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## Stable splittings associated with Chevalley groups, I

MARK FESHBACH and STEWART PRIDDY<sup>1</sup>

In recent years stable splittings have been studied for the classifying spaces of various finite groups, for example: elementary abelian  $p$ -groups [MP1], abelian groups [HK], dihedral and quaternion groups [MP2], etc. In this paper we continue this study; here we consider groups  $E$  which are extensions of an elementary abelian 2-group  $V$  by a cyclic group of order 2. These groups are among those of symplectic type [T, 2.4]; examples are the extra-special 2-groups [G, H]. A quadratic form  $Q$  is naturally associated with such an extension and the outer automorphisms of  $E$  which fix the center are precisely those automorphisms of  $V$  which preserve this form. Thus one of the classical orthogonal groups  $O(V, Q)$  acts on  $BE$  (up to homotopy) and we can use idempotents from the group ring to stably split  $BE$ . In particular since the commutator subgroups of these groups are Chevalley groups, they have a  $BN$  pair and an associated Steinberg idempotent  $e$ . We determine the stable summand  $eBE$ . The degenerate case where  $E$  itself is an elementary abelian 2-group was studied in [MP1]. These cases cover the four systems of Chevalley groups  $A_m$ ,  $B_m$ ,  $D_m$  defined over  $\mathbf{F}_2$  and the twisted group  ${}^2D_m(\mathbf{F}_4)$ .

It is well known that the orthogonal groups  $O(V, Q)$  over  $\mathbf{F}_2$  are determined by the dimension of  $V$  and the Arf invariant of  $Q$ . There exists three types of forms: one if  $\dim V$  is odd and two if  $\dim V$  is even. The latter cases are distinguished by  $\text{Arf}(Q) = 0$  or 1. In this paper we set up machinery for handling the general cases but give specific analysis only for the  $\text{Arf}(Q) = 0$  case. Here our main result (Theorem 4.1) is that  $BE$  contains  $2^{m(m-1)}$  wedge summands, each equivalent to

$$eBE = M(m) \vee L(m) \vee eT(\Delta_{2m})$$

where  $2m = \dim V$ ,  $M(m)$  and  $L(m)$  are wedge summands of  $B(\mathbf{Z}/2)^m$  and  $T(\Delta_{2m})$  is the Thom spectrum associated to an irreducible representation  $\Delta_{2m}$  of  $E$ . In Part II, we study the remaining cases.

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The paper is organized as follows: Section 1 consists of some preliminaries on  $E$ , quadratic forms and Quillen’s computation of  $H^*BE$ . The homotopy action of  $O(V, Q)$  on  $BE$  is explained in Section 2. In Section 3 we describe the structure of  $O(V, Q)$  as a Chevalley group and determine the Steinberg idempotent  $e$ . The cohomology of  $H^*(eBE)$  is determined in Section 4. This leads to a proof of the main splitting in Theorem 4.1. In Section 5, we give a splitting of  $BE$  for  $|E| = 32$  and  $\text{Arf } Q = 0$ . In what follows all spaces are localized at 2 and all cohomology groups are taken with simple coefficients in  $\mathbf{F}_2$ .

It is a pleasure to thank Dave Benson for several helpful conversations on this material.

**§1. Preliminaries**

In this section we recall some preliminaries on quadratic forms, the groups  $E$  and their cohomology.

We begin with some standard facts about quadratic forms over  $\mathbf{F}_2 [Q]$ . Let  $V$  be a vector space over  $\mathbf{F}_2$ . A *quadratic* form  $Q : V \rightarrow \mathbf{F}_2$  is a function such that  $Q(x + y) = Q(x) + Q(y) + B(x, y)$  for  $x, y \in V$  and some bilinear form  $B$ . Necessarily  $B$  is *symplectic*, i.e.  $B(x, x) = 0$ . Let  $V_0$  be the set of  $x \in V$  such that  $B(x, y) = 0$  for all  $y \in V$ . Then  $Q$  is said to be *non-degenerate* if  $Q(x) \neq 0$  for all  $x \neq 0$  in  $V_0$ . Throughout this paper we will assume all quadratic forms to be non-degenerate.

Let  $n = \dim V$ . According to Dickson [Dk] there are, up to isomorphism three types of non-degenerate quadratic forms:

$$\begin{aligned}
 \text{If } n = 2m \quad Q &= \sum_{i=1}^m x_i x_{-i} && \text{(real case)} \\
 Q &= \sum_{i=1}^{m-1} x_i x_{-i} + x_m^2 + x_m x_{-m} + x_{-m}^2 && \text{(quaternion case)} \tag{1.0}
 \end{aligned}$$

for some choice of basis  $\{x_1, \dots, x_m, x_{-1}, \dots, x_{-m}\} \subset V^*$

$$\text{If } n = 2m + 1 \quad Q = x_0^2 + \sum_{i=1}^m x_i x_{-i} \quad \text{(complex case)}$$

for some choice of basis  $\{x_0, x_1, \dots, x_m, x_{-1}, \dots, x_{-m}\} \subset V^*$ . In the first two

cases we have  $\text{Arf } Q = 0, 1$  respectively, where we recall

$$\text{Arf } Q = \begin{cases} 0 & \text{if } |Q^{-1}(0)| > \frac{1}{2} |V| \\ 1 & \text{if } |Q^{-1}(0)| < \frac{1}{2} |V|. \end{cases}$$

For convenience, however, we will use Quillen’s terminology  $[Q]$  of *real* and *quaternion*; similarly we will call the third case *complex*.

Now suppose a group  $E$  is given as a central extension

$$\mathbf{Z}/2 \xrightarrow{i} E \xrightarrow{\pi} V \tag{1.1}$$

If  $n = \dim V$  we shall often write  $E = E(n)$ . The associated quadratic and bilinear forms are given by

$$\begin{aligned} Q(x) &= \bar{x}^2 && \text{where } \pi(\bar{x}) = x \\ B(x, y) &= \bar{x}\bar{y}\bar{x}^{-1}\bar{y}^{-1} && \text{where } \pi(\bar{x}) = x, \pi(\bar{y}) = y \end{aligned}$$

For  $n = 2$  in the real case  $E \approx D_8$ , the dihedral group of order 8 while in the quaternion case  $E \approx Q_8$ , the quaternion group of order 8. In general if  $n$  is even,  $E(n)$  can be built up from the central product  $(G \circ G' \approx G \times G'$  with centers identified). It is known that  $D_8 \circ D_8 \approx D_8 \circ Q_8$ . It is also straightforward to check

**PROPOSITION 1.2.** *If  $n = 2m$*

$$\begin{aligned} E(n) &\approx D_8 \overset{\circ}{\circ} \overset{\circ}{\circ} \dots \overset{\circ}{\circ} \overset{\circ}{\circ} D_8 && \text{(real case)} \\ &\approx D_8 \overset{\circ}{\circ} \overset{\circ}{\circ} \dots \overset{\circ}{\circ} \overset{\circ}{\circ} D_8 \circ Q_8 && \text{(quaternion case)} \end{aligned}$$

*In the real and quaternion cases,  $E$  is an extra-special 2-group.*

(1.3) It will be convenient to specify generators of  $E$ : let  $b_1, \dots, b_m, b_{-1}, \dots, b_{-m}$  (and  $b_0$  in the complex case) be elements of  $E$  such that  $\{v_{\pm i} = \pi(b_{\pm i})\}$  is dual to the basis  $\{x_{\pm i}\}$  of  $V^*$ . Then  $E$  is generated by  $\{b_{\pm i}, c\}$  where  $c$  is the non-trivial element of  $\ker \pi$ . (By convention  $b_{\pm 0} = b_0$  in the complex case.) Using (1.0) a set of relations is seen to be given by commutators and squares.

(1.4) We now turn to  $H^*BE$ . A subspace  $W$  of  $V$  is called *isotropic* if  $Q(W) = 0$ . Now assume  $W$  is a maximal isotropic subspace or equivalently

$\tilde{W} = \pi^{-1}(W)$  is a maximal elementary abelian subgroup. Let  $\chi: \tilde{W} \rightarrow \mathbf{Z}/2$  be a representation which is non-trivial on  $\ker \pi = \mathbf{Z}/2$  and consider  $\Delta = \text{Ind}_{\tilde{W}}^E(\chi)$ , that is,  $\Delta$  is the real representation induced from  $\tilde{W}$  to  $E$ . [Q; §5] shows that  $\Delta$  is the unique irreducible real representation which is non-trivial on  $\ker \pi$ .

**THEOREM 1.5.** [Q; Th. 4.6]. *Given an extension (1.1) and the associated bilinear form  $Q$ , then*

$$H^*(BE) = S(V^*)/J \otimes \mathbf{F}_2[w_{2^h}]$$

where  $J$  is the ideal generated by the regular sequence  $Q, Sq^1Q, Sq^2Sq^1Q, \dots, Sq^{2^{h-2}} \cdots Sq^2Sq^1Q$ ;  $h$  is the codimension of a maximal isotropic subspace of  $V$  and  $w_{2^h} = w_{2^h}(\Delta)$  is the  $2^h$ -th Stiefel–Whitney class of  $\Delta$ .

*Remark 1.6.* For reference we record the values of  $h$  [Q; §2].

Case	$\dim V$	$h$
real	$2m$	$m$
complex	$2m + 1$	$m + 1$
quaternion	$2m$	$m + 1$

(1.7) Since the dimension of  $\Delta$  is  $2^h$  and  $\ker \pi = \mathbf{Z}/2$  acts as  $-1$  on  $\Delta$ ,  $\Delta$  restricted to  $\ker \pi$  is  $2^h \cdot \eta$ , where  $\eta$  is the non-trivial real character on  $\mathbf{Z}/2$ . It follows that  $i^*(w_{2^h}) \neq 0$  and that any element with this property can be taken as a generator in place of  $w_{2^h}$ .

**§2. Classical groups acting on  $H^*BE$**

Since conjugation is homotopic to the identity on the classifying space  $BG$  of any group  $G$ , the outer automorphism group  $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$  acts up to homotopy on  $BG$ , i.e. there is a homomorphism

$$\text{Out}(G) \rightarrow \text{Aut}_{H_0}(BG)$$

where  $\text{Aut}_{H_0}(BG)$  is the group of base point preserving equivalences in the homotopy category.

The referee points out that  $\text{Out}(E)$  can be made to act on  $BE$  (not just up to homotopy). There is a group extension (which is not necessarily split)

$$1 \rightarrow E \rightarrow G_0 \rightarrow \text{Out}(E) \rightarrow 1$$

with  $G_0/\langle c \rangle \approx \text{Aut}(E)$ . Thus if  $X$  is a contractible CW-complex on which  $G_0$  acts freely, then  $X/E$  is a model for  $BE$  on which  $\text{Out}(E)$  acts as required.

Let  $\text{Out}_z(G)$  be the subgroup of  $\text{Out}(G)$  consisting of automorphisms which are the identity on the center of  $G$ . For  $G = E$  as in (1.1) we have

**PROPOSITION 2.1.**  $\text{Out}_z(E) \approx O(V, Q)$

*Proof.* It is clear from the definitions that  $\pi$  induces a homomorphism  $\text{Out}_z(E) \rightarrow O(V, Q)$ . This map is surjective by (1.3) and so any orthogonal automorphism of  $V$  can be lifted to an automorphism of  $E$ . That the center is fixed follows from examining the types of  $Q$  in (1.0). Conversely, suppose  $\beta \in \text{Out}_z(E)$  induces the identity on  $V$ . Then for  $b \in E$ ,  $\beta(b) = b$  or  $bc$  where  $\langle c \rangle = \ker \pi$ . Let  $\{v_i, v'_j\}$  be a basis for  $V$  such that  $B(v_i, v'_j) \neq 0$  for at most one  $j$  for each  $i$  (e.g. in the real case  $v_i$  is dual to  $x_i$  and  $v'_j$  to  $x_{-j}$ ). Let  $\{b_i, b_j\}$  satisfy  $\pi(b_i) = v_i$ ,  $\pi(b'_j) = v'_j$  and let  $\varepsilon$  be the product in any order of those  $b'_j$ 's for which  $\beta(b_i) = b_i c$  and  $B(v_i, v'_j) \neq 0$  for some  $i$ . Then  $\beta(b_i) = \varepsilon b_i \varepsilon^{-1}$ . Similarly let  $\varepsilon'$  be the product in any order of those  $b_i$ 's for which  $\beta(b'_j) = b'_j c$  and  $B(v_i, v'_j) \neq 0$  for some  $j$ . Then  $\beta(b'_j) = \varepsilon' b'_j \varepsilon'^{-1}$ . Consequently  $\beta$  is conjugation by  $\varepsilon \varepsilon'$ .

*Remark.* In the real and quaternion cases,  $\text{Out}_z(V, Q) = O(V, Q)$  since the center is  $\mathbf{Z}/2$ . In the complex case the center is  $\mathbf{Z}/4$  generated by an element  $b_0$  such that  $\pi(b_0)$  is dual to  $x_0$ . Here  $\text{Out}(E) = \mathbf{Z}/2 \times \text{Out}_z(E)$  where the extra automorphism is given by  $b_0 \mapsto b_0^3$ .

We now turn to the action of  $O(V, Q)$  on  $H^*BE$  and the resulting invariants. The uniqueness of  $\Delta$  (1.4) implies that its Stiefel–Whitney classes are invariants. In this connection Quillen has shown

**THEOREM 2.2** [Q, Th. 5.1]. *The non-zero positive dimensional Stiefel–Whitney classes of  $\Delta_n$  are  $\omega_{2^h}, \omega_{2^h-2^r}, \omega_{2^h-2^{r+1}}, \dots, \omega_{2^h-2^{h-1}}$  where  $r = 0, 1, 2$  in the real, complex, and quaternion cases resp. Further, these classes form a regular sequence of maximal length in  $H^*BE$  and hence form a polynomial ring over which  $H^*BE$  is a free finitely generated module.*

Quillen further remarks, without proof, that in the real case these classes generate all of the invariants. We will prove a slightly sharper result. For

convenience we use the following notation

$$O(V, Q) = \begin{cases} O_{2m}^+(\mathbf{F}_2) & \text{if } n = 2m, \text{ real case} \\ O_{2m}^-(\mathbf{F}_2) & n = 2m, \text{ quaternion case} \\ O_{2m+1}(\mathbf{F}_2) & n = 2m + 1, \text{ complex case} \end{cases} \quad (2.3)$$

where  $n = \dim V$ . Let  $\Omega_{2m}^\pm(\mathbf{F}_2)$  denote the commutator subgroup of  $O_{2m}^\pm(\mathbf{F}_2)$ .

**THEOREM 2.4.** *In the real case*

$$H^*BE^{\Omega_{2m}^+} = \mathbf{F}_2[\omega_{2m}, \omega_{2m-1}, \dots, \omega_{2^{m-1}}].$$

The proof depends on three lemmas, the first of which holds for a general  $V$  and  $Q$ .

**LEMMA 2.5.**  *$O(V, Q)$  acts transitively on  $\{A < E : A \text{ is a maximal elementary abelian group}\}$ .*

*Proof.*  $O(V, Q)$  acts transitively on  $\{W < V : W \text{ is a maximal isotropic subspace}\}$ . This is a result of Arf [A] in the real and quaternion cases. In the complex case  $O_{2m+1}(\mathbf{F}_2) \approx Sp_{2m}(\mathbf{F}_2)$  and a proof can be found in [Dd]. The lemma follows since  $\pi$  induces an isomorphism between maximal elementary abelian subgroups of  $E$  and maximal isotropic subspaces of  $V$ .

Let  $H : GL_m(\mathbf{F}_2) \rightarrow O_{2m}^+(\mathbf{F}_2)$  be the *hyperbolic map* given by

$$H(M) = \begin{bmatrix} M & O \\ O & {}^tM^{-1} \end{bmatrix}$$

(see [F-P; p. 152–154]). The appropriate quadratic form for the range is of the real type.

**LEMMA 2.6.**  $H : GL_m(\mathbf{F}_2) \rightarrow \Omega_{2m}^+(\mathbf{F}_2)$

*Proof.* Since  $\Omega_{2m}^+ = \ker d$  where  $d : O_{2m}^+(\mathbf{F}_2) \rightarrow \mathbf{Z}/2$  is the Dickson invariant, we need only check  $d \circ H = 0$ . This follows from the formula for  $d$  [Dd; p. 64].

**LEMMA 2.7.** *Let  $A \xrightarrow{j} E$  be the inclusion of a maximal elementary abelian subgroup. Then  $j^*(H^*(BE)^{\Omega_{2m}^+}) = \text{Im}(j^* \Delta^*)$ .*

*Proof.* The inclusion  $\supset$  follows from the inclusion  $H^*(BE)^{\Omega_{2m}^+} \supset \text{Im} \Delta^*$  noted

above. For the other inclusion it suffices by Theorem 1.5 to consider  $x \in H^*(BE)^{\Omega_{2m}^+}$  in the image of  $\pi^*: H^*BV \rightarrow H^*BE$ . By Lemma 2.5, (1.4) and the normality of  $\Omega_{2m}^+$ , it suffices to prove the result for one maximal elementary abelian subgroup  $A$ . Let  $A = \langle b_1, \dots, b_m, c \rangle \xrightarrow{\iota} E$ ; we can write  $A = A' \oplus C$  where  $C = \langle c \rangle = \ker \pi$ . Let  $M \in GL_m(\mathbb{F}_2)$ . Then for  $j^*(x) = y \otimes 1 \in H^*BA' \otimes H^*BC$ , we have

$$(y \otimes 1)H(M) = yM \otimes 1$$

Hence  $y \in H^*(BA')^{GL(A')}$ . By [Wk; 4.1],  $H^*(BA')^{GL(A')} = \text{Im}(\text{reg}(A')^*)$  for the regular representation of  $A'$ . Since  $\Delta j = \text{reg}(A') \otimes \chi$  on  $A' \oplus C$  [Q; 5.1], we have  $j^*(x) = y \otimes 1 \in \text{Im}(j^*\Delta^*)$  using the formula for the Stiefel–Whitney classes of  $\text{reg}(A') \otimes \chi$  [Q; 5.6].

*Proof of Theorem 2.4.* By [Q; Th. 5.10],  $H^*BE$  is detected by elementary abelian subgroups. Hence the result follows directly from Lemma 2.7.

**COROLLARY 2.8.**  $H^*(BE)^{O_{2m}^+} = H^*(BE)^{\Omega_{2m}^+}$ .

### §3. $O_n(\mathbb{F}_2)$ as Chevalley groups

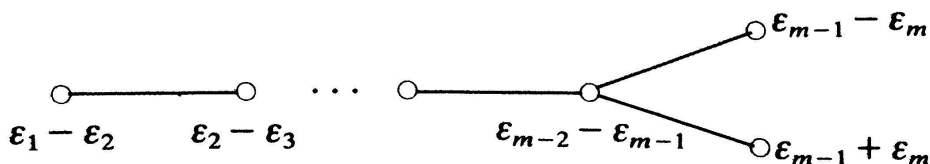
Our goal in this section is to describe what we need about the Steinberg idempotent for the orthogonal group. A good general reference is R. Carter’s book [C]. For each simple Lie algebra  $L$  over  $\mathbb{C}$  and each field  $K$ , Chevalley has constructed a group  $L(K)$ . Later Steinberg, Tits and Hertzog discovered additional twisted versions of these groups. For the simple Lie algebras of type  $A_m$ ,  $B_m$ ,  $C_m$  and  $D_m$  and for  $K$  finite, Ree has identified these Chevalley groups with classical groups. We state the result for  $K = \mathbb{F}_2$ .

**THEOREM 3.1** (Ree [C; Th. 11.3.2])

- i)  $A_m(\mathbb{F}_2) \approx GL_{m+1}(\mathbb{F}_2)$
- ii)  $B_m(\mathbb{F}_2) \approx O_{2m+1}(\mathbb{F}_2)$
- iii)  $C_m(\mathbb{F}_2) \approx B_m(\mathbb{F}_2)$
- iv)  $D_m(\mathbb{F}_2) \approx \Omega_{2m}^+(\mathbb{F}_2)$

The group  $\Omega_{2m}^-(\mathbb{F}_2)$  occurs as a twisted Chevalley group and will be treated at the end of this section.

3.2 *The real case:* The Dynkin diagram for  $D_m$ ,  $m > 1$ , is



where  $\varepsilon_1, \dots, \varepsilon_m$  is the standard basis for  $\mathbf{R}^m$ .

Let  $e_{ij}$  be the  $2m$  square matrix with 1 in the  $(i, j)$  position and 0's elsewhere. Let  $u_{ij} = I + e_{ij} + e_{-j, -i} \in GL_{2m}(\mathbf{F}_2)$ . Then the unipotent subgroup  $U_{2m} < \Omega_{2m}^+(\mathbf{F}_2)$  is generated by

$$\{u_{i,j}, u_{i,-j} : 1 \leq i < j \leq m\}$$

(We recall that the underlying vector space  $V$  has basis  $\{v_1, \dots, v_m, v_{-1}, \dots, v_{-m}\}$  over  $\mathbf{F}_2$ .) The Weyl group  $W_{2m}^+ < \Omega_{2m}^+(\mathbf{F}_2)$  is generated by

$$\{\sigma_{ij} = u_{i,j}u_{-i,-j}u_{i,j}, \sigma_{i,-j} = u_{i,-j}u_{-i,j}u_{i,-j} : 1 \leq i < j \leq m\}.$$

Abstractly  $W_{2m}^+ \approx (\mathbf{Z}/2)^{m-1} \rtimes \Sigma_m$  (permutations together with an even number of sign changes).

Finally  $\Omega_{2m}^+(\mathbf{F}_2)$  is generated by  $U_{2m}$  and  $V_{2m}$  where  $V_{2m}$  is generated by  $\{u_{-i,-j}, u_{-i,j} : 1 \leq i < j \leq m\}$ .

(3.3) The Steinberg idempotent  $e \in \mathbf{F}_2\Omega_{2m}^+(\mathbf{F}_2)$  is defined by

$$e = \sum u\sigma \quad u \in U_{2m}, \sigma \in W_{2m}^+.$$

For computational purposes, it will be convenient to use another expression for  $e$ . For each of the simple roots  $\{\varepsilon_i - \varepsilon_{i+1}\}$  in the Dynkin diagram let  $e_i$  be the idempotent

$$e_i = (1 + u_{i,i+1})(1 + \sigma_{i,i+1}) \quad 1 \leq i \leq m - 1$$

For the last root  $\varepsilon_{m-1} + \varepsilon_m$  let

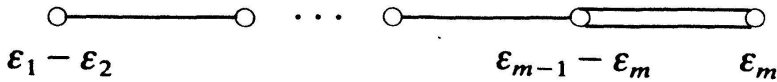
$$e_m = (1 + u_{m-1,-m})(1 + \sigma_{m-1,-m})$$

Kuhn [K] has shown that  $e$  can be expressed as a product of the  $e_i$ ,  $i = 1, 2, \dots, m$ . Moreover

**THEOREM 3.4.** [K, Th. 1.3] *Let  $M$  be a right  $\mathbf{F}_2\Omega_{2m}^+(\mathbf{F}_2)$  module. Then*

$$Me = \bigcap_{i=1}^m Me_i.$$

(3.5) *The complex case:* The Dynkin diagram for  $B_m$  is



Let

$$u_{ij} = I + e_{ij} + e_{-j,-i} \quad i \neq j$$

$$u_{ii} = I + e_{0,-i} + e_{i,-i} \quad i \neq 0$$

( $V$  has basis  $v_0, v_1, \dots, v_m, v_{-1}, \dots, v_{-m}$ ). The unipotent subgroup  $U_{2m+1} < O_{2m+1}(\mathbf{F}_2)$  is generated by

$$\{u_{ij}, u_{i,-j}, u_{ii} : 1 \leq i < j \leq m\}$$

The Weyl group  $W_{2m+1} < O_{2m+1}(\mathbf{F}_2)$  is generated by

$$\begin{cases} \sigma_{ij} = u_{-i,-j}u_{i,j}u_{-i,-j} & 1 \leq i < j < m \\ \sigma_{i,-j} = u_{-i,j}u_{i,-j}u_{-i,j} \\ \sigma_{ii} = u_{-i,-i}u_{ii}u_{-i,-i} & 1 \leq i \leq m \end{cases}$$

Then  $O_{2m+1}(\mathbf{F}_2)$  is generated by  $U_{2m+1}$  and  $V_{2m+1}$  where  $V_{2m+1}$  is generated by

$$\{u_{-i,-j}, u_{-i,j}, u_{-i,-i} : 1 \leq i < j \leq m\}.$$

The Steinberg idempotent  $e \in \mathbf{F}_2O_{2m+1}(\mathbf{F}_2)$  is defined by

$$e = \sum u\sigma \quad u \in U_{2m+1}, \sigma \in W_{2m+1}$$

In this case Kuhn [K] has shown that  $e$  can be expressed as a product of the

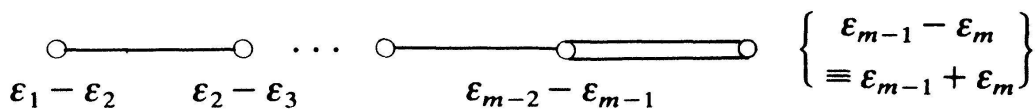
following idempotents

$$e_i = (1 + u_{i,i+1})(1 + \sigma_{i,i+1}) \quad 1 \leq i \leq m - 1$$

$$e_m = (1 + u_{m,m})(1 + \sigma_{mm})$$

and the analog of Theorem 3.4 holds.

3.6 *The quaternion case:* The group  $\Omega_{2m}^-(\mathbb{F}_2)$  is isomorphic to the twisted Chevalley group  ${}^2D_m(\mathbb{F}_4)$  [C; Th. 14.5.2] with Dynkin diagram of type  $B_{m-1}$



It is a projection of the diagram for  $D_m$  in (3.2). For details of this group see Chapters 13, 14 of [C].

Let

$$\tau_m = I + e_{m-1,m} + e_{m-1,-(m-1)} + e_{m-1,-m} + e_{m,-(m-1)} + e_{-m,-(m-1)}$$

$$\gamma_m = I - e_{m-1,m} + e_{m-1,-(m-1)} + e_{-m,-(m-1)}$$

$$\tau'_m = I - e_{-(m-1),m-1} + e_{-(m-1),m} + e_{-m,m-1}$$

$$\gamma'_m = I + e_{m,m-1} + e_{-(m-1),m-1} + e_{-(m-1),m} + e_{-(m-1),-m} + e_{m,m-1}$$

The unipotent subgroup  $U_{2m}^- < \Omega_{2m}^-(\mathbb{F}_2)$  is generated by  $\{\tau_m, \gamma_m\} \cup \{u_{i,j}, u_{i,-j} : 1 \leq i < j \leq m - 1\}$ .  $V_{2m}^-$  is generated by  $\{\tau'_m, \gamma'_m\} \cup \{u_{-i,-j}, u_{-i,j} : 1 \leq i < j \leq m - 1\}$ .  $\Omega_{2m}^-(\mathbb{F}_2)$  is generated by  $U_{2m}^-$  and  $V_{2m}^-$ . Let  $B_{2m}^-$  be the normalizer of  $U_{2m}^-$  in  $\Omega_{2m}^-(\mathbb{F}_2)$ .

The Weyl group  $W_{2m}^-$  of  $\Omega_{2m}^-(\mathbb{F}_2)$  is generated by  $\{\sigma_{ij}, \sigma_{i,-j} : 1 \leq i < j \leq m\} \cup \{\tau_m \tau'_m \tau_m = W_m\}$ . The Steinberg idempotent  $e \in \mathbb{F}_2 \Omega_{2m}^-(\mathbb{F}_2)$  is defined by  $e = \sum b \sigma$   $b \in B_{2m}^-, \sigma \in W_{2m}^-$ .

In this case Kuhn [K] has shown that  $e$  can be expressed as a product of the idempotents corresponding to the nodes in the Dynkin diagram for  $B_{m-1}$ . These are

$$e_i = (1 + u_{i,i+1})(1 + \sigma_{i,i+1}) \quad 1 \leq i \leq m - 2$$

and the idempotent  $e'_m$  corresponding to the last node

$$e'_m = (1 + \tau_m)(1 + \gamma_m)(1 + H_m + H_m^2)(1 + W_m)$$

where  $H_m = I + e_{m,m} + e_{m,-m} + e_{-m,m}$  and  $W_m = I + e_{m-1,m-1} + e_{m-1,-(m-1)} + e_{m,-m} + e_{-(m-1),m-1} + e_{-(m-1),-(m-1)}$ .

**§4. The Steinberg wedge summand: the real case**

For  $n = 2m$  let  $E = E(n)$  denote the extra-special 2-group of real type. Let  $\tilde{M}(n)$  be the stable summand

$$\tilde{M}(n) = eBE$$

corresponding to the Steinberg idempotent of (3.3). Our main result is

**THEOREM 4.1.** *Stably, for  $m \geq 2$ ,  $BE$  contains  $2^{m(m-1)}$  copies of  $\tilde{M}(n) = M(m) \vee L(m) \vee eT(\Delta_n)$ .*

Here  $M(m)$  is the Steinberg summand of  $B(\mathbf{Z}/2)^m$  [MP1],  $L(m) = \Sigma^{-m} Sp^{2^m} S^0 / Sp^{2^{m-1}} S^0$ , and  $T(\Delta_n)$  is the Thom spectrum of the bundle  $B\Delta_n$  over  $BE$ . As a spectrum  $M(m) = L(m) \vee L(m-1)$ .

(4.2) The uniqueness of  $\Delta_n$  (1.4) implies that the homotopy action of  $O_n^+(\mathbf{F}_2)$  on  $BE$  preserves the isomorphism type of  $\Delta_n$  and hence induces a homotopy action of  $O_n^+(\mathbf{F}_2)$  on  $T(\Delta_n)$ . The summand  $eT(\Delta_n)$  is defined with respect to this action.

On the way to proving Theorem 4.1 we first determine  $H^*\tilde{M}(n)$ . Let

$$\alpha = \alpha_m = \sum x_{i_1}^{-1} x_{i_2}^{-1} \cdots x_{i_m}^{-1},$$

$i_j = \pm j$  with an even number of minus signs occurring

$$\beta = \beta_m = \sum x_{i_1}^{-1} x_{i_2}^{-1} \cdots x_{i_m}^{-1},$$

$i_j = \pm j$  with an odd number of minus signs occurring.

These elements belong to  $S_V$ , that is,  $S = H^*BV$  with the inverses of all non-zero linear elements adjoined. The action of  $O_n^+(\mathbf{F}_2)$  on  $H^*BV$  extends to  $S_V$ .

**LEMMA 4.3.**  $\alpha e = \alpha, \beta e = \beta$ .

*Proof.* By 3.4 it suffices to show  $\alpha$  and  $\beta$  are fixed by  $e_i, i = 1, \dots, m$ . Write

$$\alpha = (x_i^{-1} x_{i+1}^{-1} + x_{-i}^{-1} x_{-(i+1)}^{-1}) \hat{\alpha}_i + (x_{-i}^{-1} x_{i+1}^{-1} + x_i^{-1} x_{-(i+1)}^{-1}) \hat{\beta}_i$$

where  $\hat{\alpha}_i$  (resp.  $\hat{\beta}_i$ ) is the sum of those terms  $x_{j_1}^{-1} \cdots x_{j_{m-2}}^{-1}$  not containing  $x_{\pm i}^{-1}$ ,

$x_{\pm(i+1)}^{-1}$  and having an even (resp. odd) number of minus signs. By 3.3,

$$e_i = (1 + u_{i,i+1})(1 + \sigma_{i,i+1}) \quad 1 \leq i < m$$

$$e_m = (1 + u_{m-1,-m})(1 + \sigma_{m-1,-m})$$

where the action of  $u_{i,j}$  is  $x_i \rightarrow x_i + x_j$ ,  $x_{-j} \rightarrow x_{-i} + x_{-j}$ ,  $x_k \rightarrow x_k$  otherwise and the action of  $\sigma_{i,j}$  is  $x_{\pm i} \rightarrow x_{\pm j}$ ,  $x_{\pm j} \rightarrow x_{\pm i}$ . Hence for  $1 \leq i < m$ ,

$$\begin{aligned} \alpha e_i &= \alpha + [(x_i + x_{i+1})^{-1}x_{i+1}^{-1} + x_{-i}^{-1}(x_{-i} + x_{-(i+1)})^{-1}] \hat{\alpha}_i \\ &\quad + [(x_i + x_{i+1})^{-1}(x_{-i} + x_{-(i+1)})^{-1} + x_{-i}^{-1}x_{i+1}^{-1}] \hat{\beta}_i \\ &\quad + [x_i^{-1}x_{i+1}^{-1} + x_{-i}^{-1}x_{-(i+1)}^{-1}] \hat{\alpha}_i \\ &\quad + [x_{-i}^{-1}x_{i+1}^{-1} + x_i^{-1}x_{-(i+1)}^{-1}] \hat{\beta}_i + [(x_i + x_{i+1})^{-1}x_i^{-1} \\ &\quad + x_{-(i+1)}^{-1}(x_{-i} + x_{-(i+1)})^{-1}] \hat{\alpha}_i \\ &\quad + [(x_i + x_{i+1})^{-1}(x_{-i} + x_{-(i+1)})^{-1} + x_i^{-1}x_{-(i+1)}^{-1}] \hat{\beta}_i = \alpha. \end{aligned}$$

For  $i = m$  we have

$$\begin{aligned} \alpha e_m &= \alpha + [(x_{m-1} + x_{-m})^{-1}(x_m + x_{-(m-1)})^{-1} + x_{-(m-1)}^{-1}x_{-m}^{-1}] \hat{\alpha}_{m-1} \\ &\quad + [(x_{m-1} + x_{-m})^{-1}x_{-m}^{-1} + x_{-(m-1)}^{-1}(x_m + x_{-(m-1)})^{-1}] \hat{\beta}_{m-1} \\ &\quad + [x_{-m}^{-1}x_{-(m-1)}^{-1} + x_m^{-1}x_{m-1}^{-1}] \hat{\alpha}_{m-1} \\ &\quad + [x_{-m}^{-1}x_{m-1}^{-1} + x_m^{-1}x_{-(m-1)}^{-1}] \hat{\beta}_{m-1} + [(x_{-m} + x_{m-1})^{-1}(x_{-(m-1)} + x_m)^{-1} \\ &\quad + x_m^{-1}x_{m-1}^{-1}] \hat{\alpha}_{m-1} \\ &\quad + [(x_{-m} + x_{m-1})^{-1}x_{m-1}^{-1} + x_m^{-1}(x_{-(m-1)} + x_m)^{-1}] \hat{\beta}_{m-1} = \alpha \end{aligned}$$

A similar calculation shows  $\beta e = \beta$ .

LEMMA 4.4.  $Sq^1 \alpha = Sq^1 \beta$ .

The proof is straightforward calculation using  $Sq^1 x^{-1} = 1$ . Now let

$$A = \mathbf{F}_2 \langle Sq^l \alpha, Sq^l \beta : l \text{ admissible, } l(I) = m \rangle$$

$$B = \mathbf{F}_2 \langle Sq^j Sq^1 \alpha + Sq^j Sq^1 \beta : (j, 1) \text{ admissible, } l(J) = m - 1 \rangle$$

THEOREM 4.5. i)  $H^* \tilde{M}(n) = (A/B) \otimes \mathbf{F}_2[\omega_{2^m}]$

ii)  $H^*(eBV) = (A/B) \otimes \mathbf{F}_2[Q, Sq^1 Q, \dots, Sq^{2^m-2} \dots Sq^2 Sq^1 Q]$

*Proof.* In discussing i) and ii) we will implicitly use the commutative diagram

$$\begin{array}{ccc} H^*BV & \xrightarrow{\epsilon} & H^*BV \\ \downarrow \pi^* & & \downarrow \pi^* \\ H^*BE & \xrightarrow{\epsilon} & H^*BE \end{array}$$

The elements  $Sq^I\alpha, Sq^J\beta \in H^*(eBV)$  by Lemma 4.3 and the relations  $B$  hold by Lemma 4.4. A basis for  $A/B \subset H^*(eBV)$  is given by

$$\{Sq^I\alpha, Sq^J\beta : I, J \text{ admissible, } l(I) = m, l(J) = m, j_m > 1\} \tag{4.6}$$

Restricting to the subgroups  $\langle b_1, b_2, \dots, b_m \rangle, \langle b_{-1}, b_2, \dots, b_m \rangle$  shows these elements remain linearly independent in  $H^*BE$ . Thus

$$(A/B) \otimes \mathbb{F}_2[\omega_{2^m}] \subset H^*\tilde{M}(n) \tag{4.7i}$$

since  $\omega_{2^m}$  is invariant under  $\Omega_n^+(\mathbb{F}_2)$ . By Theorem 1.5,  $Q, Sq^1Q, \dots, Sq^{2^{m-2}} \dots Sq^2Sq^1Q \subset H^*BV$  is a regular sequence of invariants; therefore a theorem of P. Baum [B, 3.5] implies

$$(A/B) \otimes \mathbb{F}_2[Q, Sq^1Q, \dots, Sq^{2^{m-2}} \dots Sq^2Sq^1Q] \subset H^*(eBV). \tag{4.7ii}$$

It remains to check equality of the Poincaré series of these modules. The proof is by induction on  $n = 2m$ .

For this we first treat the case  $n = 4$ . It is readily seen that  $\Omega_4^+(\mathbb{F}_2) \approx GL_2(\mathbb{F}_2) \times GL_2(\mathbb{F}_2)$  with generators  $\{u_{12}, \sigma_{12}\}$  for the first factor and  $\{u_{1,-2}, \sigma_{1,-2}\}$  for the second. Then  $f_1 = (1 + u_{12})(1 + \sigma_{12})$  corresponds to the Steinberg idempotent for  $GL_2(\mathbb{F}_2)$  [MP1] and

$$1 = f_0 + f_1 + f_2 \tag{4.8}$$

is an orthogonal decomposition into primitive idempotents, where  $f_0 = 1 + u_{12}\sigma_{12} + (u_{12}\sigma_{12})^2$  and  $f_2 = (1 + \sigma_{12})(1 + u_{12})$ . Similarly in the second factor, let

$$1 = f'_0 + f'_1 + f'_2 \tag{4.9}$$

be the corresponding decomposition. Then  $f_1f'_1$  is the Steinberg idempotent for  $\mathbb{F}_2\Omega_4^+(\mathbb{F}_2)$ .

Consider  $V = V_4$ , the vector space with dual basis  $x_1, x_2, x_{-1}, x_{-2}$ . Then  $(H^*BV)f_0f'_0 = H^*BV^{\mathbb{Z}/3 \times \mathbb{Z}/3}$  since  $u_{12}\sigma_{12}$  and  $u_{1,-2}\sigma_{1,-2}$  have order three. A

simple application of Molien's series [M] computes the Poincaré series

$$P.S.(H^*BVf_0f'_0) = \frac{(1+t^3)^2}{(1-t^2)^2(1-t^3)^2}$$

Similarly  $(H^*BV)f_0 = H^*BV^{Z/3}$  and Molien's series yields

$$P.S.(H^*BVf_0) = \frac{1+2t^2+6t^3+2t^4+t^6}{(1-t^2)^2(1-t^3)^2}$$

Since  $f_1$  and  $f_2$  are conjugate as well as  $f'_1$  and  $f'_2$ , (4.8) then implies

$$P.S.(H^*BVf_1) = \frac{2t+3t^2+2t^3+3t^4+2t^5}{(1-t^2)^2(1-t^3)^2}$$

Now  $f_0 = f_0f'_0 + f_0f'_1 + f_0f'_2$ ; hence

$$P.S.(H^*BVf_0f'_1) = \frac{t^2+2t^3+t^4}{(1-t^2)^2(1-t^3)^2}$$

Therefore

$$P.S.(H^*BVf_1f'_1) = \frac{t+t^2+t^4+t^5}{(1-t^2)^2(1-t^3)^2}$$

which, by 4.6 ( $m=2$ ), equals the Poincaré series for  $(A/B) \otimes \mathbf{F}_2[Q, Sq^1Q]$ . Hence, we have equality in 4.7ii ( $m=2$ ). Since  $\omega_4$  is an invariant, equality in 4.7i ( $m=2$ ) follows from Theorem 2.2.

We now turn to the general case part i),  $n=2m$ , assuming by induction both parts of case  $2m-2$ . To compute  $H^*\widetilde{M}(n)$  as a module over  $\mathbf{F}_2[\omega_{2m}]$  we consider the commutative diagram

$$\begin{array}{ccc} H^*BE & \xrightarrow{\bar{e}} & H^*BE \\ \uparrow \pi^* & & \uparrow \pi^* \\ H^*BV & \xrightarrow{\bar{e}} & H^*BV \end{array}$$

where  $\bar{e} \in \mathbf{F}_2\Omega_{2m}^+(\mathbf{F}_2)$  is the image of the Steinberg idempotent for  $\Omega_{2m-2}^+(\mathbf{F}_2)$  acting on the last  $2m-2$  co-ordinates. Since  $\text{Im } e \subset \text{Im } \bar{e}$  by Theorem 3.4,

induction and the relations

$$Q \equiv x_1 x_{-1} + \sum_{i=2}^m x_i x_{-i}, Sq^1 Q, \dots, Sq^{2^{m-2}} \dots Sq^2 Sq^1 Q$$

of  $H^*BE$  imply  $\text{Im } e$  is generated by elements of the form

$$\omega(Sq^l \alpha'_{m-1}), \quad \omega(Sq^l \beta'_{m-1}) \tag{4.10}$$

where  $\alpha'_{m-1}, \beta'_{m-1}$  are  $\alpha_{m-1}, \beta_{m-1}$  on the last  $2m - 2$  co-ordinates,  $l(I) = m - 1$  and  $\omega = \omega(x_1, x_{-1})$  is a homogeneous polynomial in  $x_1, x_{-1}$ . The remainder of the proof of this inductive step consists of two steps 4.11, 12.

(4.11) Suppose  $z \in \text{Im } e$  is a linear combination of terms from (4.10). Restriction to the subgroups  $\langle b_1, \dots, b_m \rangle$  (resp.  $\langle b_1, \dots, b_{m-1}, b_{-m} \rangle$ ) detects the summands  $\omega Sq^l \alpha'_{m-1}$  (resp.  $\omega Sq^l \beta'_{m-1}$ ) of  $z$  with some  $\omega$  a polynomial in  $x_1$ . Invariance of  $\text{Im } e$  under the Weyl group  $W_{2m}^+$  then shows  $z$  is a linear combination of terms  $Sq^K \alpha_m, Sq^K \beta_m, l(K) = m$ . A similar argument shows the same conclusion holds if  $\omega$  is a polynomial in  $x_{-1}$  alone. Thus  $\text{Im } e$  consists of  $(A/B) \otimes \mathbb{F}_2[\omega_{2^m}]$  plus possibly terms from (4.10) with  $\omega$  divisible by  $x_1 x_{-1}$ . It remains to eliminate the possibly of such terms.

(4.12) We shall need to recall some facts about Molien's series [M]. Let  $G$  be a finite group and  $N$  a graded  $\mathbb{F}_2 G$  module. As usual the Poincaré series of  $N$  is given by  $P.S.(N) = F(N; t) = \sum (\dim_{\mathbb{F}_2} N_i) t^i$ . For an irreducible  $\mathbb{F}_2 G$  module  $E$ , we also consider the series

$$F(N, G, E; t) = \sum a_i t^i$$

where  $a_i$  is the multiplicity of  $E$  as a composition factor in  $N_i$ . Finally, let

$$\chi(N; t) = \sum \chi_{N_i} t^i$$

be the modular character series where  $\chi_{N_i}$  is the modular (or Brauer) character of  $N_i$  defined on the  $p$ -regular elements  $G_{\text{reg}}$  of  $G$  ([S]).

In the present situation let  $G = \Omega_{2^m}^+(\mathbb{F}_2)$ ,  $R = H^*BE$  and

$$R' = \mathbb{F}_2[\omega_{2^m}, \omega_{2^{m-2^i}}, i = 0, 1, \dots, m - 1].$$

We note  $R' = R^{\Omega_{2^m}^+}$  by Theorem 3.4. Let  $M = R \otimes_{R'} \mathbb{F}_2$ . Then in each dimension

$R$  and  $R' \otimes M$  have the same composition series by Theorem 2.2 and the proof of [M, 1.3]. Hence

$$F(R, G, St; t) = F(M, G, St; t)F(R', t) \tag{4.13}$$

where  $St$  is the Steinberg module  $St = e\mathbf{F}_2G$ . By [M; 1.2b] and 4.13 we have

$$F(Re; t) = F(R, G, St; t) \tag{4.14}$$

Now the orthogonality relations for modular characters [S, M] imply

$$F(Re; t) = \frac{1}{|G|} \sum_{g \in G_{\text{reg}}} \chi_{St}(g^{-1})\chi(R; t)(g) \tag{4.15}$$

where  $|G| = (2^m - 1)\prod_{i=1}^{m-1} (2^{2^i} - 1)2^{2^i}$  by [Dk; p. 206]. To evaluate this series we use

LEMMA 4.16.

$$\chi(R; t)(g) = \frac{(1 - t^2)(1 - t^3) \cdots (1 - t^{2^{m-1}+1})}{[\prod_i^{2^m} (1 - \lambda_i(g)t)](1 - t^{2^m})}$$

where  $\{\lambda_i(g)\}$  are the eigenvalues of  $g$  acting on  $V$ .

*Proof.* Let  $S = S(V^*)$  be the symmetric algebra of  $V^*$ . Then  $R = N \otimes \mathbf{F}_2[\omega_{2^m}]$  where  $N = S \otimes_P \mathbf{F}_2$  and  $P = \mathbf{F}_2[Q, Sq^1Q, \dots, Sq^{2^{m-2}} \cdots Sq^1Q]$ . The generators of  $P$  form a regular sequence on  $S$  by Theorem 1.5. Hence by [B; 3.5],  $S \approx P \otimes N$ . Thus

$$\chi(S; t) = \chi(P; t)\chi(N, t)$$

or

$$\prod_1^{2^m} (1 - \lambda_i t) = \prod_{i=0}^{m-1} (1 - t^{2^i+1})\chi(N; t)$$

and the lemma follows since  $\chi(\mathbf{F}_2[\omega_{2^m}]) = (1 - t^{2^m})^{-1}$ .

From 4.6

$$\begin{aligned}
 F(A/B \otimes \mathbf{F}_2[\omega_{2^m}]; t) &= \frac{2t^{2^{m+1}-2-m}}{\prod_{i=1}^m (1-t^{2^i-1})(1-t^{2^m})} + \frac{t^{2^m-2-(m-1)}}{\prod_{i=1}^{m-1} (1-t^{2^i-1})(1-t^{2^m})} \\
 &= \frac{(t^{2^{m+1}-2-m} + t^{2^m-1-m})\prod_{k=1}^{m-1} Q_k(t)}{(\prod_{i=0}^{m-1} (1-t^{2^m-2^i}))(1-t^{2^m})} = f(t)F(R'; t)
 \end{aligned}$$

where  $Q_k(t) = \prod_{i=0}^{k-1} (1 + t^{2^i(2^{m-k}-1)})$  and  $f(t) = (t^{2^{m+1}-2-m} + t^{2^m-1-m})\prod_{k=1}^{m-1} Q_k(t)$ . Combining 4.14, 15 and Lemma 4.16 we have

$$F(Re; t) = g(t)F(R'; t)$$

where

$$g(t) = \frac{1}{|G|} \sum \chi_{St}(g^{-1}) \frac{\prod_{i=0}^{m-1} (1-t^{2^i+1})\prod_{j=0}^{m-1} (1-t^{2^m-2^j})}{\prod_{i=1}^n (1-\lambda_i(g) \cdot t)}.$$

By 4.7i

$$f(t)F(R'; t) = F(A/B \otimes \mathbf{F}_2[\omega_{2^m}]; t) \leq F(Re; t) = g(t)F(R'; t).$$

Thus  $f(t) \leq g(t)$  since the  $R'$  indecomposable classes of  $A/B$  remain indecomposable in  $\text{Im } e$ . This is seen by restricting to  $\langle b_1, \dots, b_m \rangle, \langle b_{-1}, b_2, \dots, b_m \rangle$  where the elements of 4.10 with  $\omega$  divisible by  $x_1x_{-1}$  restrict to zero and using the known indecomposable classes of  $M(m)$  [M; 3.11 ( $p = 2$ )]. The Stiefel–Whitney classes  $\omega_{2^m-2^i}$  of  $\Delta_n$  restrict to  $\omega_{2^m-2^i}$  of  $\text{reg}$  on these subgroups by [Q, 5.1]. Now  $f(t), g(t)$  are polynomials with positive integer coefficients. For  $t = 1$  all terms in  $g(t)$  vanish unless  $g = 1$ . Since  $\chi_{St}(1) = \dim St = |U_{2^m}| = 2^{m(m-1)}$ ,  $f(1) = 2^{\binom{m}{2}+1} = g(1)$ . Thus  $f(t) \leq g(t)$  implies  $f(t) = g(t)$  and so 4.7i) is an equality.

To prove part ii) of the Theorem we observe that  $Q, Sq^1Q, \dots, Sq^{2^m-2} \dots Sq^2Sq^1Q$  is a regular sequence in  $H^*BV$ ; hence the same Molien’s series argument implies equality in 4.7ii). This completes the proof of Theorem 4.5.

*Remark.* A similar proof for computing  $H^*M(n)$  was outlined in [M]; however, the argument is incomplete because of divisibility questions.

*Remark.* It is immediate from Theorem 4.5 that the Poincaré series of

$H^*\tilde{M}(2m)$  is

$$P.S.(H^*\tilde{M}(2m)) = \frac{2t^{2^{m+1}-2-m}}{[\prod_{i=1}^m (1-t^{2^i-1})](1-t^{2^m})} + \frac{t^{2^m-2-(m-1)}}{[\prod_{i=1}^{m-1} (1-t^{2^i-1})](1-t^{2^m})}.$$

*Proof of Theorem 4.1.* Since the Steinberg module is irreducible and projective, it lies in a matrix ring block; since its dimension equals  $2^{m(m-1)}$ , it follows that  $2^{m(m-1)}$  summands appear (see [MP1]).

It remains to produce the desired splitting  $\tilde{M}(2m)$ . Let  $U = \langle u_1, \dots, u_m \rangle$  be a vector space of dimension  $m$  over  $\mathbb{F}_2$ . For  $I = \{i_1, \dots, i_m\}$ ,  $i_j = \pm j$  define

$$\pi_I: V \rightarrow U$$

by

$$\begin{aligned} \pi_I(v_{i_j}) &= u_j \\ \pi_I(v_k) &= 0 \quad k \notin I. \end{aligned}$$

Define stable maps

$$\pi_\alpha = \sum \pi_I \pi: BE \rightarrow BU$$

$$\pi_\beta = \sum \pi_I \pi: BE \rightarrow BU$$

where sums are taken over those sequences  $I$  with an even (resp. odd) number of negative integers. By (4.2) it follows that  $\Omega_n^+(\mathbb{F}_2)$  also acts on  $T(\Delta_n)$  up to homotopy.

Finally let

$$f_1: \tilde{M}(n) \xrightarrow{i} BE \xrightarrow{\pi_\alpha} BU \xrightarrow{\pi} M(m)$$

$$f_2: \tilde{M}(n) \xrightarrow{i} BE \xrightarrow{\pi_\beta} BU \xrightarrow{\pi} L(m)$$

$$f_3: \tilde{M}(n) \xrightarrow{i} BE \xrightarrow{t} T(\Delta_n) \xrightarrow{\pi} eT(\Delta_n)$$

where  $t$  is the transfer [MP1; 3.7] and  $\pi$  is projection onto a stable summand. We

will show that

$$f = f_1 \vee f_2 \vee f_3: \tilde{M}(n) \rightarrow M(m) \vee L(m) \vee eT(\Delta_n)$$

is a 2-local equivalence.

As modules,

$$H^*M(m) = \mathbb{F}_2 \langle Sq^I(x_1^{-1} \cdots x_m^{-1}) \rangle$$

$$H^*L(m) = \mathbb{F}_2 \langle Sq^J(x_1^{-1} \cdots x_m^{-1}) \rangle$$

([MP1]) with the same restrictions on  $I, J$  as in (4.6). Using the Cartan formula it follows that  $Sq^I(x_1^{-1} \cdots x_m^{-1})$  is polynomial in  $x_1, \dots, x_m$  (i.e. there are no negative powers). Hence

$$f_1^*(Sq^I(x_1^{-1} \cdots x_m^{-1})) = Sq^I(\alpha)$$

and analogously

$$f_2^*(Sq^J(x_1^{-1} \cdots x_m^{-1})) = Sq^J(\beta)$$

Since  $\Omega_n^+(\mathbb{F}_2)$  preserves the Euler class  $\omega_{2^m}$  of  $\Delta_n$ , it commutes with the Thom isomorphism

$$H^*BE \xrightarrow{\cong} H^*T(\Delta_n) = [H^*BE]\omega_{2^m}$$

Hence we have

$$H^*eT(\Delta_n) = [(H^*BE)e]\omega_{2^m} = [H^*\tilde{M}(n)]\omega_{2^m}$$

Under these identifications  $t^*: H^*T(\Delta_n) \rightarrow H^*BE$  is the obvious inclusion. Hence  $f_3^*$  is an inclusion with image  $[H^*\tilde{M}(n)]\omega_{2^m}$ . The result follows from Theorem 4.5 and (4.6).

### §5. Splitting $BE(4)$

Let  $E = E(4)$ , the extra-special 2-group of real type and of order 32. The Chevalley group  $\Omega_4^+(\mathbb{F}_2)$  acts on  $BE$  up to homotopy; thus an orthogonal idempotent decomposition of 1 in  $\mathbb{F}_2\Omega_4^+(\mathbb{F}_2)$  will provide a splitting of  $BE$ . One

summand of this splitting is  $BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$  where  $E \approx Q_8 \circ Q_8$  is a 2-Sylow subgroup of  $SL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$ .

Corresponding to the two factors of  $\Omega_4^+(\mathbf{F}_2) \approx GL_2(\mathbf{F}_2) \times GL_2(\mathbf{F}_2)$  there are two orthogonal idempotent decompositions (4.8–9)

$$\begin{aligned} 1 &= f_0 + f_1 + f_2 \\ 1 &= f'_0 + f'_1 + f'_2 \end{aligned}$$

Thus in  $\mathbf{F}_2\Omega_4^+(\mathbf{F}_2)$  we have the orthogonal idempotent decomposition

$$1 = f_0f'_0 + (f_1f'_1 + f_1f'_2 + f_2f'_1 + f_2f'_2) + (f_0f'_1 + f_0f'_2 + f_1f'_0 + f_2f'_0) \tag{5.1}$$

where  $f_1f'_1$  is the Steinberg idempotent.

**THEOREM 5.2.** *Corresponding to (5.1) there is a stable 2-local decomposition*

$$BE \cong BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3) \vee 4(M(2) \vee L(2) \vee eT(\Delta_4)) \vee 4X$$

where  $X = f_0f'_1BE$  is a spectrum with Poincaré series  $(t^2 + t^3)/(1 - t)(1 - t^3)(1 - t^4)$ .

*Proof.* The idempotents  $f_1, f_2$  are conjugate [MP2] as are  $f'_1$  and  $f'_2$ . Hence the summands corresponding to  $f_1f'_1, f_1f'_2, f_2f'_1$  and  $f_2f'_2$  are equivalent. By Theorem 4.1, each is equivalent to  $M(2) \vee L(2) \vee eT(\Delta_4)$ . Similarly  $f_0$  and  $f'_0$  are conjugate. Thus there are four summands equivalent to  $X$ . By comparing Poincaré series, the result now follows from part *i*) of

- PROPOSITION 5.3.** *i)  $f_0f'_0BE \cong BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$*   
*ii) For  $\mathbf{Z}/3 \times \mathbf{Z}/3 \subset \Omega_4^+(\mathbf{F}_2)$ ,  $H^*SL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3) \approx H^*(E)^{\mathbf{Z}/3 \times \mathbf{Z}/3}$*   
*More explicitly,*

$$H^*BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3) = \mathbf{F}_2[v_2, v_3, x_3, \bar{x}_3, \omega_4]/R$$

where

$$R = \left( \begin{aligned} &v_2^3 + v_3^2 + x_3^2 + v_3x_3 \\ &v_2^3 + v_3^2 + \bar{x}_3^2 + v_3\bar{x}_3 \end{aligned} \right)$$

and

$$i^*(v_2) = x_1^2 + x_1x_{-1} + x_{-1}^2$$

$$i^*(v_3) = x_1x_{-1}^2 + x_1^2x_{-1}$$

$$i^*(x_3) = x_1^2x_{-1} + x_1^3 + x_{-1}^3$$

$$i^*(\bar{x}_3) = x_2^2x_{-2} + x_2^3 + x_{-2}^3$$

under the inclusion  $i: E \approx Q_8 \circ Q_8 \rightarrow SL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$ .

*Proof.* Part i) follows immediately from ii) since  $f_0f'_0$  is the trace over  $\mathbf{Z}/3 \times \mathbf{Z}/3$ , i.e.  $f_0f'_0 = \sum g$ ,  $g \in \mathbf{Z}/3 \times \mathbf{Z}/3$ . Part ii) is a straightforward generalization of that for  $H^*BPSL_2(\mathbf{F}_3)$  [MP2]. One considers the map of fibrations

$$\begin{array}{ccccc} B\mathbf{Z}/2 & \longrightarrow & BQ_8 \times Q_8 & \longrightarrow & BQ_8 \circ Q_8 \\ \downarrow & & \downarrow^{Bi \times i} & & \downarrow^{Bi} \\ B\mathbf{Z}/2 & \longrightarrow & BSL_2(\mathbf{F}_3) \times SL_2(\mathbf{F}_3) & \longrightarrow & BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3) \end{array}$$

and the corresponding map of spectral sequences.

*Remark.* The Poincaré series for  $H^*BSL_2(\mathbf{F}_3) \circ SL_2(\mathbf{F}_3)$  is easily seen to be  $(1+t^3)^2/(1-t^2)(1-t^3)(1-t^4)$ .

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