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Maximal subalgebras in the algebra of foliated vector fields of a Riemannian foliation

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The problem of determining all finite codimensional subalgebras of a given algebra of global vector fields is an old one. For the first time it was posed and solved for the algebra of all vectors fields by L. E. Purcell and M. E. Shanks in [8]. A whose series of papers followed, e.g. [1] and [2]. In this short note we describe all finite codimensional subalgebras of the algebra of foliated vector fields $\bar{\mathscr{X}}(M, F)$ of a Riemannian foliation F on a compact manifold M. For simplicity's sake we assume that all objects are smooth, i.e. of class C^∞.

We begin with some general properties of foliated vector fields. The basic foliation F_b by the closures of leaves is a singular Riemannian foliation (SRF), (cf. [5, 6]). The leaves of F_b can be grouped into classes (types) relative to their transverse structures, and the manifold itself is stratified, (cf. [3], [7]), i.e. two points belong to the same stratum iff the leaves to which they belong are of the same type. The strata are embedded submanifolds of M.

LEMMA 1. *Any global foliated vector field is tangent to the strata of the manifold M.*

Proof. Let us take a representative of a foliated vector field. Its flow preserves the foliation and maps closures of leaves into closures of leaves. Thus it must map each stratum M_α into itself, and therefore any representative must be tangent to the strata.

Moreover, we have the following proposition, (cf. [7]).

PROPOSITION 1. *Let (M, F) be a Riemannian foliation. Then the space of global foliated vector fields is transverse in each stratum to the closures of leaves.*

We shall call the closure of a leaf (or a leaf of the basic foliation) *isolated* if in some neighbourhood of this submanifold there are no other closures of the same type.

LEMMA 2. *There is only a finite number of isolated leaves of the basic foliation.*

Proof. It is a direct consequence of Molino's description of a neighbourhood of a leaf in *SRF*, (cf, [5, 6]).

For any leaf L of the foliation F_b, by A_L we denote the following subalgebra

$$\{X \in \bar{\mathscr{X}}(M, F) : X_{|L} \text{ is tangent to } L\}.$$

If the leaf is not isolated, then owing to Proposition 1, A_L is a proper subalgebra of finite codimension; moreover, Lemma 1 ensures that any foliated vector field is tangent to the isolated leaves of F_b.

Let L be a leaf of the basic foliation. The manifold L is foliated by the leaves of F, and the Lie algebra of all foliated vector fields of this foliated manifold we denote by $\bar{\mathscr{X}}(L, F)$. For Riemannian foliations we can prove the following:

PROPOSITION 2. *Any foliated vector field of the Lie algebra $\bar{\mathscr{X}}(L, F)$ can be extended to a global foliated vector field on (M, F).*

Proof. Let us first consider the local problem: how to prolong a foliated vector field X on (L, F) onto an open neighbourhood of the leaf L. To accomplish that we shall use Haefliger's local model of the holonomy pseudogroup, (cf. [4]). According to this theorem, in a neighbourhood of the leaf L the holonomy pseudogroup is equivalent to the pseudogroup generated by the action on the left of a dense subgroup Γ of G on the space $G \times_H V$, where V is a $\bar{q}$-dimensional vector space and H a compact subgroup of G. It is well-known that foliated vector fields on this neighbourhood correspond bijectively to Γ-invariant vector fields on $G \times_H V$ and the leaf L corresponds to the closure of the orbit $[(x, 0)]$, i.e. $G/H \subset G \times_H V$. Therefore the foliated vector field X defines a Γ-invariant vector field $\hat{X}$ on G/H. Using a G-invariant connection in the H-bundle $G \rightarrow G/H$ we lift the vector field $\hat{X}$ to a Γ-invariant vector field on G and next to the whole space $G \times_H V$. Thus we have constructed an extension $\tilde{X}$ of the vector field X to a foliated neighbourhood U of the leaf L.

Now, we would like to find a global foliated vector field $\bar{X}$ which is equal to $\tilde{X}$ on a foliated neighbourhood of L. To do that we need to construct a special basic function.

LEMMA 3. *Let U be any foliated open neighbourhood of L. Then there exists a smooth basic function λ whose support is contained in U and which is equal to* 1 *on a neighbourhood of L.*

Proof. Let $N(L)$ be the normal bundle of the compact submanifold L (the closure of a leaf of the foliation F). As the basic foliation F_b is *SRF*, other leaves of F_b, near to L, are on the sphere bundles of L. In fact, let $D_\varepsilon(L) =$

$\{v \in N(L): \|v\| < \varepsilon\}$ and $S_r(L) = \{v \in N(L): \|v\| = r\}$. From the compactness of L it follows that there exists $\varepsilon > 0$ such that the exponential mapping restricted to $D_\varepsilon(L)$ is a diffeomorphism onto the image. The neighbourhood $\exp(D_\varepsilon(L)) = D_\varepsilon^L$ of L is saturated, and leaves of F_b contained in D_ε^L are, in fact, in $\exp(S_r(L))$ for $r < \varepsilon$.

Since the leaf L is compact there exists r, $0 < r < \varepsilon$ such that $D_r^L \subset U$. Let $\delta < r$ and let us choose a smooth function $\lambda^\delta: [0, r^2] \to [0, 1]$.

$$\lambda^\delta(t) = \begin{cases} 1 & t \in [0, \delta^2] \\ 0 & t \in [(\delta + r - \delta/2)^2, r^2]. \\ \text{smooth elsewhere} & \end{cases}$$

Then we put

$$\lambda_L^{\delta r}(x) = \begin{cases} \lambda^\delta(\|\exp^{-1}(x)\|^2) & x \in D_r^L \\ 0 & x \in \backslash D_r^L. \end{cases}$$

The support of the function $\lambda_L^{\delta r}$ is contained in D_r^L and the function $\lambda_L^{\delta r}$ is equal to 1 on D_δ^L.

We continue the proof of Proposition 2. The support of $\tilde{X}$ is a neighbourhood of the compact leaf L. Therefore there exists $r < \varepsilon$ such that D_r^L is contained in $\operatorname{supp} \tilde{X}$. Thus the global foliated vector field $\lambda_L^{\delta r}\tilde{X}$ is the one we have been looking for.

Using Lemma 3 we can construct partitions of unity with the following properties.

LEMMA 4. *For any locally finite covering $\mathcal{U}$ by open (saturated) subsets there exists a partition of unity $\{\lambda_i\}$ subordinated to $\mathcal{U}$ such that for any basic function λ_i there exist a leaf L_i of the basic foliation F_b and $\varepsilon_i > 0$ for which supp $\lambda_i \subset D_{\varepsilon_i}^{L_i}$.*

Let B be a proper subalgebra of finite codimension of the Lie algebra $\mathfrak{X}(M, F)$. An open saturated subset U satisfying the following condition (P):

(P) there are a global foliated vector field Y of B and a basic function f such that $Y(f)$ does not vanish on U.

is called *admissible* for the subalgebra B. Using the same method as in [1] we can prove that for any admissible subset U for B the subalgebra A_U, $A_U = \{X \in \mathfrak{X}(M, F): \operatorname{supp} X \subset U\}$ is a subalgebra of B. The main point in the proof of this fact is the following geometrical property.

LEMMA 5. *Let U be an admissible open subset. Then for any foliated vector field X with support contained in U there exists a global basic function g with support contained in U and equal to $1/Y(f)$ on a neighbourhood of support X.*

Proof. Using the partition of unity from Lemma 4, we can reduce our considerations to $U = D_\varepsilon^L$. Then, supp X being compact is contained in some D_δ^L, for $\delta < \varepsilon$. Thus the following function g

$$g = \begin{cases} 1/Y(f) \cdot \lambda_L^{\delta\varepsilon} & \text{on } D_\varepsilon^L \\ 0 & \text{on } \backslash D_\varepsilon^L \end{cases}$$

is the required one.

Remark. In our considerations we shall meet only admissible subsets of the form D_ε^L.

Now we turn our attention to subalgebras of finite codimension.

LEMMA 6. *Let B be a subalgebra of finite codimension of $\bar{\mathscr{X}}(M, F)$. If there exists an element Y of B which is not zero on some non-isolated leaf L of F_b, then there exists an open saturated neighbourhood of L which is admissible for B.*

Proof. There is a representative of Y, denoted by the same letter, which is non-zero on some neighbourhood U_0 of L. Since the manifold L is compact there exists an $\varepsilon > 0$ such that $\bar{D}_\varepsilon^L$ is contained in U_0. It is easy to find a foliated function f for which the function $Y(f)$ is non-vanishing on D_ε^L. In fact, as L is a non-isolated closure, then through D_ε^L pass other leaves of the same type; the trace of this stratum is of the form $\exp(D_\varepsilon(L) \cap V)$ where V is a vector subbundle of $N(L)$. The vector field Y is tangent to $\exp(D_\varepsilon(L) \cap V)$. Then for one of the functions $f(y) = \operatorname{dist}(y, L_0)^2$, where L_0 is a leaf of the same type as L, $Y(f)(x) \neq 0 \; \forall x \in L$. Then by choosing a smaller neighbourhood of U we obtain that the foliated function $Y(f)$ does not vanish on U.

Using a similar argument as in [1] we can demonstrate the following characterization of subalgebras of finite codimension.

THEOREM 1. *Let (M, F) a Riemannian foliation on a compact manifold. Let B be a proper subalgebra of finite codimension of $\bar{\mathscr{X}}(M, F)$. Then B is a proper subalgebra of A_L, L a leaf of the basic foliation F_b associated to F, and the restrictions of vector fields of B to L form a finite dimensional Lie algebra.*

The proof is an adaptation of the argument of Amemiya in which we use the following two facts:

i) if U is an admissible subset, then $A_U \subset B$;

ii) for any locally finite covering of the manifold M by F_b-saturated subsets there exists a partition of unity consisting of foliated functions subordinated to this covering.

Let us assume that for any isolated leaf L of F_b the restrictions of elements of B form the whole Lie algebra $\bar{\mathscr{X}}(L, F)$. Then we shall show that there exists a non-isolated leaf L for which $B \subseteq A_L$. Assume the contrary. Then any foliated vector field X can be represented as $X_0 + \sum_{i=1}^{\infty} X_i$ where $X_i, X_0 \in B$, and there exists a locally finite covering $\mathscr{U} = \{U_i\}$ of the set $M - \bigcup L_j$, L_j-the isolated leaves of F_b, such that $\operatorname{supp} X_i \subset U_i$, for $i = 1, \ldots, n, \ldots$. Since the subalgebra B is of finite codimension, for some k, the foliated vector field $\sum_{i=k}^{\infty} X_i$ must belong to B. Hence X is an element of B. Contradiction.

Remark. We cannot obtain the fact that the vector fields of the subalgebra B vanish on some leaf of F or F_b. It is sufficient to take a subalgebra $\tilde{A}_L = \{X \in \bar{\mathscr{X}}(M, F) : X_{|L} \equiv 0\}$ and any subalgebra C of foliated vector fields on L. C defines a subalgebra $\tilde{C}$ of global foliated vector fields extending the vector fields of the subalgebra C. Then the subalgebra generated by $\tilde{A}_L \oplus \tilde{C}$ is contained in A_L and of finite codimension. This leads us to the following theorem.

THEOREM 2. *Let B a maximal proper subalgebra of $\bar{\mathscr{X}}(M, F)$. Then B is equal to A_L for some non-singular leaf L of the basic foliation F_b, or the restrictions of vector fields of B to a singular leaf L form a maximal proper subalgebra of $\bar{\mathscr{X}}(L, F)$.*

The final remark. The same characterization of maximal subalgebras can be obtained for complete G-foliations whose central sheaf has compact orbits, G a Lie group of finite type, (cf. [9]). The proof mimicks the one just presented.

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