Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 64 (1989)

Artikel: Stable splittings associated with Chevalley groups, I.
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DOI: https://doi.org/10.5169/seals-48958

Comment. Math. Helvetici 64 (1989) 474–495 0010–2571/89/030474–22\$01.50 + 0.20/0

Stable splittings associated with Chevalley groups, I

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In recent years stable splittings have been studied for the classifying spaces of various finite groups, for example: elementary abelian p-groups [MP1], abelian groups [HK], dihedral and quaternion groups [MP2], etc. In this paper we continue this study; here we consider groups E which are extensions of an elementary abelian 2-group V by a cyclic group of order 2. These groups are among those of symplectic type [T, 2.4]; examples are the extra-special 2-groups [G, H]. A quadratic form Q is naturally associated with such an extension and the outer automorphisms of E which fix the center are precisely those automorphisms of V which preserve this form. Thus one of the classical orthogonal groups $O(V, Q)$ acts on BE (up to homotopy) and we can use idempotents from the group ring to stably split BE. In particular since the commutator subgroups of these groups are Chevalley groups, they have a BN pair and an associated Steinberg idempotent e. We determine the stable summand eBE. The degenerate case where E itself is an elementary abelian 2-group was studied in [MP1]. These cases cover the four systems of Chevalley groups A_m, B_m, D_m defined over $\mathbf{F}_2$ and the twisted group ${}^2D_m(\mathbf{F}_4)$.

It is well known that the orthogonal groups $O(V, Q)$ over $\mathbf{F}_2$ are determined by the dimension of V and the Arf invariant of Q. There exists three types of forms: one if dim V is odd and two if dim V is even. The latter cases are distinguished by Arf $(Q) = 0$ or 1. In this paper we set up machinery for handling the general cases but give specific analysis only for the Arf $(Q) = 0$ case. Here our main result (Theorem 4.1) is that BE contains $2^{m(m-1)}$ wedge summands, each equivalent to

$$eBE = M(m) \vee L(m) \vee eT(\Delta_{2m})$$

where $2m = \dim V$, $M(m)$ and $L(m)$ are wedge summands of $B(\mathbf{Z}/2)^m$ and $T(\Delta_{2m})$ is the Thom spectrum associated to an irreducible representation Δ_{2m} of E. In Part II, we study the remaining cases.

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[1] The authors are partially supported by NSF Grants.

The paper is organized as follows: Section 1 consists of some preliminaries on E, quadratic forms and Quillen's computation of H^*BE. The homotopy action of $O(V, Q)$ on BE is explained in Section 2. In Section 3 we describe the structure of $O(V, Q)$ as a Chevalley group and determine the Steinberg idempotent e. The cohomology of $H^*(eBE)$ is determined in Section 4. This leads to a proof of the main splitting in Theorem 4.1. In Section 5, we give a splitting of BE for $|E| = 32$ and $\operatorname{Arf} Q = 0$. In what follows all spaces are localized at 2 and all cohomology groups are taken with simple coefficients in $\mathbf{F}_2$.

It is a pleasure to thank Dave Benson for several helpful conversations on this material.

§1. Preliminaries

In this section we recall some preliminaries on quadratic forms, the groups E and their cohomology.

We begin with some standard facts about quadratic forms over $\mathbf{F}_2$ [Q]. Let V be a vector space over $\mathbf{F}_2$. A *quadratic* form $Q: V \to \mathbf{F}_2$ is a function such that $Q(x+y) = Q(x) + Q(y) + B(x, y)$ for $x, y \in V$ and some bilinear form B. Necessarily B is *symplectic*, i.e. $B(x, x) = 0$. Let V_0 be the set of $x \in V$ such that $B(x, y) = 0$ for all $y \in V$. Then Q is said to be *non-degenerate* if $Q(x) \neq 0$ for all $x \neq 0$ in V_0. Throughout this paper we will assume all quadratic forms to be non-degenerate.

Let $n = \dim V$. According to Dickson [Dk] there are, up to isomorphism three types of non-degenerate quadratic forms:

$$\text{If } n = 2m \qquad Q = \sum_{i=1}^{m} x_i x_{-i} \qquad \text{(real case)}$$

$$Q = \sum_{i=1}^{m-1} x_i x_{-i} + x_m^2 + x_m x_{-m} + x_{-m}^2 \qquad \text{(quaternion case)} \tag{1.0}$$

for some choice of basis $\{x_1, \ldots, x_m, x_{-1}, \ldots, x_{-m}\} \subset V^*$

$$\text{If } n = 2m+1 \qquad Q = x_0^2 + \sum_{i=1}^{m} x_i x_{-i} \qquad \text{(complex case)}$$

for some choice of basis $\{x_0, x_1, \ldots, x_m, x_{-1}, \ldots, x_{-m}\} \subset V^*$. In the first two cases we have $\operatorname{Arf} Q = 0, 1$ respectively, where we recall

$$\operatorname{Arf} Q = \begin{cases} 0 & \text{if } |Q^{-1}(0)| > \frac{1}{2}|V| \\ 1 & \text{if } |Q^{-1}(0)| < \frac{1}{2}|V|. \end{cases}$$

For convenience, however, we will use Quillen's terminology [Q] of *real* and *quaternion*; similarly we will call the third case *complex*.

Now suppose a group E is given as a central extension

$$\mathbf{Z}/2 \xrightarrow{i} E \xrightarrow{\pi} V \tag{1.1}$$

If $n = \dim V$ we shall often write $E = E(n)$. The associated quadratic and bilinear forms are given by

$$Q(x) = \tilde{x}^2 \qquad \text{where } \pi(\tilde{x}) = x$$
$$B(x, y) = \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} \qquad \text{where } \pi(\tilde{x}) = x,\ \pi(\tilde{y}) = y$$

For $n = 2$ in the real case $E \approx D_8$, the dihedral group of order 8 while in the quaternion case $E \approx Q_8$, the quaternion group of order 8. In general if n is even, $E(n)$ can be built up from the central product ($G \circ G' \approx G \times G'$ with centers identified). It is known that $D_8 \circ D_8 \approx D_8 \circ Q_8$. It is also straightforward to check

PROPOSITION 1.2. *If $n = 2m$*

$$E(n) \approx D_8 \overset{--m--}{\circ \cdots \circ} D_8 \qquad (\textit{real case})$$

$$\approx D_8 \overset{--m-1--}{\circ \cdots \circ} D_8 \circ Q_8 \qquad (\textit{quaternion case})$$

In the real and quaternion cases, E is an extra-special 2-group.

(1.3) It will be convenient to specify generators of E: let $b_1, \ldots, b_m$, $b_{-1}, \ldots, b_{-m}$ (and b_0 in the complex case) be elements of E such that $\{v_{\pm i} = \pi(b_{\pm i})\}$ is dual to the basis $\{x_{\pm i}\}$ of V^*. Then E is generated by $\{b_{\pm i}, c\}$ where c is the non-trivial element of $\ker \pi$. (By convention $b_{\pm 0} = b_0$ in the complex case.) Using (1.0) a set of relations is seen to be given by commutators and squares.

(1.4) We now turn to H^*BE. A subspace W of V is called *isotropic* if $Q(W) = 0$. Now assume W is a maximal isotropic subspace or equivalently

$\tilde{W} = \pi^{-1}(W)$ is a maximal elementary abelian subgroup. Let $\chi: \tilde{W} \to \mathbf{Z}/2$ be a representation which is non-trivial on $\ker \pi = \mathbf{Z}/2$ and consider $\Delta = \mathrm{Ind}_{\tilde{W}}^{E}(\chi)$, that is, Δ is the real representation induced from $\tilde{W}$ to E. [Q; §5] shows that Δ is the unique irreducible real representation which is non-trivial on $\ker \pi$.

THEOREM 1.5. [Q; Th. 4.6]. *Given an extension* (1.1) *and the associated bilinear form Q, then*

$$H^*(BE) = S(V^*)/J \otimes \mathbf{F}_2[w_{2^h}]$$

where J is the ideal generated by the regular sequence Q, Sq^1Q, $Sq^2Sq^1Q, \ldots,$ $Sq^{2^{h-2}} \cdots Sq^2Sq^1Q$; h is the codimension of a maximal isotropic subspace of V and $w_{2^h} = w_{2^h}(\Delta)$ is the 2^h-th Stiefel–Whitney class of Δ.

Remark 1.6. For reference we record the values of h [Q; §2].

Case	dim V	h
real	$2m$	m
complex	$2m+1$	$m+1$
quaternion	$2m$	$m+1$

(1.7) Since the dimension of Δ is 2^h and $\ker \pi = \mathbf{Z}/2$ acts as -1 on Δ, Δ restricted to $\ker \pi$ is $2^h \cdot \eta$, where η is the non-trivial real character on $\mathbf{Z}/2$. It follows that $i^*(w_{2^h}) \neq 0$ and that any element with this property can be taken as a generator in place of w_{2^h}.

§2. Classical groups acting on H^*BE

Since conjugation is homotopic to the identity on the classifying space BG of any group G, the outer automorphism group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ acts up to homotopy on BG, i.e. there is a homomorphism

$$\mathrm{Out}(G) \to \mathrm{Aut}_{H_0}(BG)$$

where $\mathrm{Aut}_{H_0}(BG)$ is the group of base point preserving equivalences in the homotopy category.

The referee points out that Out (E) can be made to act on BE (not just up to homotopy). There is a group extension (which is not necessarily split)

$$1 \to E \to G_0 \to \mathrm{Out}\,(E) \to 1$$

with $G_0/\langle c\rangle \approx \mathrm{Aut}\,(E)$. Thus if X is a contractible CW-complex on which G_0 acts freely, then X/E is a model for BE on which Out (E) acts as required.

Let $\mathrm{Out}_z\,(G)$ be the subgroup of Out (G) consisting of automorphisms which are the identity on the center of G. For $G = E$ as in (1.1) we have

PROPOSITION 2.1. *$\mathrm{Out}_z\,(E) \approx O(V, Q)$*

Proof. It is clear from the definitions that π induces a homomorphism $\mathrm{Out}_z\,(E) \to O(V, Q)$. This map is surjective by (1.3) and so any orthogonal automorphism of V can be lifted to an automorphism of E. That the center is fixed follows from examining the types of Q in (1.0). Conversely, suppose $\beta \in \mathrm{Out}_z\,(E)$ induces the identity on V. Then for $b \in E$, $\beta(b) = b$ or bc where $\langle c\rangle = \ker \pi$. Let $\{v_i, v_j'\}$ be a basis for V such that $B(v_i, v_j') \neq 0$ for at most one j for each i (e.g. in the real case v_i is dual to x_i and v_j' to x_{-j}). Let $\{b_i, b_j\}$ satisfy $\pi(b_i) = v_i$, $\pi(b_j') = v_j$ and let ε be the product in any order of those b_j''s for which $\beta(b_i) = b_i c$ and $B(v_i, v_j') \neq 0$ for some i. Then $\beta(b_i) = \varepsilon b_i \varepsilon^{-1}$. Similarly let ε' be the product in any order of those b_i's for which $\beta(b_j') = b_j' c$ and $B(v_i, v_j') \neq 0$ for some j. Then $\beta(b_j') = \varepsilon' b_j' \varepsilon'^{-1}$. Consequently β is conjugation by $\varepsilon\varepsilon'$.

Remark. In the real and quaternion cases, $\mathrm{Out}_z\,(V, Q) = O(V, Q)$ since the center is $\mathbf{Z}/2$. In the complex case the center is $\mathbf{Z}/4$ generated by an element b_0 such that $\pi(b_0)$ is dual to x_0. Here $\mathrm{Out}\,(E) = \mathbf{Z}/2 \times \mathrm{Out}_z\,(E)$ where the extra automorphism is given by $b_0 \mapsto b_0^3$.

We now turn to the action of $O(V, Q)$ on H^*BE and the resulting invariants. The uniqueness of Δ (1.4) implies that its Stiefel–Whitney classes are invariants. In this connection Quillen has shown

THEOREM 2.2 [Q, Th. 5.1]. *The non-zero positive dimensional Stiefel–Whitney classes of Δ_n are $\omega_{2^h}, \omega_{2^h-2^r}, \omega_{2^h-2^{r+1}}, \ldots, \omega_{2^h-2^{h-1}}$ where $r = 0, 1, 2$ in the real, complex, and quaternion cases resp. Further, these classes form a regular sequence of maximal length in H^*BE and hence form a polynomial ring over which H^*BE is a free finitely generated module.*

Quillen further remarks, without proof, that in the real case these classes generate all of the invariants. We will prove a slightly sharper result. For convenience we use the following notation

$$O(V, Q) = \begin{cases} O^{+}_{2m}(\mathbf{F}_2) & \text{if } n = 2m, \text{ real case} \\ O^{-}_{2m}(\mathbf{F}_2) & n = 2m, \text{ quaternion case} \\ O_{2m+1}(\mathbf{F}_2) & n = 2m+1, \text{ complex case} \end{cases} \tag{2.3}$$

where $n = \dim V$. Let $\Omega^{\pm}_{2m}(\mathbf{F}_2)$ denote the commutator subgroup of $O^{\pm}_{2m}(\mathbf{F}_2)$.

THEOREM 2.4. *In the real case*

$$H^*BE^{\Omega^{+}_{2m}} = \mathbf{F}_2[\omega_{2^m}, \omega_{2^m-1}, \ldots, \omega_{2^{m-1}}].$$

The proof depends on three lemmas, the first of which holds for a general V and Q.

LEMMA 2.5. *$O(V, Q)$ acts transitively on $\{A < E : A$ is a maximal elementary abelian group$\}$.*

Proof. $O(V, Q)$ acts transitively on $\{W < V : W$ is a maximal isotropic subspace$\}$. This is a result of Arf [A] in the real and quaternion cases. In the complex case $O_{2m+1}(\mathbf{F}_2) \approx Sp_{2m}(\mathbf{F}_2)$ and a proof can be found in [Dd]. The lemma follows since π induces an isomorphism between maximal elementary abelian subgroups of E and maximal isotropic subspaces of V.

Let $H : GL_m(\mathbf{F}_2) \to O^{+}_{2m}(\mathbf{F}_2)$ be the *hyperbolic map* given by

$$H(M) = \begin{bmatrix} M & O \\ O & {}^t M^{-1} \end{bmatrix}$$

(see [F-P; p. 152–154]). The appropriate quadratic form for the range is of the real type.

LEMMA 2.6. $H : GL_m(\mathbf{F}_2) \to \Omega^{+}_{2m}(\mathbf{F}_2)$

Proof. Since $\Omega^{+}_{2m} = \ker d$ where $d : O^{+}_{2m}(\mathbf{F}_2) \to \mathbf{Z}/2$ is the Dickson invariant, we need only check $d \circ H = 0$. This follows from the formula for d [Dd; p. 64].

LEMMA 2.7. *Let $A \xrightarrow{j} E$ be the inclusion of a maximal elementary abelian subgroup. Then $j^*(H^*(BE)^{\Omega^{+}_{2m}}) = \mathrm{Im}\,(j^*\Delta^*)$.*

Proof. The inclusion $\supset$ follows from the inclusion $H^*(BE)^{\Omega^{+}_{2m}} \supset \mathrm{Im}\,\Delta^*$ noted above. For the other inclusion it suffices by Theorem 1.5 to consider $x \in H^*(BE)^{\Omega_{2m}^+}$ in the image of $\pi^*: H^*BV \to H^*BE$. By Lemma 2.5, (1.4) and the normality of Ω_{2m}^+, it suffices to prove the result for one maximal elementary abelian subgroup A. Let $A = \langle b_1, \ldots, b_m, c \rangle \overset{i}{\hookrightarrow} E$; we can write $A = A' \oplus C$ where $C = \langle c \rangle = \ker \pi$. Let $M \in GL_m(\mathbf{F}_2)$. Then for $j^*(x) = y \otimes 1 \in H^*BA' \otimes H^*BC$, we have

$$(y \otimes 1)H(M) = yM \otimes 1$$

Hence $y \in H^*(BA')^{GL(A')}$. By [Wk; 4.1], $H^*(BA')^{GL(A')} = \operatorname{Im}(\operatorname{reg}(A')^*)$ for the regular representation of A'. Since $\Delta j = \operatorname{reg}(A') \otimes \chi$ on $A' \oplus C$ [Q; 5.1], we have $j^*(x) = y \otimes 1 \in \operatorname{Im}(j^*\Delta^*)$ using the formula for the Stiefel–Whitney classes of $\operatorname{reg}(A') \otimes \chi$ [Q; 5.6].

Proof of Theorem 2.4. By [Q; Th. 5.10], H^*BE is detected by elementary abelian subgroups. Hence the result follows directly from Lemma 2.7.

COROLLARY 2.8. $H^*(BE)^{O_{2m}^+} = H^*(BE)^{\Omega_{2m}^+}$.

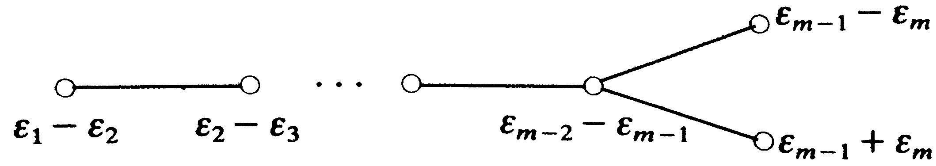
§3. $O_n(\mathbf{F}_2)$ as Chevalley groups

Our goal in this section is to describe what we need about the Steinberg idempotent for the orthogonal group. A good general reference is R. Carter's book [C]. For each simple Lie algebra L over $\mathbf{C}$ and each field K, Chevalley has constructed a group $L(K)$. Later Steinberg, Tits and Hertzig discovered additional twisted versions of these groups. For the simple Lie algebras of type A_m, B_m, C_m and D_m and for K finite, Ree has identified these Chevalley groups with classical groups. We state the result for $K = \mathbf{F}_2$.

THEOREM 3.1 (*Ree* [*C*; Th. 11.3.2])
i) $A_m(\mathbf{F}_2) \approx GL_{m+1}(\mathbf{F}_2)$
ii) $B_m(\mathbf{F}_2) \approx O_{2m+1}(\mathbf{F}_2)$
iii) $C_m(\mathbf{F}_2) \approx B_m(\mathbf{F}_2)$
iv) $D_m(\mathrm{D}_2) \approx \Omega_{2m}^+(\mathbf{F}_2)$

The group $\Omega_{2m}^-(\mathbf{F}_2)$ occurs as a twisted Chevalley group and will be treated at the end of this section.

3.2 *The real case*: The Dynkin diagram for D_m, $m>1$, is



where $\varepsilon_1,\ldots,\varepsilon_m$ is the standard basis for $\mathbf{R}^m$.

Let e_{ij} be the $2m$ square matrix with 1 in the (i,j) position and 0's elsewhere. Let $u_{ij}=I+e_{ij}+e_{-j,-i}\in GL_{2m}(\mathbf{F}_2)$. Then the *unipotent subgroup* $U_{2m}<\Omega^+_{2m}(\mathbf{F}_2)$ is generated by

$$\{u_{i,j},\ u_{i,-j}\colon 1\leqslant i<j\leqslant m\}$$

(We recall that the underlying vector space V has basis $\{v_1,\ldots,v_m, v_{-1},\ldots,v_{-m}\}$ over $\mathbf{F}_2$.) The *Weyl group* $W^+_{2m}<\Omega^+_{2m}(\mathbf{F}_2)$ is generated by

$$\{\sigma_{ij}=u_{i,j}u_{-i,-j}u_{i,j},\ \sigma_{i,-j}=u_{i,-j}u_{-i,j}u_{i,-j}\colon 1\leqslant i<j\leqslant m\}.$$

Abstractly $W^+_{2m}\approx(\mathbf{Z}/2)^{m-1}\rtimes\Sigma_m$ (permutations together with an even number of sign changes).

Finally $\Omega^+_{2m}(\mathbf{F}_2)$ is generated by U_{2m} and V_{2m} where V_{2m} is generated by $\{u_{-i,-j},\ u_{-i,j}\colon 1\leqslant i<j\leqslant m\}$.

(3.3) The *Steinberg idempotent* $e\in\mathbf{F}_2\Omega^+_{2m}(\mathbf{F}_2)$ is defined by

$$e=\sum u\sigma \qquad u\in U_{2m},\ \sigma\in W^+_{2m}.$$

For computational purposes, it will be convenient to use another expression for e. For each of the simple roots $\{\varepsilon_i-\varepsilon_{i+1}\}$ in the Dynkin diagram let e_i be the idempotent

$$e_i=(1+u_{i,i+1})(1+\sigma_{i,i+1}) \qquad 1\leq i\leq m-1$$

For the last root $\varepsilon_{m-1}+\varepsilon_m$ let

$$e_m=(1+u_{m-1,-m})(1+\sigma_{m-1,-m})$$

Kuhn [K] has shown that e can be expressed as a product of the e_i, $i=1,2,\ldots,m$. Moreover

THEOREM 3.4. [K, Th. 1.3] *Let M be a right $\mathbf{F}_2\Omega^+_{2m}(\mathbf{F}_2)$ module. Then*

$$Me = \bigcap_{i=1}^{m} Me_i.$$

(3.5) *The complex case*: The Dynkin diagram for B_m is

$\varepsilon_1 - \varepsilon_2$ $\qquad$ $\varepsilon_{m-1} - \varepsilon_m$ $\qquad$ ε_m

Let

$$\begin{aligned} u_{ij} &= I + e_{ij} + e_{-j,-i} & i \neq j \\ u_{ii} &= I + e_{0,-i} + e_{i,-i} & i \neq 0 \end{aligned}$$

(V has basis $v_0, v_1, \ldots, v_m, v_{-1}, \ldots, v_{-m}$). The *unipotent subgroup* $U_{2m+1} < O_{2m+1}(\mathbf{F}_2)$ is generated by

$$\{u_{ij}, u_{i,-j}, u_{ii}: 1 \leq i < j \leq m\}$$

The *Weyl group* $W_{2m+1} < O_{2m+1}(\mathbf{F}_2)$ is generated by

$$\begin{cases} \sigma_{ij} = u_{-i,-j}u_{i,j}u_{-i,-j} & \\ & 1 \leq i < j < m \\ \sigma_{i,-j} = u_{-i,j}u_{i,-j}u_{-i,j} & \\ \sigma_{ii} = u_{-i,-i}u_{ii}u_{-i,-i} & 1 \leq i \leq m \end{cases}$$

Then $O_{2m+1}(\mathbf{F}_2)$ is generated by U_{2m+1} and V_{2m+1} where V_{2m+1} is generated by

$$\{u_{-i,-j}, u_{-i,j}, u_{-i,-i}: 1 \leq i < j \leq m\}.$$

The *Steinberg idempotent* $e \in \mathbf{F}_2O_{2m+1}(\mathbf{F}_2)$ is defined by

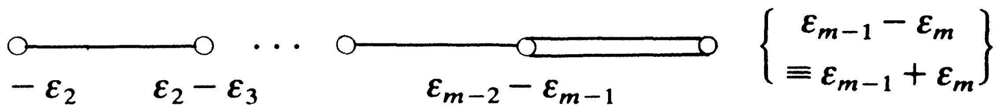
$$e = \sum u\sigma \qquad u \in U_{2m+1}, \sigma \in W_{2m+1}$$

In this case Kuhn [K] has shown that e can be expressed as a product of the following idempotents

$$e_i = (1 + u_{i,i+1})(1 + \sigma_{i,i+1}) \qquad 1 \le i \le m-1$$
$$e_m = (1 + u_{m,m})(1 + \sigma_{mm})$$

and the analog of Theorem 3.4 holds.

3.6 *The quaternion case*: The group $\Omega_{2m}^{-}(\mathbf{F}_2)$ is isomorphic to the twisted Chevalley group ${}^2D_m(\mathbf{F}_4)$ [C; Th. 14.5.2] with Dynkin diagram of type B_{m-1}



It is a projection of the diagram for D_m in (3.2). For details of this group see Chapters 13, 14 of [C].

Let

$$\tau_m = I + e_{m-1,m} + e_{m-1,-(m-1)} + e_{m-1,-m} + e_{m,-(m-1)} + e_{-m,-(m-1)}$$
$$\gamma_m = I - e_{m-1,m} + e_{m-1,-(m-1)} + e_{-m,-(m-1)}$$
$$\tau'_m = I - e_{-(m-1),m-1} + e_{-(m-1),m} + e_{-m,m-1}$$
$$\gamma'_m = I + e_{m,m-1} + e_{-(m-1),m-1} + e_{-(m-1),m} + e_{-(m-1),-m} + e_{m,m-1}$$

The *unipotent subgroup* $U_{2m}^{-} < \Omega_{2m}^{-}(\mathbf{F}_2)$ is generated by $\{\tau_m, \gamma_m\} \cup \{u_{i,j}, u_{i,-j} : 1 \le i < j \le m-1\}$. V_{2m}^{-} is generated by $\{\tau'_m, \gamma'_m\} \cup \{u_{-i,-j}, u_{-i,j} : 1 \le i < j \le m-1\}$. $\Omega_{2m}^{-}(\mathbf{F}_2)$ is generated by U_{2m}^{-} and V_{2m}^{-}. Let B_{2m}^{-} be the normalizer of U_{2m}^{-} in $\Omega_{2m}^{-}(\mathbf{F}_2)$.

The *Weyl group* W_{2m}^{-} of $\Omega_{2m}^{-}(\mathbf{F}_2)$ is generated by $\{\sigma_{ij}, \sigma_{i,-j} : 1 \le i < j \le m\} \cup \{\tau_m \tau'_m \tau_m = W_m\}$. The *Steinberg idempotent* $e \in \mathbf{F}_2\Omega_{2m}^{-}(\mathbf{F}_2)$ is defined by $e = \sum b\sigma$ $b \in B_{2m}^{-}$, $\sigma \in W_{2m}^{-}$.

In this case Kuhn [K] has shown that e can be expressed as a product of the idempotents corresponding to the nodes in the Dynkin diagram for B_{m-1}. These are

$$e_i = (1 + u_{i,i+1})(1 + \sigma_{i,i+1}) \qquad 1 \le i \le m-2$$

and the idempotent e'_m corresponding to the last node

$$e'_m = (1 + \tau_m)(1 + \gamma_m)(1 + H_m + H_m^2)(1 + W_m)$$

where $H_m = I + e_{m,m} + e_{m,-m} + e_{-m,m}$ and $W_m = I + e_{m-1,m-1} + e_{m-1,-(m-1)} + e_{m,-m} + e_{-(m-1),m-1} + e_{-(m-1),-(m-1)}$.

§4. The Steinberg wedge summand: the real case

For $n = 2m$ let $E = E(n)$ denote the extra-special 2-group of real type. Let $\tilde{M}(n)$ be the stable summand

$$\tilde{M}(n) = eBE$$

corresponding to the Steinberg idempotent of (3.3). Our main result is

THEOREM 4.1. *Stably, for $m \geq 2$, BE contains $2^{m(m-1)}$ copies of $\tilde{M}(n) = M(m) \vee L(m) \vee eT(\Delta_n)$.*

Here $M(m)$ is the Steinberg summand of $B(\mathbf{Z}/2)^m$ [MP1], $L(m) = \Sigma^{-m} Sp^{2^m}S^0/Sp^{2^{m-1}}S^0$, and $T(\Delta_n)$ is the Thom spectrum of the bundle $B\Delta_n$ over BE. As a spectrum $M(m) = L(m) \vee L(m-1)$.

(4.2) The uniqueness of Δ_n (1.4) implies that the homotopy action of $O_n^+(\mathbf{F}_2)$ on BE preserves the isomorphism type of Δ_n and hence induces a homotopy action of $O_n^+(\mathbf{F}_2)$ on $T(\Delta_n)$. The summand $eT(\Delta_n)$ is defined with respect to this action.

On the way to proving Theorem 4.1 we first determine $H^*\tilde{M}(n)$. Let

$$\alpha = \alpha_m = \sum x_{i_1}^{-1} x_{i_2}^{-1} \cdots x_{i_m}^{-1},$$

$i_j = \pm j$ with an even number of minus signs occurring

$$\beta = \beta_m = \sum x_{i_1}^{-1} x_{i_2}^{-1} \cdots x_{i_m}^{-1},$$

$i_j = \pm j$ with an odd number of minus signs occurring.

These elements belong to S_V, that is, $S = H^*BV$ with the inverses of all non-zero linear elements adjoined. The action of $O_n^+(\mathbf{F}_2)$ on H^*BV extends to S_V.

LEMMA 4.3. *$\alpha e = \alpha$, $\beta e = \beta$.*

Proof. By 3.4 it suffices to show α and β are fixed by e_i, $i = 1, \ldots, m$. Write

$$\alpha = (x_i^{-1} x_{i+1}^{-1} + x_{-i}^{-1} x_{-(i+1)}^{-1}) \hat{\alpha}_i + (x_{-i}^{-1} x_{i+1}^{-1} + x_i^{-1} x_{-(i+1)}^{-1}) \hat{\beta}_i$$

where $\hat{\alpha}_i$ (resp. $\hat{\beta}_i$) is the sum of those terms $x_{j_1}^{-1} \cdots x_{j_{m-2}}^{-1}$ not containing $x_{\pm i}^{-1}$,

$x^{-1}_{\pm(i+1)}$ and having an even (resp. odd) number of minus signs. By 3.3,

$$\begin{aligned} e_i &= (1+u_{i,i+1})(1+\sigma_{i,i+1}) \qquad 1 \le i < m \\ e_m &= (1+u_{m-1,-m})(1+\sigma_{m-1,-m}) \end{aligned}$$

where the action of $u_{i,j}$ is $x_i \to x_i + x_j$, $x_{-j} \to x_{-i} + x_{-j}$, $x_k \to x_k$ otherwise and the action of $\sigma_{i,j}$ is $x_{\pm i} \to x_{\pm j}$, $x_{\pm j} \to x_{\pm i}$. Hence for $1 \le i < m$,

$$\begin{aligned} \alpha e_i = \alpha &+ [(x_i + x_{i+1})^{-1} x_{i+1}^{-1} + x_{-i}^{-1}(x_{-i} + x_{-(i+1)})^{-1}]\hat{\alpha}_i \\ &+ [(x_i + x_{i+1})^{-1}(x_{-i} + x_{-(i+1)})^{-1} + x_{-i}^{-1} x_{i+1}^{-1}]\hat{\beta}_i \\ &+ [x_i^{-1} x_{i+1}^{-1} + x_{-i}^{-1} x_{-(i+1)}^{-1}]\hat{\alpha}_i \\ &+ [x_{-i}^{-1} x_{i+1}^{-1} + x_i^{-1} x_{-(i+1)}^{-1}]\hat{\beta}_i + [(x_i + x_{i+1})^{-1} x_i^{-1} \\ &+ x_{-(i+1)}^{-1}(x_{-i} + x_{-(i+1)})^{-1}]\hat{\alpha}_i \\ &+ [(x_i + x_{i+1})^{-1}(x_{-i} + x_{-(i+1)})^{-1} + x_i^{-1} x_{-(i+1)}^{-1}]\hat{\beta}_i = \alpha. \end{aligned}$$

For $i = m$ we have

$$\begin{aligned} \alpha e_m = \alpha &+ [(x_{m-1} + x_{-m})^{-1}(x_m + x_{-(m-1)})^{-1} + x_{-(m-1)}^{-1} x_{-m}^{-1}]\hat{\alpha}_{m-1} \\ &+ [(x_{m-1} + x_{-m})^{-1} x_{-m}^{-1} + x_{-(m-1)}^{-1}(x_m + x_{-(m-1)})^{-1}]\hat{\beta}_{m-1} \\ &+ [x_{-m}^{-1} x_{-(m-1)}^{-1} + x_m^{-1} x_{m-1}^{-1}]\hat{\alpha}_{m-1} \\ &+ [x_{-m}^{-1} x_{m-1}^{-1} + x_m^{-1} x_{-(m-1)}^{-1}]\hat{\beta}_{m-1} + [(x_{-m} + x_{m-1})^{-1}(x_{-(m-1)} + x_m)^{-1} \\ &+ x_m^{-1} x_{m-1}^{-1}]\hat{\alpha}_{m-1} \\ &+ [(x_{-m} + x_{m-1})^{-1} x_{m-1}^{-1} + x_m^{-1}(x_{-(m-1)} + x_m)^{-1}]\hat{\beta}_{m-1} = \alpha \end{aligned}$$

A similar calculation shows $\beta e = \beta$.

LEMMA 4.4. $Sq^1\alpha = Sq^1\beta$.

The proof is straightforward calculation using $Sq^1 x^{-1} = 1$. Now let

$$\begin{aligned} A &= \mathbf{F}_2\langle Sq^I\alpha,\ Sq^I\beta : I \text{ admissible},\ l(I) = m\rangle \\ B &= \mathbf{F}_2\langle Sq^J Sq^1\alpha + Sq^J Sq^1\beta : (J,1) \text{ admissible},\ l(J) = m-1\rangle \end{aligned}$$

THEOREM 4.5. i) $H^*\tilde{M}(n) = (A/B) \otimes \mathbf{F}_2[\omega_{2^m}]$

ii) $H^*(eBV) = (A/B) \otimes \mathbf{F}_2[Q, Sq^1 Q, \ldots, Sq^{2^{m-2}} \cdots Sq^2 Sq^1 Q]$

Proof. In discussing i) and ii) we will implicitly use the commutative diagram

$$\begin{tikzcd} H^*BV \arrow[r, "e"] \arrow[d, "\pi^*"] & H^*BV \arrow[d, "\pi^*"] \\ H^*BE \arrow[r, "e"] & H^*BE \end{tikzcd}$$

The elements $Sq^I\alpha$, $Sq^I\beta \in H^*(eBV)$ by Lemma 4.3 and the relations B hold by Lemma 4.4. A basis for $A/B \subset H^*(eBV)$ is given by

$$\{Sq^I\alpha, Sq^J\beta : I, J \text{ admissible}, \, l(I) = m, \, l(J) = m, \, j_m > 1\} \tag{4.6}$$

Restricting to the subgroups $\langle b_1, b_2, \ldots, b_m \rangle$, $\langle b_{-1}, b_2, \ldots, b_m \rangle$ shows these elements remain linearly independent in H^*BE. Thus

$$(A/B) \otimes \mathbf{F}_2[\omega_{2^m}] \subset H^*\tilde{M}(n) \tag{4.7i}$$

since ω_{2^m} is invariant under $\Omega_n^+(\mathbf{F}_2)$. By Theorem 1.5, $Q, Sq^1Q, \ldots, Sq^{2^{m-2}} \cdots Sq^2Sq^1Q \subset H^*BV$ is a regular sequence of invariants; therefore a theorem of P. Baum [B, 3.5] implies

$$(A/B) \otimes \mathbf{F}_2[Q, Sq^1Q, \ldots, Sq^{2^{m-2}} \cdots Sq^2Sq^1Q] \subset H^*(eBV). \tag{4.7ii}$$

It remains to check equality of the Poincaré series of these modules. The proof is by induction on $n = 2m$.

For this we first treat the case $n = 4$. It is readily seen that $\Omega_4^+(\mathbf{F}_2) \approx GL_2(\mathbf{F}_2) \times GL_2(\mathbf{F}_2)$ with generators $\{u_{12}, \sigma_{12}\}$ for the first factor and $\{u_{1,-2}, \sigma_{1,-2}\}$ for the second. Then $f_1 = (1 + u_{12})(1 + \sigma_{12})$ corresponds to the Steinberg idempotent for $GL_2(\mathbf{F}_2)$ [MP1] and

$$1 = f_0 + f_1 + f_2 \tag{4.8}$$

is an orthogonal decomposition into primitive idempotents, where $f_0 = 1 + u_{12}\sigma_{12} + (u_{12}\sigma_{12})^2$ and $f_2 = (1 + \sigma_{12})(1 + u_{12})$. Similarly in the second factor, let

$$1 = f_0' + f_1' + f_2' \tag{4.9}$$

be the corresponding decomposition. Then $f_1 f_1'$ is the Steinberg idempotent for $\mathbf{F}_2\Omega_4^+(\mathbf{F}_2)$.

Consider $V = V_4$, the vector space with dual basis x_1, x_2, x_{-1}, x_{-2}. Then $(H^*BV)f_0f_0' = H^*BV^{\mathbf{Z}/3 \times \mathbf{Z}/3}$ since $u_{12}\sigma_{12}$ and $u_{1,-2}\sigma_{1,-2}$ have order three. A simple application of Molien's series [M] computes the Poincaré series

$$P.S.(H^*BVf_0f'_0)=\frac{(1+t^3)^2}{(1-t^2)^2(1-t^3)^2}$$

Similarly $(H^*BV)f_0=H^*BV^{\mathbf{Z}/3}$ and Molien's series yields

$$P.S.(H^*BVf_0)=\frac{1+2t^2+6t^3+2t^4+t^6}{(1-t^2)^2(1-t^3)^2}$$

Since f_1 and f_2 are conjugate as well as f'_1 and f'_2, (4.8) then implies

$$P.S.(H^*BVf_1)=\frac{2t+3t^2+2t^3+3t^4+2t^5}{(1-t^2)^2(1-t^3)^2}$$

Now $f_0=f_0f'_0+f_0f'_1+f_0f'_2$; hence

$$P.S.(H^*BVf_0f'_1)=\frac{t^2+2t^3+t^4}{(1-t^2)^2(1-t^3)^2}$$

Therefore

$$P.S.(H^*BVf_1f'_1)=\frac{t+t^2+t^4+t^5}{(1-t^2)^2(1-t^3)^2}$$

which, by 4.6 ($m=2$), equals the Poincaré series for $(A/B)\otimes\mathbf{F}_2[Q, Sq^1Q]$. Hence, we have equality in 4.7ii ($m=2$). Since ω_4 is an invariant, equality in 4.7i ($m=2$) follows from Theorem 2.2.

We now turn to the general case part i), $n=2m$, assuming by induction both parts of case $2m-2$. To compute $H^*\widetilde{M(n)}$ as a module over $\mathbf{F}_2[\omega_{2^m}]$ we consider the commutative diagram

$$\begin{array}{ccc} H^*BE & \xrightarrow{\bar{e}} & H^*BE \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle \pi^*} \\ H^*BV & \xrightarrow{\bar{e}} & H^*BV \end{array}$$

where $\bar{e}\in\mathbf{F}_2\Omega^+_{2m}(\mathbf{F}_2)$ is the image of the Steinberg idempotent for $\Omega^+_{2m-2}(\mathbf{F}_2)$ acting on the last $2m-2$ co-ordinates. Since $\operatorname{Im} e\subset\operatorname{Im}\bar{e}$ by Theorem 3.4, induction and the relations

$$Q \equiv x_1x_{-1} + \sum_{i=2}^{m} x_ix_{-i},\ Sq^1Q, \dots, Sq^{2^{m-2}} \cdots Sq^2Sq^1Q$$

of H^*BE imply $\operatorname{Im} e$ is generated by elements of the form

$$\omega(Sq^I\alpha'_{m-1}), \qquad \omega(Sq^I\beta'_{m-1}) \tag{4.10}$$

where α'_{m-1}, β'_{m-1} are α_{m-1}, β_{m-1} on the last $2m-2$ co-ordinates, $l(I) = m-1$ and $\omega = \omega(x_1, x_{-1})$ is a homogeneous polynomial in x_1, x_{-1}. The remainder of the proof of this inductive step consists of two steps 4.11, 12.

(4.11) Suppose $z \in \operatorname{Im} e$ is a linear combination of terms from (4.10). Restriction to the subgroups $\langle b_1, \dots, b_m\rangle$ (resp. $\langle b_1, \dots, b_{m-1}, b_{-m}\rangle$) detects the summands $\omega Sq^I\alpha'_{m-1}$ (resp. $\omega Sq^I\beta'_{m-1}$) of z with some ω a polynomial in x_1. Invariance of $\operatorname{Im} e$ under the Weyl group W^+_{2m} then shows z is a linear combination of terms $Sq^K\alpha_m$, $Sq^K\beta_m$, $l(K) = m$. A similar argument shows the same conclusion holds if ω is a polynomial in x_{-1} alone. Thus $\operatorname{Im} e$ consists of $(A/B) \otimes \mathbf{F}_2[\omega_{2^m}]$ plus possibly terms from (4.10) with ω divisible by x_1x_{-1}. It remains to eliminate the possibly of such terms.

(4.12) We shall need to recall some facts about Molien's series [M]. Let G be a finite group and N a graded $\mathbf{F}_2G$ module. As usual the Poincaré series of N is given by $P.S.(N) = F(N; t) = \sum (\dim_{\mathbf{F}_2} N_i)t^i$. For an irreducible $\mathbf{F}_2G$ module E, we also consider the series

$$F(N, G, E; t) = \sum a_it^i$$

where a_i is the multiplicity of E as a composition factor in N_i. Finally, let

$$\chi(N; t) = \sum \chi_{N_i}t^i$$

be the modular character series where χ_{N_i} is the modular (or Brauer) character of N_i defined on the p-regular elements G_{reg} of G ([S]).

In the present situation let $G = \Omega^+_{2m}(\mathbf{F}_2)$, $R = H^*BE$ and

$$R' = \mathbf{F}_2[\omega_{2^m}, \omega_{2^m-2^i}, i = 0, 1, \dots, m-1].$$

We note $R' = R^{\Omega^+_{2m}}$ by Theorem 3.4. Let $M = R \otimes_{R'} \mathbf{F}_2$. Then in each dimension

R and $R' \otimes M$ have the same composition series by Theorem 2.2 and the proof of [M, 1.3]. Hence

$$F(R, G, St; t) = F(M, G, St; t)F(R', t) \tag{4.13}$$

where St is the Steinberg module $St = e\mathbf{F}_2G$. By [M; 1.2b] and 4.13 we have

$$F(Re; t) = F(R, G, St; t) \tag{4.14}$$

Now the orthogonality relations for modular characters [S, M] imply

$$F(Re; t) = \frac{1}{|G|} \sum_{g \in G_{\text{reg}}} \chi_{St}(g^{-1})\chi(R; t)(g) \tag{4.15}$$

where $|G| = (2^m - 1)\Pi_{i=1}^{m-1} (2^{2i} - 1)2^{2i}$ by [Dk; p. 206]. To evaluate this series we use

LEMMA 4.16.

$$\chi(R; t)(g) = \frac{(1 - t^2)(1 - t^3) \cdots (1 - t^{2^{m-1}+1})}{[\Pi_i^{2m} (1 - \lambda_i(g)t)](1 - t^{2^m})}$$

where $\{\lambda_i(g)\}$ *are the eigenvalues of* g *acting on* V.

Proof. Let $S = S(V^*)$ be the symmetric algebra of V^*. Then $R = N \otimes \mathbf{F}_2[\omega_{2^m}]$ where $N = S \otimes_P \mathbf{F}_2$ and $P = \mathbf{F}_2[Q, Sq^1Q, \ldots, Sq^{2^{m-2}} \cdots Sq^1Q]$. The generators of P form a regular sequence on S by Theorem 1.5. Hence by [B; 3.5], $S \approx P \otimes N$. Thus

$$\chi(S; t) = \chi(P; t)\chi(N, t)$$

or

$$\prod_1^{2m} (1 - \lambda_i t) = \prod_{i=0}^{m-1} (1 - t^{2^i+1})\chi(N; t)$$

and the lemma follows since $\chi(\mathbf{F}_2[\omega_{2^m}]) = (1 - t^{2^m})^{-1}$.

From 4.6

$$
\begin{aligned}
F(A/B\otimes \mathbf{F}_2[\omega_{2^m}];t)&=\frac{2t^{2^{m+1}-2-m}}{\prod_{i=1}^{m}(1-t^{2^i-1})(1-t^{2^m})}+\frac{t^{2^m-2-(m-1)}}{\prod_{i=1}^{m-1}(1-t^{2^i-1})(1-t^{2^m})}\\
&=\frac{(t^{2^{m+1}-2-m}+t^{2^m-1-m})\prod_{k=1}^{m-1}Q_k(t)}{(\prod_{i=0}^{m-1}(1-t^{2^m-2^i}))(1-t^{2^m})}=f(t)F(R';t)
\end{aligned}
$$

where $Q_k(t)=\prod_{i=0}^{k-1}(1+t^{2^i(2^{m-k}-1)})$ and $f(t)=(t^{2^{m+1}-2-m}+t^{2^m-1-m})\prod_{k=1}^{m-1}Q_k(t)$. Combining 4.14, 15 and Lemma 4.16 we have

$$F(Re;t)=g(t)F(R';t)$$

where

$$g(t)=\frac{1}{|G|}\sum \chi_{St}(g^{-1})\frac{\prod_{i=0}^{m-1}(1-t^{2^i+1})\prod_{j=0}^{m-1}(1-t^{2^m-2^j})}{\prod_{i=1}^{n}(1-\lambda_i(g)\cdot t)}.$$

By 4.7i

$$f(t)F(R';t)=F(A/B\otimes \mathbf{F}_2[\omega_{2^m}];t)\le F(Re;t)=g(t)F(R';t).$$

Thus $f(t)\le g(t)$ since the R' indecomposable classes of A/B remain indecomposable in $\operatorname{Im} e$. This is seen by restricting to $\langle b_1,\dots,b_m\rangle$, $\langle b_{-1},b_2,\dots,b_m\rangle$ where the elements of 4.10 with ω divisible by x_1x_{-1} restrict to zero and using the known indecomposable classes of $M(m)$ [M; 3.11 ($p=2$)]. The Stiefel–Whitney classes $\omega_{2^m-2^i}$ of Δ_n restrict to $\omega_{2^m-2^i}$ of reg on these subgroups by [Q, 5.1]. Now $f(t)$, $g(t)$ are polynomials with positive integer coefficients. For $t=1$ all terms in $g(t)$ vanish unless $g=1$. Since $\chi_{st}(1)=\dim St=|U_{2m}|=2^{m(m-1)}$, $f(1)=2^{\binom{m}{2}+1}=g(1)$. Thus $f(t)\le g(t)$ implies $f(t)=g(t)$ and so 4.7i) is an equality.

To prove part ii) of the Theorem we observe that $Q, Sq^1Q,\dots, Sq^{2^{m-2}}\cdots Sq^2Sq^1Q$ is a regular sequence in H^*BV; hence the same Molien's series argument implies equality in 4.7ii). This completes the proof of Theorem 4.5.

Remark. A similar proof for computing $H^*M(n)$ was outlined in [M]; however, the argument is incomplete because of divisibility questions.

Remark. It is immediate from Theorem 4.5 that the Poincaré series of

$H^*\tilde{M}(2m)$ is

$$P.S.(H^*\tilde{M}(2m)) = \frac{2t^{2^{m+1}-2-m}}{[\prod_{i=1}^{m}(1-t^{2^i-1})](1-t^{2^m})} + \frac{t^{2^m-2-(m-1)}}{[\prod_{i=1}^{m-1}(1-t^{2^i-1})](1-t^{2^m})}.$$

Proof of Theorem 4.1. Since the Steinberg module is irreducible and projective, it lies in a matrix ring block; since its dimension equals $2^{m(m-1)}$, it follows that $2^{m(m-1)}$ summands appear (see [MP1]).

It remains to produce the desired splitting $\tilde{M}(2m)$. Let $U = \langle u_1, \dots, u_m \rangle$ be a vector space of dimension m over $\mathbf{F}_2$. For $I = \{i_1, \dots, i_m\}$, $i_j = \pm j$ define

$$\pi_I : V \to U$$

by

$$\pi_I(v_{i_j}) = u_j$$
$$\pi_I(v_k) = 0 \qquad k \notin I.$$

Define stable maps

$$\pi_\alpha = \sum \pi_I \pi : BE \to BU$$

$$\pi_\beta = \sum \pi_I \pi : BE \to BU$$

where sums are taken over those sequences I with an even (resp. odd) number of negative integers. By (4.2) it follows that $\Omega_n^+(\mathbf{F}_2)$ also acts on $T(\Delta_n)$ up to homotopy.

Finally let

$$f_1 : \tilde{M}(n) \xrightarrow{i} BE \xrightarrow{\pi_\alpha} BU \xrightarrow{\pi} M(m)$$

$$f_2 : \tilde{M}(n) \xrightarrow{i} BE \xrightarrow{\pi_\beta} BU \xrightarrow{\pi} L(m)$$

$$f_3 : \tilde{M}(n) \xrightarrow{i} BE \xrightarrow{t} T(\Delta_n) \xrightarrow{\pi} eT(\Delta_n)$$

where t is the transfer [MP1; 3.7] and π is projection onto a stable summand. We will show that

$$f = f_1 \vee f_2 \vee f_3 : \tilde{M}(n) \to M(m) \vee L(m) \vee eT(\Delta_n)$$

is a 2-local equivalence.

As modules,

$$H^*M(m) = \mathbf{F}_2\langle Sq^I(x_1^{-1} \cdots x_m^{-1})\rangle$$
$$H^*L(m) = \mathbf{F}_2\langle Sq^J(x_1^{-1} \cdots x_m^{-1})\rangle$$

([MP1]) with the same restrictions on I, J as in (4.6). Using the Cartan formula it follows that $Sq^I(x_1^{-1} \cdots x_m^{-1})$ is polynomial in $x_1, \ldots, x_m$ (i.e. there are no negative powers). Hence

$$f_1^*(Sq^I(x_1^{-1} \cdots x_m^{-1})) = Sq^I(\alpha)$$

and analogously

$$f_2^*(Sq^J(x_1^{-1} \cdots x_m^{-1})) = Sq^J(\beta)$$

Since $\Omega_n^+(\mathbf{F}_2)$ preserves the Euler class ω_{2^m} of Δ_n, it commutes with the Thom isomorphism

$$H^*BE \xrightarrow{\sim} H^*T(\Delta_n) = [H^*BE]\omega_{2^m}$$

Hence we have

$$H^*eT(\Delta_n) = [(H^*BE)e]\omega_{2^m} = [H^*\tilde{M}(n)]\omega_{2^m}$$

Under these identifications $t^* : H^*T(\Delta_n) \to H^*BE$ is the obvious inclusion. Hence f_3^* is an inclusion with image $[H^*\tilde{M}(n)]\omega_{2^m}$. The result follows from Theorem 4.5 and (4.6).

§5. Splitting $BE(4)$

Let $E = E(4)$, the extra-special 2-group of real type and of order 32. The Chevalley group $\Omega_4^+(\mathbf{F}_2)$ acts on BE up to homotopy; thus an orthogonal idempotent decomposition of 1 in $\mathbf{F}_2\Omega_4^+(\mathbf{F}_2)$ will provide a splitting of BE. One summand of this splitting is $BSL_2(\mathbf{F}_3)\circ SL_2(\mathbf{F}_3)$ where $E\approx Q_8\circ Q_8$ is a 2-Sylow subgroup of $SL_2(\mathbf{F}_3)\circ SL_2(\mathbf{F}_3)$.

Corresponding to the two factors of $\Omega_4^+(\mathbf{F}_2)\approx GL_2(\mathbf{F}_2)\times GL_2(\mathbf{F}_2)$ there are two orthogonal idempotent decompositions (4.8–9)

$$\begin{aligned}1&=f_0+f_1+f_2\\1&=f'_0+f'_1+f'_2\end{aligned}$$

Thus in $\mathbf{F}_2\Omega_4^+(\mathbf{F}_2)$ we have the orthogonal idempotent decomposition

$$1=f_0f'_0+(f_1f'_1+f_1f'_2+f_2f'_1+f_2f'_2)+(f_0f'_1+f_0f'_2+f_1f'_0+f_2f'_0) \tag{5.1}$$

where $f_1f'_1$ is the Steinberg idempotent.

THEOREM 5.2. *Corresponding to* (5.1) *there is a stable 2-local decomposition*

$$BE\cong BSL_2(\mathbf{F}_3)\circ SL_2(\mathbf{F}_3)\vee 4(M(2)\vee L(2)\vee eT(\Delta_4))\vee 4X$$

where $X=f_0f'_1BE$ is a *spectrum with Poincaré series* $(t^2+t^3)/(1-t)(1-t^3)(1-t^4)$.

Proof. The idempotents f_1,f_2 are conjugate [MP2] as are f'_1 and f'_2. Hence the summands corresponding to $f_1f'_1$, $f_1f'_2$, $f_2f'_1$ and $f_2f'_2$ are equivalent. By Theorem 4.1, each is equivalent to $M(2)\vee L(2)\vee eT(\Delta_4)$. Similarly f_0 and f'_0 are conjugate. Thus there are four summands equivalent to X. By comparing Poincaré series, the result now follows from part *i*) of

PROPOSITION 5.3. i) $f_0f'_0BE\cong BSL_2(\mathbf{F}_3)\circ SL_2(\mathbf{F}_3)$
ii) *For* $\mathbf{Z}/3\times\mathbf{Z}/3\subset\Omega_4^+(\mathbf{F}_2)$, $H^*SL_2(\mathbf{F}_3)\circ SL_2(\mathbf{F}_3)\approx H^*(E)^{\mathbf{Z}/3\times\mathbf{Z}/3}$
More explicitly,

$$H^*BSL_2(\mathbf{F}_3)\circ SL_2(\mathbf{F}_3)=\mathbf{F}_2[v_2,v_3,x_3,\bar{x}_3,\omega_4]/R$$

where

$$R=\begin{pmatrix}v_2^3+v_3^2+x_3^2+v_3x_3\\v_2^3+v_3^2+\bar{x}_3^2+v_3\bar{x}_3\end{pmatrix}$$

and

$$i^*(v_2) = x_1^2 + x_1x_{-1} + x_{-1}^2$$
$$i^*(v_3) = x_1x_{-1}^2 + x_1^2x_{-1}$$
$$i^*(x_3) = x_1^2x_{-1} + x_1^3 + x_{-1}^3$$
$$i^*(\bar{x}_3) = x_2^2x_{-2} + x_2^3 + x_{-2}^3$$

under the inclusion $i: E \approx Q_8 \circ Q_8 \to SL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$.

Proof. Part i) follows immediately from ii) since $f_0f'_0$ is the trace over $\mathbf{Z}/3 \times \mathbf{Z}/3$, i.e. $f_0f'_0 = \Sigma\, g$, $g \in \mathbf{Z}/3 \times \mathbf{Z}/3$. Part ii) is a straightforward generalization of that for $H^*BPSL_2(\mathbf{F}_3)$ [MP2]. One considers the map of fibrations

$$\begin{tikzcd}
B\mathbf{Z}/2 \arrow[r] \arrow[d] & BQ_8 \times Q_8 \arrow[r] \arrow[d, "Bi\times i"] & BQ_8 \circ Q_8 \arrow[d, "Bi"] \\
B\mathbf{Z}/2 \arrow[r] & BSL_2(\mathbf{F}_3) \times SL_2(F_3) \arrow[r] & BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)
\end{tikzcd}$$

and the corresponding map of spectral sequences.

Remark. The Poincaré series for $H^*BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$ is easily seen to be $(1+t^3)^2/(1-t^2)(1-t^3)(1-t^4)$.

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Received May 25 1987