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# Spanning homogeneous vector bundles

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Let  $G$  be a semisimple complex Lie group, let  $P$  be a parabolic subgroup and let  $E$  be a rational  $P$ -module. In this note we give a simple criterion to determine whether a homogeneous vector bundle  $\mathbf{E} = G \times_P E$  over the projective rational base  $G/P$  is spanned by global sections, or equivalently whether the evaluation map of the induced  $G$ -module,  $E|_G \rightarrow E$ , is surjective. This result complements earlier work [7] in which a formula for the ampleness of homogeneous vector bundles is derived, and generalizes results obtained in [5] for the case  $\text{rank } G = 1$ .

The criterion for spanning is as follows, see Corollary 2. Given a  $P$ -module  $E$ , we canonically associate to each simple root  $\alpha$  a string of integers called the  $\alpha$ -indices of  $E$  which are derived from the decomposition of  $E$  as a  $G_\alpha$ -module. Then  $\mathbf{E}$  is spanned by global sections if and only if the  $\alpha$ -indices are non-negative for all simple roots  $\alpha$ . The criterion is actually phrased in slightly more general terms for Schubert varieties, see Theorem 2.

A condition on a vector bundle  $\mathbf{E}$  which is weaker than being spanned, but nevertheless quite useful, is to have some power of the tautological line bundle  $\xi_{\mathbf{E}}$  over the projectivized bundle  $\mathbf{P}(\mathbf{E})$  be spanned. A consequence of the above criterion for homogeneous vector bundles is that the condition of  $\xi_{\mathbf{E}}^n$  being spanned is in fact equivalent to  $\mathbf{E}$  being spanned, see Theorem 3. This equivalence simplifies both the statement and proof of [7, Theorem 2.1].

## 1. Preliminaries

All algebraic groups and varieties are assumed to be defined over the complex numbers.

**1.1. Desingularization of a Schubert variety.** References for this paragraph are [1], [2], [6]. Let  $G$  be a semisimple complex Lie group,  $B$  a Borel subgroup generated by the *negative* roots of  $G$ ,  $P$  a parabolic subgroup, and  $W$  the Weyl group of  $G$ . Let  $w \in W$  have a reduced expression  $s_{i_1} \dots s_{i_n}$  where  $s_j$  denotes the

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simple reflection associated to the simple root  $\alpha_j$ . The Schubert variety in  $G/B$  associated to  $w$ , denoted by  $X_w$ , is defined to be the closure of  $BwB$  in  $G/B$ . Let  $P_i$  be the parabolic subgroup generated by the simple root  $\alpha_i$ . A desingularization of  $X_w$  can be obtained as a quotient

$$Z_w = P_{i_1} \times \cdots \times P_{i_n} / B \times \cdots \times B$$

where the  $n$ -fold product  $B \times \cdots \times B$  acts on  $P_{i_1} \times \cdots \times P_{i_n}$  on the right via

$$\begin{aligned} (p_1, \dots, p_n) \cdot (b_1, \dots, b_n) \\ = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{n-1}^{-1} p_n b_n), \quad p_j \in P_{i_j}, b_j \in B. \end{aligned}$$

The desingularization map  $\phi_w: Z_w \rightarrow X_w$  is induced by the multiplication  $(p_1, \dots, p_n) \rightarrow p_1 \cdots p_n$ . There is also the map  $f_n: Z_w \rightarrow Z_{ws_n}$  induced from the projection  $(p_1, \dots, p_n) \rightarrow (p_1, \dots, p_{n-1})$ .

**1.2. Homogeneous vector bundles.** Let  $E$  be a  $P$ -module and  $\mathbf{E} = G \times_P E$  the associated homogeneous vector bundle. Then  $\mathbf{E}$  is spanned by global sections if and only if the evaluation map of the induced  $G$ -module,  $E|_G \rightarrow E$ , is surjective. (Recall that  $E|_G$  is defined to be the module of all  $P$ -equivariant algebraic maps  $G \rightarrow E$ , and the evaluation map sends a map  $v: G \rightarrow E$  to  $v(1)$ .) Since  $E|_G$  is the same  $G$ -module whether we induce from  $P$  or from  $B$ , see e.g. [3],

$$G \times_P E \text{ is spanned by global sections if and only if } G \times_B E \text{ is.} \quad (1)$$

For this reason, we usually let  $E$  stand for a  $B$ -module and  $\mathbf{E} = G \times_B E$  for the associated homogeneous vector bundle over  $G/B$ . The restriction of  $\mathbf{E}$  to  $X_w$  is denoted by  $\mathbf{E}_w$  and the pull-back  $\phi_w^* \mathbf{E}_w$  by  $\tilde{\mathbf{E}}_w$ . These bundles satisfy the following isomorphisms:

$$H^i(Z_w, \tilde{\mathbf{E}}_w) \cong H^i(X_w, \mathbf{E}_w), \quad i \geq 0, \quad (2)$$

$$f_{n*} \tilde{\mathbf{E}}_w \cong \tilde{\mathbf{H}}_{ws_{i_n}} \text{ where } H \text{ is the } B\text{-module } H^0(P_{i_n}/B, \mathbf{E}_{s_{i_n}}) = E|_{P_{i_n}}, \quad (3)$$

see [2, Theorem 3.1, Lemma 1.4]. Through these isomorphisms and standard Leray spectral sequences based on the tower of  $\mathbf{P}^1$ -bundles  $Z_w \rightarrow Z_{ws_n} \rightarrow \cdots \rightarrow Z_{s_n} \cong \mathbf{P}^1$  we also obtain:

$$H^0(X_w, \mathbf{E}_w) \cong E|_{P_{i_1} \cdots P_{i_n}}, \quad (4)$$

where  $E|^{P_1 \cdots P_n}$  is the module obtained by successively restricting to  $B$  and inducing to  $P_j$ ,  $j = i_1, \dots, i_n$ , see [4].

**1.3. Rank one subgroups.** Let  $G_\alpha$  be the rank one simple subgroup of  $G$  generated by the positive root  $\alpha$ , and let  $B_\alpha$  be the intersection of  $G_\alpha$  with  $B$ ,  $B_\alpha = T_\alpha U_{-\alpha}$ , where  $T_\alpha$  is a maximal torus of  $G_\alpha$  and  $U_{-\alpha}$  is the unipotent subgroup generated by  $-\alpha$ . Let  $E$  be a  $B$ -module. If we consider  $E$  as a  $U_{-\alpha}$ -module, then it is well known that  $E$  extends to a  $G_\alpha$ -module and has a unique (up to order of factors) decomposition into a direct sum of  $G_\alpha$ -modules:  $E = E_1 \oplus \cdots \oplus E_k$  where  $E_i = m_{i,\alpha} \lambda_\alpha | G_\alpha$  is the  $G_\alpha$ -module induced from a non-negative multiple of the fundamental dominant weight  $\lambda_\alpha$  (considered either as a weight of  $G_\alpha$  or of  $G$ ), see [5], [8]. Note that  $\dim E_i = m_{i,\alpha} + 1$ . In particular, the 'zero' weight induces a one dimensional trivial module. Furthermore, each factor  $E_i$  is invariant under  $T_\alpha$  with highest weight  $t_{i,\alpha} \lambda_\alpha$ ,  $1 \leq i \leq k$ . Thus, as a  $B_\alpha$ -module,  $E_i = m_{i,\alpha} \lambda_\alpha |^{G_\alpha} \otimes n_{i,\alpha} \lambda_\alpha$ , where  $n_{i,\alpha} = t_{i,\alpha} - m_{i,\alpha}$ , see [5].

**DEFINITION.** Let  $E$  be a  $B$ -module. For each positive root  $\alpha$ , the  $\alpha$ -indices of  $E$  are defined to be the string of integers  $n_{i,\alpha}$ ,  $1 \leq i \leq k$ .

## 2. Criterion for spanning homogeneous vector bundles

The main results on spanning homogeneous vector bundles are consequences of the following lemma about  $B$ -modules induced to minimal parabolics.

**LEMMA.** Let  $E$  be a  $B$ -module, and let  $P_\alpha$  be the minimal parabolic generated by a simple root  $\alpha$ .

(1) The evaluation map  $E|^{P_\alpha} \rightarrow E$  is surjective if and only if the  $\alpha$ -indices of  $E$  are non-negative.

(2) Let  $\alpha, \beta$  be two distinct simple roots. If the  $\alpha$ -indices and the  $\beta$ -indices of  $E$  are non-negative, then they are also non-negative for the induced module  $E|^{P_\alpha}$ .

*Proof.* (1) The induced module  $E|^{P_\alpha}$  is isomorphic to the space of sections of the homogeneous bundle  $P_\alpha \times_B E = G_\alpha \times_{B_\alpha} E$ , and thus  $E|^{P_\alpha} = E|^{G_\alpha}$ . As in 1.3, we write  $E$  as a  $B_\alpha$ -module direct sum,  $E = E_1 \oplus \cdots \oplus E_k$ , with  $E_i = m_{i,\alpha} \lambda_\alpha |^{G_\alpha} \otimes n_{i,\alpha} \lambda_\alpha$ ,  $1 \leq i \leq k$ . Since

$$E_i |^{G_\alpha} = m_{i,\alpha} \lambda_\alpha |^{G_\alpha} \otimes n_{i,\alpha} \lambda_\alpha |^{G_\alpha}$$

(see e.g. [3]), it is clear that  $E|^{P_\alpha} \rightarrow E$  is surjective if and only if  $n_{i,\alpha}\lambda_\alpha|^{G_\alpha} \neq 0$ , i.e.  $n_{i,\alpha} \geq 0$ ,  $i = 1, \dots, k$ .

(2) The  $\alpha$ -indices of  $E|^{P_\alpha}$  are obviously zero. To see why the  $\beta$ -indices remain non-negative in this induced module, let us determine explicitly the action of  $b \in B_\beta$  on a  $B$ -equivariant morphism  $s: P_\alpha \rightarrow E$  (i.e.  $s \in E|^{P_\alpha}$ ). Let  $\mathfrak{u}_\alpha$  be the Lie algebra of  $U_\alpha$ , and let  $u: \mathfrak{u}_\alpha \rightarrow U_\alpha$  be the exponential map which in this case is an algebraic isomorphism of groups. We can use  $z \in \mathfrak{u}_\alpha \cong \mathbb{C}$  as a parameter for  $\mathbb{P}^1 \cong P_\alpha/B$  via the correspondence  $z \leftrightarrow u(z)B \in P_\alpha/B$ . Express  $b$  as  $b = \mu_\beta(t)w$  where  $w$  is in the root group  $U_{-\beta}$  and  $\mu_\beta: \mathbb{C}^* \rightarrow G$  is a one-parameter subgroup with image  $T_\beta \subset G_\beta$  such that  $\lambda_\beta(\mu_\beta(t)) = t$  for all  $t \in \mathbb{C}^*$ . Then the action of  $b$  on  $P_\alpha/B$  is given by

$$bu(z)B = \mu_\beta(t)wu(z)B = \mu_\beta(t)u(z)\mu_\beta(t)^{-1}B = u(\alpha(\mu_\beta(t))z)B = u(t^{\langle \alpha, \beta \rangle}z)B,$$

since  $w$  and  $u(z)$  always commute. Thus, in terms of the parameter  $z$  for  $\mathbb{P}^1$ , the action is simply  $z \rightarrow t^{\langle \alpha, \beta \rangle}z$ .

Now let  $E = E_1 \oplus \dots \oplus E_q$  be the decomposition of  $E$  as a  $B_\beta$ -module with  $E_\nu = m_{\nu,\beta}\lambda_\beta|^{G_\beta} \otimes n_{\nu,\beta}\lambda_\beta$ . Let  $\rho_\nu$  denote the representation of  $G_\beta$  on  $m_{\nu,\beta}\lambda_\beta|^{G_\beta}$ . We may view  $s \in E|^{P_\alpha}$  as a section of a direct sum of line bundles  $\mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_r)$  on  $\mathbb{P}^1$ , where  $r = \dim E$  and each  $k_j$  is one of the  $\alpha$ -indices  $n_{i,\alpha} \geq 0$ , see [2], [5]. Therefore, we write  $s = \sum_{1 \leq \nu \leq q} s_\nu$ ,  $s_\nu = s_{\nu,1}e_{\nu,1} + \dots + s_{\nu,j(\nu)}e_{\nu,j(\nu)}$ , where  $e_{\nu,1}, \dots, e_{\nu,j(\nu)}$  is a basis for  $E_\nu|^{P_\alpha}$ ,  $\nu = 1, \dots, q$ . We consider each component function  $s_{\nu,v}$  to be a polynomial of degree  $k(\nu, v)$  (i.e. one of the above  $k_j$ 's, depending on  $\nu, v$ ) in the parameter  $z \in \mathbb{P}^1$ :  $s_{\nu,v}(z) = \sum_\eta c_{\nu,v}^\eta z^\eta$ . Now the action of  $b = \mu_\beta(t)w$  on  $s$  is given by:  $(b.s)(z) = s(b^{-1}.z) = b.s(t^{-\langle \alpha, \beta \rangle}z)$ . Note that on the left side of this equation  $b$  is acting in  $E|^{P_\alpha}$  and on the right side the action is in  $E$ . Continuing to expand this expression further, we find

$$(b.s)(z) = \sum_{\nu=1}^q \sum_{v=1}^{j(\nu)} \sum_{\eta=0}^{k(\nu,v)} t^{n_{\nu,\beta} - \eta\langle \alpha, \beta \rangle} c_{\nu,v}^\eta z^\eta \rho_\nu(b) e_{\nu,v}$$

From this expression it is clear that the  $\beta$ -indices of  $E|^{P_\alpha}$  are of the form  $n_{\nu,\beta} - \eta\langle \alpha, \beta \rangle$ , i.e. only non-negative multiples of  $-\langle \alpha, \beta \rangle \geq 0$  added to the original  $\beta$ -indices of  $E$ .  $\square$

As in section 1, let  $w \in W$  and fix a reduced expression  $w = s_{i_1} \dots s_{i_n}$ . Let  $I$  be the set of simple roots corresponding to this sequence of reflections,  $I = \{\alpha_j \mid j = i_1, \dots, i_n\}$ . Let  $E$  be a  $B$ -module,  $\mathbf{E} = G \times_B E$  the induced homogeneous vector bundle on  $G/B$ ,  $X_w$  the Schubert variety associated to  $w$  in  $G/B$ , and  $\mathbf{E}_w$  the bundle  $\mathbf{E}$  restricted to  $X_w$ .

**THEOREM.** *The vector bundle  $\mathbf{E}_w$  is spanned by global sections if and only if the  $\alpha$ -indices of  $E$  are non-negative for all simple roots  $\alpha \in I$ .*

*Proof.* First the necessity of the condition: If  $\mathbf{E}_w$  is spanned, then so is the bundle restricted to the  $G_\alpha$  orbit  $G_\alpha/B_\alpha \cong \mathbf{P}^1 \subset X_w \subset G/B$  for any  $\alpha \in I$ . Now the restricted bundle,  $G_\alpha \times_{B_\alpha} E$ , is spanned if and only if  $E|^{P_\alpha} \rightarrow E$  is surjective. By the Lemma, this happens only when the  $\alpha$ -indices of  $E$  are non-negative.

The sufficiency of the condition follows from the isomorphism 1.2(4) and repeated application of the previous Lemma.  $\square$

An obvious consequence of the Theorem is the following:

**COROLLARY.** *A homogeneous vector bundle  $\mathbf{E}$  is spanned by global sections if and only if the  $\alpha$ -indices of  $E$  are non-negative for all simple roots  $\alpha$ .*

### 3. The tautological line bundle

Let  $\mathbf{E}$  be a vector bundle over a variety  $X$ . The projectivization of  $\mathbf{E}$ , denoted  $\mathbf{P}(\mathbf{E})$ , is the bundle over  $X$  defined as the space of 1-dimensional subspaces in the fibers of the dual bundle  $\mathbf{E}^*$ . Let  $\xi_{\mathbf{E}}$  be the tautological line bundle over  $\mathbf{P}(\mathbf{E})$  whose restriction to the fiber  $\mathbf{P}(E)$  is  $\mathcal{O}(1)$ . There is a canonical isomorphism of sheaves  $\pi_* \xi_{\mathbf{E}} \cong \mathbf{E}$  where  $\pi: \mathbf{P}(\mathbf{E}) \rightarrow X$  is the bundle map. If the zero sections are removed, the two spaces are isomorphic:  $\xi_{\mathbf{E}} \setminus \mathbf{P}(\mathbf{E}) \cong \mathbf{E} \setminus X$ , and  $\mathbf{E}$  is spanned if and only if  $\xi_{\mathbf{E}}$  is spanned. More generally, there is an isomorphism  $\pi_* \xi_{\mathbf{E}}^n \cong S^n(\mathbf{E})$  where  $S^n(\cdot)$  denotes the  $n$ th symmetric power. In this case, however,  $\xi_{\mathbf{E}}^n$  being spanned does not necessarily imply that  $S^n(\mathbf{E})$ , or even  $\mathbf{E}$ , is spanned. As an application of the criterion in section 2, we prove that this implication does hold for homogeneous bundles:

**THEOREM.** *Let  $\mathbf{E} = G \times_P E$  be a homogeneous vector bundle over a projective rational homogeneous space  $G/P$ . Then the following are equivalent:*

- (1)  $\mathbf{E}$  is spanned by global sections.
- (2)  $\xi_{\mathbf{E}}$  is spanned by global sections.
- (3)  $\xi_{\mathbf{E}}^n$  is spanned by global sections for some  $n > 0$ .
- (4)  $S^n(\mathbf{E})$  is spanned by global sections for some  $n > 0$ .

*Proof.* The equivalence (1)  $\Leftrightarrow$  (2) is well-known and the implications (1)  $\Rightarrow$  (4)  $\Rightarrow$  (3) are obvious. Therefore it is sufficient to prove (3)  $\Rightarrow$  (2). Also, by 1.2(1), we may assume  $P = B$ .

Assume  $\xi_{\mathbf{E}}$  is *not* spanned. Then by Theorem 2, there is a simple root  $\alpha$  with a

negative  $\alpha$ -index,  $n_{i,\alpha} < 0$  for some integer  $i$ . Let  $E_i = m_{i,\alpha}\lambda_\alpha|^{G_\alpha} \otimes n_{i,\alpha}\lambda_\alpha$  be the  $B_\alpha$ -invariant submodule of  $E$  corresponding to this negative  $\alpha$ -index. Let  $\mathbf{F}$  be the restriction of  $\mathbf{E}_i$  to the orbit under  $G_\alpha$  of the identity coset:  $G_\alpha/B_\alpha \subset G/B$ , i.e.  $\mathbf{F} = G_\alpha \times_{B_\alpha} E_i$ . Let  $v$  be a weight vector in  $E_i$  of weight  $(m_{i,\alpha} + n_{i,\alpha})\lambda_\alpha$  and let  $p = 1 \times [v] \in \mathbf{P}(\mathbf{F}) = G_\alpha \times_{B_\alpha} \mathbf{P}(E_i)$ , so that  $p$  is a  $B_\alpha$ -fixed point in  $\mathbf{P}(\mathbf{F})$  and  $G_{\alpha \cdot p} \cong G_\alpha/B_\alpha$ . If  $L$  denotes the restriction of  $\xi_{\mathbf{F}}$  to  $G_{\alpha \cdot p}$ , then  $L = G_\alpha \times_{B_\alpha} n_{i,\alpha}\lambda_\alpha$ . Now, if  $\xi_{\mathbf{E}}^n$  were spanned, then  $L^n$  would also be spanned, but this is impossible since  $n_{i,\alpha} < 0$ , so that no power of  $L$  has sections.  $\square$

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