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Stable splittings associated with Chevalley groups, I

MARK FESHBACH and STEWART PRIDDY¹

In recent years stable splittings have been studied for the classifying spaces of various finite groups, for example: elementary abelian p -groups [MP1], abelian groups [HK], dihedral and quaternion groups [MP2], etc. In this paper we continue this study; here we consider groups E which are extensions of an elementary abelian 2-group V by a cyclic group of order 2. These groups are among those of symplectic type [T, 2.4]; examples are the extra-special 2-groups [G, H]. A quadratic form Q is naturally associated with such an extension and the outer automorphisms of E which fix the center are precisely those automorphisms of V which preserve this form. Thus one of the classical orthogonal groups $O(V, Q)$ acts on BE (up to homotopy) and we can use idempotents from the group ring to stably split BE . In particular since the commutator subgroups of these groups are Chevalley groups, they have a BN pair and an associated Steinberg idempotent e . We determine the stable summand eBE . The degenerate case where E itself is an elementary abelian 2-group was studied in [MP1]. These cases cover the four systems of Chevalley groups A_m , B_m , D_m defined over \mathbf{F}_2 and the twisted group ${}^2D_m(\mathbf{F}_4)$.

It is well known that the orthogonal groups $O(V, Q)$ over \mathbf{F}_2 are determined by the dimension of V and the Arf invariant of Q . There exists three types of forms: one if $\dim V$ is odd and two if $\dim V$ is even. The latter cases are distinguished by $\text{Arf}(Q) = 0$ or 1. In this paper we set up machinery for handling the general cases but give specific analysis only for the $\text{Arf}(Q) = 0$ case. Here our main result (Theorem 4.1) is that BE contains $2^{m(m-1)}$ wedge summands, each equivalent to

$$eBE = M(m) \vee L(m) \vee eT(\Delta_{2m})$$

where $2m = \dim V$, $M(m)$ and $L(m)$ are wedge summands of $B(\mathbf{Z}/2)^m$ and $T(\Delta_{2m})$ is the Thom spectrum associated to an irreducible representation Δ_{2m} of E . In Part II, we study the remaining cases.

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The paper is organized as follows: Section 1 consists of some preliminaries on E , quadratic forms and Quillen's computation of H^*BE . The homotopy action of $O(V, Q)$ on BE is explained in Section 2. In Section 3 we describe the structure of $O(V, Q)$ as a Chevalley group and determine the Steinberg idempotent e . The cohomology of $H^*(eBE)$ is determined in Section 4. This leads to a proof of the main splitting in Theorem 4.1. In Section 5, we give a splitting of BE for $|E| = 32$ and $\text{Arf } Q = 0$. In what follows all spaces are localized at 2 and all cohomology groups are taken with simple coefficients in \mathbf{F}_2 .

It is a pleasure to thank Dave Benson for several helpful conversations on this material.

§1. Preliminaries

In this section we recall some preliminaries on quadratic forms, the groups E and their cohomology.

We begin with some standard facts about quadratic forms over \mathbf{F}_2 [Q]. Let V be a vector space over \mathbf{F}_2 . A *quadratic form* $Q: V \rightarrow \mathbf{F}_2$ is a function such that $Q(x+y) = Q(x) + Q(y) + B(x, y)$ for $x, y \in V$ and some bilinear form B . Necessarily B is *symplectic*, i.e. $B(x, x) = 0$. Let V_0 be the set of $x \in V$ such that $B(x, y) = 0$ for all $y \in V$. Then Q is said to be *non-degenerate* if $Q(x) \neq 0$ for all $x \neq 0$ in V_0 . Throughout this paper we will assume all quadratic forms to be non-degenerate.

Let $n = \dim V$. According to Dickson [Dk] there are, up to isomorphism three types of non-degenerate quadratic forms:

$$\begin{aligned} \text{If } n = 2m \quad Q &= \sum_{i=1}^m x_i x_{-i} && \text{(real case)} \\ Q &= \sum_{i=1}^{m-1} x_i x_{-i} + x_m^2 + x_m x_{-m} + x_{-m}^2 && \text{(quaternion case)} \end{aligned} \tag{1.0}$$

for some choice of basis $\{x_1, \dots, x_m, x_{-1}, \dots, x_{-m}\} \subset V^*$

$$\text{If } n = 2m + 1 \quad Q = x_0^2 + \sum_{i=1}^m x_i x_{-i} \quad \text{(complex case)}$$

for some choice of basis $\{x_0, x_1, \dots, x_m, x_{-1}, \dots, x_{-m}\} \subset V^*$. In the first two

cases we have $\text{Arf } Q = 0, 1$ respectively, where we recall

$$\text{Arf } Q = \begin{cases} 0 & \text{if } |Q^{-1}(0)| > \frac{1}{2} |V| \\ 1 & \text{if } |Q^{-1}(0)| < \frac{1}{2} |V|. \end{cases}$$

For convenience, however, we will use Quillen's terminology [Q] of *real* and *quaternion*; similarly we will call the third case *complex*.

Now suppose a group E is given as a central extension

$$\mathbf{Z}/2 \xrightarrow{i} E \xrightarrow{\pi} V \tag{1.1}$$

If $n = \dim V$ we shall often write $E = E(n)$. The associated quadratic and bilinear forms are given by

$$\begin{aligned} Q(x) &= \tilde{x}^2 & \text{where } \pi(\tilde{x}) = x \\ B(x, y) &= \tilde{x}\tilde{y}\tilde{x}^{-1}\tilde{y}^{-1} & \text{where } \pi(\tilde{x}) = x, \pi(\tilde{y}) = y \end{aligned}$$

For $n = 2$ in the real case $E \approx D_8$, the dihedral group of order 8 while in the quaternion case $E \approx Q_8$, the quaternion group of order 8. In general if n is even, $E(n)$ can be built up from the central product $(G \circ G' \approx G \times G'$ with centers identified). It is known that $D_8 \circ D_8 \approx D_8 \circ Q_8$. It is also straightforward to check

PROPOSITION 1.2. *If $n = 2m$*

$$\begin{aligned} E(n) &\approx D_8 \overset{\sim}{\circ} \cdots \overset{\sim}{\circ} D_8 & (\text{real case}) \\ &\approx D_8 \overset{\sim}{\circ} \cdots \overset{\sim}{\circ} D_8 \circ Q_8 & (\text{quaternion case}) \end{aligned}$$

In the real and quaternion cases, E is an extra-special 2-group.

(1.3) It will be convenient to specify generators of E : let $b_1, \dots, b_m, b_{-1}, \dots, b_{-m}$ (and b_0 in the complex case) be elements of E such that $\{v_{\pm i} = \pi(b_{\pm i})\}$ is dual to the basis $\{x_{\pm i}\}$ of V^* . Then E is generated by $\{b_{\pm i}, c\}$ where c is the non-trivial element of $\ker \pi$. (By convention $b_{\pm 0} = b_0$ in the complex case.) Using (1.0) a set of relations is seen to be given by commutators and squares.

(1.4) We now turn to H^*BE . A subspace W of V is called *isotropic* if $Q(W) = 0$. Now assume W is a maximal isotropic subspace or equivalently

$\tilde{W} = \pi^{-1}(W)$ is a maximal elementary abelian subgroup. Let $\chi: \tilde{W} \rightarrow \mathbf{Z}/2$ be a representation which is non-trivial on $\ker \pi = \mathbf{Z}/2$ and consider $\Delta = \text{Ind}_{\tilde{W}}^E(\chi)$, that is, Δ is the real representation induced from \tilde{W} to E . [Q; §5] shows that Δ is the unique irreducible real representation which is non-trivial on $\ker \pi$.

THEOREM 1.5. [Q; Th. 4.6]. *Given an extension (1.1) and the associated bilinear form Q , then*

$$H^*(BE) = S(V^*)/J \otimes \mathbf{F}_2[w_{2^h}]$$

where J is the ideal generated by the regular sequence $Q, Sq^1Q, Sq^2Sq^1Q, \dots, Sq^{2^{h-2}} \cdots Sq^2Sq^1Q$; h is the codimension of a maximal isotropic subspace of V and $w_{2^h} = w_{2^h}(\Delta)$ is the 2^h -th Stiefel–Whitney class of Δ .

Remark 1.6. For reference we record the values of h [Q; §2].

Case	$\dim V$	h
real	$2m$	m
complex	$2m + 1$	$m + 1$
quaternion	$2m$	$m + 1$

(1.7) Since the dimension of Δ is 2^h and $\ker \pi = \mathbf{Z}/2$ acts as -1 on Δ , Δ restricted to $\ker \pi$ is $2^h \cdot \eta$, where η is the non-trivial real character on $\mathbf{Z}/2$. It follows that $i^*(w_{2^h}) \neq 0$ and that any element with this property can be taken as a generator in place of w_{2^h} .

§2. Classical groups acting on H^*BE

Since conjugation is homotopic to the identity on the classifying space BG of any group G , the outer automorphism group $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$ acts up to homotopy on BG , i.e. there is a homomorphism

$$\text{Out}(G) \rightarrow \text{Aut}_{H_0}(BG)$$

where $\text{Aut}_{H_0}(BG)$ is the group of base point preserving equivalences in the homotopy category.

The referee points out that $\text{Out}(E)$ can be made to act on BE (not just up to homotopy). There is a group extension (which is not necessarily split)

$$1 \rightarrow E \rightarrow G_0 \rightarrow \text{Out}(E) \rightarrow 1$$

with $G_0/\langle c \rangle \approx \text{Aut}(E)$. Thus if X is a contractible CW-complex on which G_0 acts freely, then X/E is a model for BE on which $\text{Out}(E)$ acts as required.

Let $\text{Out}_z(G)$ be the subgroup of $\text{Out}(G)$ consisting of automorphisms which are the identity on the center of G . For $G = E$ as in (1.1) we have

PROPOSITION 2.1. $\text{Out}_z(E) \approx O(V, Q)$

Proof. It is clear from the definitions that π induces a homomorphism $\text{Out}_z(E) \rightarrow O(V, Q)$. This map is surjective by (1.3) and so any orthogonal automorphism of V can be lifted to an automorphism of E . That the center is fixed follows from examining the types of Q in (1.0). Conversely, suppose $\beta \in \text{Out}_z(E)$ induces the identity on V . Then for $b \in E$, $\beta(b) = b$ or bc where $\langle c \rangle = \ker \pi$. Let $\{v_i, v'_j\}$ be a basis for V such that $B(v_i, v'_j) \neq 0$ for at most one j for each i (e.g. in the real case v_i is dual to x_i and v'_j to x_{-j}). Let $\{b_i, b'_j\}$ satisfy $\pi(b_i) = v_i$, $\pi(b'_j) = v'_j$ and let ε be the product in any order of those b'_j 's for which $\beta(b_i) = b_i c$ and $B(v_i, v'_j) \neq 0$ for some i . Then $\beta(b_i) = \varepsilon b_i \varepsilon^{-1}$. Similarly let ε' be the product in any order of those b_i 's for which $\beta(b'_j) = b'_j c$ and $B(v_i, v'_j) \neq 0$ for some j . Then $\beta(b'_j) = \varepsilon' b'_j \varepsilon'^{-1}$. Consequently β is conjugation by $\varepsilon \varepsilon'$.

Remark. In the real and quaternion cases, $\text{Out}_z(V, Q) = O(V, Q)$ since the center is $\mathbf{Z}/2$. In the complex case the center is $\mathbf{Z}/4$ generated by an element b_0 such that $\pi(b_0)$ is dual to x_0 . Here $\text{Out}(E) = \mathbf{Z}/2 \times \text{Out}_z(E)$ where the extra automorphism is given by $b_0 \mapsto b_0^3$.

We now turn to the action of $O(V, Q)$ on H^*BE and the resulting invariants. The uniqueness of Δ (1.4) implies that its Stiefel–Whitney classes are invariants. In this connection Quillen has shown

THEOREM 2.2 [Q, Th. 5.1]. *The non-zero positive dimensional Stiefel–Whitney classes of Δ_n are ω_{2^h} , $\omega_{2^h-2^r}$, $\omega_{2^h-2^r+1}$, \dots , $\omega_{2^h-2^{h-1}}$ where $r = 0, 1, 2$ in the real, complex, and quaternion cases resp. Further, these classes form a regular sequence of maximal length in H^*BE and hence form a polynomial ring over which H^*BE is a free finitely generated module.*

Quillen further remarks, without proof, that in the real case these classes generate all of the invariants. We will prove a slightly sharper result. For

convenience we use the following notation

$$O(V, Q) = \begin{cases} O_{2m}^+(\mathbf{F}_2) & \text{if } n = 2m, \text{ real case} \\ O_{2m}^-(\mathbf{F}_2) & n = 2m, \text{ quaternion case} \\ O_{2m+1}(\mathbf{F}_2) & n = 2m + 1, \text{ complex case} \end{cases} \quad (2.3)$$

where $n = \dim V$. Let $\Omega_{2m}^\pm(\mathbf{F}_2)$ denote the commutator subgroup of $O_{2m}^\pm(\mathbf{F}_2)$.

THEOREM 2.4. *In the real case*

$$H^*BE^{\Omega_{2m}^+} = \mathbf{F}_2[\omega_{2m}, \omega_{2m-1}, \dots, \omega_{2^m-1}].$$

The proof depends on three lemmas, the first of which holds for a general V and Q .

LEMMA 2.5. *$O(V, Q)$ acts transitively on $\{A < E : A$ is a maximal elementary abelian group $\}$.*

Proof. $O(V, Q)$ acts transitively on $\{W < V : W$ is a maximal isotropic subspace $\}$. This is a result of Arf [A] in the real and quaternion cases. In the complex case $O_{2m+1}(\mathbf{F}_2) \approx Sp_{2m}(\mathbf{F}_2)$ and a proof can be found in [Dd]. The lemma follows since π induces an isomorphism between maximal elementary abelian subgroups of E and maximal isotropic subspaces of V .

Let $H: GL_m(\mathbf{F}_2) \rightarrow O_{2m}^+(\mathbf{F}_2)$ be the *hyperbolic map* given by

$$H(M) = \begin{bmatrix} M & O \\ O & M^{-1} \end{bmatrix}$$

(see [F-P; p. 152–154]). The appropriate quadratic form for the range is of the real type.

LEMMA 2.6. $H: GL_m(\mathbf{F}_2) \rightarrow \Omega_{2m}^+(\mathbf{F}_2)$

Proof. Since $\Omega_{2m}^+ = \ker d$ where $d: O_{2m}^+(\mathbf{F}_2) \rightarrow \mathbf{Z}/2$ is the Dickson invariant, we need only check $d \circ H = 0$. This follows from the formula for d [Dd; p. 64].

LEMMA 2.7. *Let $A \xrightarrow{j} E$ be the inclusion of a maximal elementary abelian subgroup. Then $j^*(H^*(BE)^{\Omega_{2m}^+}) = \text{Im } (j^* \Delta^*)$.*

Proof. The inclusion \supset follows from the inclusion $H^*(BE)^{\Omega_{2m}^+} \supset \text{Im } \Delta^*$ noted

above. For the other inclusion it suffices by Theorem 1.5 to consider $x \in H^*(BE)^{\Omega_{2m}^+}$ in the image of $\pi^*: H^*BV \rightarrow H^*BE$. By Lemma 2.5, (1.4) and the normality of Ω_{2m}^+ , it suffices to prove the result for one maximal elementary abelian subgroup A . Let $A = \langle b_1, \dots, b_m, c \rangle \triangleleft E$; we can write $A = A' \oplus C$ where $C = \langle c \rangle = \ker \pi$. Let $M \in GL_m(\mathbf{F}_2)$. Then for $j^*(x) = y \otimes 1 \in H^*BA' \otimes H^*BC$, we have

$$(y \otimes 1)H(M) = yM \otimes 1$$

Hence $y \in H^*(BA')^{GL(A')}$. By [Wk; 4.1], $H^*(BA')^{GL(A')} = \text{Im}(\text{reg}(A')^*)$ for the regular representation of A' . Since $\Delta j = \text{reg}(A') \otimes \chi$ on $A' \oplus C$ [Q; 5.1], we have $j^*(x) = y \otimes 1 \in \text{Im}(j^*\Delta^*)$ using the formula for the Stiefel–Whitney classes of $\text{reg}(A') \otimes \chi$ [Q; 5.6].

Proof of Theorem 2.4. By [Q; Th. 5.10], H^*BE is detected by elementary abelian subgroups. Hence the result follows directly from Lemma 2.7.

COROLLARY 2.8. $H^*(BE)^{O_{2m}^+} = H^*(BE)^{\Omega_{2m}^+}$.

§3. $O_n(\mathbf{F}_2)$ as Chevalley groups

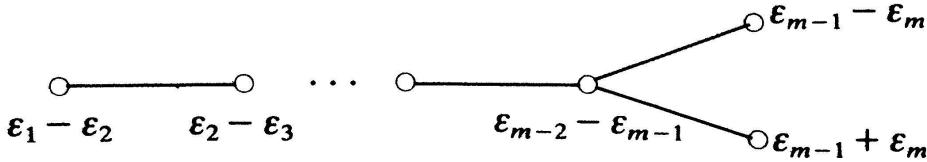
Our goal in this section is to describe what we need about the Steinberg idempotent for the orthogonal group. A good general reference is R. Carter's book [C]. For each simple Lie algebra L over \mathbf{C} and each field K , Chevalley has constructed a group $L(K)$. Later Steinberg, Tits and Hertzig discovered additional twisted versions of these groups. For the simple Lie algebras of type A_m , B_m , C_m and D_m and for K finite, Ree has identified these Chevalley groups with classical groups. We state the result for $K = \mathbf{F}_2$.

THEOREM 3.1 (Ree [C; Th. 11.3.2])

- i) $A_m(\mathbf{F}_2) \approx GL_{m+1}(\mathbf{F}_2)$
- ii) $B_m(\mathbf{F}_2) \approx O_{2m+1}(\mathbf{F}_2)$
- iii) $C_m(\mathbf{F}_2) \approx B_m(\mathbf{F}_2)$
- iv) $D_m(\mathbf{F}_2) \approx \Omega_{2m}^+(\mathbf{F}_2)$

The group $\Omega_{2m}^-(\mathbf{F}_2)$ occurs as a twisted Chevalley group and will be treated at the end of this section.

3.2 *The real case:* The Dynkin diagram for D_m , $m > 1$, is



where $\epsilon_1, \dots, \epsilon_m$ is the standard basis for \mathbf{R}^m .

Let e_{ij} be the $2m$ square matrix with 1 in the (i, j) position and 0's elsewhere. Let $u_{ij} = I + e_{ij} + e_{-j, -i} \in GL_{2m}(\mathbf{F}_2)$. Then the *unipotent subgroup* $U_{2m} < \Omega_{2m}^+(\mathbf{F}_2)$ is generated by

$$\{u_{i,j}, u_{i,-j} : 1 \leq i < j \leq m\}$$

(We recall that the underlying vector space V has basis $\{v_1, \dots, v_m, v_{-1}, \dots, v_{-m}\}$ over \mathbf{F}_2 .) The *Weyl group* $W_{2m}^+ < \Omega_{2m}^+(\mathbf{F}_2)$ is generated by

$$\{\sigma_{ij} = u_{i,j}u_{-i,-j}u_{i,j}, \sigma_{i,-j} = u_{i,-j}u_{-i,j}u_{i,-j} : 1 \leq i < j \leq m\}.$$

Abstractly $W_{2m}^+ \approx (\mathbf{Z}/2)^{m-1} \rtimes \Sigma_m$ (permutations together with an even number of sign changes).

Finally $\Omega_{2m}^+(\mathbf{F}_2)$ is generated by U_{2m} and V_{2m} where V_{2m} is generated by $\{u_{-i,-j}, u_{-i,j} : 1 \leq i < j \leq m\}$.

(3.3) The *Steinberg idempotent* $e \in \mathbf{F}_2 \Omega_{2m}^+(\mathbf{F}_2)$ is defined by

$$e = \sum u\sigma \quad u \in U_{2m}, \sigma \in W_{2m}^+.$$

For computational purposes, it will be convenient to use another expression for e . For each of the simple roots $\{\epsilon_i - \epsilon_{i+1}\}$ in the Dynkin diagram let e_i be the idempotent

$$e_i = (1 + u_{i,i+1})(1 + \sigma_{i,i+1}) \quad 1 \leq i \leq m-1$$

For the last root $\epsilon_{m-1} + \epsilon_m$ let

$$e_m = (1 + u_{m-1,-m})(1 + \sigma_{m-1,-m})$$

Kuhn [K] has shown that e can be expressed as a product of the e_i , $i = 1, 2, \dots, m$. Moreover

THEOREM 3.4. [K, Th. 1.3] *Let M be a right $\mathbf{F}_2\Omega_{2m}^+(\mathbf{F}_2)$ module. Then*

$$Me = \bigcap_{i=1}^m Me_i.$$

(3.5) *The complex case:* The Dynkin diagram for B_m is

$$\begin{array}{ccccccc} \circ & \text{---} & \circ & \cdots & \circ & \text{---} & \circ \\ \varepsilon_1 - \varepsilon_2 & & & & \varepsilon_{m-1} - \varepsilon_m & & \varepsilon_m \end{array}$$

Let

$$u_{ij} = I + e_{ij} + e_{-j, -i} \quad i \neq j$$

$$u_{ii} = I + e_{0, -i} + e_{i, -i} \quad i \neq 0$$

(V has basis $v_0, v_1, \dots, v_m, v_{-1}, \dots, v_{-m}$). The *unipotent subgroup* $U_{2m+1} < O_{2m+1}(\mathbf{F}_2)$ is generated by

$$\{u_{ij}, u_{i, -j}, u_{ii} : 1 \leq i < j \leq m\}$$

The *Weyl group* $W_{2m+1} < O_{2m+1}(\mathbf{F}_2)$ is generated by

$$\left\{ \begin{array}{ll} \sigma_{ij} = u_{-i, -j} u_{i, j} u_{-i, -j} & 1 \leq i < j < m \\ \sigma_{i, -j} = u_{-i, j} u_{i, -j} u_{-i, j} & \\ \sigma_{ii} = u_{-i, -i} u_{ii} u_{-i, -i} & 1 \leq i \leq m \end{array} \right.$$

Then $O_{2m+1}(\mathbf{F}_2)$ is generated by U_{2m+1} and V_{2m+1} where V_{2m+1} is generated by

$$\{u_{-i, -j}, u_{-i, j}, u_{-i, -i} : 1 \leq i < j \leq m\}.$$

The *Steinberg idempotent* $e \in \mathbf{F}_2 O_{2m+1}(\mathbf{F}_2)$ is defined by

$$e = \sum u\sigma \quad u \in U_{2m+1}, \sigma \in W_{2m+1}$$

In this case Kuhn [K] has shown that e can be expressed as a product of the

following idempotents

$$e_i = (1 + u_{i,i+1})(1 + \sigma_{i,i+1}) \quad 1 \leq i \leq m-1$$

$$e_m = (1 + u_{m,m})(1 + \sigma_{mm})$$

and the analog of Theorem 3.4 holds.

3.6 The quaternion case: The group $\Omega_{2m}^-(\mathbf{F}_2)$ is isomorphic to the twisted Chevalley group ${}^2D_m(\mathbf{F}_4)$ [C; Th. 14.5.2] with Dynkin diagram of type B_{m-1}

$$\begin{array}{ccccccc} \circ & \text{---} & \circ & \dots & \circ & \text{---} & \circ \\ \varepsilon_1 - \varepsilon_2 & & \varepsilon_2 - \varepsilon_3 & & \varepsilon_{m-2} - \varepsilon_{m-1} & & \left\{ \begin{array}{l} \varepsilon_{m-1} - \varepsilon_m \\ \equiv \varepsilon_{m-1} + \varepsilon_m \end{array} \right\} \end{array}$$

It is a projection of the diagram for D_m in (3.2). For details of this group see Chapters 13, 14 of [C].

Let

$$\begin{aligned} \tau_m &= I + e_{m-1,m} + e_{m-1,-(m-1)} + e_{m-1,-m} + e_{m,-(m-1)} + e_{-m,-(m-1)} \\ \gamma_m &= I - e_{m-1,m} + e_{m-1,-(m-1)} + e_{-m,-(m-1)} \\ \tau'_m &= I - e_{-(m-1),m-1} + e_{-(m-1),m} + e_{-m,m-1} \\ \gamma'_m &= I + e_{m,m-1} + e_{-(m-1),m-1} + e_{-(m-1),m} + e_{-(m-1),-m} + e_{m,m-1} \end{aligned}$$

The unipotent subgroup $U_{2m}^- < \Omega_{2m}^-(\mathbf{F}_2)$ is generated by $\{\tau_m, \gamma_m\} \cup \{u_{i,j}, u_{i,-j} : 1 \leq i < j \leq m-1\}$. V_{2m}^- is generated by $\{\tau'_m, \gamma'_m\} \cup \{u_{-i,-j}, u_{-i,j} : 1 \leq i < j \leq m-1\}$. $\Omega_{2m}^-(\mathbf{F}_2)$ is generated by U_{2m}^- and V_{2m}^- . Let B_{2m}^- be the normalizer of U_{2m}^- in $\Omega_{2m}^-(\mathbf{F}_2)$.

The Weyl group W_{2m}^- of $\Omega_{2m}^-(\mathbf{F}_2)$ is generated by $\{\sigma_{ij}, \sigma_{i,-j} : 1 \leq i < j \leq m\} \cup \{\tau_m \tau'_m \tau_m = W_m\}$. The Steinberg idempotent $e \in \mathbf{F}_2 \Omega_{2m}^-(\mathbf{F}_2)$ is defined by $e = \sum b \sigma$ $b \in B_{2m}^-$, $\sigma \in W_{2m}^-$.

In this case Kuhn [K] has shown that e can be expressed as a product of the idempotents corresponding to the nodes in the Dynkin diagram for B_{m-1} . These are

$$e_i = (1 + u_{i,i+1})(1 + \sigma_{i,i+1}) \quad 1 \leq i \leq m-2$$

and the idempotent e'_m corresponding to the last node

$$e'_m = (1 + \tau_m)(1 + \gamma_m)(1 + H_m + H_m^2)(1 + W_m)$$

where $H_m = I + e_{m,m} + e_{m,-m} + e_{-m,m}$ and $W_m = I + e_{m-1,m-1} + e_{m-1,-(m-1)} + e_{m,-m} + e_{-(m-1),m-1} + e_{-(m-1),-(m-1)}$.

§4. The Steinberg wedge summand: the real case

For $n = 2m$ let $E = E(n)$ denote the extra-special 2-group of real type. Let $\tilde{M}(n)$ be the stable summand

$$\tilde{M}(n) = eBE$$

corresponding to the Steinberg idempotent of (3.3). Our main result is

THEOREM 4.1. *Stably, for $m \geq 2$, BE contains $2^{m(m-1)}$ copies of $\tilde{M}(n) = M(m) \vee L(m) \vee eT(\Delta_n)$.*

Here $M(m)$ is the Steinberg summand of $B(\mathbf{Z}/2)^m$ [MP1], $L(m) = \sum_{-m}^{+m} Sp^{2m}S^0/Sp^{2m-1}S^0$, and $T(\Delta_n)$ is the Thom spectrum of the bundle $B\Delta_n$ over BE . As a spectrum $M(m) = L(m) \vee L(m-1)$.

(4.2) The uniqueness of Δ_n (1.4) implies that the homotopy action of $O_n^+(\mathbf{F}_2)$ on BE preserves the isomorphism type of Δ_n and hence induces a homotopy action of $O_n^+(\mathbf{F}_2)$ on $T(\Delta_n)$. The summand $eT(\Delta_n)$ is defined with respect to this action.

On the way to proving Theorem 4.1 we first determine $H^*\tilde{M}(n)$. Let

$$\alpha = \alpha_m = \sum x_{i_1}^{-1} x_{i_2}^{-1} \cdots x_{i_m}^{-1},$$

$i_j = \pm j$ with an even number of minus signs occurring

$$\beta = \beta_m = \sum x_{i_1}^{-1} x_{i_2}^{-1} \cdots x_{i_m}^{-1},$$

$i_j = \pm j$ with an odd number of minus signs occurring.

These elements belong to S_V , that is, $S = H^*BV$ with the inverses of all non-zero linear elements adjoined. The action of $O_n^+(\mathbf{F}_2)$ on H^*BV extends to S_V .

LEMMA 4.3. $\alpha e = \alpha$, $\beta e = \beta$.

Proof. By 3.4 it suffices to show α and β are fixed by e_i , $i = 1, \dots, m$. Write

$$\alpha = (x_i^{-1}x_{i+1}^{-1} + x_{-i}^{-1}x_{-(i+1)}^{-1})\hat{\alpha}_i + (x_{-i}^{-1}x_{i+1}^{-1} + x_i^{-1}x_{-(i+1)}^{-1})\hat{\beta}_i$$

where $\hat{\alpha}_i$ (resp. $\hat{\beta}_i$) is the sum of those terms $x_{j_1}^{-1} \cdots x_{j_{m-2}}^{-1}$ not containing $x_{\pm i}^{-1}$,

$x_{\pm(i+1)}^{-1}$ and having an even (resp. odd) number of minus signs. By 3.3,

$$e_i = (1 + u_{i,i+1})(1 + \sigma_{i,i+1}) \quad 1 \leq i < m$$

$$e_m = (1 + u_{m-1,-m})(1 + \sigma_{m-1,-m})$$

where the action of $u_{i,j}$ is $x_i \rightarrow x_i + x_j$, $x_{-j} \rightarrow x_{-i} + x_{-j}$, $x_k \rightarrow x_k$ otherwise and the action of $\sigma_{i,j}$ is $x_{\pm i} \rightarrow x_{\pm j}$, $x_{\pm j} \rightarrow x_{\pm i}$. Hence for $1 \leq i < m$,

$$\begin{aligned} \alpha e_i &= \alpha + [(x_i + x_{i+1})^{-1} x_{i+1}^{-1} + x_{-i}^{-1} (x_{-i} + x_{-(i+1)})^{-1}] \hat{\alpha}_i \\ &\quad + [(x_i + x_{i+1})^{-1} (x_{-i} + x_{-(i+1)})^{-1} + x_{-i}^{-1} x_{i+1}^{-1}] \hat{\beta}_i \\ &\quad + [x_i^{-1} x_{i+1}^{-1} + x_{-i}^{-1} x_{-(i+1)}^{-1}] \hat{\alpha}_i \\ &\quad + [x_{-i}^{-1} x_{i+1}^{-1} + x_i^{-1} x_{-(i+1)}^{-1}] \hat{\beta}_i + [(x_i + x_{i+1})^{-1} x_i^{-1} \\ &\quad + x_{-(i+1)}^{-1} (x_{-i} + x_{-(i+1)})^{-1}] \hat{\alpha}_i \\ &\quad + [(x_i + x_{i+1})^{-1} (x_{-i} + x_{-(i+1)})^{-1} + x_i^{-1} x_{-(i+1)}^{-1}] \hat{\beta}_i = \alpha. \end{aligned}$$

For $i = m$ we have

$$\begin{aligned} \alpha e_m &= \alpha + [(x_{m-1} + x_{-m})^{-1} (x_m + x_{-(m-1)})^{-1} + x_{-(m-1)}^{-1} x_{-m}^{-1}] \hat{\alpha}_{m-1} \\ &\quad + [(x_{m-1} + x_{-m})^{-1} x_{-m}^{-1} + x_{-(m-1)}^{-1} (x_m + x_{-(m-1)})^{-1}] \hat{\beta}_{m-1} \\ &\quad + [x_{-m}^{-1} x_{-(m-1)}^{-1} + x_m^{-1} x_{m-1}^{-1}] \hat{\alpha}_{m-1} \\ &\quad + [x_{-m}^{-1} x_{m-1}^{-1} + x_m^{-1} x_{-(m-1)}^{-1}] \hat{\beta}_{m-1} + [(x_{-m} + x_{m-1})^{-1} (x_{-(m-1)} + x_m)^{-1} \\ &\quad + x_m^{-1} x_{m-1}^{-1}] \hat{\alpha}_{m-1} \\ &\quad + [(x_{-m} + x_{m-1})^{-1} x_{m-1}^{-1} + x_m^{-1} (x_{-(m-1)} + x_m)^{-1}] \hat{\beta}_{m-1} = \alpha \end{aligned}$$

A similar calculation shows $\beta e = \beta$.

LEMMA 4.4. $Sq^1 \alpha = Sq^1 \beta$.

The proof is straightforward calculation using $Sq^1 x^{-1} = 1$. Now let

$$A = \mathbf{F}_2 \langle Sq^1 \alpha, Sq^1 \beta : I \text{ admissible, } l(I) = m \rangle$$

$$B = \mathbf{F}_2 \langle Sq^1 Sq^1 \alpha + Sq^1 Sq^1 \beta : (J, 1) \text{ admissible, } l(J) = m-1 \rangle$$

THEOREM 4.5. i) $H^* \tilde{M}(n) = (A/B) \otimes \mathbf{F}_2[\omega_{2^m}]$

ii) $H^*(eBV) = (A/B) \otimes \mathbf{F}_2[Q, Sq^1 Q, \dots, Sq^{2^m-2} \cdots Sq^2 Sq^1 Q]$

Proof. In discussing i) and ii) we will implicitly use the commutative diagram

$$\begin{array}{ccc} H^*BV & \xrightarrow{\epsilon} & H^*BV \\ \downarrow \pi^* & & \downarrow \pi^* \\ H^*BE & \xrightarrow{\epsilon} & H^*BE \end{array}$$

The elements $Sq^I\alpha, Sq^I\beta \in H^*(eBV)$ by Lemma 4.3 and the relations B hold by Lemma 4.4. A basis for $A/B \subset H^*(eBV)$ is given by

$$\{Sq^I\alpha, Sq^I\beta : I, J \text{ admissible, } l(I) = m, l(J) = m, j_m > 1\} \quad (4.6)$$

Restricting to the subgroups $\langle b_1, b_2, \dots, b_m \rangle, \langle b_{-1}, b_2, \dots, b_m \rangle$ shows these elements remain linearly independent in H^*BE . Thus

$$(A/B) \otimes \mathbf{F}_2[\omega_{2^m}] \subset H^*\tilde{M}(n) \quad (4.7i)$$

since ω_{2^m} is invariant under $\Omega_n^+(\mathbf{F}_2)$. By Theorem 1.5, $Q, Sq^1Q, \dots, Sq^{2^{m-2}} \cdots Sq^2Sq^1Q \subset H^*BV$ is a regular sequence of invariants; therefore a theorem of P. Baum [B, 3.5] implies

$$(A/B) \otimes \mathbf{F}_2[Q, Sq^1Q, \dots, Sq^{2^{m-2}} \cdots Sq^2Sq^1Q] \subset H^*(eBV). \quad (4.7ii)$$

It remains to check equality of the Poincaré series of these modules. The proof is by induction on $n = 2m$.

For this we first treat the case $n = 4$. It is readily seen that $\Omega_4^+(\mathbf{F}_2) \approx GL_2(\mathbf{F}_2) \times GL_2(\mathbf{F}_2)$ with generators $\{u_{12}, \sigma_{12}\}$ for the first factor and $\{u_{1,-2}, \sigma_{1,-2}\}$ for the second. Then $f_1 = (1 + u_{12})(1 + \sigma_{12})$ corresponds to the Steinberg idempotent for $GL_2(\mathbf{F}_2)$ [MP1] and

$$1 = f_0 + f_1 + f_2 \quad (4.8)$$

is an orthogonal decomposition into primitive idempotents, where $f_0 = 1 + u_{12}\sigma_{12} + (u_{12}\sigma_{12})^2$ and $f_2 = (1 + \sigma_{12})(1 + u_{12})$. Similarly in the second factor, let

$$1 = f'_0 + f'_1 + f'_2 \quad (4.9)$$

be the corresponding decomposition. Then $f_1f'_1$ is the Steinberg idempotent for $\mathbf{F}_2\Omega_4^+(\mathbf{F}_2)$.

Consider $V = V_4$, the vector space with dual basis x_1, x_2, x_{-1}, x_{-2} . Then $(H^*BV)f_0f'_0 = H^*BV^{\mathbf{Z}/3 \times \mathbf{Z}/3}$ since $u_{12}\sigma_{12}$ and $u_{1,-2}\sigma_{1,-2}$ have order three. A

simple application of Molien's series [M] computes the Poincaré series

$$P.S.(H^*BVf_0f'_0) = \frac{(1+t^3)^2}{(1-t^2)^2(1-t^3)^2}$$

Similarly $(H^*BV)f_0 = H^*BV^{\mathbf{Z}/3}$ and Molien's series yields

$$P.S.(H^*BVf_0) = \frac{1+2t^2+6t^3+2t^4+t^6}{(1-t^2)^2(1-t^3)^2}$$

Since f_1 and f_2 are conjugate as well as f'_1 and f'_2 , (4.8) then implies

$$P.S.(H^*BVf_1) = \frac{2t+3t^2+2t^3+3t^4+2t^5}{(1-t^2)^2(1-t^3)^2}$$

Now $f_0 = f_0f'_0 + f_0f'_1 + f_0f'_2$; hence

$$P.S.(H^*BVf_0f'_1) = \frac{t^2+2t^3+t^4}{(1-t^2)^2(1-t^3)^2}$$

Therefore

$$P.S.(H^*BVf_1f'_1) = \frac{t+t^2+t^4+t^5}{(1-t^2)^2(1-t^3)^2}$$

which, by 4.6 ($m = 2$), equals the Poincaré series for $(A/B) \otimes \mathbf{F}_2[Q, Sq^1Q]$. Hence, we have equality in 4.7ii ($m = 2$). Since ω_4 is an invariant, equality in 4.7i ($m = 2$) follows from Theorem 2.2.

We now turn to the general case part i), $n = 2m$, assuming by induction both parts of case $2m - 2$. To compute $H^*\widetilde{M(n)}$ as a module over $\mathbf{F}_2[\omega_{2^m}]$ we consider the commutative diagram

$$\begin{array}{ccc} H^*BE & \xrightarrow{\bar{e}} & H^*BE \\ \uparrow \pi^* & & \uparrow \pi^* \\ H^*BV & \xrightarrow{\bar{e}} & H^*BV \end{array}$$

where $\bar{e} \in \mathbf{F}_2\Omega_{2m}^+(\mathbf{F}_2)$ is the image of the Steinberg idempotent for $\Omega_{2m-2}^+(\mathbf{F}_2)$ acting on the last $2m - 2$ co-ordinates. Since $\text{Im } e \subset \text{Im } \bar{e}$ by Theorem 3.4,

induction and the relations

$$Q \equiv x_1 x_{-1} + \sum_{i=2}^m x_i x_{-i}, Sq^1 Q, \dots, Sq^{2^{m-2}} \dots Sq^2 Sq^1 Q$$

of H^*BE imply $\text{Im } e$ is generated by elements of the form

$$\omega(Sq^I \alpha'_{m-1}), \quad \omega(Sq^I \beta'_{m-1}) \quad (4.10)$$

where $\alpha'_{m-1}, \beta'_{m-1}$ are $\alpha_{m-1}, \beta_{m-1}$ on the last $2m-2$ co-ordinates, $l(I) = m-1$ and $\omega = \omega(x_1, x_{-1})$ is a homogeneous polynomial in x_1, x_{-1} . The remainder of the proof of this inductive step consists of two steps 4.11, 12.

(4.11) Suppose $z \in \text{Im } e$ is a linear combination of terms from (4.10). Restriction to the subgroups $\langle b_1, \dots, b_m \rangle$ (resp. $\langle b_1, \dots, b_{m-1}, b_{-m} \rangle$) detects the summands $\omega Sq^I \alpha'_{m-1}$ (resp. $\omega Sq^I \beta'_{m-1}$) of z with some ω a polynomial in x_1 . Invariance of $\text{Im } e$ under the Weyl group W_{2m}^+ then shows z is a linear combination of terms $Sq^K \alpha_m, Sq^K \beta_m$, $l(K) = m$. A similar argument shows the same conclusion holds if ω is a polynomial in x_{-1} alone. Thus $\text{Im } e$ consists of $(A/B) \otimes \mathbf{F}_2[\omega_{2^m}]$ plus possibly terms from (4.10) with ω divisible by $x_1 x_{-1}$. It remains to eliminate the possibly of such terms.

(4.12) We shall need to recall some facts about Molien's series [M]. Let G be a finite group and N a graded $\mathbf{F}_2 G$ module. As usual the Poincaré series of N is given by $P.S.(N) = F(N; t) = \sum (\dim_{\mathbf{F}_2} N_i) t^i$. For an irreducible $\mathbf{F}_2 G$ module E , we also consider the series

$$F(N, G, E; t) = \sum a_i t^i$$

where a_i is the multiplicity of E as a composition factor in N_i . Finally, let

$$\chi(N; t) = \sum \chi_{N_i} t^i$$

be the modular character series where χ_{N_i} is the modular (or Brauer) character of N_i defined on the p -regular elements G_{reg} of G ([S]).

In the present situation let $G = \Omega_{2m}^+(\mathbf{F}_2)$, $R = H^*BE$ and

$$R' = \mathbf{F}_2[\omega_{2^m}, \omega_{2^m-2^i}, i = 0, 1, \dots, m-1].$$

We note $R' = R^{\Omega_{2^m}^+}$ by Theorem 3.4. Let $M = R \otimes_{R'} \mathbf{F}_2$. Then in each dimension

R and $R' \otimes M$ have the same composition series by Theorem 2.2 and the proof of [M, 1.3]. Hence

$$F(R, G, St; t) = F(M, G, St; t)F(R', t) \quad (4.13)$$

where St is the Steinberg module $St = e\mathbf{F}_2 G$. By [M; 1.2b] and 4.13 we have

$$F(Re; t) = F(R, G, St; t) \quad (4.14)$$

Now the orthogonality relations for modular characters [S, M] imply

$$F(Re; t) = \frac{1}{|G|} \sum_{g \in G_{\text{reg}}} \chi_{St}(g^{-1}) \chi(R; t)(g) \quad (4.15)$$

where $|G| = (2^m - 1) \prod_{i=1}^{m-1} (2^{2i} - 1) 2^{2i}$ by [Dk; p. 206]. To evaluate this series we use

LEMMA 4.16.

$$\chi(R; t)(g) = \frac{(1 - t^2)(1 - t^3) \cdots (1 - t^{2^{m-1}+1})}{[\prod_i^{2m} (1 - \lambda_i(g)t)](1 - t^{2^m})}$$

where $\{\lambda_i(g)\}$ are the eigenvalues of g acting on V .

Proof. Let $S = S(V^*)$ be the symmetric algebra of V^* . Then $R = N \otimes \mathbf{F}_2[\omega_{2^m}]$ where $N = S \otimes_P \mathbf{F}_2$ and $P = \mathbf{F}_2[Q, Sq^1Q, \dots, Sq^{2^{m-2}} \cdots Sq^1Q]$. The generators of P form a regular sequence on S by Theorem 1.5. Hence by [B; 3.5], $S \approx P \otimes N$. Thus

$$\chi(S; t) = \chi(P; t)\chi(N; t)$$

or

$$\prod_1^{2m} (1 - \lambda_i t) = \prod_{i=0}^{m-1} (1 - t^{2^i+1}) \chi(N; t)$$

and the lemma follows since $\chi(\mathbf{F}_2[\omega_{2^m}]) = (1 - t^{2^m})^{-1}$.

From 4.6

$$\begin{aligned}
F(A/B \otimes \mathbf{F}_2[\omega_{2^m}]; t) &= \frac{2t^{2^{m+1}-2-m}}{\prod_{i=1}^m (1-t^{2^i-1})(1-t^{2^m})} + \frac{t^{2^m-2-(m-1)}}{\prod_{i=1}^{m-1} (1-t^{2^i-1})(1-t^{2^m})} \\
&= \frac{(t^{2^{m+1}-2-m} + t^{2^m-1-m}) \prod_{k=1}^{m-1} Q_k(t)}{(\prod_{i=0}^{m-1} (1-t^{2^m-2^i}))(1-t^{2^m})} = f(t)F(R'; t)
\end{aligned}$$

where $Q_k(t) = \prod_{i=0}^{k-1} (1+t^{2^i(2^{m-k}-1)})$ and $f(t) = (t^{2^{m+1}-2-m} + t^{2^m-1-m}) \prod_{k=1}^{m-1} Q_k(t)$. Combining 4.14, 15 and Lemma 4.16 we have

$$F(Re; t) = g(t)F(R'; t)$$

where

$$g(t) = \frac{1}{|G|} \sum \chi_{St}(g^{-1}) \frac{\prod_{i=0}^{m-1} (1-t^{2^i+1}) \prod_{j=0}^{m-1} (1-t^{2^m-2^j})}{\prod_{i=1}^n (1-\lambda_i(g) \cdot t)}.$$

By 4.7i

$$f(t)F(R'; t) = F(A/B \otimes \mathbf{F}_2[\omega_{2^m}]; t) \leq F(Re; t) = g(t)F(R'; t).$$

Thus $f(t) \leq g(t)$ since the R' indecomposable classes of A/B remain indecomposable in $\text{Im } e$. This is seen by restricting to $\langle b_1, \dots, b_m \rangle$, $\langle b_{-1}, b_2, \dots, b_m \rangle$ where the elements of 4.10 with ω divisible by $x_1 x_{-1}$ restrict to zero and using the known indecomposable classes of $M(m)$ [M; 3.11 ($p=2$)]. The Stiefel–Whitney classes $\omega_{2^m-2^i}$ of Δ_n restrict to $\omega_{2^m-2^i}$ of reg on these subgroups by [Q, 5.1]. Now $f(t)$, $g(t)$ are polynomials with positive integer coefficients. For $t=1$ all terms in $g(t)$ vanish unless $g=1$. Since $\chi_{St}(1) = \dim St = |U_{2m}| = 2^{m(m-1)}$, $f(1) = 2^{\binom{m}{2}+1} = g(1)$. Thus $f(t) \leq g(t)$ implies $f(t) = g(t)$ and so 4.7i) is an equality.

To prove part ii) of the Theorem we observe that $Q, Sq^1Q, \dots, Sq^{2^{m-2}} \dots Sq^2 Sq^1 Q$ is a regular sequence in H^*BV ; hence the same Molien's series argument implies equality in 4.7ii). This completes the proof of Theorem 4.5.

Remark. A similar proof for computing $H^*M(n)$ was outlined in [M]; however, the argument is incomplete because of divisibility questions.

Remark. It is immediate from Theorem 4.5 that the Poincaré series of

$H^* \tilde{M}(2m)$ is

$$P.S.(H^* \tilde{M}(2m)) = \frac{2t^{2^{m+1}-2-m}}{[\prod_{i=1}^m (1-t^{2^{i-1}})](1-t^{2^m})} + \frac{t^{2^m-2-(m-1)}}{[\prod_{i=1}^{m-1} (1-t^{2^{i-1}})](1-t^{2^m})}.$$

Proof of Theorem 4.1. Since the Steinberg module is irreducible and projective, it lies in a matrix ring block; since its dimension equals $2^{m(m-1)}$, it follows that $2^{m(m-1)}$ summands appear (see [MP1]).

It remains to produce the desired splitting $\tilde{M}(2m)$. Let $U = \langle u_1, \dots, u_m \rangle$ be a vector space of dimension m over \mathbf{F}_2 . For $I = \{i_1, \dots, i_m\}$, $i_j = \pm j$ define

$$\pi_I: V \rightarrow U$$

by

$$\begin{aligned} \pi_I(v_{i_j}) &= u_j \\ \pi_I(v_k) &= 0 \quad k \notin I. \end{aligned}$$

Define stable maps

$$\pi_\alpha = \sum \pi_I \pi: BE \rightarrow BU$$

$$\pi_\beta = \sum \pi_I \pi: BE \rightarrow BU$$

where sums are taken over those sequences I with an even (resp. odd) number of negative integers. By (4.2) it follows that $\Omega_n^+(\mathbf{F}_2)$ also acts on $T(\Delta_n)$ up to homotopy.

Finally let

$$f_1: \tilde{M}(n) \xrightarrow{i} BE \xrightarrow{\pi_\alpha} BU \xrightarrow{\pi} M(m)$$

$$f_2: \tilde{M}(n) \xrightarrow{i} BE \xrightarrow{\pi_\beta} BU \xrightarrow{\pi} L(m)$$

$$f_3: \tilde{M}(n) \xrightarrow{i} BE \xrightarrow{t} T(\Delta_n) \xrightarrow{\pi} eT(\Delta_n)$$

where t is the transfer [MP1; 3.7] and π is projection onto a stable summand. We

will show that

$$f = f_1 \vee f_2 \vee f_3 : \tilde{M}(n) \rightarrow M(m) \vee L(m) \vee eT(\Delta_n)$$

is a 2-local equivalence.

As modules,

$$H^*M(m) = \mathbf{F}_2 \langle Sq^I(x_1^{-1} \cdots x_m^{-1}) \rangle$$

$$H^*L(m) = \mathbf{F}_2 \langle Sq^J(x_1^{-1} \cdots x_m^{-1}) \rangle$$

([MP1]) with the same restrictions on I, J as in (4.6). Using the Cartan formula it follows that $Sq^I(x_1^{-1} \cdots x_m^{-1})$ is polynomial in x_1, \dots, x_m (i.e. there are no negative powers). Hence

$$f_1^*(Sq^I(x_1^{-1} \cdots x_m^{-1})) = Sq^I(\alpha)$$

and analogously

$$f_2^*(Sq^J(x_1^{-1} \cdots x_m^{-1})) = Sq^J(\beta)$$

Since $\Omega_n^+(\mathbf{F}_2)$ preserves the Euler class ω_{2^m} of Δ_n , it commutes with the Thom isomorphism

$$H^*BE \xrightarrow{\sim} H^*T(\Delta_n) = [H^*BE]\omega_{2^m}$$

Hence we have

$$H^*eT(\Delta_n) = [(H^*BE)e]\omega_{2^m} = [H^*\tilde{M}(n)]\omega_{2^m}$$

Under these identifications $t^* : H^*T(\Delta_n) \rightarrow H^*BE$ is the obvious inclusion. Hence f_3^* is an inclusion with image $[H^*\tilde{M}(n)]\omega_{2^m}$. The result follows from Theorem 4.5 and (4.6).

§5. Splitting $BE(4)$

Let $E = E(4)$, the extra-special 2-group of real type and of order 32. The Chevalley group $\Omega_4^+(\mathbf{F}_2)$ acts on BE up to homotopy; thus an orthogonal idempotent decomposition of 1 in $\mathbf{F}_2\Omega_4^+(\mathbf{F}_2)$ will provide a splitting of BE . One

summand of this splitting is $BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$ where $E \approx Q_8 \circ Q_8$ is a 2-Sylow subgroup of $SL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$.

Corresponding to the two factors of $\Omega_4^+(\mathbf{F}_2) \approx GL_2(\mathbf{F}_2) \times GL_2(\mathbf{F}_2)$ there are two orthogonal idempotent decompositions (4.8–9)

$$1 = f_0 + f_1 + f_2$$

$$1 = f'_0 + f'_1 + f'_2$$

Thus in $\mathbf{F}_2\Omega_4^+(\mathbf{F}_2)$ we have the orthogonal idempotent decomposition

$$1 = f_0f'_0 + (f_1f'_1 + f_1f'_2 + f_2f'_1 + f_2f'_2) + (f_0f'_1 + f_0f'_2 + f_1f'_0 + f_2f'_0) \quad (5.1)$$

where $f_1f'_1$ is the Steinberg idempotent.

THEOREM 5.2. *Corresponding to (5.1) there is a stable 2-local decomposition*

$$BE \cong BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3) \vee 4(M(2) \vee L(2) \vee eT(\Delta_4)) \vee 4X$$

where $X = f_0f'_1BE$ is a *spectrum with Poincaré series* $(t^2 + t^3)/(1 - t)(1 - t^3)(1 - t^4)$.

Proof. The idempotents f_1, f_2 are conjugate [MP2] as are f'_1 and f'_2 . Hence the summands corresponding to $f_1f'_1, f_1f'_2, f_2f'_1$ and $f_2f'_2$ are equivalent. By Theorem 4.1, each is equivalent to $M(2) \vee L(2) \vee eT(\Delta_4)$. Similarly f_0 and f'_0 are conjugate. Thus there are four summands equivalent to X . By comparing Poincaré series, the result now follows from part *i*) of

PROPOSITION 5.3. i) $f_0f'_0BE \cong BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$
ii) For $\mathbf{Z}/3 \times \mathbf{Z}/3 \subset \Omega_4^+(\mathbf{F}_2)$, $H^*SL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3) \approx H^*(E)^{\mathbf{Z}/3 \times \mathbf{Z}/3}$

More explicitly,

$$H^*BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3) = \mathbf{F}_2[v_2, v_3, x_3, \bar{x}_3, \omega_4]/R$$

where

$$R = \begin{pmatrix} v_2^3 + v_3^2 + x_3^2 + v_3x_3 \\ v_2^3 + v_3^2 + \bar{x}_3^2 + v_3\bar{x}_3 \end{pmatrix}$$

and

$$i^*(v_2) = x_1^2 + x_1x_{-1} + x_{-1}^2$$

$$i^*(v_3) = x_1x_{-1}^2 + x_1^2x_{-1}$$

$$i^*(x_3) = x_1^2x_{-1} + x_1^3 + x_{-1}^3$$

$$i^*(\bar{x}_3) = x_2^2x_{-2} + x_2^3 + x_{-2}^3$$

under the inclusion $i: E \approx Q_8 \circ Q_8 \rightarrow SL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$.

Proof. Part i) follows immediately from ii) since $f_0 f'_0$ is the trace over $\mathbf{Z}/3 \times \mathbf{Z}/3$, i.e. $f_0 f'_0 = \sum g, g \in \mathbf{Z}/3 \times \mathbf{Z}/3$. Part ii) is a straightforward generalization of that for $H^*BPSL_2(\mathbf{F}_3)$ [MP2]. One considers the map of fibrations

$$\begin{array}{ccccccc} B\mathbf{Z}/2 & \longrightarrow & BQ_8 \times Q_8 & \longrightarrow & BQ_8 \circ Q_8 & & \\ \downarrow & & \downarrow Bi \times i & & \downarrow Bi & & \\ B\mathbf{Z}/2 & \longrightarrow & BSL_2(\mathbf{F}_3) \times SL_2(\mathbf{F}_3) & \longrightarrow & BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3) & & \end{array}$$

and the corresponding map of spectral sequences.

Remark. The Poincaré series for $H^*BPSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$ is easily seen to be $(1 + t^3)^2 / (1 - t^2)(1 - t^3)(1 - t^4)$.

REFERENCES

- [A] C. ARF, *Untersuchungen über quadratische Formen in Körpern der Characteristic 2*, J. reine angew. Math. 183 (1941) 148–167.
- [B] P. BAUM, *On the cohomology of homogeneous spaces*, Topology 7 (1968) 15–38.
- [C] R. CARTER, *Simple groups of Lie type*, John Wiley and Sons, New York (1972).
- [Dk] L. DICKSON, *Linear Groups*, Dover, New York (1958).
- [Dd] J. DIEUDONNÉ, *Le géométrie des groupes classiques*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 5, Springer (1955).
- [FP] Z. FREDOROWICZ and S. PRIDDY, *Homology of classical groups*, Springer Lecture Notes in Math., no. 674 (1978).
- [G] D. GORENSTEIN, *Finite groups*, Harper and Row, New York (1968).
- [H] P. HALL, *Lecture Notes* (unpublished).
- [HK] J. HARRIS and N. KUHN, *Stable decompositions of classifying spaces of finite abelian p -groups* (to appear).
- [K] N. KUHN, *The modular Hecke algebra and Steinberg representation of finite Chevalley groups*. J. of Algebra 91 (1984) 125–141.
- [M] S. MITCHELL, *Finite complexes with $A(n)$ -free cohomology*, Topology 24 (1985) 227–248.
- [MP1] S. MITCHELL and S. PRIDDY, *Stable splittings derived from the Steinberg module*, Topology 22 (1983) 285–298.

- [MP2] S. MITCHELL and S. PRIDDY, *Symmetric product spectra and splittings of classifying spaces*, Amer. J. Math., 106 (1984) 219–232.
- [MS] W. MASSEY and F. PETERSON, *The cohomology of certain fiber spaces I*, Topology 4 (1965) 47–65.
- [Q] D. QUILLEN, *The mod-2 cohomology rings of extra-special 2-groups and the spinor groups*, Math. Ann. 194 (1971) 197–212.
- [S] J.-P. SERRE, *Linear Representations of Finite Groups*, Springer-Verlag, New York, 1977.
- [T] J. G. THOMPSON, *Nonsolvable finite groups all of whose local subgroups are solvable*, Bull. Amer. Math. Soc. 74 (1968) 383–437.
- [W] P. WELCHER, *Symmetric products and the stable Hurewicz homomorphism*, Ill. J. Math. 24 (1980) 527–542.
- [Wk] C. WILKERSON, *A primer on the Dickson invariants*, Proceedings of the Northwestern Homotopy Theory Conference, Contemporary Mathematics 19 (1983), 421–434.

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