Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 64 (1989)

Artikel: Stability of meromorphic vector fields in projective spaces.
Autor: Gómez-Mont, Xavier / Kempf, George
DOI: https://doi.org/10.5169/seals-48957

Comment. Math. Helvetici 64 (1989) 462–473 0010–2571/89/030462–12$01.50 + 0.20/0

Stability of meromorphic vector fields in projective spaces[1]

XAVIER GÓMEZ-MONT and GEORGE KEMPF

A meromorphic vector field of degree r on the complex projective space $\mathbb{P}^n$ is a bundle map $\alpha: L_{-r} \to T\mathbb{P}^n$ from the line bundle of Chern class $-r$ to the tangent bundle on $\mathbb{P}^n$, defined up to multiplication by a non-zero scalar. The group PGL(n) of automorphisms of $\mathbb{P}^n$ acts on the space $\mathcal{MVec}_r = \operatorname{Proj} H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n} \otimes L_r)$ of meromorphic vector fields of degree r. We will say that a meromorphic vector field is non-degenerate if all its zeroes have multiplicity one.

In this paper we analyse the stability properties of the action of PGL(n) on $\mathcal{MVec}_r$ in the sense of Mumford ([17]). We prove:

THEOREM. *Let α be a non-degenerate meromorphic vector field of degree $r > 0$, then:*

(1) *α is completely determined by its zero set.*
(2) *α is* PGL(n)*-stable.*
(3) *The zero set of α is* PGL(n)*-stable*

A finite collection of points in $\mathbb{P}^n$ have to be in special position to be the zero set of a meromorphic vector field, but this position is of general type in Mumford's sense. The main ingredients in the proof are Bott's computations of the cohomology of homogeneous bundles ([1]), Mumford's numerical criterium of stability ([7]) and the Koszul resolution associated to the zero set Z of α:

$$0 \to \Lambda^n \Omega_{\mathbb{P}^n}(-nr) \to \cdots \to \Omega_{\mathbb{P}^n}(-r) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Z \to 0.$$

1. Meromorphic vector fields in $\mathbb{P}^n$

Let $\mathbb{P}^n$ be the projective space over the complex numbers $\mathbb{C}$, and let $\mathcal{O}_{\mathbb{P}^n}$, $\Theta_{\mathbb{P}^n}$, $\Omega_{\mathbb{P}^n}$ and $\mathcal{L}$ be the structure, tangent, cotangent and hyperplane sheaves on $\mathbb{P}^n$,

---

[1] Supported by CONACYT at UNAM.

respectively. If $\mathscr{E}$ is an $\mathcal{O}_{\mathbb{P}^n}$-sheaf, we will use the notation $\mathscr{E}(r)$ for $\mathscr{E}\otimes\mathscr{L}^{\otimes r}$ if $r\geq 0$ and $\mathscr{E}\otimes(\mathscr{L}^*)^{\otimes|r|}$ if $r\leq 0$. The space $\mathscr{MV}ec_r$ of meromorphic vector fields of degree $r\geq -1$ is the projective space of lines through 0 in $H^0(P^n, \Theta_{\mathbb{P}^n}(r)) = H^0(P^n, \mathscr{H}om(\mathscr{L}_{-r}, \Theta_{\mathbb{P}^n}))$. A meromorphic vector field on $\mathbb{P}^n$ is a non identically zero $\mathcal{O}_{\mathbb{P}^n}$-morphism $\alpha:\mathscr{L}_{-r}\to\Theta_{\mathbb{P}^n}$, defined up to multiplication by non-zero scalar (see [2], [4]). The twisted Euler sequence ([5] p. 176)

$$0\to\mathcal{O}_{\mathbb{P}^n}(r)\to\mathcal{O}_{\mathbb{P}^n}(r+1)^{\oplus(n+1)}\to\Theta_{\mathbb{P}^n}(r)\to 0 \tag{1.1}$$

gives us a way to represent meromorphic vector fields on $\mathbb{P}^n$ as homogeneous polynomial vector fields of degree $r+1$ in $\mathbb{C}^{n+1}$. The group $\mathrm{PGL}(n)$ of automorphisms of $\mathbb{P}^n$ acts on $\mathscr{MV}ec_r$

$$\mathrm{PGL}(n)\times\mathscr{MV}ec_r\to\mathscr{MV}ec_r \qquad (g,\alpha)\mapsto(Dg\cdot\alpha)\circ g^{-1} \tag{1.2}$$

The universal family of meromorphic vector fields of degree r is the tautological morphism on $\mathscr{MV}ec_r\times P^n$

$$A:\Pi_1^*\mathscr{H}^*\otimes\Pi_2^*\mathscr{L}_{-r}\to\Pi_2^*\Theta_{\mathbb{P}^n} \tag{1.3}$$

where Π_1 and Π_2 are the projections to the factors and $\mathscr{H}$ is the sheaf of hyperplanes in $\mathscr{MV}ec_r$. Let Z be the subvariety of $\mathscr{MV}ec_r\times\mathbb{P}^n$ defined by $A=0$.

LEMMA 1.1. *Z is a smooth subvariety of $\mathscr{MV}ec_r\times\mathbb{P}^n$ of codimension n. The restriction of Π_2 to Z is a projective space bundle over $\mathbb{P}^n$ and $\Pi_1:Z\to\mathscr{MV}ec_r$ is generically finite.*

Proof. Let $\mathscr{I}$ be the ideal defining the diagonal in $\mathbb{P}^n\times\mathbb{P}^n$, where ρ_1 and ρ_2 are the two projections. The subsheaf

$$\rho_{1*}[\rho_2^*\Theta_{\mathbb{P}^n}(r)\otimes\mathscr{I}]\hookrightarrow\rho_{1*}\rho_2^*\Theta_{\mathbb{P}^n}(r)=H^0(\mathbb{P}^n,\Theta_{\mathbb{P}^n}(r))\otimes_{\mathbb{C}}\mathcal{O}_{\mathbb{P}^n} \tag{1.4}$$

has as fiber over $p\in\mathbb{P}^n$ the sections of $h^0(\mathbb{P}^n,\Theta_{\mathbb{P}^n}(r))$ that vanish on p and it is locally free, since $\mathrm{PGL}(n)$ acts transitively on $\mathbb{P}^n$. The projective bundle over $\mathbb{P}^n$ associated to (1.4) shows that $Z\hookrightarrow\mathscr{MV}ec_r\times\mathbb{P}^n$ is a projective sub-bundle of Π_2.

Using the representation (1.1), it is easy to see that the line field given by

$$\sum_{i=0}^{n} z_i^{r+1}\frac{\partial}{\partial z_i} \tag{1.5}$$

has isolated zeroes, hence $\Pi_1: Z \to \mathcal{MV}ec_r$ is generically finite. One checks that (1.5) has

$$c_n(\Theta_{\mathbb{P}^n}(r)) = \sum_{j=0}^{n} \binom{n+1}{j} r^{n-j} = \frac{(r+1)^{n+1}-1}{r} \tag{1.6}$$

zeroes of multiplicity one, where c_n is the n-th Chern class.

We will now exhibit some PGL(n) invariant divisors in $\mathcal{MV}ec_r$.

LEMMA 1.2. *Let Q be a non-zero* GL(n) *homogeneous invariant polynomial defined on the space $M_{n\times n}$ of n by n matrices. Then for $r>0$, the space*

$$Z_Q = \{\alpha \in \mathcal{MV}ec_r / \exists p \in \mathbb{P}^n \text{ with } \alpha(p)=0,\ Q(D\alpha(p))=0\}$$

is a PGL(n)*-invariant divisor in $\mathcal{MV}ec_r$; where $D\alpha(p)$: $T_p\mathbb{P}^n \to T_p\mathbb{P}^n$ is the linear part of α at p.*

Proof. Let $\tilde{Z}_Q = \{(\alpha, p) \in Z / Q(D\alpha(p)) = 0\}$. The projection map $\Pi_2: \tilde{Z}_Q \to \mathbb{P}^n$ has a structure of a fibre bundle, since PGL(n) acts transitively on $\mathbb{P}^n$, and has as fiber $\{\alpha \in \mathcal{MV}ec_r / \alpha(p_0)=0,\ Q(D\alpha(p_0))=0\}$ which has codimension 1 in $\{\alpha \in \mathcal{MV}ec_r / \alpha(p_0)=0\}$, since the derivative at p_0

$$D_{p_0}: \{\alpha \in H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r)) / \alpha(P_0) = 0\} \to M_{n\times n}$$

is surjective. This shows that $\tilde{Z}_Q$ has codimension 1 in Z.

To finish the proof of the lemma, we will show that $\Pi_1: \tilde{Z}_Q \to \mathcal{MV}ec_r$ is generically finite. To see this, it suffices to show that the codimension of $Z_1 = \{(\alpha, p) \in Z / \dim_p \{\alpha = 0\} > 0\}$ has codimension bigger than 1 in Z. We have

$$Z_1 \subset \tilde{Z}_{\det} \subset Z \tag{1.7}$$

where $\tilde{Z}_{\det}$ is obtained by setting Q = determinant above. $\tilde{Z}_{\det}$ is irreducible of codimension 1 in Z. It is easy to see that the first inclusion in (1.7) is proper (i.e. let $\mathbb{A}^n \subset \mathbb{P}^n$ be an affine chart, we may find a homogeneous field $\sum_{i=1}^{n} F_i(\partial/\partial z_i)$ in $\mathbb{A}^n$ with 0 as only singular point in $\mathbb{A}^n$, with high multiplicity, and only singular points of multiplicity 1 on $\mathbb{P}^n - \mathbb{A}^n$), and hence the codimension of Z_1 in Z is bigger than 1.

We will say that a zero p of a meromorphic vector field $\alpha: \mathcal{L}_{-r} \to \Theta_{\mathbb{P}^n}$ is non-degenerate if $\det(D\alpha(p)) \neq 0$. A meromorphic vector field α with only non-degenerate zeroes will be said to be non-degenerate.

Our main technical tool will be the Koszul resolution $K_* \to \mathcal{O}_Z \to 0$ associated to the zero set Z of a meromorphic vector field $\alpha: \mathcal{L}_{-r} \to \Theta_{\mathbb{P}^n}$ vanishing on a finite number of points:

$$0 \to \Lambda^n \Omega_{\mathbb{P}^n}(-nr) \to \cdots \to \Lambda^2 \Omega_{\mathbb{P}^n}(-2r) \to \Omega_{\mathbb{P}^n}(-r) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_Z \to 0 \tag{1.8}$$

We will also use the computations of some cohomology groups of homogeneous bundles in $\mathbb{P}^n$ which may be deduced from Bott's Theorem ([1]). We will sketch a proof of the following proposition in an appendix:

PROPOSITION 1.3. (1) *If $j < n$ and $r > 0$, then*

$$H^j(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n}(-r)) = 0$$

(2) *Let $\mathcal{L}_i(r) = (\Lambda^i \Omega_{\mathbb{P}^n}) \otimes \Theta_{\mathbb{P}^n}((1-i)r)$, where $r > 0$, $0 \le i \le n$ and $n \ge 2$; then $H^j(\mathbb{P}^n, \mathcal{L}_i(r)) = 0$ if $j < i$, except for $H^0(\mathbb{P}^n, \mathcal{L}_1(r))$, which is one dimensional.*

2. Stable meromorphic vector fields

In this section we will show that if $\alpha \in \mathcal{MV}ec_r$, $r > 0$ has only non-degenerate zeroes, then α is PGL(n)-stable in the sense of Mumford (see [7]).

PROPOSITION 2.1. *Let $\alpha \in \mathcal{MV}ec_r$ be a non-degenerate meromorphic vector field with $r > 0$; then the zeroes of α span $\mathbb{P}^n$.*

Proof. If Z denotes the zero set of α, we have to show that Z is not contained in any hyperplane of $\mathbb{P}^n$, or equivalently, that the restriction map $\rho: \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(1)) \to \Gamma(Z, \mathcal{O}_{\mathbb{P}^n}(1)|_Z)$ is injective. Tensoring (1.8) with $\mathcal{O}_{\mathbb{P}^n}(1)$, an easy diagram chase shows that it suffices to prove that $H^j(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n}(-ir+1)) = 0$ for $j < i \le n$, which follows from part 1 of Proposition 1.3.

Remark. If α has isolated singularities, the proof of the Proposition still holds, where the span is the span of the zeroes of α with multiplicities.

PROPOSITION 2.2. *Let $\alpha \in \Gamma(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r))$ be a non-degenerate meromorphic vector field and $r > 0$. If α is an eigenvector for the action of a one-parameter subgroup $\lambda: G_m \to GL(n+1)$, then λ factors through the center of $GL(n+1)$.*

We will first prove:

LEMMA 2.3. *Let α' be a section of $\Theta_{\mathbb{A}^n}$ which is an eigenvector for the action of a one-parameter subgroup $\lambda': G_m \to GL(n)$. Then its eigenvalue is* 1 *if α' has an isolated zero at p_0 with non-nilpotent linear part $D\alpha'(p_0)$.*

Proof. p_0 is a fixed point of λ' and let $\lambda'(t)_*\alpha' = t^m\alpha'$. Then the linear part of $\lambda'(t)\alpha'$ is on the one hand $t^m D\alpha'(p_0)$, and on the other

$$D_{p_0}\lambda'(t) \circ D\alpha'(p_0) \circ D_{p_0}\lambda'(t)^{-1}$$

For the first expression, the eigenvalues are multiplied by t^m, and for the second one they remain the same, hence $m = 0$.

Proof of proposition. Choose coordinates $(x_0: \ldots : x_n)$ of $\mathbb{P}^n$ so that $\lambda(t) = \operatorname{diag}(t^{m_0}, \ldots, t^{m_n})$. Then $\alpha' = x_i^{-r}\alpha$ is a section of $\Theta_{D(x_i)}$ which is an eigenvector for the action of G_m on the affine space $D(x_i) = \{x_i \neq 0\}$. By Proposition 2.1, α' has a zero in $D(x_i)$, and by Lemma 2.3 we obtain (eigenvalue α) = (eigenvalue $x_i)^r$, and hence $m_i = m_j$.

PROPOSITION 2.4. *Let $\alpha \in \mathcal{MV}ec_r$ be a non-degenerate meromorphic vector field and $r > 0$; then the stabilizer S of α in* PGL(n) *is finite.*

Proof. We will first show that S does not contain a connected unipotent subgroup. Since such groups are extensions of $G_a = \mathbb{C}$, it will suffice to show that if $\mu: \mathbb{G}_a \to \mathrm{PGL}(n)$ stabilizes α, then it is the identity. The fixed points of μ in $\mathbb{P}^n$ form a linear subspace, and since all the zeroes of α are fixed by μ, their span is contained in this linear subspace. Hence by Proposition 2.1 we conclude that μ is the identity. Hence S is a reductive group. By Proposition 2.2 a maximal torus of S is trivial, so S is a finite group.

THEOREM 2.5. *The set of* PGL(n)*-stable meromorphic vector fields in $\mathcal{MV}ec_r$, $r > 0$, in $\mathbb{P}^n$ contains the open set formed by non-degenerate meromorphic vector fields.*

Proof. $Z_{\det}$ is a PGL(n)-invariant divisor in $\mathcal{MV}ec_r$, and its complement is the set of non-degenerate meromorphic vector fields. By Proposition 2.4 the stabilizers are finite, hence they are all PGL(n)-stable (see [7]).

Remark. Note that by Lemma 1.2, those meromorphic vector fields such that all its singular points have non-nilpotent linear parts are semistable.

We will now show that a non-degenerate meromorphic vector field $\alpha \in \mathcal{M}\mathcal{V}ec_r$, $r > 0$, is determined by its singular points:

THEOREM 2.6. *Let $\alpha \in \mathcal{M}\mathcal{V}ec_r$, $r > 0$, be a non-degenerate meromorphic vector field with zero set Z, and let $\alpha' \in \mathcal{M}\mathcal{V}ec_r$ be another meromorphic vector field vanishing on Z, then $\alpha' = k\alpha$ with $k \in \mathbb{C}$.*

Proof. Tensor the sequence (1.8) with $\Theta_{\mathbb{P}^n}(r)$. To prove the theorem it suffices to show that the map of global sections

$$H^0(\mathbb{P}^n, \Theta_{\mathbb{P}^n}(r)) \to H^0(Z, \Theta_{\mathbb{P}^n}(r)|_Z)$$

has one dimensional kernel. A diagram chase on the above sequence $K_* \otimes \Theta_{\mathbb{P}^n}(r)$ shows that it is sufficient to prove that for $1 < i \leq n$ and $j < i$ we have $H^j(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n} \otimes \Theta_{\mathbb{P}^n}((1-i)r) = 0$ and $H^0(\mathbb{P}^n, \Omega_{\mathbb{P}^n} \otimes \Theta_{\mathbb{P}^n}) = \mathbb{C} \cdot Id$. This follows from part 2 of Proposition 1.3.

Remark. The above argument generalizes to: Let M be a projective manifold such that the only global endomorphisms of the tangent sheaf are multiples of the identity; i.e. $H^0(M, \Omega_M \otimes \Theta_M) = \mathbb{C} \cdot \mathrm{Id}$. If $\mathcal{L}$ is an ample sheaf on M denote $\mathrm{Proj}\, H^0(M, \Theta_M \otimes \mathcal{L}^r)$ by $\mathcal{M}\mathcal{V}ec_r(M, \mathcal{L})$. Then for r sufficiently large, the zero set of a non-degenerate meromorphic vector field $\alpha \in \mathcal{M}\mathcal{V}ec_r(M, \mathcal{L})$ determines α uniquely. If furthermore $H^0(M, \Omega_M \otimes \Theta_M \otimes \mathcal{M}) = 0$ for any line bundle $\mathcal{M} \neq \mathcal{O}_M$ with Chern class zero, then for r sufficiently large the zero set of a non-degenerate meromorphic vector field $\alpha \in \mathcal{M}\mathcal{V}ec_r(M, \mathcal{L})$ determines α uniquely in $\bigcup_{\mathcal{L}'} \mathcal{M}\mathcal{V}ec_r(M, \mathcal{L}')$, where $\mathcal{L}'$ are line bundles on M with the same Chern class as $\mathcal{L}$. For the proof, replace in the above argument Bott's computations by the Kodaira–Nakano vanishing theorem.

3. Stability of the zeroes of a meromorphic vector field

In this section we will show that the zero set of a non-degenerate meromorphic vector field in $\mathbb{P}^n$ is PGL(n)-stable. We will rely on Mumford's numerical criterium ([7], p. 76):

LEMMA 3.1. *The set of points $(p_1, \ldots, p_m) \in \mathbb{P}^n \times \cdots \times \mathbb{P}^n = (\mathbb{P}^n)^m$ such that for every proper linear subspace $L \subset \mathbb{P}^n$*

$$(\textit{number of } p_i \textit{ in } L) < \left(\frac{\dim L + 1}{n+1}\right) m \tag{3.1}$$

is PGL(n)*-stable.*

We begin with the following generic result:

THEOREM 3.2. *Let $\mathscr{E}$ be a locally free sheaf of rank n on $\mathbb{P}^n$, then there exists a k_0 such that for $k > k_0$ there is a Zarizki dense open subset in* $\operatorname{Proj} H^0(\mathbb{P}^n, \mathscr{E}(k))$ *formed of sections σ such that the zero set $\{\sigma = 0\}$ is* $\mathrm{PGL}(n)$*-stable.*

Proof. Let k_0 be such that for $k > k_0$ we have:

(1) $\mathscr{E}(k)$ has global sections with only zeroes of multiplicity 1.

(2) If Grass_s is the Grassmanian of s-dimensional subspaces of $\mathbb{P}^n$, $0 < s \leq n$, and V is the tautological subvariety of $\mathrm{Grass}_s \times \mathbb{P}^n$, then for $q \neq s$ and $j = 1, \ldots, n$ we have

$$R^q \Pi_{1*}(\Pi_2^*(\Lambda^j \mathscr{E}(-jk)) \otimes \mathcal{O}_V) = 0 \tag{3.2}$$

This is possible by Bertini's theorem and a parameter version of the Kodaira–Nakano vanishing theorem (see [4] Theorem 6.7 and [5] p. 252).

Let $k > k_0$, $\sigma \in H^0(\mathbb{P}^n, \mathscr{E}(k))$ with only zeroes of multiplicity 1 and $K_* \to \mathcal{O}_Z \to 0$ be the Koszul resolution of the zero set Z of σ, where $K_* \to 0$ is the complex of sheaves

$$0 \to \Lambda^n \mathscr{E}(-nk) \to \cdots \to \mathscr{E}(-k) \to \mathcal{O}_{\mathbb{P}^n} \to 0 \tag{3.3}$$

with grading $K_j = \Lambda^{n-j} \mathscr{E}(-(n-j)k)$. Let $L \subset \mathbb{P}^n$ be an s-dimensional subspace, and denote by $\mathbb{H}^j$ the hypercohomolgy groups of the complex of sheaves $K_* \otimes \mathcal{O}_L \to 0$ (see [3]). The cohomology sheaves $\mathscr{H}^q$ of (3.3) have support on $Z \cap L$ and $\mathscr{H}^n = \mathcal{O}_Z \otimes \mathcal{O}_L$. Hence the dimension of $\mathbb{H}^n$ is the cardinality of $Z \cap L$.

For the other spectral sequence of $K_* \otimes \mathcal{O}_L \to 0$ we have ${}_1E^{p,q} = H^q(L, \Lambda^p \mathscr{E}(-(n-p)k)|_L)$, and by hypothesis on k then ${}_1E^{p,q} = 0$ except if $q = s$ or ${}_1E^{n,0} = \mathbb{C}$. Directly from the spectral sequence, we obtain

$$\begin{aligned} \dim \mathbb{H}^n &\leq \dim {}_2E^{n-s,s} + 1 \leq \dim {}_1E^{n-s,s} + 1 \\ &= \dim H^s(L, \Lambda^{n-s} \mathscr{E}(-sk)|_L) + 1 \end{aligned}$$

By the vanishing hypothesis (3.2), we have that

$$\#(Z \cap L) = \dim \mathbb{H}^n \leq |\chi(L, \Lambda^{n-s} \mathscr{E}(-sk)|_L)| + 1 = \sum_{j=0}^{s} C_j k^j$$

where χ is the holomorphic Euler–Poincaré characteristic. By setting $L = P^n$ we obtain

$$\#(Z) \leq \sum_{j=0}^{n} D_j k^j \tag{3.1}$$

Hence, for probably larger k, Mumford's numerical criterium holds since for large k we will have

$$\sum_{j=0}^{s} C_j k^j < \frac{s+1}{n+1} \sum_{j=0}^{n} D_j k^j$$

Hence by Lemma 3.1 Z is PGL(n)-stable.

Remark. Hypercohomolgy of the complex of sheaves $K_* \to 0$ in (1.8) gives a way to compute the number (1.6) of zeroes of a non-degenerate meromorphic vector field $\alpha \in \mathcal{MV}ec_r$. By one of the spectral sequences, we see that all hypercohmology vanishes except $\mathbb{H}^n = H^0(Z, \mathcal{O}_Z)$. The ${}_1E$ terms of the other spectral sequence vanish except ${}_1E^{n,0} = \mathbb{C}$ and ${}_1E^{j,n} = H^n(P^n, \Lambda^{n-j}\Omega_{\mathbb{P}^n}((j-n)r)$. The ${}_1E$-terms form an exact sequence

$$0 \to {}_2E^{j,n} \to H^n(\mathbb{P}^n, \Lambda^n\Omega_{\mathbb{P}^n}(-nr)) \to \cdots \to H^n(\mathbb{P}^n, \Omega_{\mathbb{P}^n}(-r)) \to 0$$

and $\mathbb{H}^n = \mathbb{C} \oplus {}_2E^{j,n}$. Hence α vanishes on

$$\sum_{j=0}^{n-1} (-1)^j \chi(\mathbb{P}^n, \Lambda^{n-j}\Omega_{\mathbb{P}^n}((j-n)r)) + 1$$

which is computable by the Riemann–Roch theorem.

Theorem 3.2 shows that for large r, the zero set of non-degenerate meromorphic vector field in $\mathcal{MV}ec_r$ are PGL(n)-stable. We will give another argument for every $r > 0$, using Mumford's numerical criterium (3.1). We will begin with:

LEMMA 3.3. *Let $\alpha \in \mathcal{MV}ec_r$, $r > 0$, be a non-degenerate meromorphic vector field in $\mathbb{P}^n$, then for every s-dimensional linear subspace $L \subset \mathbb{P}^n$ there are at most $r^{-1}[(r-1)^{s+2} - 1]$ points of $Z = \{\alpha = 0\}$ in L.*

Proof. $L \hookrightarrow \mathbb{P}^n$ induces an exact sequence of sheaves on L

$$0 \to \Theta_L \to \Theta_{\mathbb{P}^n}|_L \to \mathcal{N}(L, \mathbb{P}^n) \to 0 \tag{3.4}$$

where $\mathcal{N}(L, \mathbb{P}^n)$ is the normal sheaf to L in $\mathbb{P}^n$. If L' is a linear sub-space of $\mathbb{P}^n$ of dimension $n-s-1$ and disjoint from L', then the linear projection of $\mathbb{P}^n - L'$ from L' to L induces a splitting of (3.4). Using this splitting, we may obtain from $\alpha: \mathcal{O}_{\mathbb{P}^n}(-r) \to \Theta_{\mathbb{P}^n}$ by restricting to L and then projection to Θ_L a morphism $\alpha': \mathcal{O}_L(-r) \to \Theta_L$. We will show that we may choose L' in such a way that α' has isolated zeroes. By (1.6) this number is bounded by $r^{-1}[(r+1)^{s+1}-1]$. Since all the zeroes of α are zeroes of α', this is enough to prove the lemma.

To see how one chooses L', note that if $p \in L - Z$ then $\alpha'(p) = 0$ if and only if the line l_p passing through p with tangent direction $\alpha(p) \subset T_p\mathbb{P}^n$ intersects L'. Hence, we have to show that we may choose L' in such a way that only a finite number of lines l_p with $p \in L - Z$ intersect L'. Let $A \subset L \times \mathbb{P}^n$ be the closure of

$$\{(p, q) \in (L-Z) \times \mathbb{P}^n \mid q \in l_p\}$$

A is an irreducible variety of dimension $n+1$. Let B be the projection of A to the second factor: $B = \Pi_2(A)$. It is an irreducible variety which contains L; hence B has dimension n or $n+1$.

If B has dimension n, then $B = L$ and we have that $l_p \subset L$ for $p \in L - Z$. Hence the map α restricted to L takes values in Θ_L; i.e. $\alpha|_L: \mathcal{O}_L(-r) \to \Theta_L$. In this case it is not necessary to choose L' since we may choose $\alpha' = \alpha|_L$. If B has dimension $n+1$, then there is a proper subvariety B' of B such that $\Pi_2: A - \Pi_2^{-1}(B') \to B - B'$ is a finite morphism. Then choose L' disjoint from B' and L. This proves the Lemma.

THEOREM 3.4. *The zero set of a non-degenerate meromorphic vector field in* $\mathcal{MV}ec_r$, $r > 0$, *is* PGL(n)*-stable.*

Proof. Let α be a non-degenerate meromorphic vector field in $\mathcal{MV}ec_r$, $r > 0$, Z has $r^{-1}[(r+1)^{n+1}-1]$ points. Let $L \subset \mathbb{P}^n$ be a proper linear subspace of dimension s. By Lemma 3.3, L contains at most $r^{-1}[(r+1)^{s+1}-1]$ singular points of Z. Now

$$\frac{s+1}{(n+1)r}[(r+1)^{n+1}-1] - \frac{1}{r}[(r+1)^{s+1}-1]$$

$$= \sum_{j=0}^{s}\left[\frac{s+1}{n+1}\binom{n+1}{j+1} - \binom{s+1}{j+1}\right]r^j + \frac{s+1}{n+1}\sum_{j=s+1}^{n}\binom{n+1}{j+1}r^j > 0$$

Hence, Mumford's numerical criterium applies.

4. Appendix

In this appendix we will prove Proposition 1.3 using elements from Representation Theory.

Let $T=\{(t_0,\dots,t_n)\in\mathbb{C}^{n+1}\}$ be the diagonal subgroup of $\mathrm{GL}(n+1)$. The characters of T have the form $t^m=t_0^{m_0}\cdots t_n^{m_n}$, where $m=(m_0,\dots,m_n)$ is a $(n+1)$-vector of integers. The character t^m is studied by looking at the vector $m'=(m_i')$ where $m_i'=m_i+n-i$. If all the entries of m' are distinct, we say that the character is non-singular and the index of t^m is the length of the vector permutation σ of $[0,n]$ such that $m'_{\sigma(0)},\dots,m'_{\sigma(n)}$ is strictly decreasing. Otherwise t^m is called singular.

We may regard the projective space $\mathbb{P}^n$ as the homogeneous space $\mathrm{GL}(n+1)/P$, where P is the subgroup of matrices of the form

$$\left(\begin{array}{c|c} a & c \\ \hline 0 & \\ 0 & b \\ 0 & \end{array}\right)$$

which is the stabilizer of the first coordinate line l in A^{n+1}. A homogeneous bundle W on $\mathbb{P}^n$ is determined by a representation ρ of $\mathbb{P}$ on the fiber $W(l)$ of W over l. The bundle W is irreducible if ρ is an irreducible representation. There is a one to one correspondence between irreducible homogeneous bundles and some subset of characters of T. Let $t^{m(W)}$ be the character corresponding to such a bundle W. Then $t^{m(W)}$ is the T-eigenvalue of the unique B-invariant line in $W(l)$, where B is the group of upper diagonal matrices in $\mathrm{GL}(n+1)$. The characteristic property of such characters t^m is that the sequence $m_0,\dots,m_n$ is non-increasing.

THEOREM 4.1 (Bott [1]). *Let W be an irreducible homogeneous bundle on $\mathbb{P}^n$, and let $m=m(W)$ and $\mathcal{O}(W)$ be the sheaf of holomorphic sections of W, then:*

(1) *If t^m is singular, then $H^i(\mathbb{P}^n,\mathcal{O}(W))=0$ for all i, and*

(2) *If t^m is non-singular, then $H^i(\mathbb{P}^n,\mathcal{O}(W))=0$ if $i\neq\mathrm{index}(m)$ and $H^{\mathrm{index}(m)}(\mathbb{P}^n,\mathcal{O}(W))$ is an irreducible representation with highest weight t^r, where $r_i=m'_{\sigma(i)}-n+i$.*

EXAMPLE. 4.2. The tangent bundle $T_{\mathbb{P}^n}$ of $\mathbb{P}^n$ is irreducible with $m(T_{\mathbb{P}^n})=t_0t_n^{-1}$. In this case the index is zero, as $T_{\mathbb{P}^n}$ has global sections.

EXAMPLE. 4.3. The bundle of i-forms $\Lambda^i T^*_{\mathbb{P}^n}$ of $\mathbb{P}^n$ is irreducible and $m(\Lambda^i T^*_{\mathbb{P}^n}) = t_0^{-i} t_1 \cdots t_i$. Here the index is i as $H^i(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n}) \neq 0$.

EXAMPLE. 4.4. $m(\mathcal{O}(r))$ corresponds to $(r, 0, \cdots, 0)$.

LEMMA 4.5. *Let* $m = m(\Lambda^i T^*_{\mathbb{P}^n})$ *with* $0 \leq i \leq n$. *Denoting* $m - (r, 0, \ldots, 0)$ *by* $m(r)$, *we have;*

(1) $t^{m(r)}$ *is singular if* $1 \leq r \leq n - i$.

(2) *The index of* $t^{m(r)}$ *is* n *if* $r > n - i$.

Proof. as $m(r)' = (n - i - r, n, \ldots, n - i + 1, n - i - 1, \ldots, 0)$, the lemma is clear.

We are ready to prove part 1 of Proposition 1.3.

COROLLARY 4.6. *If* $r > 0$, *then* $H^j(\mathbb{P}^n, \Lambda^i \Omega_{\mathbb{P}^n}(-r)) = 0$ *for* $j < n$.

Proof. Recalling that L_{-r} denotes the line bundle on $\mathbb{P}^n$ with Chern class $-r$, we have that $m(\Lambda^i T^*_{\mathbb{P}^n} \otimes l_{-r}) = m(r)$. So the corollary follows from Lemma 4.5.

LEMMA 4.7. *For* $0 \leq i \leq n$ *and* $r \geq 0$, *let* W^i_r *be the irreducible homogeneous bundle on* $\mathbb{P}^n$ *with associated character*

$$\chi = \chi_r(W^i_r) = t_r^{m(W^i_r)} = t_0^{1-i-r} t_1 \cdots t_i t_n^{-1}, \text{ then}$$

(1) *For* $0 < i < n$, *we have:*

(a) χ *is singular for* $r = 0$, $2 \leq r \leq n - i$, *or* $r = n - i + 2$.

(b) *The index of* χ *is* i *for* $r = 1$; $n - 1$ *for* $r = n - i + 1$ *and* n *for* $r > n - i + 2$.

(2) *For* $i = 0$, *we have:*

(a) χ *is singular for* $2 \leq r \leq n$ *or* $r = n + 2$.

(b) *The index of* χ *is* 0 *for* $r = 0, 1$; $n - 1$ *for* $r = n + 1$ *and* n *for* $r > n + 2$.

(3) *For* $i = n$, *we have:*

(a) χ *is singular for* $r = 1$.

(b) *The index of* χ *is* $n - 1$ *for* $r = 0$ *and* n *for* $r > 1$.

Proof. $m(W^i_r)' = (1 - i - r + n, n, n - 1, \ldots, n - i + 1, n - i - 1, \ldots, 1, -1)$, so the Lemma is clear.

COROLLARY 4.8. *For* $(0 \leq i \leq n - 1$ *and* $r \geq 0)$ *or* $(i = n$ *and* $r > 0)$ *we have* $H^j(\mathbb{P}^n, \mathcal{O}(W^i_r)) = 0$ *for all* $j < i$.

Proof. For $i=0$ the statement is vacuous and for $0<i\leq n$, the corollary follows from Lemma 4.7 and Theorem 4.1.

We now prove the second part of Proposition 1.3:

PROPOSITION 4.9. *Let* $\mathscr{L}_i(r)=(\Lambda^i\Omega_{\mathbb{P}^n})\otimes\Theta((1-i)r)$, *with* $r>0$ *and* $0\leq i\leq n$, *then* $H^j(\mathbb{P}^n, \mathscr{L}_i(r))=0$ *if* $j<i$ *and* $n\geq 2$; *except for* $H^0(\mathbb{P}^n, \mathscr{L}_1(r))$, *which is one dimensional.*

Proof. The statement is vacuous for $i=0$. For $i=n$ we have $\mathscr{L}_n(r)=\Theta(r(1-n)-n-1)=\mathcal{O}(W^0_{r(n-1)+n+1})$. Hence the proposition follows from Lemma 4.7 since $n\geq 2$ and $r>0$.

For $0<i<n-1$, we have by Littlewood–Richardson [6] that

$$\mathscr{L}_i(r)=\mathcal{O}(W^i_{(i-1)r})\oplus\Lambda^{i-1}\Omega((1-i)r)$$

where $(i-1)r\geq 0$. The vanishing of the j^{th}-cohomology groups of the first summand, $j<i$, follows from Corollary 4.8 and the vanishing of the second summand follows from Corollary 4.6, except for $i=1$ and $j=0$, which is one dimensional.

REFERENCES

[1] R. Bott. *Homogeneous vector bundles,* Ann. of Math. *66*, 203–248 (1957).

[2] P. Deligne. *Equations Différentielles à Points Singulers Reguliers,* Springer Lecture Notes 163, 1970.

[3] R. Godement. *Théorie des Faisceaux, Act. Scient. et Ind. 1252,* Herman, 1964.

[4] X. Gómez-Mont. *The transverse dynamics of a holomorphic flow,* Ann. of Math. *127*, (1988), 49–92.

[5] R. Hartshorne. *Algebraic Geometry,* Springer-Verlag, 1977.

[6] D. E. Littlewood, A. R. Richardson. *Group characters and algebra,* Philos. Trans. Roy. Soc. London, Ser. *A233* (1934) 49–141.

[7] D. Mumford. *Geometric Invariant Theory,* Springer-Verlag, 1965.

Instituto de Matemáticas
Universidad Nacional Autónoma de México
Mexico, 04510, D.F.

Department of Mathematics
The Johns Hopkins University
Baltimore, Maryland.

Received November 10, 1987