarrow{\cup e} H^2B$ is injective. Hence from the Gysin sequence,

$$H^1M = H^1E \cong H^1B \cong Z^4,$$

generated by the classes of the closed 1-forms

$$\alpha_1,\ \alpha_2,\ \beta_1 \text{ and } \beta_2.$$

By contrast, the map $H^1B \xrightarrow{\cup e} H^3B$ is zero. Visibly,

$$\underline{\alpha}_1(-\underline{\alpha}_1\underline{\beta}_2) = 0 = \underline{\beta}_2(-\underline{\alpha}_1\underline{\beta}_2).$$

But also,

$$\underline{\alpha}_2(-\underline{\alpha}_1\underline{\beta}_2) = d(\underline{\alpha}_1\underline{\gamma}_1), \text{ and}$$
$$\underline{\beta}_1(-\underline{\alpha}_1\underline{\beta}_2) = d(\underline{\beta}_2\underline{\gamma}_1).$$

So we extract from the Gysin sequence the fairly short exact sequence

$$0 \longrightarrow H^0B \xrightarrow{\cup e} H^2B \xrightarrow{\pi^*} H^2E \xrightarrow{\Delta} H^1B \longrightarrow 0.$$

We've already calculated the integral cohomology of the base space $B = L$. We have:

$H^0B \cong Z$, with generator 1.

$H^1B \cong Z^4$, with generators $\underline{\alpha}_1$, $\underline{\alpha}_2$, $\underline{\beta}_1$, $\underline{\beta}_2$.

$H^2B \cong Z^5$, with generators $\underline{\alpha}_1\underline{\alpha}_2$, $\underline{\alpha}_1\underline{\beta}_1$, $\underline{\alpha}_1\underline{\beta}_2$, $\underline{\alpha}_2\underline{\beta}_1$, $\underline{\beta}_1\underline{\beta}_2$.

Cupping with the Euler class $e = -\underline{\alpha}_1\underline{\beta}_2$ takes the generator 1 of H^0B to the negative of one of the listed generators of H^2B. So from the portion of the Gysin sequence highlighted above, we conclude that

$$H^2M = H^2E \cong Z^8,$$

and that half of a basis is represented by the closed 2-forms

$$\alpha_1\alpha_2,\ \alpha_1\beta_1,\ \alpha_2\beta_1 \text{ and } \beta_1\beta_2.$$

The other half is represented by closed 2-forms which map by Δ to the basis elements for H^1B listed above.

We make a provisional choice of these remaining basis elements as follows. Since $\alpha_1\gamma_2$ is closed and Δ sends it to the basis element $\underline{\alpha}_1$ of H^1B, we tentatively add $\alpha_1\gamma_2$ to our basis for H^2E. Likewise, we include $\beta_2\gamma_2$. By contrast, $\alpha_2\gamma_2$ is *not* closed, but $\alpha_1\gamma_1 + \alpha_2\gamma_2$ is closed, and Δ sends it to $\underline{\alpha}_2$. So we include $\alpha_1\gamma_1 + \alpha_2\gamma_2$ in our provisional basis. Likewise, we include $\beta_1\gamma_2 + \beta_2\gamma_1$.

These eight closed 2-forms on the total space $E = M$ certainly form a basis for the 2-dimensional cohomology over the reals. Indeed, we have already seen this in §2. The first four of these closed 2-forms represent integral classes, since they come from integral classes on the base. But the last four may not represent integral classes, and may have to be adjusted by adding combinations of the first four in order to produce integral classes. As we will see, this is precisely what happens.

We switch for a moment to homology.

We saw in the previous section that the closed left-invariant 2-currents

$$X_1X_2,\ Y_1Y_2,\ X_2Y_1,$$
$$X_1Y_1 - X_2Y_2 + (1/2)X_1Z_1 + (1/2)Y_1Z_2,$$
$$X_1Z_1,\ X_1Z_2,\ Y_1Z_2 \text{ and } Y_2Z_2$$

represent *integral* homology classes, and constitute a basis for the *real homology* $H_2(M; R)$.

We will see now that these classes are a basis for the integral homology $H_2(M; Z)$.

To that end, consider the closed 2-forms on M which represent our provisional basis for H^2M:

$$\alpha_1\alpha_2,\ \beta_1\beta_2,\ \alpha_2\beta_1,\ \alpha_1\beta_1,\ \alpha_1\gamma_1 + \alpha_2\gamma_2,\ \alpha_1\gamma_2,\ \beta_1\gamma_2 + \beta_2\gamma_1 \text{ and } \beta_2\gamma_2.$$

The first four are part of an integer basis for H^2M. The second four will have to be altered by linear combinations of the first four in order to complete this integer basis. Note that this passage from provisional to final basis for H^2M will be unimodular.

If we had this final integer basis for H^2M, we could evaluate it on each of the integral homology classes above and take the determinant of the resulting 8 by 8 matrix. If this determinant were ± 1, then the homology classes would form an integral basis for H_2M.

Since the change from provisional to final basis for H^2M is unimodular, we can use the provisional basis (which we know) instead of the final basis (which we don't) in carrying out the above integrality test.

A quick calculation reveals that the eight left-invariant closed 2-forms on M which represent the provisional basis for H^2M are almost perfectly dual to the eight left-invariant closed 2-currents given above. Indeed, the corresponding 8 by 8 matrix of evaluations has 1's down the diagonal, and only two nonzero off-diagonal terms: the 2-forms

$$\alpha_1\gamma_1 + \alpha_2\gamma_2 \quad \text{and} \quad \beta_1\gamma_2 + \beta_2\gamma_1$$

both take the value 1/2 on the "double torus" 2-cycle

$$X_1Y_1 - X_2Y_2 + (1/2)X_1Z_1 + (1/2)Y_1Z_2.$$

The determinant is clearly 1. Hence these eight closed left-invariant 2-currents (concretely represented by seven tori and a double torus) represent an *integer* basis for H_2M, as claimed.

We now return to cohomology.

We simply take the eight closed left-invariant 2-forms listed above. We subtract $(1/2)\alpha_1\beta_1$ from both $\alpha_1\gamma_1+\alpha_2\gamma_2$ and $\beta_1\gamma_2+\beta_2\gamma_1$, and leave the other six 2-forms alone. What results is a basis for cohomology dual to the integer homology basis given above. Hence we have our integer cohomology basis.

With this topological description of M in hand, we now aim to see how the geometry changes as the metric g_t varies.

5. How to find the smallest cycles in a homology class

We define the "comass" of a form and the "mass" of a current, following Federer [Fe1], and begin in a linear algebra setting.

Let V be a finite dimensional real vector space with an inner product. The inner product extends in a natural way to the space $\wedge^k V$ of k-vectors, and to the space $\wedge^k V^*$ of k-forms. In particular, it provides norms on these spaces.

Given a k-form φ, its *comass* is

$$\|\varphi\|^* = \sup\{\varphi(U): U \text{ a } \textit{simple } k\text{-vector of norm } 1\},$$

"simple" meaning "decomposable as an exterior product of vectors". For example, let $V=R^4$, with orthonormal basis $e_1, \ldots, e_4$, and dual orthonormal basis $e_1^*, \ldots, e_4^*$ for V^*. Then the 2-form $e_1^*e_2^*+e_3^*e_4^*$ has comass 1, and takes this maximum value on the 2-vector e_1e_2, as well as on any other 2-vector corresponding to a complex line in C^2. More generally, the comass of the 2-form

$$ae_1^*e_2^*+be_3^*e_4^*$$

is $\max\{|a|, |b|\}$.

Given a k-vector U, its *mass* is

$$\|U\| = \sup\{\varphi(U): \varphi \text{ a } k\text{-form of comass } 1\}.$$

For example, the mass of the 2-vector $e_1e_2+e_3e_4$ is 2, and this maximum is achieved when the 2-vector is evaluated against the 2-form $e_1^*e_2^*+e_3^*e_4^*$ of comass

1. More generally, the mass of the 2-vector

$$ae_1e_2 + be_3e_4$$

is $|a| + |b|$.

These ideas carry over from the linear algebra setting to that of forms and currents on a compact Riemannian manifold M.

Given a smooth k-form φ on M, its *comass* is

$$\|\varphi\|^* = \sup\{\|\varphi_x\|^* : x \in M\}.$$

Given a k-current U on M, its *mass* is

$$\|U\| = \sup\{U(\varphi) : \varphi \text{ a smooth } k\text{-form of comass } 1\}.$$

One checks that if the k-current U corresponds to integration over a classical k-chain, then its mass is the k-dimensional area of the chain.

If we restrict ourselves to currents on M of finite mass whose boundaries also have finite mass (the so-called *normal currents*), then their homology coincides, by a theorem of Federer and Fleming [Fe–Fl], with the real homology $H_*(M; R)$.

By the mass of a real homology class (informally defined in the introduction), we mean the minimum mass of any closed current in that class. Note that "mass" is a norm on homology: it is subadditive and is linear on rays.

We will still use this definition when the homology class happens to be integral, though one might also consider the minimum mass of just the classical cycles therein. This minimum may be larger. For example, take a flat rectangle of length 1 and paste its left and right sides together to form a Möbius band B. The distance around the center of the band is 1; the distance around the boundary ∂B is 2. Now introduce a little bit of positive curvature, so that the distance around the center remains 1, but the distance around the boundary decreases to 1.9. Consider the integral 1-dimensional homology class corresponding to once around the Möbius band. If we restrict to classical cycles, the minimum mass is 1. If we allow the more general currents, then "half the boundary" (that is, the current defined by $\varphi \mapsto (1/2)\int_{\partial B} \varphi$, for any 1-form φ) is admissible, and has mass 0.95. The mass of this homology class, by our definition, is 0.95.

These two competing measurements of an integral homology class are related by a theorem of Federer [Fe2, §5.8]. The mass of the integral class $[U]$, that is, the minimum mass of any closed current in it, is equal to the limit, as $m \to \infty$, of $(1/m)$ times the minimum mass of any classical cycle in the class $m[U]$.

Frequently, the mass of a homology class and the corresponding minimizing currents therein can be found with the aid of a "calibrating" form.

A closed k-form φ of comass 1 on a Riemannian manifold M is called a *calibration.* A closed k-current U on M, for which $U(\varphi)$ coincides with the mass of U, is said to be *calibrated* by φ. The simplest example of such a U is a smooth oriented k-dimensional submanifold of M, on which φ restricts to the volume form.

The principal observation is:

A closed k-current U which is calibrated by some form φ must be mass minimizing in its homology class.

For if U' is another closed k-current in the same class, then

$$\text{Mass}\,(U) = U(\varphi) = U'(\varphi) \leq \text{Mass}\,(U').$$

The first equality is because φ calibrates U. The second is because φ is closed, and hence Stokes' Theorem may be applied. The final inequality is because φ has comass one. Note that equality holds if and only if U' is also calibrated by φ.

The standard examples of calibrations are provided by the normalized powers of the Kähler form on a Kähler manifold. The classical cycles so calibrated are just the complex subvarieties, which are thereby seen to be mass minimizing in their homology classes. Many more examples are given in [Ha–La].

6. Closed geodesics

In this and the following section, we return to our 6-dimensional nilmanifold M, together with the metric g_t on it at time t, and use calibrations by invariant differential forms to identify mass minimizing cycles, and to calculate the masses of homology classes.

The classical cycles of minimum length in the 1-dimensional integer homology classes are, of course, closed geodesics. It is well known (see, for example, [Du–Gu] and [CdV]) that under certain generic conditions, the Laplace spectrum of a Riemannian manifold determines the length spectrum, that is, the collection of lengths of closed geodesics. While the nilmanifolds studied here do not satisfy that generic condition, the length spectrum of (M, g_t) is nonetheless independent of t. In fact, it is shown in [Go] that for each free homotopy class α of closed curves in M, there exists a bijection $T: A(\alpha) \to A_t(\alpha)$, from the set $A(\alpha)$ of closed geodesics in the metric g which lie in the class α, to the corresponding set $A_t(\alpha)$ in the metric g_t. This bijection carries closed geodesics of a given length to ones of the same length. In particular, the manifolds (M, g_t) have the same length spectrum, and so can not be distinguished this way.

We will see below that the manifolds (M, g_t) can be distinguished by the relative positions of the closed geodesics in certain homology classes. This phenomenon was exhibited by a pair of isospectral surfaces constructed by Brooks and Tse [Br–Ts]; see also [Br].

THEOREM C. *There is a 5-dimensional submanifold $P = \{x_1 = 0\}$ of (M, g_t) foliated by circles of length 1 which are integral curves of Y_1. They are all calibrated by the closed 1-form β_1, and hence are length minimizing in the Y_1 homology class. There are no other classical cycles which minimize length in this class.*

Likewise, there is a 4-dimensional submanifold $Q_t = \{x_1 = t, x_2 = 0\}$ of (M, g_t), foliated by circles of length 1 which are integral curves of Y_2. They are all calibrated by the closed 1-form β_2, and hence are length minimizing in the Y_2 homology class. No other classical cycles minimize length in this class.

REMARK. The distance in (M, g_t) from P to Q_t is the distance from t to the nearest integer. By the Proposition of §1, this distance parametrizes the isometry classes of the manifolds (M, g_t).

The two parts of the above theorem have similar proofs. We do only the second part, which is more interesting.

Recall that in the metric g_t on G, we have an orthonormal basis of left-invariant vector fields:

$$X_1, X_2, Y_1, Y_2(t) = Y_2 - tZ_2, Z_1 \text{ and } Z_2.$$

The dual left-invariant 1-forms are:

$$\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1 \quad \text{and} \quad \gamma_2(t) = \gamma_2 + t\beta_2.$$

These cover "left-invariant" vector fields and 1-forms down on M.

In (M, g_t), the closed 1-form β_2 calibrates the closed current

$$\begin{aligned} Y_2(t) &= Y_2 - tZ_2 \\ &= \partial/\partial y_2 + x_2\, \partial/\partial z_1 + (x_1 - t)\, \partial/\partial z_2, \end{aligned}$$

whose mass of 1 is therefore the minimum possible in its homology class. This homology class is independent of t,

$$[Y_2(t)] = [Y_2] - t[Z_2] = [Y_2],$$

since Z_2 bounds.

We now seek the geodesics of length 1 in this class. The integral curve of $Y_2(t)$ passing through the identity of G is given by $s \mapsto (0, 0, 0, s, 0, -ts)$, as one sees from the local coordinate expression for $Y_2(t)$. Hence the integral curve of $Y_2(t)$ passing through the point $(x_1, x_2, y_1, y_2, z_1, z_2)$ of G is given by

$$\begin{aligned} h_t(s) &= (x_1, x_2, y_1, y_2, z_1, z_2)(0, 0, 0, s, 0, -ts) \\ &= (x_1, x_2, y_1, y_2 + s, z_1 + x_2 s, z_2 + (x_1 - t)s). \end{aligned}$$

This will descend to a circle of length 1 in $M = \Gamma \backslash G$ if and only if there is an element $(a_1, a_2, b_1, b_2, c_1, c_2)$ in Γ such that

$$(a_1, a_2, b_1, b_2, c_1, c_2)h_t(s) = h_t(s+1).$$

This vector equation is equivalent to the six scalar equations

$$a_1 = a_2 = b_1 = 0, \qquad b_2 = 1, \qquad c_1 = x_2, \qquad c_2 = x_1 - t.$$

Since Γ is the integer lattice of G, this can be satisfied if and only if both $x_1 - t$ and x_2 are integers.

Left translating by an appropriate element of the lattice Γ, we can assume that $x_1 - t = 0$ and $x_2 = 0$. Thus we get a 4-dimensional submanifold $Q_t = \{x_1 = t,\ x_2 = 0\}$ of M, foliated by circles of length 1 in the metric g_t, which are of minimum length in their homology class $[Y_2]$. They are the only classical cycles which minimize length in this homology class.

This completes the proof of Theorem C.

7. Area-minimizing surfaces

In the previous section, we saw that the manifolds (M, g_t) can be distinguished by the distance between the closed geodesics in the Y_1 and Y_2 homology classes. We now expect, for reasons sketched below, that these manifolds can also be distinguished by the area of the smallest cycle in the 2-dimensional $Y_1 Y_2$ homology class.

At time 0, the flat 2-torus

$$T_0 = \{x_1 = x_2 = z_1 = z_2 = 0\}$$

is easily seen to be area minimizing in the $Y_1 Y_2$ class. At time t, suppose that T_t is a surface in this homology class. Visualize this surface as a torus (this is only a heuristic argument). Intersecting T_t with the 5-cycle $\{y_2 = 0\}$, we must get curves in the Y_1 homology class. Think of these as "meridians" on T_t. Likewise we get "longitudes" on T_t by intersecting with the 5-cycle $\{y_1 = 0\}$, and these lie in the Y_2 homology class. In similar fashion, we get curves on T_t in each homology class $[m_1 Y_1 + m_2 Y_2]$, where m_1 and m_2 are integers. These 1-dimensional cycles must have length at least $(m_1^2 + m_2^2)^{1/2}$, measured in the metric g_t, since the calibrating 1-form

$$(m_1^2 + m_2^2)^{-1/2}(m_1\beta_1 + m_2\beta_2)$$

shows this to be the minimum length of any 1-cycle in this class. In other words, all of the homologically non-trivial curves on the torus T_t are at least as long as their minimizing counterparts on T_0. It follows (with thanks to Chris Croke) that the area of T_t is at least as large as that of the flat torus T_0.

At time 0, the minimum length meridians and longitudes intersect. But as t increases, a unit-length Y_1 geodesic no longer intersects a unit-length Y_2 geodesic, and so T_t can no longer have both meridians and longitudes of length 1. As a consequence, the area of T_t must be larger than that of T_0.

The actual proof will use calibrations.

THEOREM D. *When $|t| < 2$, there is a 4-dimensional submanifold $\{x_1 = t/2,\ x_2 = 0\}$ of (M, g_t), foliated by flat 2-dimensional tori of area $1 + t^2/4$ running in the $Y_1 Y_2$ direction. They are all calibrated by the closed 2-form*

$$(1 + t^2/4)\beta_1\beta_2 + (t/2)(\beta_1\gamma_2 + \beta_2\gamma_1),$$

and are hence area-minimizing in the $Y_1 Y_2$ homology class. There are no other classical cycles which minimize area in this homology class.

Thus the manifolds (M, g_t) can be distinguished by the mass $1+t^2/4$ of the Y_1Y_2 homology class. There is no contradiction here with the fact that (M, g_t) and (M, g_r) are isometric whenever t and r have the same distance to their nearest integers. The isometry simply does not preserve the Y_1Y_2 homology class.

THEOREM E. *On (M, g_t), the left-invariant closed 2-current*

$$(X_1Y_1 - X_2Y_2) - (t/2)(X_1Z_1 - X_2Z_2)$$

has mass $\sqrt{4+t^2}$. It is calibrated by the closed 2-form

$$1/\sqrt{1+t^2/4}\{(\alpha_1\beta_1 - \alpha_2\beta_2) - (t/2)(\alpha_1\gamma_1 + \alpha_2\gamma_2) - (t^2/2)\alpha_2\beta_2\},$$

and therefore has minimum mass in its homology class, which is the same as the homology class of $X_1Y_1 - X_2Y_2$, since $X_1Z_1 - X_2Z_2$ is a boundary.

REMARK. The $X_1Y_1 - X_2Y_2$ homology class is not integral, but twice it is.

QUESTION. *Is there a classical cycle in the homology class $2[X_1Y_1 - X_2Y_2]$ with the minimum possible area $2\sqrt{4+t^2}$?*

We prove Theorem D.
We will show that the closed left-invariant 2-form

$$\varphi = (1+t^2/4)\beta_1\beta_2 + (t/2)(\beta_1\gamma_2 + \beta_2\gamma_1)$$

1) has comass 1 in the metric g_t, and
2) calibrates the closed 2-current

$$U = (Y_1 - (t/2)Z_1)(Y_2 - (t/2)Z_2).$$

Multiplying out, we get

$$U = Y_1Y_2 - (t/2)(Y_1Z_2 - Y_2Z_1) + (t^2/4)Z_1Z_2.$$

We saw in §2 that the 2-currents $Y_1Z_2 - Y_2Z_1$ and Z_1Z_2 are both boundaries. Hence U lies in the same homology class as Y_1Y_2.

To evaluate the comass of φ in the metric g_t, we first express it in terms of orthonormal coordinates with respect to that metric. That is, we replace γ_2 by $\gamma_2(t) - t\beta_2$, getting

$$\varphi = (1 - t^2/4)\beta_1\beta_2 + (t/2)(\beta_1\gamma_2(t) + \beta_2\gamma_1).$$

For the time being, we write

$$\varphi = a\beta_1\beta_2 + b(\beta_1\gamma_2(t) + \beta_2\gamma_1),$$

and will determine the coefficients a and b so as to satisfy conditions 1) and 2) above.

First, notice that $\varphi \wedge \varphi \wedge \varphi = 0$. Hence there are orthonormal left-invariant 1-forms ε_1, ε_2, ε_3 and ε_4, such that

$$\varphi = j\varepsilon_1\varepsilon_2 + k\varepsilon_3\varepsilon_4, \qquad j \geq k \geq 0.$$

In these coordinates, we have

$$\begin{aligned} \text{comass of } \varphi &= |\varphi|^* = j \\ \text{norm of } \varphi &= |\varphi| = \sqrt{j^2 + k^2} \\ \text{norm of } \varphi \wedge \varphi &= |\varphi \wedge \varphi| = 2jk. \end{aligned}$$

From the earlier coordinates, we have

$$\begin{aligned} |\varphi| &= \sqrt{a^2 + 2b^2} \\ |\varphi \wedge \varphi| &= 2b^2. \end{aligned}$$

To make φ have comass 1, we must therefore satisfy the equations

$$j = 1, \qquad j^2 + k^2 = a^2 + 2b^2, \qquad jk = b^2.$$

Thus

$$k = b^2 \quad \text{and} \quad 1 + b^4 = a^2 + 2b^2.$$

In other words, we guarantee that φ has comass 1 if we choose a and b so that $|b| \leq 1$ and $|a| = 1 - b^2$.

Now we want to arrange that φ calibrates the 2-current U in the metric g_t. We begin by expressing U in terms of orthonormal coordinates with respect to that metric. That is, we replace Y_2 by $Y_2(t)+tZ_2$, getting

$$U=(Y_1-(t/2)Z_1)(Y_2(t)+(t/2)Z_2).$$

Multiplying out, we get

$$U=Y_1Y_2(t)+(t/2)Y_1Z_2+(t/2)Y_2(t)Z_1-(t^2/4)Z_1Z_2.$$

Hence the norm of U in the metric g_t is

$$\begin{aligned}|U|&=\sqrt{1+(t/2)^2+(t/2)^2+(t^2/4)^2}\\&=1+t^2/4.\end{aligned}$$

For φ to calibrate U, we must have $\varphi(U)=|U|$. Now

$$\varphi(U)=a+b(t/2)+b(t/2)=a+bt.$$

Setting this equal to the norm of U, as calculated above, we get

$$a+bt=1+t^2/4.$$

So a and b must satisfy this equation, in addition to

$$|a|=1-b^2.$$

Solving, we get

$$a=1-t^2/4 \quad \text{and} \quad b=t/2$$

for the coefficients of φ. Note that $|t|<2$ implies $|b|<1$. Hence this 2-form φ calibrates the 2-current U, as claimed. It follows that, in (M, g_t), U has minimum mass in its homology class, which as we observed above is the same as the homology class of Y_1Y_2.

We now seek the classical cycles which minimize area in this homology class.

Recall the orthonormal left invariant 1-forms ε_1, ε_2, ε_3 and ε_4 such that

$$\varphi=j\varepsilon_1\varepsilon_2+k\varepsilon_3\varepsilon_4.$$

In determining φ, we arranged that $j=1$. The restriction $|t|<2$ guarantees that $0 \le k < 1$. It follows that, at each point, φ calibrates the 2-plane corresponding to $\varepsilon_1\varepsilon_2$, and nothing else. Hence the minimizing classical cycles which we seek, since they must also be calibrated by φ, must be tangent to this field of 2-planes.

Note that the Lie bracket

$$[Y_1-(t/2)Z_1,\ Y_2-(t/2)Z_2]=0,$$

so that this field of 2-planes provides a 2-dimensional foliation of M. Since we know that φ calibrates U, these 2-planes must be the ones corresponding to $\varepsilon_1\varepsilon_2$. Therefore the minimizing classical cycles will appear as compact leaves of this foliation.

Lift this foliation to a foliation on the Lie group G. The leaf through the identity of G is given by

$$(s_1, s_2) \to (0, 0, s_1, s_2, -ts_1/2, -ts_2/2).$$

Hence the leaf through the point $(x_1, x_2, y_1, y_2, z_1, z_2)$ of G is given by

$$\begin{aligned} h(s_1, s_2) &= (x_1, x_2, y_1, y_2, z_1, z_2)(0, 0, s_1, s_2, -ts_1/2, -ts_2/2) \\ &= (x_1, x_2, y_1+s_1, y_2+s_2, z_1+(x_1-t/2)s_1+x_2s_2, z_2+(x_1-t/2)s_2). \end{aligned}$$

This leaf projects to a closed surface in M in the homology class $[Y_1Y_2]$ if and only if there exist γ_1 and γ_2 in the lattice Γ such that $h(s_1+1, s_2)=\gamma_1 h(s_1, s_2)$ and $h(s_1, s_2+1)=\gamma_2 h(s_1, s_2)$ for all real s_1 and s_2. Now

$$h(s_1+1, s_2)=(0, 0, 1, 0, x_1-t/2, 0)\ h(s_1, s_2) \text{ and}$$
$$h(s_1, s_2+1)=(0, 0, 0, 1, x_2, x_1-t/2)\ h(s_1, s_2).$$

Thus the leaf descends to a compact surface in $[Y_1Y_2]$ if and only if $x_1-t/2$ and x_2 are integers. When this condition holds, we may left translate the leaf in G by an appropriate element of Γ so as to arrange that $x_1-t/2=0$ and $x_2=0$. Hence the leaf in G is given by

$$h(s_1, s_2)=(t/2, 0, y_1+s_1, y_2+s_2, z_1, z_2).$$

Dividing this leaf by the lattice

$$\{(0, 0, b_1, b_2, 0, 0): b_i \in Z\},$$

we obtain a torus leaf in M. Indeed this is just one of the flat tori running in the Y_1Y_2 direction, which fill the 4-dimensional submanifold $\{x_1 = t/2,\ x_2 = 0\}$.

These are the only classical cycles in M which lie in the homology class $[Y_1Y_2]$ and have minimum area $1 + t^2/4$ in the metric g_t.

This completes the proof of Theorem D.

Now we prove Theorem E.

We must show that the closed left invariant 2-form

$$\varphi = 1/\sqrt{1+t^2/4}\{(\alpha_1\beta_1 - \alpha_2\beta_2) - (t/2)(\alpha_1\gamma_1 + \alpha_2\gamma_2) - (t^2/2)\alpha_2\beta_2\}$$

1) has comass 1 in the metric g_t, and
2) calibrates the closed 2-current

$$U = (X_1Y_1 - X_2Y_2) - (t/2)(X_1Z_1 - X_2Z_2).$$

Writing

$$U = X_1(Y_1 - (t/2)Z_1) - X_2(Y_2(t) + (t/2)Z_2),$$

we see that U has mass $2\sqrt{1+(t/2)^2} = \sqrt{4+t^2}$ in the metric g_t. We also compute that $U(\varphi) = \sqrt{4+t^2}$. Thus we need only show that φ has comass one.

To evaluate the comass of φ in the metric g_t, we first express φ in terms of orthonormal coordinates with respect to that metric. That is, we replace γ_2 by $\gamma_2(t) - t\beta_2$, getting

$$\begin{aligned}\varphi &= 1/\sqrt{1+t^2/4}\{(\alpha_1\beta_1 - \alpha_2\beta_2) - (t/2)(\alpha_1\gamma_1 + \alpha_2\gamma_2(t))\}\\ &= \alpha_1(\beta_1 - (t/2)\gamma_1)/\sqrt{1+t^2/4} - \alpha_2(\beta_2 + (t/2)\gamma_2(t))/\sqrt{1+t^2/4}.\end{aligned}$$

This has the form $e_1^*e_2^* - e_3^*e_4^*$, where the e_i are orthonormal in the metric g_t, and therefore has comass 1.

It follows that the current U has minimum mass in its homology class, which coincides with the homology class of $X_1Y_1 - X_2Y_2$ because $X_1Z_1 - X_2Z_2$ is a boundary.

This completes the proof of Theorem E.

Of course, either Theorem D or Theorem E implies Theorem A.

To prove Theorem B, simply note that the area spectrum is countable, while the mass of the integral homology class $[Y_1Y_2]$ in $(M,\ g_t)$ is $\sqrt{1+t^2}$. It follows that the area spectrum must vary with t.

REFERENCES

[Be] P. BÉRARD, Spectral geometry: direct and inverse problems, Springer Lecture Notes No. 1207 (1986).

[Br] R. BROOKS, *Constructing isospectral manifolds,* Amer. Math. Monthly *95* (1988), 823–839.

[Br–Ts] R. BROOKS and R. TSE, *Isospectral surfaces of small genus,* Nagoya Math J. *107* (1987), 13–24.

[CdV] Y. COLIN DE VERDIÈRE, *Spectre du Laplacien et longuers des géodésiques périodiques,* II, Compositio Math. *27* (1973), 159–184.

[DeT–Go] D. DETURCK and C. S. GORDON, *Isospectral deformations I: Riemannian structures on two-step nilspaces,* Comm. Pure and Appl. Math. *40* (1987), 367–387.

[Du–Gu] J. DUISTERMAAT and V. GUILLEMIN, *The spectrum of positive elliptic operators and periodic bicharacteristics,* Invent. math. *29* (1975), 39–79.

[Fe1] H. FEDERER, Geometric measure theory, Springer-Verlag, New York (1969).

[Fe2] H. FEDERER, *Real flat chains, cochains and variational problems,* Indiana U. Math. J *24* (1974), 351–407.

[Fe–Fl] H. FEDERER and W. FLEMING, *Normal and integral currents,* Ann. of Math. *72* (1960), 458–520.

[Go] C. S. GORDON, *The Laplace spectra versus the length spectra of Riemannian manifolds,* Contemp. Math. *51* (1986) 63–80.

[Go–Wi] C. S. GORDON and E. N. WILSON, *Isospectral deformations of compact solvmanifolds,* J. Diff. Geom. *19* (1984), 241–256.

[Ha–La] R. HARVEY and H. B. LAWSON, Jr., *Calibrated geometries,* Acta Math. *147* (1981), 47–157.

[Ka] M. KAC, *Can one hear the shape of a drum?,* Amer. Math. Monthly *73* (1966), 1–23.

[Kn] M. KNESER, *Lineare Relationen zwischen Darstellungen quadratischer Formen,* Math. Ann. *168* (1967), 31–39.

[Mi] J. MILNOR, *Eigenvalues of the Laplace operator of certain manifolds,* Proc. Nat. Acad. Sci. USA *51* (1964), 542.

[No] K. NOMIZU, *On the cohomology of homogeneous spaces of nilpotent Lie groups,* Ann. of Math. *59* (1954), 531–538.

[Wt] E. WITT, *Eine Identität zwischen Modulformen zweiten Grades,* Abhandl. Math. Seminar Hamburg Univ., *14* (1941), 323–337.

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Received October 25, 1987