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Stability of meromorphic vector fields in projective spaces¹

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A meromorphic vector field of degree r on the complex projective space \mathbb{P}^n is a bundle map $\alpha: L_{-r} \rightarrow T\mathbb{P}^n$ from the line bundle of Chern class $-r$ to the tangent bundle on \mathbb{P}^n , defined up to multiplication by a non-zero scalar. The group $\text{PGL}(n)$ of automorphisms of \mathbb{P}^n acts on the space $\mathcal{M}\mathcal{V}ec_r = \text{Proj } H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n} \otimes L_r)$ of meromorphic vector fields of degree r . We will say that a meromorphic vector field is non-degenerate if all its zeroes have multiplicity one.

In this paper we analyse the stability properties of the action of $\text{PGL}(n)$ on $\mathcal{M}\mathcal{V}ec_r$ in the sense of Mumford ([17]). We prove:

THEOREM. *Let α be a non-degenerate meromorphic vector field of degree $r > 0$, then:*

- (1) α is completely determined by its zero set.
- (2) α is $\text{PGL}(n)$ -stable.
- (3) The zero set of α is $\text{PGL}(n)$ -stable

A finite collection of points in \mathbb{P}^n have to be in special position to be the zero set of a meromorphic vector field, but this position is of general type in Mumford's sense. The main ingredients in the proof are Bott's computations of the cohomology of homogeneous bundles ([1]), Mumford's numerical criterium of stability ([7]) and the Koszul resolution associated to the zero set Z of α :

$$0 \rightarrow \Lambda^n \Omega_{\mathbb{P}^n}(-nr) \rightarrow \dots \rightarrow \Omega_{\mathbb{P}^n}(-r) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_Z \rightarrow 0.$$

1. Meromorphic vector fields in \mathbb{P}^n

Let \mathbb{P}^n be the projective space over the complex numbers \mathbb{C} , and let $\mathcal{O}_{\mathbb{P}^n}$, $\Theta_{\mathbb{P}^n}$, $\Omega_{\mathbb{P}^n}$ and \mathcal{L} be the structure, tangent, cotangent and hyperplane sheaves on \mathbb{P}^n ,

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respectively. If \mathcal{E} is an $\mathcal{O}_{\mathbb{P}^n}$ -sheaf, we will use the notation $\mathcal{E}(r)$ for $\mathcal{E} \otimes \mathcal{L}^{\otimes r}$ if $r \geq 0$ and $\mathcal{E} \otimes (\mathcal{L}^*)^{\otimes |r|}$ if $r \leq 0$. The space $\mathcal{M}Vec_r$ of meromorphic vector fields of degree $r \geq -1$ is the projective space of lines through 0 in $H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r)) = H^0(\mathbb{P}^n, \mathcal{H}om(\mathcal{L}_{-r}, \Theta_{\mathbb{P}^n}))$. A meromorphic vector field on \mathbb{P}^n is a non identically zero $\mathcal{O}_{\mathbb{P}^n}$ -morphism $\alpha: \mathcal{L}_{-r} \rightarrow \Theta_{\mathbb{P}^n}$, defined up to multiplication by non-zero scalar (see [2], [4]). The twisted Euler sequence ([5] p. 176)

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n}(r) \rightarrow \mathcal{O}_{\mathbb{P}^n}(r+1)^{\oplus(n+1)} \rightarrow \Theta_{\mathbb{P}^n}(r) \rightarrow 0 \tag{1.1}$$

gives us a way to represent meromorphic vector fields on \mathbb{P}^n as homogeneous polynomial vector fields of degree $r+1$ in \mathbb{C}^{n+1} . The group $\text{PGL}(n)$ of automorphisms of \mathbb{P}^n acts on $\mathcal{M}Vec_r$

$$\text{PGL}(n) \times \mathcal{M}Vec_r \rightarrow \mathcal{M}Vec_r \quad (g, \alpha) \mapsto (Dg \cdot \alpha) \circ g^{-1} \tag{1.2}$$

The universal family of meromorphic vector fields of degree r is the tautological morphism on $\mathcal{M}Vec_r \times \mathbb{P}^n$

$$A: \Pi_1^* \mathcal{H}^* \otimes \Pi_2^* \mathcal{L}_{-r} \rightarrow \Pi_2^* \Theta_{\mathbb{P}^n} \tag{1.3}$$

where Π_1 and Π_2 are the projections to the factors and \mathcal{H} is the sheaf of hyperplanes in $\mathcal{M}Vec_r$. Let Z be the subvariety of $\mathcal{M}Vec_r \times \mathbb{P}^n$ defined by $A = 0$.

LEMMA 1.1. *Z is a smooth subvariety of $\mathcal{M}Vec_r \times \mathbb{P}^n$ of codimension n. The restriction of Π_2 to Z is a projective space bundle over \mathbb{P}^n and $\Pi_1: Z \rightarrow \mathcal{M}Vec_r$ is generically finite.*

Proof. Let \mathcal{I} be the ideal defining the diagonal in $\mathbb{P}^n \times \mathbb{P}^n$, where ρ_1 and ρ_2 are the two projections. The subsheaf

$$\rho_{1*}[\rho_2^* \Theta_{\mathbb{P}^n}(r) \otimes \mathcal{I}] \hookrightarrow \rho_{1*} \rho_2^* \Theta_{\mathbb{P}^n}(r) = H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r)) \otimes_{\mathbb{C}} \mathcal{O}_{\mathbb{P}^n} \tag{1.4}$$

has as fiber over $p \in \mathbb{P}^n$ the sections of $h^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r))$ that vanish on p and it is locally free, since $\text{PGL}(n)$ acts transitively on \mathbb{P}^n . The projective bundle over \mathbb{P}^n associated to (1.4) shows that $Z \hookrightarrow \mathcal{M}Vec_r \times \mathbb{P}^n$ is a projective sub-bundle of Π_2 .

Using the representation (1.1), it is easy to see that the line field given by

$$\sum_{i=0}^n z_i^{r+1} \frac{\partial}{\partial z_i} \tag{1.5}$$

has isolated zeroes, hence $\Pi_1: Z \rightarrow \mathcal{M}Vec_r$ is generically finite. One checks that (1.5) has

$$c_n(\Theta_{\mathbb{P}^n}(r)) = \sum_{j=0}^n \binom{n+1}{j} r^{n-j} = \frac{(r+1)^{n+1} - 1}{r} \tag{1.6}$$

zeroes of multiplicity one, where c_n is the n -th Chern class.

We will now exhibit some $\text{PGL}(n)$ invariant divisors in $\mathcal{M}Vec_r$.

LEMMA 1.2. *Let Q be a non-zero $\text{GL}(n)$ homogeneous invariant polynomial defined on the space $M_{n \times n}$ of n by n matrices. Then for $r > 0$, the space*

$$Z_Q = \{ \alpha \in \mathcal{M}Vec_r / \exists p \in \mathbb{P}^n \text{ with } \alpha(p) = 0, Q(D\alpha(p)) = 0 \}$$

is a $\text{PGL}(n)$ -invariant divisor in $\mathcal{M}Vec_r$; where $D\alpha(p): T_p\mathbb{P}^n \rightarrow T_p\mathbb{P}^n$ is the linear part of α at p .

Proof. Let $\tilde{Z}_Q = \{ (\alpha, p) \in Z / Q(D\alpha(p)) = 0 \}$. The projection map $\Pi_2: \tilde{Z}_Q \rightarrow \mathbb{P}^n$ has a structure of a fibre bundle, since $\text{PGL}(n)$ acts transitively on \mathbb{P}^n , and has as fiber $\{ \alpha \in \mathcal{M}Vec_r / \alpha(p_0) = 0, Q(D\alpha(p_0)) = 0 \}$ which has codimension 1 in $\{ \alpha \in \mathcal{M}Vec_r / \alpha(p_0) = 0 \}$, since the derivative at p_0

$$D_{p_0}: \{ \alpha \in H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r)) / \alpha(P_0) = 0 \} \rightarrow M_{n \times n}$$

is surjective. This shows that \tilde{Z}_Q has codimension 1 in Z .

To finish the proof of the lemma, we will show that $\Pi_1: \tilde{Z}_Q \rightarrow \mathcal{M}Vec_r$ is generically finite. To see this, it suffices to show that the codimension of $Z_1 = \{ (\alpha, p) \in Z / \dim_p \{ \alpha = 0 \} > 0 \}$ has codimension bigger than 1 in Z . We have

$$Z_1 \subset \tilde{Z}_{\det} \subset Z \tag{1.7}$$

where \tilde{Z}_{\det} is obtained by setting $Q = \text{determinant}$ above. \tilde{Z}_{\det} is irreducible of codimension 1 in Z . It is easy to see that the first inclusion in (1.7) is proper (i.e. let $\mathbb{A}^n \subset \mathbb{P}^n$ be an affine chart, we may find a homogeneous field $\sum_{i=1}^n F_i(\partial/\partial z_i)$ in \mathbb{A}^n with 0 as only singular point in \mathbb{A}^n , with high multiplicity, and only singular points of multiplicity 1 on $\mathbb{P}^n - \mathbb{A}^n$), and hence the codimension of Z_1 in Z is bigger than 1.

We will say that a zero p of a meromorphic vector field $\alpha: \mathcal{L}_{-r} \rightarrow \Theta_{\mathbb{P}^n}$ is non-degenerate if $\det(D\alpha(p)) \neq 0$. A meromorphic vector field α with only non-degenerate zeroes will be said to be non-degenerate.

Our main technical tool will be the Koszul resolution $K_* \rightarrow \mathcal{O}_Z \rightarrow 0$ associated to the zero set Z of a meromorphic vector field $\alpha: \mathcal{L}_{-r} \rightarrow \Theta_{\mathbb{P}^n}$ vanishing on a finite number of points:

$$0 \rightarrow \Lambda^n \Omega_{\mathbb{P}^n}(-nr) \rightarrow \dots \rightarrow \Lambda^2 \Omega_{\mathbb{P}^n}(-2r) \rightarrow \Omega_{\mathbb{P}^n}(-r) \xrightarrow{\alpha} \Theta_{\mathbb{P}^n} \rightarrow \mathcal{O}_Z \rightarrow 0 \tag{1.8}$$

We will also use the computations of some cohomology groups of homogeneous bundles in \mathbb{P}^n which may be deduced from Bott's Theorem ([1]). We will sketch a proof of the following proposition in an appendix:

PROPOSITION 1.3. (1) *If $j < n$ and $r > 0$, then*

$$H^j(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n}(-r)) = 0$$

(2) *Let $\mathcal{L}_i(r) = (\Lambda^i \Omega_{\mathbb{P}^n}) \otimes \Theta_{\mathbb{P}^n}((1-i)r)$, where $r > 0$, $0 \leq i \leq n$ and $n \geq 2$; then $H^j(\mathbb{P}^n, \mathcal{L}_i(r)) = 0$ if $j < i$, except for $H^0(\mathbb{P}^n, \mathcal{L}_1(r))$, which is one dimensional.*

2. Stable meromorphic vector fields

In this section we will show that if $\alpha \in \mathcal{M}Vec_r$, $r > 0$ has only non-degenerate zeroes, then α is $PGL(n)$ -stable in the sense of Mumford (see [7]).

PROPOSITION 2.1. *Let $\alpha \in \mathcal{M}Vec_r$ be a non-degenerate meromorphic vector field with $r > 0$; then the zeroes of α span \mathbb{P}^n .*

Proof. If Z denotes the zero set of α , we have to show that Z is not contained in any hyperplane of \mathbb{P}^n , or equivalently, that the restriction map $\rho: \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \rightarrow \Gamma(Z, \mathcal{O}_{\mathbb{P}^n}(1)|_Z)$ is injective. Tensoring (1.8) with $\mathcal{O}_{\mathbb{P}^n}(1)$, an easy diagram chase shows that it suffices to prove that $H^j(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n}(-ir + 1)) = 0$ for $j < i \leq n$, which follows from part 1 of Proposition 1.3.

Remark. If α has isolated singularities, the proof of the Proposition still holds, where the span is the span of the zeroes of α with multiplicities.

PROPOSITION 2.2. *Let $\alpha \in \Gamma(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r))$ be a non-degenerate meromorphic vector field and $r > 0$. If α is an eigenvector for the action of a one-parameter subgroup $\lambda: G_m \rightarrow GL(n + 1)$, then λ factors through the center of $GL(n + 1)$.*

We will first prove:

LEMMA 2.3. *Let α' be a section of $\Theta_{\mathbb{A}^n}$ which is an eigenvector for the action of a one-parameter subgroup $\lambda' : G_m \rightarrow GL(n)$. Then its eigenvalue is 1 if α' has an isolated zero at p_0 with non-nilpotent linear part $D\alpha'(p_0)$.*

Proof. p_0 is a fixed point of λ' and let $\lambda'(t)_* \alpha' = t^m \alpha'$. Then the linear part of $\lambda'(t)\alpha'$ is on the one hand $t^m D\alpha'(p_0)$, and on the other

$$D_{p_0} \lambda'(t) \circ D\alpha'(p_0) \circ D_{p_0} \lambda'(t)^{-1}$$

For the first expression, the eigenvalues are multiplied by t^m , and for the second one they remain the same, hence $m = 0$.

Proof of proposition. Choose coordinates $(x_0 : \dots : x_n)$ of \mathbb{P}^n so that $\lambda(t) = \text{diag}(t^{m_0}, \dots, t^{m_n})$. Then $\alpha' = x_i^{-r} \alpha$ is a section of $\Theta_{D(x_i)}$ which is an eigenvector for the action of G_m on the affine space $D(x_i) = \{x_i \neq 0\}$. By Proposition 2.1, α' has a zero in $D(x_i)$, and by Lemma 2.3 we obtain (eigenvalue α) = (eigenvalue x_i)^r, and hence $m_i = m_j$.

PROPOSITION 2.4. *Let $\alpha \in \mathcal{M}Vec_r$ be a non-degenerate meromorphic vector field and $r > 0$; then the stabilizer S of α in $\text{PGL}(n)$ is finite.*

Proof. We will first show that S does not contain a connected unipotent subgroup. Since such groups are extensions of $G_a = \mathbb{C}$, it will suffice to show that if $\mu : G_a \rightarrow \text{PGL}(n)$ stabilizes α , then it is the identity. The fixed points of μ in \mathbb{P}^n form a linear subspace, and since all the zeroes of α are fixed by μ , their span is contained in this linear subspace. Hence by Proposition 2.1 we conclude that μ is the identity. Hence S is a reductive group. By Proposition 2.2 a maximal torus of S is trivial, so S is a finite group.

THEOREM 2.5. *The set of $\text{PGL}(n)$ -stable meromorphic vector fields in $\mathcal{M}Vec_r$, $r > 0$, in \mathbb{P}^n contains the open set formed by non-degenerate meromorphic vector fields.*

Proof. Z_{\det} is a $\text{PGL}(n)$ -invariant divisor in $\mathcal{M}Vec_r$, and its complement is the set of non-degenerate meromorphic vector fields. By Proposition 2.4 the stabilizers are finite, hence they are all $\text{PGL}(n)$ -stable (see [7]).

Remark. Note that by Lemma 1.2, those meromorphic vector fields such that all its singular points have non-nilpotent linear parts are semistable.

We will now show that a non-degenerate meromorphic vector field $\alpha \in \mathcal{M}Vec_r$, $r > 0$, is determined by its singular points:

THEOREM 2.6. *Let $\alpha \in \mathcal{M}Vec_r$, $r > 0$, be a non-degenerate meromorphic vector field with zero set Z , and let $\alpha' \in \mathcal{M}Vec_r$ be another meromorphic vector field vanishing on Z , then $\alpha' = k\alpha$ with $k \in \mathbb{C}$.*

Proof. Tensor the sequence (1.8) with $\Theta_{\mathbb{P}^n}(r)$. To prove the theorem it suffices to show that the map of global sections

$$H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r)) \rightarrow H^0(Z, \Theta_{\mathbb{P}^n}(r)|_Z)$$

has one dimensional kernel. A diagram chase on the above sequence $K_* \otimes \Theta_{\mathbb{P}^n}(r)$ shows that it is sufficient to prove that for $1 < i \leq n$ and $j < i$ we have $H^i(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n} \otimes \Theta_{\mathbb{P}^n}((1-i)r)) = 0$ and $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n} \otimes \Theta_{\mathbb{P}^n}) = \mathbb{C} \cdot Id$. This follows from part 2 of Proposition 1.3.

Remark. The above argument generalizes to: Let M be a projective manifold such that the only global endomorphisms of the tangent sheaf are multiples of the identity; i.e. $H^0(M, \Omega_M \otimes \Theta_M) = \mathbb{C} \cdot Id$. If \mathcal{L} is an ample sheaf on M denote $\text{Proj } H^0(M, \Theta_M \otimes \mathcal{L}^r)$ by $\mathcal{M}Vec_r(M, \mathcal{L})$. Then for r sufficiently large, the zero set of a non-degenerate meromorphic vector field $\alpha \in \mathcal{M}Vec_r(M, \mathcal{L})$ determines α uniquely. If furthermore $H^0(M, \Omega_M \otimes \Theta_M \otimes \mathcal{M}) = 0$ for any line bundle $\mathcal{M} \neq \mathcal{O}_M$ with Chern class zero, then for r sufficiently large the zero set of a non-degenerate meromorphic vector field $\alpha \in \mathcal{M}Vec_r(M, \mathcal{L})$ determines α uniquely in $\bigcup_{\mathcal{L}'} \mathcal{M}Vec_r(M, \mathcal{L}')$, where \mathcal{L}' are line bundles on M with the same Chern class as \mathcal{L} . For the proof, replace in the above argument Bott's computations by the Kodaira–Nakano vanishing theorem.

3. Stability of the zeroes of a meromorphic vector field

In this section we will show that the zero set of a non-degenerate meromorphic vector field in \mathbb{P}^n is $\text{PGL}(n)$ -stable. We will rely on Mumford's numerical criterium ([7], p. 76):

LEMMA 3.1. *The set of points $(p_1, \dots, p_m) \in \mathbb{P}^n \times \dots \times \mathbb{P}^n = (\mathbb{P}^n)^m$ such that for every proper linear subspace $L \subset \mathbb{P}^n$*

$$(\text{number of } p_i \text{ in } L) < \left(\frac{\dim L + 1}{n + 1} \right) m \tag{3.1}$$

is $\text{PGL}(n)$ -stable.

We begin with the following generic result:

THEOREM 3.2. *Let \mathcal{E} be a locally free sheaf of rank n on \mathbb{P}^n , then there exists a k_0 such that for $k > k_0$ there is a Zariski dense open subset in $\text{Proj } H^0(\mathbb{P}^n, \mathcal{E}(k))$ formed of sections σ such that the zero set $\{\sigma = 0\}$ is $\text{PGL}(n)$ -stable.*

Proof. Let k_0 be such that for $k > k_0$ we have:

- (1) $\mathcal{E}(k)$ has global sections with only zeroes of multiplicity 1.
- (2) If Grass_s is the Grassmanian of s -dimensional subspaces of \mathbb{P}^n , $0 < s \leq n$, and V is the tautological subvariety of $\text{Grass}_s \times \mathbb{P}^n$, then for $q \neq s$ and $j = 1, \dots, n$ we have

$$R^q \Pi_{1*}(\Pi_2^*(\Lambda^j \mathcal{E}(-jk)) \otimes \mathcal{O}_V) = 0 \tag{3.2}$$

This is possible by Bertini’s theorem and a parameter version of the Kodaira–Nakano vanishing theorem (see [4] Theorem 6.7 and [5] p. 252).

Let $k > k_0$, $\sigma \in H^0(\mathbb{P}^n, \mathcal{E}(k))$ with only zeroes of multiplicity 1 and $K_* \rightarrow \mathcal{O}_Z \rightarrow 0$ be the Koszul resolution of the zero set Z of σ , where $K_* \rightarrow 0$ is the complex of sheaves

$$0 \rightarrow \Lambda^n \mathcal{E}(-nk) \rightarrow \dots \rightarrow \mathcal{E}(-k) \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow 0 \tag{3.3}$$

with grading $K_j = \Lambda^{n-j} \mathcal{E}(-(n-j)k)$. Let $L \subset \mathbb{P}^n$ be an s -dimensional subspace, and denote by \mathbb{H}^j the hypercohomology groups of the complex of sheaves $K_* \otimes \mathcal{O}_L \rightarrow 0$ (see [3]). The cohomology sheaves \mathcal{H}^q of (3.3) have support on $Z \cap L$ and $\mathcal{H}^n = \mathcal{O}_Z \otimes \mathcal{O}_L$. Hence the dimension of \mathbb{H}^n is the cardinality of $Z \cap L$.

For the other spectral sequence of $K_* \otimes \mathcal{O}_L \rightarrow 0$ we have ${}_1 E^{p,q} = H^q(L, \Lambda^p \mathcal{E}(-(n-p)k)|_L)$, and by hypothesis on k then ${}_1 E^{p,q} = 0$ except if $q = s$ or ${}_1 E^{n,0} = \mathbb{C}$. Directly from the spectral sequence, we obtain

$$\begin{aligned} \dim \mathbb{H}^n &\leq \dim {}_2 E^{n-s,s} + 1 \leq \dim {}_1 E^{n-s,s} + 1 \\ &= \dim H^s(L, \Lambda^{n-s} \mathcal{E}(-sk)|_L) + 1 \end{aligned}$$

By the vanishing hypothesis (3.2), we have that

$$\#(Z \cap L) = \dim \mathbb{H}^n \leq |\chi(L, \Lambda^{n-s} \mathcal{E}(-sk)|_L)| + 1 = \sum_{j=0}^s C_j k^j$$

where χ is the holomorphic Euler–Poincaré characteristic. By setting $L = P^n$ we obtain

$$\#(Z) \leq \sum_{j=0}^n D_j k^j \quad (3.1)$$

Hence, for probably larger k , Mumford’s numerical criterium holds since for large k we will have

$$\sum_{j=0}^s C_j k^j < \frac{s+1}{n+1} \sum_{j=0}^n D_j k^j$$

Hence by Lemma 3.1 Z is $\text{PGL}(n)$ -stable.

Remark. Hypercohomology of the complex of sheaves $K_* \rightarrow 0$ in (1.8) gives a way to compute the number (1.6) of zeroes of a non-degenerate meromorphic vector field $\alpha \in \mathcal{M}\mathcal{V}ec_r$. By one of the spectral sequences, we see that all hypercohomology vanishes except $\mathbb{H}^n = H^0(Z, \mathcal{O}_Z)$. The ${}_1E$ terms of the other spectral sequence vanish except ${}_1E^{n,0} = \mathbb{C}$ and ${}_1E^{j,n} = H^n(P^n, \Lambda^{n-j} \Omega_{P^n}((j-n)r))$. The ${}_1E$ -terms form an exact sequence

$$0 \rightarrow {}_2E^{j,n} \rightarrow H^n(\mathbb{P}^n, \Lambda^n \Omega_{\mathbb{P}^n}(-nr)) \rightarrow \cdots \rightarrow H^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(-r)) \rightarrow 0$$

and $\mathbb{H}^n = \mathbb{C} \oplus {}_2E^{j,n}$. Hence α vanishes on

$$\sum_{j=0}^{n-1} (-1)^j \chi(\mathbb{P}^n, \Lambda^{n-j} \Omega_{\mathbb{P}^n}((j-n)r)) + 1$$

which is computable by the Riemann–Roch theorem.

Theorem 3.2 shows that for large r , the zero set of non-degenerate meromorphic vector field in $\mathcal{M}\mathcal{V}ec_r$ are $\text{PGL}(n)$ -stable. We will give another argument for every $r > 0$, using Mumford’s numerical criterium (3.1). We will begin with:

LEMMA 3.3. *Let $\alpha \in \mathcal{M}\mathcal{V}ec_r$, $r > 0$, be a non-degenerate meromorphic vector field in \mathbb{P}^n , then for every s -dimensional linear subspace $L \subset \mathbb{P}^n$ there are at most $r^{-1}[(r-1)^{s+2} - 1]$ points of $Z = \{\alpha = 0\}$ in L .*

Proof. $L \hookrightarrow \mathbb{P}^n$ induces an exact sequence of sheaves on L

$$0 \rightarrow \Theta_L \rightarrow \Theta_{\mathbb{P}^n}|_L \rightarrow \mathcal{N}(L, \mathbb{P}^n) \rightarrow 0 \quad (3.4)$$

where $\mathcal{N}(L, \mathbb{P}^n)$ is the normal sheaf to L in \mathbb{P}^n . If L' is a linear sub-space of \mathbb{P}^n of dimension $n - s - 1$ and disjoint from L , then the linear projection of $\mathbb{P}^n - L'$ from L' to L induces a splitting of (3.4). Using this splitting, we may obtain from $\alpha: \mathcal{O}_{\mathbb{P}^n}(-r) \rightarrow \mathcal{O}_{\mathbb{P}^n}$ by restricting to L and then projection to \mathcal{O}_L a morphism $\alpha': \mathcal{O}_L(-r) \rightarrow \mathcal{O}_L$. We will show that we may choose L' in such a way that α' has isolated zeroes. By (1.6) this number is bounded by $r^{-1}[(r + 1)^{s+1} - 1]$. Since all the zeroes of α are zeroes of α' , this is enough to prove the lemma.

To see how one chooses L' , note that if $p \in L - Z$ then $\alpha'(p) = 0$ if and only if the line l_p passing through p with tangent direction $\alpha(p) \subset T_p \mathbb{P}^n$ intersects L' . Hence, we have to show that we may choose L' in such a way that only a finite number of lines l_p with $p \in L - Z$ intersect L' . Let $A \subset L \times \mathbb{P}^n$ be the closure of

$$\{(p, q) \in (L - Z) \times \mathbb{P}^n \mid q \in l_p\}$$

A is an irreducible variety of dimension $n + 1$. Let B be the projection of A to the second factor: $B = \Pi_2(A)$. It is an irreducible variety which contains L ; hence B has dimension n or $n + 1$.

If B has dimension n , then $B = L$ and we have that $l_p \subset L$ for $p \in L - Z$. Hence the map α restricted to L takes values in \mathcal{O}_L ; i.e. $\alpha|_L: \mathcal{O}_L(-r) \rightarrow \mathcal{O}_L$. In this case it is not necessary to choose L' since we may choose $\alpha' = \alpha|_L$. If B has dimension $n + 1$, then there is a proper subvariety B' of B such that $\Pi_2: A - \Pi_2^{-1}(B') \rightarrow B - B'$ is a finite morphism. Then choose L' disjoint from B' and L . This proves the Lemma.

THEOREM 3.4. *The zero set of a non-degenerate meromorphic vector field in $\mathcal{M}Vec_r$, $r > 0$, is $PGL(n)$ -stable.*

Proof. Let α be a non-degenerate meromorphic vector field in $\mathcal{M}Vec_r$, $r > 0$, Z has $r^{-1}[(r + 1)^{n+1} - 1]$ points. Let $L \subset \mathbb{P}^n$ be a proper linear subspace of dimension s . By Lemma 3.3, L contains at most $r^{-1}[(r + 1)^{s+1} - 1]$ singular points of Z . Now

$$\begin{aligned} & \frac{s + 1}{(n + 1)r} [(r + 1)^{n+1} - 1] - \frac{1}{r} [(r + 1)^{s+1} - 1] \\ &= \sum_{j=0}^s \left[\frac{s + 1}{n + 1} \binom{n + 1}{j + 1} - \binom{s + 1}{j + 1} \right] r^j + \frac{s + 1}{n + 1} \sum_{j=s+1}^n \binom{n + 1}{j + 1} r^j > 0 \end{aligned}$$

Hence, Mumford's numerical criterium applies.

4. Appendix

In this appendix we will prove Proposition 1.3 using elements from Representation Theory.

Let $T = \{(t_0, \dots, t_n) \in \mathbb{C}^{n+1}\}$ be the diagonal subgroup of $GL(n + 1)$. The characters of T have the form $t^m = t_0^{m_0} \cdots t_n^{m_n}$, where $m = (m_0, \dots, m_n)$ is a $(n + 1)$ -vector of integers. The character t^m is studied by looking at the vector $m' = (m'_i)$ where $m'_i = m_i + n - i$. If all the entries of m' are distinct, we say that the character is non-singular and the index of t^m is the length of the vector permutation σ of $[0, n]$ such that $m'_{\sigma(0)}, \dots, m'_{\sigma(n)}$ is strictly decreasing. Otherwise t^m is called singular.

We may regard the projective space \mathbb{P}^n as the homogeneous space $GL(n + 1)/P$, where P is the subgroup of matrices of the form

$$\left(\begin{array}{c|c} a & c \\ \hline 0 & \\ 0 & b \\ 0 & \end{array} \right)$$

which is the stabilizer of the first coordinate line l in A^{n+1} . A homogeneous bundle W on \mathbb{P}^n is determined by a representation ρ of \mathbb{P} on the fiber $W(l)$ of W over l . The bundle W is irreducible if ρ is an irreducible representation. There is a one to one correspondence between irreducible homogeneous bundles and some subset of characters of T . Let $t^{m(W)}$ be the character corresponding to such a bundle W . Then $t^{m(W)}$ is the T -eigenvalue of the unique B -invariant line in $W(l)$, where B is the group of upper diagonal matrices in $GL(n + 1)$. The characteristic property of such characters t^m is that the sequence m_0, \dots, m_n is non-increasing.

THEOREM 4.1 (Bott [1]). *Let W be an irreducible homogeneous bundle on \mathbb{P}^n , and let $m = m(W)$ and $\mathcal{O}(W)$ be the sheaf of holomorphic sections of W , then:*

- (1) *If t^m is singular, then $H^i(\mathbb{P}^n, \mathcal{O}(W)) = 0$ for all i , and*
- (2) *If t^m is non-singular, then $H^i(\mathbb{P}^n, \mathcal{O}(W)) = 0$ if $i \neq \text{index}(m)$ and $H^{\text{index}(m)}(\mathbb{P}^n, \mathcal{O}(W))$ is an irreducible representation with highest weight t^r , where $r_i = m'_{\sigma(i)} - n + i$.*

EXAMPLE. 4.2. The tangent bundle $T_{\mathbb{P}^n}$ of \mathbb{P}^n is irreducible with $m(T_{\mathbb{P}^n}) = t_0 t_n^{-1}$. In this case the index is zero, as $T_{\mathbb{P}^n}$ has global sections.

EXAMPLE. 4.3. The bundle of i -forms $\Lambda^i T_{\mathbb{P}^n}^*$ of \mathbb{P}^n is irreducible and $m(\Lambda^i T_{\mathbb{P}^n}^*) = t_0^{-i} t_1 \cdots t_i$. Here the index is i as $H^i(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n}) \neq 0$.

EXAMPLE. 4.4. $m(\mathcal{O}(r))$ corresponds to $(r, 0, \dots, 0)$.

LEMMA 4.5. Let $m = m(\Lambda^i T_{\mathbb{P}^n}^*)$ with $0 \leq i \leq n$. Denoting $m - (r, 0, \dots, 0)$ by $m(r)$, we have;

- (1) $t^{m(r)}$ is singular if $1 \leq r \leq n - i$.
- (2) The index of $t^{m(r)}$ is n if $r > n - i$.

Proof. as $m(r)' = (n - i - r, n, \dots, n - i + 1, n - i - 1, \dots, 0)$, the lemma is clear.

We are ready to prove part 1 of Proposition 1.3.

COROLLARY 4.6. If $r > 0$, then $H^j(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n}(-r)) = 0$ for $j < n$.

Proof. Recalling that L_{-r} denotes the line bundle on \mathbb{P}^n with Chern class $-r$, we have that $m(\Lambda^i T_{\mathbb{P}^n}^* \otimes l_{-r}) = m(r)$. So the corollary follows from Lemma 4.5.

LEMMA 4.7. For $0 \leq i \leq n$ and $r \geq 0$, let W_r^i be the irreducible homogeneous bundle on \mathbb{P}^n with associated character

$$\chi = \chi_r(W_r^i) = t_r^{m(W_r^i)} = t_0^{1-i-r} t_1 \cdots t_i t_n^{-1}, \text{ then}$$

- (1) For $0 < i < n$, we have:
 - (a) χ is singular for $r = 0, 2 \leq r \leq n - i$, or $r = n - i + 2$.
 - (b) The index of χ is i for $r = 1$; $n - 1$ for $r = n - i + 1$ and n for $r > n - i + 2$.
- (2) For $i = 0$, we have:
 - (a) χ is singular for $2 \leq r \leq n$ or $r = n + 2$.
 - (b) The index of χ is 0 for $r = 0, 1$; $n - 1$ for $r = n + 1$ and n for $r > n + 2$.
- (3) For $i = n$, we have:
 - (a) χ is singular for $r = 1$.
 - (b) The index of χ is $n - 1$ for $r = 0$ and n for $r > 1$.

Proof. $m(W_r^i)' = (1 - i - r + n, n, n - 1, \dots, n - i + 1, n - i - 1, \dots, 1, -1)$, so the Lemma is clear.

COROLLARY 4.8. For $(0 \leq i \leq n - 1$ and $r \geq 0)$ or $(i = n$ and $r > 0)$ we have $H^j(\mathbb{P}^n, \mathcal{O}(W_r^i)) = 0$ for all $j < i$.

Proof. For $i = 0$ the statement is vacuous and for $0 < i \leq n$, the corollary follows from Lemma 4.7 and Theorem 4.1.

We now prove the second part of Proposition 1.3:

PROPOSITION 4.9. *Let $\mathcal{L}_i(r) = (\Lambda^i \Omega_{\mathbb{P}^n}) \otimes \Theta((1-i)r)$, with $r > 0$ and $0 \leq i \leq n$, then $H^j(\mathbb{P}^n, \mathcal{L}_i(r)) = 0$ if $j < i$ and $n \geq 2$; except for $H^0(\mathbb{P}^n, \mathcal{L}_1(r))$, which is one dimensional.*

Proof. The statement is vacuous for $i = 0$. For $i = n$ we have $\mathcal{L}_n(r) = \Theta(r(1-n) - n - 1) = \mathcal{O}(W_{r(n-1)+n+1}^0)$. Hence the proposition follows from Lemma 4.7 since $n \geq 2$ and $r > 0$.

For $0 < i < n - 1$, we have by Littlewood–Richardson [6] that

$$\mathcal{L}_i(r) = \mathcal{O}(W_{(i-1)r}^i) \oplus \Lambda^{i-1} \Omega((1-i)r)$$

where $(i-1)r \geq 0$. The vanishing of the j^{th} -cohomology groups of the first summand, $j < i$, follows from Corollary 4.8 and the vanishing of the second summand follows from Corollary 4.6, except for $i = 1$ and $j = 0$, which is one dimensional.

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