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Stability of meromorphic vector fields in projective spaces¹

XAVIER GÓMEZ-MONT and GEORGE KEMPF

A meromorphic vector field of degree r on the complex projective space \mathbb{P}^n is a bundle map $\alpha: L_{-r} \to T\mathbb{P}^n$ from the line bundle of Chern class -r to the tangent bundle on \mathbb{P}^n , defined up to multiplication by a non-zero scalar. The group $\operatorname{PGL}(n)$ of automorphisms of \mathbb{P}^n acts on the space $\operatorname{MVec}_r = \operatorname{Proj} H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n} \otimes L_r)$ of meromorphic vector fields of degree r. We will say that a meromorphic vector field is non-degenerate if all its zeroes have multiplicity one.

In this paper we analyse the stability properties of the action of PGL(n) on MV_{ec_*} in the sense of Mumford ([17]). We prove:

THEOREM. Let α be a non-degenerate meromorphic vector field of degree r > 0, then:

- (1) α is completely determined by its zero set.
- (2) α is PGL(n)-stable.
- (3) The zero set of α is PGL(n)-stable

A finite collection of points in \mathbb{P}^n have to be in special position to be the zero set of a meromorphic vector field, but this position is of general type in Mumford's sense. The main ingredients in the proof are Bott's computations of the cohomology of homogeneous bundles ([1]), Mumford's numerical criterium of stability ([7]) and the Koszul resolution associated to the zero set Z of α :

$$0 \to \Lambda^n \Omega_{\mathbf{P}^n}(-nr) \to \cdots \to \Omega_{\mathbf{P}^n}(-r) \xrightarrow{\alpha} \mathcal{O}_{\mathbf{P}^n} \to \mathcal{O}_Z \to 0.$$

1. Meromorphic vector fields in \mathbb{P}^n

Let \mathbb{P}^n be the projective space over the complex numbers \mathbb{C} , and let $\mathcal{O}_{\mathbb{P}^n}$, $\mathcal{O}_{\mathbb{P}^n}$, $\mathcal{O}_{\mathbb{P}^n}$ and \mathcal{L} be the structure, tangent, cotangent and hyperplane sheaves on \mathbb{P}^n .

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respectively. If \mathscr{E} is an \mathcal{O}_{P^n} -sheaf, we will use the notation $\mathscr{E}(r)$ for $\mathscr{E} \otimes \mathscr{L}^{\otimes r}$ if $r \geq 0$ and $\mathscr{E} \otimes (\mathscr{L}^*)^{\otimes |r|}$ if $r \leq 0$. The space \mathscr{MV}_{ec_r} of meromorphic vector fields of degree $r \geq -1$ is the projective space of lines through 0 in $H^0(P^n, \Theta_{P^n}(r)) = H^0(P^n, \mathscr{H}_{om}(\mathscr{L}_{-r}, \Theta_{P^n}))$. A meromorphic vector field on \mathbb{P}^n is a non identically zero $\mathscr{O}_{\mathbb{P}^n}$ -morphism $\alpha: \mathscr{L}_{-r} \to \Theta_{\mathbb{P}^n}$, defined up to multiplication by non-zero scalar (see [2], [4]). The twisted Euler sequence ([5] p. 176)

$$0 \to \mathcal{O}_{\mathbf{p}^n}(r) \to \mathcal{O}_{\mathbf{p}^n}(r+1)^{\oplus (n+1)} \to \mathcal{O}_{\mathbf{p}^n}(r) \to 0 \tag{1.1}$$

gives us a way to represent meromorphic vector fields on \mathbb{P}^n as homogeneous polynomial vector fields of degree r+1 in \mathbb{C}^{n+1} . The group $\operatorname{PGL}(n)$ of automorphisms of \mathbb{P}^n acts on MVec_r

$$PGL(n) \times \mathcal{MV}_{ec_*} \to \mathcal{MV}_{ec_*} \qquad (g, \alpha) \mapsto (Dg \cdot \alpha) \circ g^{-1}$$
(1.2)

The universal family of meromorphic vector fields of degree r is the tautological morphism on $\mathscr{MV}_{ec_r} \times P^n$

$$A: \Pi_1^* \mathcal{H}^* \otimes \Pi_2^* \mathcal{L}_{-r} \to \Pi_2^* \Theta_{\mathbb{P}^n} \tag{1.3}$$

where Π_1 and Π_2 are the projections to the factors and \mathcal{H} is the sheaf of hyperplanes in \mathscr{MV}_{ec_x} . Let Z be the subvariety of $\mathscr{MV}_{ec_x} \times \mathbb{P}^n$ defined by A = 0.

LEMMA 1.1. Z is a smooth subvariety of $\mathscr{MV}_{ec_*} \times \mathbb{P}^n$ of codimension n. The restriction of Π_2 to Z is a projective space bundle over \mathbb{P}^n and $\Pi_1: Z \to \mathscr{MV}_{ec_*}$ is generically finite.

Proof. Let \mathcal{I} be the ideal defining the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$, where ρ_1 and ρ_2 are the two projections. The subsheaf

$$\rho_{1^*}[\rho_2^*\Theta_{\mathbf{P}^n}(r)\otimes\mathcal{I}]\hookrightarrow\rho_{1^*}\rho_2^*\Theta_{\mathbf{P}^n}(r)=H^0(\mathbb{P}^n,\,\Theta_{\mathbf{P}^n}(r))\otimes_{\mathbb{C}}\mathcal{O}_{\mathbf{P}^n}$$
(1.4)

has as fiber over $p \in \mathbb{P}^n$ the sections of $h^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r))$ that vanish on p and it is locally free, since $\operatorname{PGL}(n)$ acts transitively on \mathbb{P}^n . The projective bundle over \mathbb{P}^n associated to (1.4) shows that $Z \hookrightarrow \mathscr{MVec}_{x} \times \mathbb{P}^n$ is a projective sub-bundle of Π_2 . Using the representation (1.1), it is easy to see that the line field given by

$$\sum_{i=0}^{n} z_i^{r+1} \frac{\partial}{\partial z_i} \tag{1.5}$$

has isolated zeroes, hence $\Pi_1: Z \to \mathcal{MV}_{ec_r}$ is generically finite. One checks that (1.5) has

$$c_n(\Theta_{\mathbf{p}^n}(r)) = \sum_{j=0}^n \binom{n+1}{j} r^{n-j} = \frac{(r+1)^{n+1} - 1}{r}$$
(1.6)

zeroes of multiplicity one, where c_n is the *n*-th Chern class.

We will now exhibit some PGL(n) invariant divisors in \mathcal{MV}_{ec_n} .

LEMMA 1.2. Let Q be a non-zero GL(n) homogeneous invariant polynomial defined on the space $M_{n\times n}$ of n by n matrices. Then for r>0, the space

$$Z_O = \{ \alpha \in \mathcal{MV}_{ec_r} / \exists p \in \mathbb{P}^n \text{ with } \alpha(p) = 0, \ Q(D\alpha(p)) = 0 \}$$

is a PGL(n)-invariant divisor in \mathscr{MV}_{ec_r} ; where $D\alpha(p): T_p\mathbb{P}^n \to T_p\mathbb{P}^n$ is the linear part of α at p.

Proof. Let $\tilde{Z}_Q = \{(\alpha, p) \in Z/Q(D\alpha(p)) = 0\}$. The projection map $\Pi_2 : \tilde{Z}_Q \to \mathbb{P}^n$ has a structure of a fibre bundle, since PGL(n) acts transitively on \mathbb{P}^n , and has as fiber $\{\alpha \in \mathscr{MVec}_r/\alpha(p_0) = 0, \ Q(D\alpha(p_0)) = 0\}$ which has codimension 1 in $\{\alpha \in \mathscr{MVec}_r/\alpha(p_0) = 0\}$, since the derivative at p_0

$$D_{p_0}: \{\alpha \in H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r)) / \alpha(P_0) = 0\} \to M_{n \times n}$$

is surjective. This shows that \tilde{Z}_Q has codimension 1 in Z.

To finish the proof of the lemma, we will show that $\Pi_1: \tilde{Z}_Q \to \mathscr{MV}_{ec_r}$ is generically finite. To see this, it suffices to show that the codimension of $Z_1 = \{(\alpha, p) \in \mathbb{Z}/\dim_p \{\alpha = 0\} > 0\}$ has codimension bigger than 1 in \mathbb{Z} . We have

$$Z_1 \subset \tilde{Z}_{\text{det}} \subset Z \tag{1.7}$$

where \tilde{Z}_{det} is obtained by setting Q = determinant above. \tilde{Z}_{det} is irreducible of codimension 1 in Z. It is easy to see that the first inclusion in (1.7) is proper (i.e. let $\mathbb{A}^n \subset \mathbb{P}^n$ be an affine chart, we may find a homogeneous field $\sum_{i=1}^n F_i(\partial/\partial z_i)$ in \mathbb{A}^n with 0 as only singular point in \mathbb{A}^n , with high multiplicity, and only singular points of multiplicity 1 on $\mathbb{P}^n - \mathbb{A}^n$), and hence the codimension of Z_1 in Z is bigger than 1.

We will say that a zero p of a meromorphic vector field $\alpha: \mathcal{L}_{-r} \to \Theta_{\mathbb{P}^n}$ is non-degenerate if $\det(D\alpha(p)) \neq 0$. A meromorphic vector field α with only non-degenerate zeroes will be said to be non-degenerate.

Our main technical tool will be the Koszul resolution $K_* \to \mathcal{O}_Z \to 0$ associated to the zero set Z of a meromorphic vector field $\alpha: \mathcal{L}_{-r} \to \Theta_{\mathbb{P}^r}$ vanishing on a finite number of points:

$$0 \to \Lambda^n \Omega_{\mathbb{P}^n}(-nr) \to \cdots \to \Lambda^2 \Omega_{\mathbb{P}^n}(-2r) \to \Omega_{\mathbb{P}^n}(-r) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Z \to 0$$
 (1.8)

We will also use the computations of some cohomology groups of homogeneous bundles in \mathbb{P}^n which may be deduced from Bott's Theorem ([1]). We will sketch a proof of the following proposition in an appendix:

PROPOSITION 1.3. (1) If j < n and r > 0, then

$$H^{j}(\mathbb{P}^{n}, \Lambda^{i}\Omega_{\mathbb{P}^{n}}(-r)) = 0$$

(2) Let $\mathcal{L}_i(r) = (\Lambda^i \Omega_{\mathbb{P}^n}) \otimes \Theta_{\mathbb{P}^n}((1-i)r)$, where r > 0, $0 \le i \le n$ and $n \ge 2$; then $H^j(\mathbb{P}^n, \mathcal{L}_i(r)) = 0$ if j < i, except for $H^0(\mathbb{P}^n, \mathcal{L}_1(r))$, which is one dimensional.

2. Stable meromorphic vector fields

In this section we will show that if $\alpha \in \mathcal{MV}_{ec_r}$, r > 0 has only non-degenerate zeroes, then α is PGL(n)-stable in the sense of Mumford (see [7]).

PROPOSITION 2.1. Let $\alpha \in \mathcal{MV}_{ec_r}$ be a non-degenerate meromorphic vector field with r > 0; then the zeroes of α span \mathbb{P}^n .

Proof. If Z denotes the zero set of α , we have to show that Z is not contained in any hyperplane of \mathbb{P}^n , or equivalently, that the restriction map $\rho: \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to \Gamma(Z, \mathcal{O}_{\mathbb{P}^n}(1)|_Z)$ is injective. Tensoring (1.8) with $\mathcal{O}_{\mathbb{P}^n}(1)$, an easy diagram chase shows that it suffices to prove that $H^j(\mathbb{P}^n, \Lambda^i\Omega_{\mathbb{P}^n}(-ir+1)) = 0$ for $j < i \le n$, which follows from part 1 of Proposition 1.3.

Remark. If α has isolated singularities, the proof of the Proposition still holds, where the span is the span of the zeroes of α with multiplicities.

PROPOSITION 2.2. Let $\alpha \in \Gamma(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r))$ be a non-degenerate meromorphic vector field and r > 0. If α is an eigenvector for the action of a one-parameter subgroup $\lambda: G_m \to GL(n+1)$, then λ factors through the center of GL(n+1).

We will first prove:

LEMMA 2.3. Let α' be a section of $\Theta_{\mathbb{A}^n}$ which is an eigenvector for the action of a one-parameter subgroup $\lambda': G_m \to GL(n)$. Then its eigenvalue is 1 if α' has an isolated zero at p_0 with non-nilpotent linear part $D\alpha'(p_0)$.

Proof. p_0 is a fixed point of λ' and let $\lambda'(t)_*\alpha' = t^m\alpha'$. Then the linear part of $\lambda'(t)\alpha'$ is on the one hand $t^mD\alpha'(p_0)$, and on the other

$$D_{p_0}\lambda'(t)\circ D\alpha'(p_0)\circ D_{p_0}\lambda'(t)^{-1}$$

For the first expression, the eigenvalues are multiplied by t^m , and for the second one they remain the same, hence m = 0.

Proof of proposition. Choose coordinates $(x_0: \ldots : x_n)$ of \mathbb{P}^n so that $\lambda(t) = \operatorname{diag}(t^{m_0}, \ldots, t^{m_n})$. Then $\alpha' = x_i^{-r}\alpha$ is a section of $\Theta_{D(x_i)}$ which is an eigenvector for the action of G_m on the affine space $D(x_i) = \{x_i \neq 0\}$. By Proposition 2.1, α' has a zero in $D(x_i)$, and by Lemma 2.3 we obtain (eigenvalue α) = (eigenvalue x_i), and hence $m_i = m_i$.

PROPOSITION 2.4. Let $\alpha \in \mathcal{MV}_{ec_r}$ be a non-degenerate meromorphic vector field and r > 0; then the stabilizer S of α in PGL(n) is finite.

Proof. We will first show that S does not contain a connected unipotent subgroup. Since such groups are extensions of $G_a = \mathbb{C}$, it will suffice to show that if $\mu: \mathbb{G}_a \to \mathrm{PGL}(n)$ stabilizes α , then it is the identity. The fixed points of μ in \mathbb{P}^n form a linear subspace, and since all the zeroes of α are fixed by μ , their span is contained in this linear subspace. Hence by Proposition 2.1 we conclude that μ is the identity. Hence S is a reductive group. By Proposition 2.2 a maximal torus of S is trivial, so S is a finite group.

THEOREM 2.5. The set of PGL(n)-stable meromorphic vector fields in \mathscr{MV}_{ec_r} , r > 0, in \mathbb{P}^n contains the open set formed by non-degenerate meromorphic vector fields.

Proof. Z_{det} is a PGL(n)-invariant divisor in \mathscr{MVec}_r , and its complement is the set of non-degenerate meromorphic vector fields. By Proposition 2.4 the stabilizers are finite, hence they are all PGL(n)-stable (see [7]).

Remark. Note that by Lemma 1.2, those meromorphic vector fields such that all its singular points have non-nilpotent linear parts are semistable.

We will now show that a non-degenerate meromorphic vector field $\alpha \in \mathcal{MV}_{ec_x}$, r > 0, is determined by its singular points:

THEOREM 2.6. Let $\alpha \in \mathcal{MV}_{ec_*}$, r > 0, be a non-degenerate meromorphic vector field with zero set Z, and let $\alpha' \in \mathcal{MV}_{ec_*}$ be another meromorphic vector field vanishing on Z, then $\alpha' = k\alpha$ with $k \in \mathbb{C}$.

Proof. Tensor the sequence (1.8) with $\Theta_{\mathbb{P}^n}(r)$. To prove the theorem it suffices to show that the map of global sections

$$H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r)) \rightarrow H^0(Z, \Theta_{\mathbb{P}^n}(r)|_Z)$$

has one dimensional kernel. A diagram chase on the above sequence $K_* \otimes \Theta_{\mathbb{P}^n}(r)$ shows that it is sufficient to prove that for $1 < i \le n$ and j < i we have $H^j(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n} \otimes \Theta_{\mathbb{P}^n}((1-i)r) = 0$ and $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n} \otimes \Theta_{\mathbb{P}^n}) = \mathbb{C} \cdot Id$. This follows from part 2 of Proposition 1.3.

Remark. The above argument generalizes to: Let M be a projective manifold such that the only global endomorphisms of the tangent sheaf are multiples of the identity; i.e. $H^0(M, \Omega_M \otimes \Theta_M) = \mathbb{C} \cdot \text{Id}$. If \mathcal{L} is an ample sheaf on M denote $\text{Proj } H^0(M, \Theta_M \otimes \mathcal{L}^r)$ by $\mathscr{MVec}_*(M, \mathcal{L})$. Then for r sufficiently large, the zero set of a non-degenerate meromorphic vector field $\alpha \in \mathscr{MVec}_*(M, \mathcal{L})$ determines α uniquely. If furthermore $H^0(M, \Omega_M \otimes \Theta_M \otimes \mathcal{M}) = 0$ for any line bundle $\mathcal{M} \neq \mathcal{O}_M$ with Chern class zero, then for r sufficiently large the zero set of a non-degenerate meromorphic vector field $\alpha \in \mathscr{MVec}_*(M, \mathcal{L})$ determines α uniquely in $\bigcup_{\mathcal{L}'} \mathscr{MVec}_*(M, \mathcal{L}')$, where \mathcal{L}' are line bundles on M with the same Chern class as \mathcal{L} . For the proof, replace in the above argument Bott's computations by the Kodaira-Nakano vanishing theorem.

3. Stability of the zeroes of a meromorphic vector field

In this section we will show that the zero set of a non-degenerate meromorphic vector field in \mathbb{P}^n is PGL(n)-stable. We will rely on Mumford's numerical criterium ([7], p. 76):

LEMMA 3.1. The set of points $(p_1, \ldots, p_m) \in \mathbb{P}^n \times \cdots \times \mathbb{P}^n = (\mathbb{P}^n)^m$ such that for every proper linear subspace $L \subset \mathbb{P}^n$

$$(number of p_i in L) < \left(\frac{\dim L + 1}{n + 1}\right)m$$
(3.1)

is PGL(n)-stable.

We begin with the following generic result:

THEOREM 3.2. Let \mathscr{E} be a locally free sheaf of rank n on \mathbb{P}^n , then there exists a k_0 such that for $k > k_0$ there is a Zarizki dense open subset in $\operatorname{Proj} H^0(\mathbb{P}^n, \mathscr{E}(k))$ formed of sections σ such that the zero set $\{\sigma = 0\}$ is $\operatorname{PGL}(n)$ -stable.

Proof. Let k_0 be such that for $k > k_0$ we have:

- (1) $\mathscr{E}(k)$ has global sections with only zeroes of multiplicity 1.
- (2) If Grass_s is the Grassmanian of s-dimensional subspaces of \mathbb{P}^n , $0 < s \le n$, and V is the tautological subvariety of $Grass_s \times \mathbb{P}^n$, then for $q \ne s$ and $j = 1, \ldots, n$ we have

$$R^{q}\Pi_{1^{*}}(\Pi_{2}^{*}(\Lambda^{j}\mathscr{C}(-jk))\otimes\mathscr{O}_{V})=0$$
(3.2)

This is possible by Bertini's theorem and a parameter version of the Kodaira-Nakano vanishing theorem (see [4] Theorem 6.7 and [5] p. 252).

Let $k > k_0$, $\sigma \in H^0(\mathbb{P}^n, \mathscr{E}(k))$ with only zeroes of multiplicity 1 and $K_* \to \mathcal{O}_Z \to 0$ be the Koszul resolution of the zero set Z of σ , where $K_* \to 0$ is the complex of sheaves

$$0 \to \Lambda^n \mathscr{E}(-nk) \to \cdots \to \mathscr{E}(-k) \to \mathscr{O}_{\mathbb{P}^n} \to 0 \tag{3.3}$$

with grading $K_j = \Lambda^{n-j} \mathscr{C}(-(n-j)k)$. Let $L \subset \mathbb{P}^n$ be an s-dimensional subspace, and denote by \mathbb{H}^j the hypercohomology groups of the complex of sheaves $K_* \otimes \mathcal{O}_L \to 0$ (see [3]). The cohomology sheaves \mathscr{H}^q of (3.3) have support on $Z \cap L$ and $\mathscr{H}^n = \mathcal{O}_Z \otimes \mathcal{O}_L$. Hence the dimension of \mathbb{H}^n is the cardinality of $Z \cap L$.

For the other spectral sequence of $K_* \otimes \mathcal{O}_L \to 0$ we have $_1E^{p,q} = H^q(L, \Lambda^p \mathscr{E}(-(n-p)k)|_L)$, and by hypothesis on k then $_1E^{p,q} = 0$ except if q = s or $_1E^{n,0} = \mathbb{C}$. Directly from the spectral sequence, we obtain

$$\dim \mathbb{H}^n \le \dim_2 E^{n-s,s} + 1 \le \dim_1 E^{n-s,s} + 1$$
$$= \dim H^s(L, \Lambda^{n-s} \mathscr{C}(-sk)|_L) + 1$$

By the vanishing hypothesis (3.2), we have that

$$\#(Z\cap L)=\dim \mathbb{H}^n\leq |\chi(L,\Lambda^{n-s}\mathscr{E}(-sk)|_L)|+1=\sum_{j=0}^s C_jk^j$$

where χ is the holomorphic Euler-Poincaré characteristic. By setting $L = P^n$ we obtain

$$\#(Z) \le \sum_{j=0}^{n} D_j k^j \tag{3.1}$$

Hence, for probably larger k, Mumford's numerical criterium holds since for large k we will have

$$\sum_{j=0}^{s} C_{j} k^{j} < \frac{s+1}{n+1} \sum_{j=0}^{n} D_{j} k^{j}$$

Hence by Lemma 3.1 Z is PGL(n)-stable.

Remark. Hypercohomolgy of the complex of sheaves $K_* \to 0$ in (1.8) gives a way to compute the number (1.6) of zeroes of a non-degenerate meromorphic vector field $\alpha \in \mathscr{MV}_{ec_r}$. By one of the spectral sequences, we see that all hypercohmology vanishes except $\mathbb{H}^n = H^0(Z, \mathcal{O}_Z)$. The $_1E$ terms of the other spectral sequence vanish except $_1E^{n,0} = \mathbb{C}$ and $_1E^{j,n} = H^n(P^n, \Lambda^{n-j}\Omega_{\mathbb{P}^n}((j-n)r)$. The $_1E$ -terms form an exact sequence

$$0 \to {}_{2}E^{j,n} \to H^{n}(\mathbb{P}^{n}, \Lambda^{n}\Omega_{\mathbb{P}^{n}}(-nr)) \to \cdots \to H^{n}(\mathbb{P}^{n}, \Omega_{\mathbb{P}^{n}}(-r)) \to 0$$

and $\mathbb{H}^n = \mathbb{C} \oplus_2 E^{j,n}$. Hence α vanishes on

$$\sum_{j=0}^{n-1} (-1)^j \chi(\mathbb{P}^n, \Lambda^{n-j} \Omega_{\mathbb{P}^n}((j-n)r)) + 1$$

which is computable by the Riemann-Roch theorem.

Theorem 3.2 shows that for large r, the zero set of non-degenerate meromorphic vector field in \mathscr{MVec}_r are PGL(n)-stable. We will give another argument for every r > 0, using Mumford's numerical criterium (3.1). We will begin with:

LEMMA 3.3. Let $\alpha \in \mathcal{MV}_{ec_r}$, r > 0, be a non-degenerate meromorphic vector field in \mathbb{P}^n , then for every s-dimensional linear subspace $L \subset \mathbb{P}^n$ there are at most $r^{-1}[(r-1)^{s+2}-1]$ points of $Z = \{\alpha = 0\}$ in L.

Proof. $L \hookrightarrow \mathbb{P}^n$ induces an exact sequence of sheaves on L

$$0 \to \Theta_L \to \Theta_{\mathbb{P}^n}|_L \to \mathcal{N}(L, \mathbb{P}^n) \to 0 \tag{3.4}$$

where $\mathcal{N}(L, \mathbb{P}^n)$ is the normal sheaf to L in \mathbb{P}^n . If L' is a linear sub-space of \mathbb{P}^n of dimension n-s-1 and disjoint from L', then the linear projection of \mathbb{P}^n-L' from L' to L induces a splitting of (3.4). Using this splitting, we may obtain from $\alpha: \mathcal{O}_{\mathbb{P}^n}(-r) \to \Theta_{\mathbb{P}^n}$ by restricting to L and then projection to Θ_L a morphism $\alpha': \mathcal{O}_L(-r) \to \Theta_L$. We will show that we may choose L' in such a way that α' has isolated zeroes. By (1.6) this number is bounded by $r^{-1}[(r+1)^{s+1}-1]$. Since all the zeroes of α are zeroes of α' , this is enough to prove the lemma.

To see how one chooses L', note that if $p \in L - Z$ then $\alpha'(p) = 0$ if and only if the line l_p passing through p with tangent direction $\alpha(p) \subset T_p \mathbb{P}^n$ intersects L'. Hence, we have to show that we may choose L' in such a way that only a finite number of lines l_p with $p \in L - Z$ intersect L'. Let $A \subset L \times \mathbb{P}^n$ be the closure of

$$\{(p, q) \in (L - Z) \times \mathbb{P}^n \mid q \in l_p\}$$

A is an irreducible variety of dimension n + 1. Let B be the projection of A to the second factor: $B = \Pi_2(A)$. It is an irreducible variety which contains L; hence B has dimension n or n + 1.

If B has dimension n, then B=L and we have that $l_p \subset L$ for $p \in L-Z$. Hence the map α restricted to L takes values in Θ_L ; i.e. $\alpha|_L : \mathcal{O}_L(-r) \to \Theta_L$. In this case it is not necessary to choose L' since we may choose $\alpha' = \alpha|_L$. If B has dimension n+1, then there is a proper subvariety B' of B such that $\Pi_2 : A - \Pi_2^{-1}(B') \to B - B'$ is a finite morphism. Then choose L' disjoint from B' and L. This proves the Lemma.

THEOREM 3.4. The zero set of a non-degenerate meromorphic vector field in \mathscr{MV}_{ec_*} , r > 0, is PGL(n)-stable.

Proof. Let α be a non-degenerate meromorphic vector field in \mathscr{MVec}_r , r > 0, Z has $r^{-1}[(r+1)^{n+1}-1]$ points. Let $L \subset \mathbb{P}^n$ be a proper linear subspace of dimension s. By Lemma 3.3, L contains at most $r^{-1}[(r+1)^{s+1}-1]$ singular points of Z. Now

$$\frac{s+1}{(n+1)r}[(r+1)^{n+1}-1] - \frac{1}{r}[(r+1)^{s+1}-1]$$

$$= \sum_{j=0}^{s} \left[\frac{s+1}{n+1} \binom{n+1}{j+1} - \binom{s+1}{j+1} \right] r^{j} + \frac{s+1}{n+1} \sum_{j=s+1}^{n} \binom{n+1}{j+1} r^{j} > 0$$

Hence, Mumford's numerical criterium applies.

4. Appendix

In this appendix we will prove Proposition 1.3 using elements from Representation Theory.

Let $T = \{(t_0, \ldots, t_n) \in \mathbb{C}^{n+1}\}$ be the diagonal subgroup of GL(n+1). The characters of T have the form $t^m = t_0^{m_0} \cdots t_n^{m_n}$, where $m = (m_0, \ldots, m_n)$ is a (n+1)-vector of integers. The character t^m is studied by looking at the vector $m' = (m'_i)$ where $m'_i = m_i + n - i$. If all the entries of m' are distinct, we say that the character is non-singular and the index of t^m is the length of the vector permutation σ of [0, n] such that $m'_{\sigma(0)}, \ldots, m'_{\sigma(n)}$ is strictly decreasing. Otherwise t^m is called singular.

We may regard the projective space \mathbb{P}^n as the homogeneous space GL(n+1)/P, where P is the subgroup of matrices of the form

$$\begin{pmatrix} a & c \\ \hline 0 & \\ 0 & b \\ 0 & \end{pmatrix}$$

which is the stabilizer of the first coordinate line l in A^{n+1} . A homogeneous bundle W on \mathbb{P}^n is determined by a representation ρ of \mathbb{P} on the fiber W(l) of W over l. The bundle W is irreducible if ρ is an irreducible representation. There is a one to one correspondence between irreducible homogeneous bundles and some subset of characters of T. Let $t^{m(W)}$ be the character corresponding to such a bundle W. Then $t^{m(W)}$ is the T-eigenvalue of the unique B-invariant line in W(l), where B is the group of upper diagonal matrices in GL(n+1). The characteristic property of such characters t^m is that the sequence m_0, \ldots, m_n is non-increasing.

THEOREM 4.1 (Bott [1]). Let W be an irreducible homogeneous bundle on \mathbb{P}^n , and let m = m(W) and $\mathcal{O}(W)$ be the sheaf of holomorphic sections of W, then:

- (1) If t^m is singular, then $H^i(\mathbb{P}^n, \mathcal{O}(W)) = 0$ for all i, and
- (2) If t^m is non-singular, then $H^i(\mathbb{P}^n, \mathcal{O}(W)) = 0$ if $i \neq \operatorname{index}(m)$ and $H^{\operatorname{index}(m)}(\mathbb{P}^n, \mathcal{O}(W))$ is an irreducible representation with highest weight t^r , where $r_i = m'_{\sigma(i)} n + i$.

EXAMPLE. 4.2. The tangent bundle $T_{\mathbb{P}^n}$ of \mathbb{P}^n is irreducible with $m(T_{\mathbb{P}^n}) = t_0 t_n^{-1}$. In this case the index is zero, as $T_{\mathbb{P}^n}$ has global sections.

EXAMPLE. 4.3. The bundle of *i*-forms $\Lambda^i T^*_{\mathbb{P}^n}$ of \mathbb{P}^n is irreducible and $m(\Lambda^i T^*_{\mathbb{P}^n}) = t_0^{-i} t_1 \cdots t_i$. Here the index is *i* as $H^i(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n}) \neq 0$.

EXAMPLE. 4.4. $m(\mathcal{O}(r))$ corresponds to $(r, 0, \dots, 0)$.

LEMMA 4.5. Let $m = m(\Lambda^i T_{p^n}^*)$ with $0 \le i \le n$. Denoting m - (r, 0, ..., 0) by m(r), we have;

- (1) $t^{m(r)}$ is singular if $1 \le r \le n i$.
- (2) The index of $t^{m(r)}$ is n if r > n i.

Proof. as $m(r)' = (n - i - r, n, \ldots, n - i + 1, n - i - 1, \ldots, 0)$, the lemma is clear.

We are ready to prove part 1 of Proposition 1.3.

COROLLARY 4.6. If
$$r > 0$$
, then $H^{j}(\mathbb{P}^{n}, \Lambda^{i}\Omega_{\mathbb{P}^{n}}(-r)) = 0$ for $j < n$.

Proof. Recalling that L_{-r} denotes the line bundle on \mathbb{P}^n with Chern class -r, we have that $m(\Lambda^i T_{\mathbb{P}^n}^* \otimes l_{-r}) = m(r)$. So the corollary follows from Lemma 4.5.

LEMMA 4.7. For $0 \le i \le n$ and $r \ge 0$, let W_r^i be the irreducible homogeneous bundle on \mathbb{P}^n with associated character

$$\chi = \chi_r(W_r^i) = t_r^{m(W_r^i)} = t_0^{1-i-r}t_1 \cdot \cdot \cdot t_i t_n^{-1}$$
, then

- (1) For 0 < i < n, we have:
 - (a) χ is singular for r = 0, $2 \le r \le n i$, or r = n i + 2.
 - (b) The index of χ is i for r = 1; n-1 for r = n-i+1 and n for r > n-i+2.
- (2) For i = 0, we have:
 - (a) χ is singular for $2 \le r \le n$ or r = n + 2.
 - (b) The index of χ is 0 for r = 0, 1; n 1 for r = n + 1 and n for r > n + 2.
- (3) For i = n, we have:
 - (a) χ is singular for r=1.
 - (b) The index of χ is n-1 for r=0 and n for r>1.

Proof. $m(W_r^i)' = (1 - i - r + n, n, n - 1, ..., n - i + 1, n - i - 1, ..., 1, -1)$, so the Lemma is clear.

COROLLARY 4.8. For $(0 \le i \le n-1 \text{ and } r \ge 0)$ or (i = n and r > 0) we have $H^{j}(\mathbb{P}^{n}, \mathcal{O}(W_{r}^{i})) = 0$ for all j < i.

Proof. For i = 0 the statement is vacuous and for $0 < i \le n$, the corollary follows from Lemma 4.7 and Theorem 4.1.

We now prove the second part of Proposition 1.3:

PROPOSITION 4.9. Let $\mathcal{L}_i(r) = (\Lambda^i \Omega_{\mathbb{P}^n}) \otimes \Theta((1-i)r)$, with r > 0 and $0 \le i \le n$, then $H^j(\mathbb{P}^n, \mathcal{L}_i(r)) = 0$ if j < i and $n \ge 2$; except for $H^0(\mathbb{P}^n, \mathcal{L}_1(r))$, which is one dimensional.

Proof. The statement is vacuous for i = 0. For i = n we have $\mathcal{L}_n(r) = \Theta(r(1 - n) - n - 1) = \mathcal{O}(W^0_{r(n-1)+n+1})$. Hence the proposition follows from Lemma 4.7 since $n \ge 2$ and r > 0.

For 0 < i < n-1, we have by Littlewood-Richardson [6] that

$$\mathcal{L}_{i}(r) = \mathcal{O}(W_{(i-1)r}^{i}) \oplus \Lambda^{i-1} \Omega((1-i)r)$$

where $(i-1)r \ge 0$. The vanishing of the j^{th} -cohomology groups of the first summand, j < i, follows from Corollary 4.8 and the vanishing of the second summand follows from Corollary 4.6, except for i = 1 and j = 0, which is one dimensional.

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