Comment. Math. Helvetici 64 (1989) 496–507 0010–2571/89/030496–12$01.50 + 0.20/0

Stable splittings associated with Chevalley groups, II

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Introduction

This is the second in a series of papers which studies the stable splittings of the classifying spaces of certain 2-groups E of symplectic type, namely those which are associated to quadratic forms Q over $\mathbf{F}_2$. These groups are central extensions of elementary abelian groups by a group of order 2. Examples are the extra-special 2-groups. The corresponding groups of outer automorphisms which fix the center are the orthogonal groups of the form Q. Since their commutator subgroups are Chevalley groups, their groups rings over F_2 contain Steinberg idempotents. In [FP1], hereafter referred to as I, we calculated the image of the resulting stable summands of BE if the quadratic form is of real type. In this paper, we calculate the image in the other two cases, where the quadratic form is of complex or quaternionic type. We refer to I throughout this paper for notation as well as for background material on the structure of the Steinberg idempotent.

In Section 1 we compute the cohomology of eBE in case the associated quadratic form is of complex type, i.e., $Q = x_0^2 + x_1x_{-1} + \cdots + x_mx_{-m}$. We also determine the cohomology of eBV for the underlying vector space V of Q. In Section 2 we extend these results to the quaternionic or Arf invariant one case, i.e., $Q = x_1x_{-1} + \cdots + x_mx_{-m} + x_m^2 + x_{-m}^2$. The proofs in this case are strikingly more complicated since the associated Chevalley groups are twisted. We begin with $m > 1$ since $E(1)^-$ is the quaternion group of order 8 treated in [MP1]. Section 3 is devoted to describing explicit splittings for $BE(3)$ and $BE(4)^-$.

All spaces are localized at 2 and all cohomology groups are taken with simple coefficients in $\mathbf{F}_2$.

§1. The complex case

Recall from I that $V = (\mathbf{Z}/2)^n$ has basis $\{v_0, v_{\pm i}, i = 1, \ldots, m\}$ with dual basis $\{x_0, x_{\pm i}, i = 1, \ldots, m\}$ where $n = 2m + 1$. The extension

$$0 \to \mathbf{Z}/2 \to E(n) \to V \to 0$$

* The authors are partially supported by NSF Grants.

is given by the quadratic form $Q = x_0^2 + \sum_{i=1}^m x_i x_{-i}$. From I (3.5) we have that the image of the Steinberg idempotent $e \in \mathbf{F}_2 0_n(\mathbf{F}_2)$ equals the intersection of the images of the idempotents e_i, $i = 1, \ldots, m$. We refer to these and all other idempotents that correspond to the nodes of a Dynkin diagram as *nodal idempotents*. Similarly, the corresponding subgroups are called *nodal subgroups*. Recall that $\alpha_m = \sum x_{i_1}^{-1} x_{i_2}^{-1} \cdots x_{i_m}^{-1}$ where $i_j = \pm j$ and the number of minus signs is even, and $\beta_m = \sum x_{i_1}^{-1} \cdots x_{i_m}^{-1}$ where $i_j = \pm j$ and the number of minus signs is odd.

LEMMA 1.1. $(\alpha + \beta)e = (\alpha + \beta)$.

Proof. $\alpha e_i = \alpha$ and $\beta e_i = \beta$ for $i = 1, \ldots, m-1$ by the proof of Lemma 4.3 in I. Hence we need only check that $(\alpha + \beta)e_m = \alpha + \beta$ where $e_m = (1 + u_{m,m})(1 + \sigma_{m,m})$. (Here $\sigma_{m,m}$ is given by $x_{\pm m} \mapsto x_{\mp m}$ and the identity on all other generators. $u_{m,m}$ is given by $x_0 \mapsto x_0 + x_{-m}$, $x_m \mapsto x_m + x_{-m}$ and the identity on all other generators.)

$$\alpha + \beta = \alpha_{m-1} x_m^{-1} + \beta_{m-1} x_{-m}^{-1} + \beta_{m-1} x_m^{-1} + \alpha_{m-1} x_{-m}^{-1}.$$

Hence

$$\begin{aligned}
(\alpha + \beta)e_m &= (\alpha + \beta)(1 + u_{m,m})(1 + \sigma_{m,m}) \\
&= \alpha_{m-1}(x_m^{-1} + x_{-m}^{-1}) + \beta_{m-1}(x_m^{-1} + x_{-m}^{-1}) + \alpha_{m-1}((x_m + x_{-m})^{-1} + x_{-m}^{-1}) \\
&\quad + \beta_{m-1}((x_m + x_{-m})^{-1} + x_{-m}^{-1}) + \alpha_{m-1}(x_m^{-1} + x_{-m}^{-1}) \\
&\quad + \beta_{m-1}(x_m^{-1} + x_{-m}^{-1}) + \alpha_{m-1}((x_m + x_{-m})^{-1} + x_m^{-1}) \\
&\quad + \beta_{m-1}((x_m + x_{-m})^{-1} + x_m^{-1}) \\
&= \alpha + \beta.
\end{aligned}$$

Let $C_m = \mathbf{F}_2\langle Sq^I \mid I \text{ admissible}, l(I) = m, i_m \geq 2\rangle$.

LEMMA 1.2. *Suppose $Sq^I \in C_m$. Then $Sq^I(x_0(\alpha_m + \beta_m)) \in im\ (e)$.*

Proof. Let $\delta = (\alpha_{m-1} + \beta_{m-1})$.

$$\begin{aligned}
x_0(\alpha + \beta)e_m &= \delta x_0(x_m^{-1} + x_{-m}^{-1})e_m \\
&= \delta[x_0(x_m^{-1} + x_{-m}^{-1}) + (x_0 + x_{-m})((x_m + x_{-m})^{-1} + x_{-m}^{-1})] \\
&\quad + [x_0(x_m^{-1} + x_{-m}^{-1}) + (x_0 + x_m)((x_m + x_{-m})^{-1} + x_{-m}^{-1})] \\
&= x_0(\alpha + \beta) + \delta.
\end{aligned}$$

Since $Sq^I\delta = 0$ by degree considerations, we have $Sq^I(x_0(\alpha+\beta)) \in \operatorname{im} e_m$. Since every element in e_i fixes x_0 for $i = 1, \ldots, m-1$, and $\alpha+\beta \in \operatorname{im} e_i$, we have $Sq^I(x_0(\alpha+\beta)) \in \operatorname{im} e$ by I (3.5).

Remark. $Sq^I(x_0(\alpha+\beta))$ only contains terms with nonnegative exponents. (This follows from [Kl, Prop. 2.12].)

THEOREM 1.3. *For* $n = 2m+1$

i) $H^*(eBE) = [C_m(\alpha+\beta) \oplus C_m(x_0(\alpha+\beta))] \otimes \mathbf{F}_2[\omega_{2^{m+1}}]$
ii) $H^*(eBV) = [C_m(\alpha+\beta) \oplus C_m(x_0(\alpha+\beta))] \otimes$

$$\mathbf{F}_2[Q, Sq^1Q, \ldots, Sq^{2^{m-1}} \cdots Sq^1Q]$$

Proof. We recall from I (1.4) that Δ_n is the unique irreducible representation of E which is nontrivial on b_0^2. Its Stiefel–Whitney classes are therefore invariant under $0_n(\mathbf{F}_2)$. Similarly Q is invariant. Hence one inclusion follows from lemmas 1.1 and 1.2. We now set about proving the other inclusion.

Let $S = \langle b_0, b_1, \ldots, b_m\rangle$ be the *top row subgroup* of E. Then $S = T \times \langle b_0\rangle$ where $\langle b_1, \ldots, b_m\rangle = T \cong (\mathbf{Z}/2)^m$ and $\langle b_0\rangle \cong \mathbf{Z}/4$. Hence $H^*(BS) = \mathbf{Z}/2[s_1, \ldots, s_m, y] \otimes E(x)$ where $\deg s_i = 1$, $\deg x = 1$, $\deg y = 2$. Here x_0 restricts to x, x_i restricts to s_i, $i = 1, \ldots, m$, and x_{-i} restricts to 0, $i = 1, \ldots, m$.

PROPOSITION 1.4. *The image of* $H^*(eBE)$ *in* $H^*(BS)$ *is* $[C_m(s_1^{-1} \cdots s_m^{-1}) \oplus xC_m(s_1^{-1} \cdots s_m^{-1})] \otimes \mathbf{F}_2[\omega'_{2^{m+1}}]$ *where* $\omega'_{2^{m+1}}$ *is the image of* $\omega_{2^{m+1}}$.

Proof. Let $\zeta: S \to E$ be the inclusion. Since x_0^2, $x_{-i} \in \ker \zeta^*$, we have

$$\zeta^*(C_m(\alpha+\beta)) = C_m(s_1^{-1} \cdots s_m^{-1})$$
$$\zeta^*(C_m x_0(\alpha+\beta)) = C_m x(s_1^{-1} \cdots s_m^{-1}).$$

There is an inclusion of GL_m in $0_n(\mathbf{F}_2)$ given by the hyperbolic map H (see I, §2; 0_{2m}^+ is included in 0_n in the natural way). Since GL_m normalizes the top row subgroup S, the following diagram commutes for $i = 1, \ldots, m-1$:

$$\begin{array}{ccc} H^*(BT) \otimes H^*(B\langle b_0\rangle) & \xrightarrow{l_i} & H^*(BT) \otimes H^*(B\langle b_0\rangle) \\ \uparrow & & \uparrow \\ H^*(BE) & \xrightarrow{e_i} & H^*(BE) \end{array}$$

Here l_i is the Steinberg idempotent for the ith node in the A_{m-1} diagram. By Kuhn [K], $\bigcap \operatorname{im} l_i = (\operatorname{im} \mathrm{St}) \otimes H^*(\langle b_0\rangle)$, where St is the Steinberg idempotent for

GL_m. Hence the image of $H^*(eBE)$ is contained in $H^*(M(m)) \otimes H^*(B\langle b_0\rangle)$, where $H^*(M(m)) = \mathbf{F}_2\langle Sq^I(s_1^{-1}\cdots s_m^{-1}) \mid l(I)=m,\ i_1 \geq 1\rangle$ by [MP].

Now consider the last idempotent e_m. A simple calculation shows that the restriction of ime_m to $H^*(BS)$ lies in the ideal generated by s_m. (This uses the relation $x_0^2 = \sum_{i=1}^m x_i x_{-i}$.) Note that every term in $Sq^I(s_1^{-1}\cdots s_m^{-1})$ is divisible by s_m iff $i_1 \geq 2$. Thus the restriction of ime to the top row subgroup lies in $C_m(s_1^{-1}\cdots s_m^{-1}) \otimes H^*(B\langle b_0\rangle)$.

Proof of Theorem 1.3 (Conclusion). We now show that the Poincaré series of $H^*(eBE(n))$ equals that of $D = [C_m(\alpha+\beta) \oplus C_m(x_0(\alpha+\beta))] \otimes \mathbf{F}_2[\omega_{2^{m+1}}]$. The method used is a variation of the argument given in I, §4. We note the main differences.

Let $R = H^*(BE(n))$ and $R' = \mathbf{F}_2[\omega_{2^{m+1}}, \omega_{2^{m+1}-2}, \ldots, \omega_{2^m}] \subset R$, where $\{\omega_{2^{m+1}}, \omega_{2^{m+1}-2^i}\}$ are the Stiefel–Whitney classes of Δ_n. Let $S = H^*(BV)$ and

$$P = \mathbf{F}_2[Q, Sq^1Q, \ldots, Sq^{2^{m-1}}\cdots Sq^1Q]. \text{ By [Q], } \{\omega_{2^m}, \ldots, \omega_{2^{m+1}}\}$$

is a regular sequence in R and $\{Q, \ldots, Sq^{2^m}\cdots Sq^1Q\}$ is a regular sequence in S. As in I, §4, it follows that the Poincaré series of the Steinberg summand is given by $F(Re;t) = g(t)F(R';t)$ where $g(t)$ is a polynomial with positive coefficients satisfying $g(1) = 2^{m(m+1)/2+1}$. Similarly $F(D;t) = f(t)F(R';t)$ where $f(t)$ is a polynomial with positive coefficients satisfying $f(1) = g(1)$. Since $F(D;t) \leq F(\mathrm{Re};t)$ we need only show that $f(t) \leq g(t)$. But this is clear since by Proposition 1.4. Any element in Re differs from one in D by an element which restricts trivially to the top row subgroup; and $\omega_{2^{m+1}-2^i}$ restricts to the square of the Dickson invariant $\omega_{2^m-2^{i-1}}$ (reg), where reg is the regular representation of $\langle b_1, \ldots, b_m\rangle$, by [Q]. Thus the R' indecomposable elements of D remain indecomposable in *ime*.

Essentially the same argument proves ii).

Remark. We note that the subgroup S detects $H^*(eBE)$. Also since $|U_{2m+1}| = 2^{m^2}$, BE contains 2^{m^2} wedge summands each equivalent to eBE (cf. proof of I, Th. 4.1).

§2. The quaternionic case

We recall from I that $V = (\mathbf{Z}/2)^n$, $n = 2m$, has basis $\{v_{\pm i},\ i = 1, \ldots, m;\ m > 1\}$ with dual basis $\{x_{\pm i},\ i = 1, \ldots, m\}$. The extension

$$0 \to \mathbf{Z}/2 \to E(n)^- \to V \to 0$$

is given by the quadratic form $Q=\sum_{i=1}^{m} x_i x_{-i}+x_m^2+x_{-m}^2$. From I (3.6), the image of the Steinberg idempotent $e_m^- \in \mathbf{F}_2 0_{2m}^-(\mathbf{F}_2)$ equals the intersection of the images of the nodal Steinberg idempotents e_i, $1 \leq i \leq m-2$ and e'_m.

For simplicity we write $E=E(n)^-$ and $e=e_m^-$. Let

$$D_m=\mathbf{F}_2\langle Sq^K \mid K \text{ admissible of length } m,\ k_m \geq 4\rangle$$

and set

$$\delta=\delta_m=(\alpha_{m-1}+\beta_{m-1})(x_m^3+x_{-m}^3+x_m^2 x_{-m})+\alpha_{m-1}(x_m^2 x_{-m}+x_m x_{-m}^2).$$

Our main result in this section is the following.

THEOREM 2.1. i) *$H^*eBE=[C_{m-1}(\alpha_{m-1}+\beta_{m-1})\oplus D_{m-1}\delta_m]\otimes \mathbf{F}_2[\omega_{2^{m+1}}]$ where $\omega_{2^{m+1}}$ is the Stiefel–Whitney class of Δ_n^-, the unique faithful irreducible representation of E.*

ii) *$H^*eBV=[C_{m-1}(\alpha_{m-1}+\beta_{m-1})\oplus D_{m-1}\delta_m]\otimes \mathbf{F}_2[Q, Sq^1Q, \ldots, Sq^{2^{m-1}}\cdots Sq^2Sq^1Q]$*

LEMMA 2.2. *The elements of $C_{m-1}(\alpha_{m-1}+\beta_{m-1})$ and $D_{m-1}\delta_m$* are in the image of e.

Proof. α_{m-1} and β_{m-1} are fixed by the nodal idempotents $e_1, \ldots, e_{m-2}$ (as in I, Lemma 4.3). Since x_m and x_{-m} are invariant under the action of the corresponding nodal subgroups, $C_{m-1}(\alpha_{m-1}+\beta_{m-1})$ and $D_{m-1}\delta_m$ are in the image of e_i for $i=1, \ldots, m-2$. By I (3.6), we need only show that these elements are in the image of the last nodal idempotent e'_m.

The calculation becomes considerably easier if we change the basis for H^*BE. Let $a=x_{m-1}$, $b=x_{-(m-1)}$, $c=x_{m-1}+x_{-(m-1)}+x_m+x_{-m}$, $d=x_{m-1}+x_{-(m-1)}+x_{-m}$, and let $r=a+b+c+d$. Note that the orbit of x_{m-1} is $\{a, b, c, d, r\}$. Indeed, using this new basis one can check easily that $0_4^-(\mathbf{F}_2)$ is isomorphic to Σ_5.

In this new basis, we have $\tau=(a, c)(d, r)$, $\gamma=(a, r)(d, c)$, $H=(c, r, d)$ and $W=(a, b)(c, r)$ where

$$e'_m=(1+\tau)(1+\gamma)(1+H+H^2)(1+W)$$

by I (3.6). It is then easy to show

$$(a^{-1})e'_m=a^{-1}+b^{-1}.$$

Since $x_{\pm i}$, $i = 1, \ldots, m-2$ are invariant under the last nodal subgroup, and

$$(\alpha_{m-1} + \beta_{m-1}) = (\alpha_{m-2} + \beta_{m-2})(a^{-1} + b^{-1})$$

we have that $C_{m-1}(\alpha_{m-1} + \beta_{m-1})$ is in ime'_m.

A somewhat more complicated calculation shows that

$$\begin{aligned} h \equiv [a^{-1}(d^2r + b^3)]e'_m &= a^{-1}(c^2d + d^2r + r^2c + b^3) \\ &+ b^{-1}(d^2c + r^2d + c^2r + a^3) + c^{-1}(r^2d + a^2r + d^2a + r^2b + d^2r + b^2d + a^3 + b^3) \\ &+ d^{-1}(a^2c + c^2r + b^2r + c^2b + r^2c + r^2a + a^3 + b^3) \\ &+ r^{-1}(c^2a + d^2c + a^2d + c^2d + b^2c + d^2b + a^3 + b^3) \end{aligned}$$

Now upon substituting $r = a + b + c + d$ in the first four expressions, $d = a + b + c + r$ in the last and changing back to the original basis we obtain

$$h = x_{m-1}^{-1}(x_{-m}^3 + x_m^3 + x_{-m}^2 x_m) + x_{-(m-1)}^{-1}(x_{-m}^3 + x_m^3 + x_m^2 x_{-m}) + t$$

where t is a sum of terms of degree 2 in $\mathbf{F}_2[x_{\pm(m-1)}, x_{\pm m}]$. In particular, no term of t contains a negative exponent.

We note that $q \equiv \sum_{i=m-1}^{m} x_i x_{-i}(x_i + x_{-i})$ is invariant under the last nodal subgroup. Hence $l \equiv (x_{m-1}^{-1} + x_{-(m-1)}^{-1})q = (x_{m-1}^{-1} + x_{-(m-1)}^{-1})(x_m^2 x_{-m} + x_m x_{-m}^2) + x_{m-1}^2 + x_{-(m-1)}^2$ is in the image of e'_m. Thus

$$\delta + (\alpha_{m-2} + \beta_{m-2})t + \beta_{m-2}(x_{m-1}^2 + x_{-(m-1)}^2) = (\alpha_{m-2} + \beta_{m-2})h + \beta_{m-2}l$$

is in the image of e'_m since α_{m-2}, β_{m-2} are invariant under e'_m.

Since $Sq^I\alpha_{m-2} = Sq^I\beta_{m-2} = 0$ if I is admissible of length $m-1$, an inductive argument with the Cartan formula shows $Sq^K((\alpha_{m-2} + \beta_{m-2})t) = Sq^K(\beta_{m-2}(x_{m-1}^2 + x_{-(m-1)}^2)) = 0$ for K admissible of length $m-1$ and $k_{m-1} \geq 4$. Hence $Sq^K(\delta) \in im\, e'_m$. The lemma follows by noting that $Sq^K(\delta) \in \mathbf{F}_2[x_{\pm 1}, \ldots, x_{\pm m}]$, i.e., that no negative exponents occur [K1; 2.12].

We now calculate the restriction of $H^*(eBE)$ to the top row subgroup $S = \langle b_1, \ldots, b_{m-1}\rangle \times \langle b_m, b_{-m}\rangle \subset E$. Here $\{b_{\pm i}\}$ are generators of E (see I (1.3)). (This subgroup is the centralizer of the maximal elementary abelian subgroup containing $\langle b_1, \ldots, b_{m-1}\rangle$ [Q].) S is isomorphic to $(\mathbf{Z}/2)^{m-1} \times Q_8$, where Q_8 denotes the quaternion group of order 8. Thus

$$H^*(BS) \approx \mathbf{F}_2[s_1, \ldots, s_{m-1}] \otimes \mathbf{F}_2[a, b]/(a^2 + ab + b^2, a^2b + ab^2) \otimes \mathbf{F}_2[c_4]$$

where $\deg s_i = \deg a = \deg b = 1$ and $\deg c_4 = 4$. Here x_i restricts to s_i, $i = 1, \ldots, m-1$, x_m and x_{-m} restrict to a and b respectively and x_{-i} restricts to 0 for $i = 1, \ldots, m-1$. The following lemma is easily verified.

LEMMA 2.3. *Let f be a polynomial in the $x_{\pm i}$, $i = 1, \ldots, m$. Then $f \equiv f_1 + f_2x_m + f_3x_{-m} + f_4x_m^2 + f_5x_mx_{-m} + f_6x_m^2x_{-m} \bmod (Q, Sq^1Q)$, where $f_1, \ldots, f_6$ are polynomials in $x_{\pm 1}, \ldots, x_{\pm(m-1)}$.*

PROPOSITION 2.4. *Let $x \in \operatorname{im} e'_m$. Then the restriction of x to the top row subgroup is a sum of terms of the following two types.*

1) ls^k, $k \geq 1$
2) ls^ka^2b, $k \geq 3$

Here l is a monomial in $\mathbf{F}_3[s_1, \ldots, s_{m-2}]$ and $s = s_{m-1}$.

Proof. One uses the following table to compute the action of e'_m. The preceding lemma and the relations $0 = a^3 = b^3 = a^2b^2 = ab^2 + ba^2 = a^2 + ab + b^2$ play key roles in limiting the cases one must check. Note that $(x^r_{m-1}x^t_{-(m-1)}x^2_mx_{-m})e'_m$ has image containing terms of type 2) iff $r \equiv 0,\ 3 \bmod 4$ and $r > 0$.

THEOREM 2.5. *The image of $H^*(eBE)$ in $H^*(BS)$ is $[C_{m-1}(s_1^{-1}\cdots s_{m-1}^{-1} \oplus D_{m-1}(s_1^{-1}\cdots s_{m-1}^{-1})a^2b] \otimes \mathbf{F}_2[\omega'_{2^{m+1}}]$ where $\omega'_{2^{m+1}}$ is the image of $\omega_{2^{m+1}}$.*

Proof. That all such elements are in the image follows from Lemma 2.2. That no other elements occur follows by noting that the Steinberg idempotent l_{m-1} for GL_{m-1} has image $\mathbf{F}_2\langle Sq^I(s_1^{-1}s_2^{-1}\cdots s_{m-1}^{-1}) \mid I \text{ admissible of length } m-1\rangle \otimes H^*(Q_8)$ when it acts on the cohomology of S. Since the nodal idempotents $e_1, \ldots, e_{m-1}$ are the images of the nodal idempotents for GL_{m-1} under the hyperbolic map $H: GL_{m-1} \to O^-_{2m}$, we have by [K] that the image of $H^*(eBE)$ must be contained in the image of the Steinberg idempotent for GL_{m-1} (as in the proof of 1.3). Applying the preceding proposition gives us our results.

Proof of Theorem 2.1. We show that the Poincaré series of $H^*(eBE)$ equals that of $[C_{m-1}(\alpha_{m-1} + \beta_{m-1}) \oplus D_{m-1}\delta] \otimes \mathbf{F}_2[\omega_{2^{m+1}}] \equiv J$. The method used is a variation of the argument given in [I, §4]. We note the main differences.

Let $R = H^*(BE(2m)^-)$ and $R' = \mathbf{F}_2[\omega_{2^{m+1}}, \omega_{2^{m+1}-4}, \ldots, \omega_{2^m}] \subset R$, where $\{\omega_{2^{m+1}-2^i}\}$ are the Stiefel–Whitney classes of Δ^-_{2m}. Since Δ^-_{2m} is the unique irreducible faithful representation of $E(2m)^-$, R' is a subring of the ring of invariants of R under O^-_{2m}. By [Q], $\{\omega_{2^m}, \ldots, \omega_{2^{m+1}}\}$ is a regular sequence in R and $\{Q, Sq^1Q, \ldots, Sq^{2^{m-1}}\cdots Sq^1Q\}$ is a regular sequence in $H^*(BV) \equiv S$. Let $P = \mathbf{F}_2[Q, \ldots, Sq^{2^{m-1}}\cdots Sq^1Q] \subset S$. As in the complex case (§1), it follows that

group element of e'_m	restriction of element applied to x_m	x_{-m}	x_{m-1}	$x_{-(m-1)}$
1	a	b	s	0
τ	a	b	$a+b+s$	0
γ	a	b	$a+s$	0
$\tau\gamma$	a	b	$b+s$	0
H	b	$a+b$	s	0
τH	b	$a+b$	$a+s$	0
γH	b	$a+b$	$b+s$	0
$\tau\gamma H$	b	$a+b$	$a+b+s$	0
H^2	$a+b$	a	s	0
τH^2	$a+b$	a	$b+s$	0
γH^2	$a+b$	a	$a+b+s$	0
$\tau\gamma H^2$	$a+b$	a	$a+s$	0
W	$a+b$	b	0	s
τW	$a+b+s$	$b+s$	$a+s$	s
γW	$a+b$	$b+s$	$a+b+s$	s
$\tau\gamma W$	$a+b+s$	b	$b+s$	s
HW	b	a	0	s
τHW	$b+s$	$a+s$	$a+b+s$	s
γHW	b	$a+s$	$b+s$	s
$\tau\gamma HW$	$b+s$	a	$a+s$	s
H^2W	a	$a+b$	0	s
τH^2W	$a+s$	$a+b+s$	$b+s$	s
γH^2W	a	$a+b+s$	$a+s$	s
$\tau\gamma H^2W$	$a+s$	$a+b$	$a+b+s$	s

the Poincaré series of Re is given by $F(\mathrm{Re}; t) = g(t)F(R'; t)$ where $g(t)$ is a polynomial with positive coefficients satisfying $g(1) = 2^{m(m+1)/2}$. Similarly $F(J; t) = f(t)F(R'; t)$ where $f(t)$ is a polynomial with positive coefficients satisfying $f(1) = g(1)$. Since $J \subset \mathrm{Re}$ we need only show that $f(t) \leq g(t)$. However, any element in Re differs from one in J by an element which restricts to zero on the top row subgroup; and $\omega_{2^{m+1}-2^i}$ restricts to the fourth power of the Dickson invariant $\omega_{2^{m-1}-2^{i-2}}$ (reg), where reg is the regular representation of $\langle b_1, \ldots, b_{m-1} \rangle$, by [Q]. So $f(t) \leq g(t)$ follows easily.

Essentially the same argument proves (ii).

Remark. Since $|U_{2m}^-| = 2^{m(m-1)}$, BE contains $2^{m(m-1)}$ wedge summands each equivalent to eBE (cf. proof of I, Th. 4.1).

§3. Splitting $BE(3)$ and $BE(4)^-$

In this section we obtain explicit splittings of $BE(3)$, $BE(4)^-$. Some of the arguments, e.g., those involving Molien's series, are analogous to those of I, §5 for $BE(4)^+$; others make use of special properties of the automorphism groups of $E(3)$, $E(4)^-$.

Splitting $BE(3)$

As noted in I, Prop. 2.1, the subgroup of outer automorphisms of $E(3)$ which act trivially on the center is isomorphic to $O_3(\mathbf{F}_2)$, the orthogonal group preserving the form $Q=x_0^2+x_1x_{-1}$. This group is isomorphic to Σ_3 and is generated by

$$\sigma: x_0 \mapsto x_0+x_1, \quad x_1 \mapsto x_1+x_{-1}, \quad x_{-1} \mapsto x_1$$

$$\tau: x_0 \mapsto x_0+x_1, \quad x_1 \mapsto x_1, \quad x_{-1} \mapsto x_1+x_{-1}$$

Let $H=\langle\sigma\rangle \approx \mathbf{Z}/3$. A Molien series argument shows that $H^*(BE(3))^H$ has Poincaré series $(1+t^3)/(1-t)(1-t^4)$, where the generators can be chosen to be $u_1=x_0+x_1+x_{-1}$, $d_3=x_1^3+x_{-1}^3+x_{-1}^2x_1$ and $\omega_4(\Delta)$, where Δ is the unique irreducible faithful representation of $E(3)$. The relation $u^6=d^2$ holds. Thus

$$H^*(BE(3))^H=\mathbf{F}_2[u, d]/(u^6-d^2) \otimes \mathbf{F}_2[\omega_4].$$

Let $e_0=1+\sigma+\sigma^2$. With $\gamma=\sigma\tau\sigma^2$, let $e_1=(1+\tau)(1+\gamma)$ and $e_2=(1+\gamma)(1+\tau)$. Then $1=e_0+e_1+e_2$ is a primitive idempotent decomposition of $1 \in \mathbf{F}_2\Sigma_3$. The idempotents e_1 and e_2 are equal to the Steinberg idempotent and its conjugate.

THEOREM 3.1.

$$BE(3)=B(E(3) \rtimes H) \vee 2e(BE(3))$$

where e is the Steinberg idempotent for $0_3(\mathbf{F}_2) \approx \Sigma_3$.

Proof. Since $e=e_1$, one need only observe that $H^*B(E(3) \rtimes H)$ is isomorphic to $H^*(BE(3))^H=H^*(BE(3))e_0$ under the restriction map.

Remark. There are many other possible groups to use in place of $E(3) \rtimes H$, e.g., $SL_2(\mathbf{F}_3) \circ \mathbf{Z}/4 = (SL_2(\mathbf{F}_3) \times \mathbf{Z}/4)/\langle (s, t) \rangle$ where s and t are elements of order 2 in the centers of $SL_2(\mathbf{F}_3)$ and $\mathbf{Z}/4$ respectively.

LEMMA 3.2.

$$eBE(3) = L(1) \vee X$$

where H^*X has Poincaré series $(t^2 + t^5)/(1-t)(1-t^4)$.

Proof. Consider the maps

$$L(1) \simeq B(\langle b_1 \rangle) \hookrightarrow BE(3) \xrightarrow{\pi} B(\mathbf{Z}/2)^3 \xrightarrow{w} B(\mathbf{Z}/2) \simeq L(1)$$

where π is given by dividing by the central $\langle b_0^2 \rangle \approx \mathbf{Z}/2$ and ω is the sum of two maps as follows. Recall from I, §1 that $(\mathbf{Z}/2)^3 = \langle v_0, v_1, v_{-1} \rangle$. Then ω is the sum of the maps $B(\mathbf{Z}/2)^3 \to B((\mathbf{Z}/2)^3/\langle v_0, v_i \rangle) \simeq B\mathbf{Z}/2$ for $i = \pm 1$, $H^*L(1) = \mathbf{F}_2\langle Sq^i(x^{-1}) \mid i \geq 2 \rangle$, where $x \in H^1(L(1))$. Further, $\pi^*\omega^*Sq^i(x^{-1}) = Sq^i(\alpha_1 + \beta_1) \in H^*(eBE(3))$ by §1. Moreover, $Sq^i(\alpha_1 + \beta_1)$ restricts to $Sq^i(x_1^{-1})$ on $L(1)$. Thus $eBE(3) = L(1) \vee X$ for some spectrum X.

The Poincaré series of $eBE(3)$ is $(1-t^2)(1-t^3)/(1-t)^3(1-t^4)$. Since the Poincaré series of $H^*L(1)$ is $t/1-t$, the Poincaré series of H^*X is as claimed.

LEMMA 3.3.

$$B(E(3) \rtimes H) = L(1) \vee Y$$

where H^*Y has Poincaré series $(t^3 + t^4)/(1-t)(1-t^4)$.

Proof. Let L be the subgroup of $E(3)$ generated by the element $b_0 b_1 b_{-1}$, which has order 2. The map

$$BL \to BE \xrightarrow{\pi} BV \xrightarrow{s} BL$$

is a homotopy equivalence. Here s is the sum of three maps induced by the projections $V \to V/\langle v_0, v_1 \rangle$, $V \to V/\langle v_0, v_{-1} \rangle$, $V \to V/\langle v_1, v_{-1} \rangle$ and the natural isomorphism with L. Furthermore, ims^* is the subring generated by the invariant element $u_1 = x_0 + x_1 + x_{-1}$. Thus $BL \simeq L(1)$ is a summand of $B(E(3) \rtimes H)$.

THEOREM 3.4.

$$BE(3) = 3L(1) \vee 2X \vee Y.$$

Remark. John Martino [Mt] has shown, using Nishida's theory of dominant summands, that this is a complete stable splitting of $BE(3)$. This is evidence that the Steinberg summand in the complex case $n = 2m + 1$ splits into $L(m) \vee X_m$, where X_m is indecomposable. That $L(m)$ is a summand follows directly.

Splitting $BE(4)^-$

To decompose $BE(4)^-$ we consider the full outer automorphism group $O_4^-(\mathbf{F}_2) \approx \Sigma_5$ as described in §2. According to James [J], Σ_5 has 3 irreducible representations over $\mathbf{F}_2$ corresponding to the 2-regular partitions [5], [3, 2], [4, 1] of dimension 1, 4, 4 resp. In this case the Cartan matrix [J] is given by

$$C = \begin{array}{c} [5] \\ [3,2] \\ [4,1] \end{array} \overset{\begin{array}{ccc}[5] & [3,2] & [4,1]\end{array}}{\begin{pmatrix} 8 & 4 & 0 \\ 4 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}}$$

Let P_1, P_2, P_3 denote the corresponding principal indecomposable modules. From C we have

$$\dim P_1 = 24 \qquad \dim P_2 = 16 \qquad \dim P_3 = 8$$

and

$$\mathbf{F}_2\Sigma_5 \approx P_1 \oplus 4P_2 \oplus 4P_3 \tag{1}$$

Let $H = \langle (a, b, c, d, r) \rangle \approx \mathbf{Z}/5$ and set $e_1 = \Sigma_{h \in H} h$.

LEMMA 3.5. $P_1 \approx e_1\mathbf{F}_2\Sigma_5$.

Proof. The trivial module $I \subset \mathbf{F}_2\Sigma_5$ is generated by $\bar{\Sigma}_5 = \Sigma_{\sigma \in \Sigma_5} \sigma$. Clearly $e_1\bar{\Sigma}_5 = \bar{\Sigma}_5$. Since P_1 is the unique principal indecomposable module with I as submodule, P_1 is a summand of $e_1\mathbf{F}_2\Sigma_5$. The dimension of $e_1\mathbf{F}_2\Sigma_5$ is 24 with basis $\{e_1x_i\}$, $x_i \in H\backslash\Sigma_5$. Thus $P_1 = e_1\mathbf{F}_2\Sigma_5$ by equality of dimension.

Let e_3 denote the Steinberg idempotent for $\Omega_4^-(\mathbf{F}_2) \approx A_5$.

LEMMA 3.6. $P_3 \approx e_3\mathbf{F}_2\Sigma_5$.

Proof. The Steinberg module $\mathrm{St} = e_3\mathbf{F}_2A_5$ is absolutely irreducible, hence absolutely indecomposable. By Green's theorem [L, Th. 11.10], $\mathrm{St}\uparrow_{A_5}^{\Sigma_5} = e_3\mathbf{F}_2\Sigma_5$ is also indecomposable. Since $\dim \mathrm{St}\uparrow_{A_5}^{\Sigma_5} = 2 \dim \mathrm{St} = 8$, $P_3 \approx e_3\mathbf{F}_2\Sigma_5$. ■

Finally, let $G = E(4)^- \rtimes H$, which exists since H is cyclic of odd order. Then $BG \simeq e_1BE(4)^-$ and $H^*BG = (H^*BE(4)^-)^H$. A Molien series argument shows the Poincaré series of H^*BG is $(1 + 3t^3 + 4t^4 + 3t^5 + t^8)/(1 - t^2)(1 - t^8)$.

THEOREM 3.7.

$$BE(4)^- = BG \vee 4X \vee 4\tilde{M}(4)^-$$

where

$\tilde{M}(4)^- = e_3BE(4)^-$ is the Steinberg summand and

$X = e_2BE(4)^-$ with Poincaré series $(t^2 + t^3 + t^4 + t^5)/(1 - t)(1 - t^8)$

where e_2 is a primitive idempotent corresponding to P_2.

Proof. Follows from (1) above, Lemmas 3.5, 6 and the Poincaré series for H^*BG and $H^*\tilde{M}(4)^-$. ■

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Received May 25, 1987