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Spanning homogeneous vector bundles

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Let G be a semisimple complex Lie group, let P be a parabolic subgroup and let E be a rational P-module. In this note we give a simple criterion to determine whether a homogeneous vector bundle $\mathbf{E} = G \times_p E$ over the projective rational base G/P is spanned by global sections, or equivalently whether the evaluation map of the induced G-module, $E|^G \to E$, is surjective. This result complements earlier work [7] in which a formula for the ampleness of homogeneous vector bundles is derived, and generalizes results obtained in [5] for the case rank G = 1.

The criterion for spanning is as follows, see Corollary 2. Given a P-module E, we canonically associate to each simple root α a string of integers called the α -indices of E which are derived from the decomposition of E as a G_{α} -module. Then E is spanned by global sections if and only if the α -indices are non-negative for all simple roots α . The criterion is actually phrased in slightly more general terms for Schubert varieties, see Theorem 2.

A condition on a vector bundle \mathbf{E} which is weaker than being spanned, but nevertheless quite useful, is to have some power of the tautological line bundle $\xi_{\mathbf{E}}$ over the projectivized bundle $\mathbf{P}(\mathbf{E})$ be spanned. A consequence of the above criterion for homogeneous vector bundles is that the condition of $\xi_{\mathbf{E}}^n$ being spanned is in fact equivalent to \mathbf{E} being spanned, see Theorem 3. This equivalence simplifies both the statement and proof of [7, Theorem 2.1].

1. Preliminaries

All algebraic groups and varieties are assumed to be defined over the complex numbers.

1.1. Desingularization of a Schubert variety. References for this paragraph are [1], [2], [6]. Let G be a semisimple complex Lie group, B a Borel subgroup generated by the negative roots of G, P a parabolic subgroup, and W the Weyl group of G. Let $w \in W$ have a reduced expression $s_{i_1} \ldots s_{i_n}$ where s_j denotes the

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simple reflection associated to the simple root α_j . The Schubert variety in G/B associated to w, denoted by X_w , is defined to be the closure of BwB in G/B. Let P_i be the parabolic subgroup generated by the simple root α_i . A desingularization of X_w can be obtained as a quotient

$$Z_w = P_{i_1} \times \cdots \times P_{i_m} / B \times \cdots \times B$$

where the *n*-fold product $B \times \cdots \times B$ acts on $P_{i_1} \times \cdots \times P_{i_n}$ on the right via

$$(p_1, \ldots, p_n) \cdot (b_1, \ldots, b_n)$$

$$= (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{n-1}^{-1} p_n b_n), \qquad p_j \in P_{i_j}, b_j \in B.$$

The desingularization map $\phi_w: Z_w \to X_w$ is induced by the multiplication $(p_1, \ldots, p_n) \to p_1 \ldots p_n$. There is also the map $f_n: Z_w \to Z_{ws_n}$ induced from the projection $(p_1, \ldots, p_n) \to (p_1, \ldots, p_{n-1})$.

1.2. Homogeneous vector bundles. Let E be a P-module and $\mathbf{E} = G \times_P E$ the associated homogeneous vector bundle. Then \mathbf{E} is spanned by global sections if and only if the evaluation map of the induced G-module, $E|^G \to E$, is surjective. (Recall that $E|^G$ is defined to be the module of all P-equivariant algebraic maps $G \to E$, and the evaluation map sends a map $v: G \to E$ to v(1).) Since $E|^G$ is the same G-module whether we induce from P or from B, see e.g. [3],

$$G \times_P E$$
 is spanned by global sections if and only if $G \times_B E$ is. (1)

For this reason, we usually let E stand for a B-module and $\mathbf{E} = G \times_B E$ for the associated homogeneous vector bundle over G/B. The restriction of \mathbf{E} to X_w is denoted by \mathbf{E}_w and the pull-back $\phi_w^* \mathbf{E}_w$ by $\tilde{\mathbf{E}}_w$. These bundles satisfy the following isomorphisms:

$$H^{i}(Z_{w}, \tilde{\mathbf{E}}_{w}) \cong H^{i}(X_{w}, \mathbf{E}_{w}), \qquad i \geq 0,$$
 (2)

$$f_{n*}\tilde{\mathbf{E}}_{w} \cong \tilde{\mathbf{H}}_{ws_{i_n}} \text{ where } H \text{ is the } B\text{-module } H^0(P_{i_n}/B, \mathbf{E}_{s_{i_n}}) = E|_{P_{i_n}},$$
 (3)

see [2, Theorem 3.1, Lemma 1.4]. Through these isomorphisms and standard Leray spectral sequences based on the tower of \mathbb{P}^1 -bundles $Z_w \to Z_{ws_n} \to \cdots \to Z_{s_n} \cong \mathbb{P}^1$ we also obtain:

$$H^0(X_w, \mathbb{E}_w) \cong E^{|P_{i_1} \dots P_{i_n}|}, \tag{4}$$

where $E|_{i_1...i_n}^{P_{i_1}...P_{i_n}}$ is the module obtained by successively restricting to B and inducing to P_i , $j = i_1, \ldots, i_n$, see [4].

1.3. Rank one subgroups. Let G_{α} be the rank one simple subgroup of G generated by the positive root α , and let B_{α} be the intersection of G_{α} with B, $B_{\alpha} = T_{\alpha}U_{-\alpha}$, where T_{α} is a maximal torus of G_{α} and $U_{-\alpha}$ is the unipotent subgroup generated by $-\alpha$. Let E be a B-module. If we consider E as a $U_{-\alpha}$ -module, then it is well known that E extends to a G_{α} -module and has a unique (up to order of factors) decomposition into a direct sum of G_{α} -modules: $E = E_1 \oplus \cdots \oplus E_k$ where $E_i = m_{i,\alpha}\lambda_{\alpha}|G_{\alpha}$ is the G_{α} -module induced from a non-negative multiple of the fundamental dominant weight λ_{α} (considered either as a weight of G_{α} or of G), see [5], [8]. Note that dim $E_i = m_{i,\alpha} + 1$. In particular, the 'zero' weight induces a one dimensional trivial module. Furthermore, each factor E_i is invariant under T_{α} with highest weight $t_{i,\alpha}\lambda_{\alpha}$, $1 \le i \le k$. Thus, as a B_{α} -module, $E_i = m_{i,\alpha}\lambda_{\alpha}|_{G_{\alpha}} \otimes n_{i,\alpha}\lambda_{\alpha}$, where $n_{i,\alpha} = t_{i,\alpha} - m_{i,\alpha}$, see [5].

DEFINITION. Let E be a B-module. For each positive root α , the α -indices of E are defined to be the string of integers $n_{i,\alpha}$, $1 \le i \le k$.

2. Criterion for spanning homogeneous vector bundles

The main results on spanning homogeneous vector bundles are consequences of the following lemma about B-modules induced to minimal parabolics.

LEMMA. Let E be a B-module, and let P_{α} be the minimal parabolic generated by a simple root α .

- (1) The evaluation map $E|^{P_{\alpha}} \to E$ is surjective if and only if the α -indices of E are non-negative.
- (2) Let α , β be two distinct simple roots. If the α -indices and the β -indices of E are non-negative, then they are also non-negative for the induced module $E|_{\alpha}$.
- *Proof.* (1) The induced module $E|^{P_{\alpha}}$ is isomorphic to the space of sections of the homogeneous bundle $P_{\alpha} \times_{B} E = G_{\alpha} \times_{B_{\alpha}} E$, and thus $E|^{P_{\alpha}} = E|^{G_{\alpha}}$. As in 1.3, we write E as a B_{α} -module direct sum, $E = E_{1} \oplus \cdots \oplus E_{k}$, with $E_{i} = m_{i,\alpha}\lambda_{\alpha}|^{G_{\alpha}} \otimes n_{i,\alpha}\lambda_{\alpha}$, $1 \leq i \leq k$. Since

$$E_i|_{G_\alpha} = m_{i,\alpha}\lambda_\alpha|_{G_\alpha} \otimes n_{i,\alpha}\lambda_\alpha|_{G_\alpha}$$

(see e.g. [3]), it is clear that $E|^{P_{\alpha}} \to E$ is surjective if and only if $n_{i,\alpha} \lambda_{\alpha}|^{G_{\alpha}} \neq 0$, i.e. $n_{i,\alpha} \geq 0$, $i = 1, \ldots, k$.

(2) The α -indices of $E|^{P_{\alpha}}$ are obviously zero. To see why the β -indices remain non-negative in this induced module, let us determine explicitly the action of $b \in B_{\beta}$ on a B-equivariant morphism $s: P_{\alpha} \to E$ (i.e. $s \in E|^{P_{\alpha}}$). Let \mathbf{u}_{α} be the Lie algebra of U_{α} , and let $u: \mathbf{u}_{\alpha} \to U_{\alpha}$ be the exponential map which in this case is an algebraic isomorphism of groups. We can use $z \in \mathbf{u}_{\alpha} \cong \mathbf{C}$ as a parameter for $\mathbf{P}^1 \cong P_{\alpha}/B$ via the correspondence $z \leftrightarrow u(z)B \in P_{\alpha}/B$. Express b as $b = \mu_{\beta}(t)w$ where w is in the root group $U_{-\beta}$ and $\mu_{\beta}: \mathbf{C}^* \to G$ is a one-parameter subgroup with image $T_{\beta} \subset G_{\beta}$ such that $\lambda_{\beta}(\mu_{\beta}(t)) = t$ for all $t \in \mathbf{C}^*$. Then the action of b on P_{α}/B is given by

$$bu(z)B = \mu_{\beta}(t)wu(z)B = \mu_{\beta}(t)u(z)\mu_{\beta}(t)^{-1}B = u(\alpha(\mu_{\beta}(t))z)B = u(t^{\langle \alpha,\beta \rangle}z)B,$$

since w and u(z) always commute. Thus, in terms of the parameter z for \mathbb{P}^1 , the action is simply $z \to t^{\langle \alpha, \beta \rangle} z$.

Now let $E = E_1 \oplus \cdots \oplus E_q$ be the decomposition of E as a B_{β} -module with $E_v = m_{v,\beta} \lambda_{\beta}|^{G_{\beta}} \otimes n_{v,\beta} \lambda_{\beta}$. Let ρ_v denote the representation of G_{β} on $m_{v,\beta} \lambda_{\beta}|^{G_{\beta}}$. We may view $s \in E|^{P_{\alpha}}$ as a section of a direct sum of line bundles $\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_r)$ on \mathbf{P}^1 , where $r = \dim E$ and each k_j is one of the α -indices $n_{i,\alpha} \geq 0$, see [2], [5]. Therefore, we write $s = \sum_{1 \leq v \leq q} s_v$, $s_v = s_{v,1} e_{v,1} + \cdots + s_{v,j(v)} e_{v,j(v)}$, where $e_{v,1}, \ldots, e_{v,j(v)}$ is a basis for $E_v|^{P_{\alpha}}$, $v = 1, \ldots, q$. We consider each component function $s_{v,v}$ to be a polynomial of degree k(v,v) (i.e. one of the above k_j 's, depending on v, v) in the parameter $z \in \mathbf{P}^1 : s_{v,v}(z) = \sum_{\eta} c_{v,v}^{\eta} z^{\eta}$. Now the action of $b = \mu_{\beta}(t)w$ on s is given by: $(b.s)(z) = s(b^{-1}.z) = b.s(t^{-(\alpha,\beta)}z)$. Note that on the left side of this equation b is acting in $E|^{P_{\alpha}}$ and on the right side the action is in E. Continuing to expand this expression further, we find

$$(b.s)(z) = \sum_{\nu=1}^{q} \sum_{\nu=1}^{j(\nu)} \sum_{\eta=0}^{k(\nu,\nu)} t^{n_{\nu,\beta}-\eta \langle \alpha,\beta \rangle} c_{\nu,\nu}^{\eta} z^{\eta} \rho_{\nu}(b) e_{\nu,\nu}$$

From this expression it is clear that the β -indices of $E|_{\alpha}$ are of the form $n_{\nu,\beta} - \eta \langle \alpha, \beta \rangle$, i.e. only non-negative multiples of $-\langle \alpha, \beta \rangle \ge 0$ added to the original β -indices of E. \square

As in section 1, let $w \in W$ and fix a reduced expression $w = s_{i_1} \dots s_{i_n}$. Let I be the set of simple roots corresponding to this sequence of reflections, $I = \{\alpha_j \mid j = i_1, \dots, i_n\}$. Let E be a B-module, $\mathbf{E} = G \times_B E$ the induced homogeneous vector bundle on G/B, X_w the Schubert variety associated to w in G/B, and \mathbf{E}_w the bundle \mathbf{E} restricted to X_w .

THEOREM. The vector bundle \mathbf{E}_w is spanned by global sections if and only if the α -indices of E are non-negative for all simple roots $\alpha \in I$.

Proof. First the necessity of the condition: If \mathbf{E}_w is spanned, then so is the bundle restricted to the G_α orbit $G_\alpha/B_\alpha \cong \mathbf{P}^1 \subset X_w \subset G/B$ for any $\alpha \in I$. Now the restricted bundle, $G_\alpha \times_{B_\alpha} E$, is spanned if and only if $E|^{P_\alpha} \to E$ is surjective. By the Lemma, this happens only when the α -indices of E are non-negative.

The sufficiency of the condition follows from the isomorphism 1.2(4) and repeated application of the previous Lemma. \Box

An obvious consequence of the Theorem is the following:

COROLLARY. A homogeneous vector bundle \mathbf{E} is spanned by global sections if and only if the α -indices of E are non-negative for all simple roots α .

3. The tautological line bundle

Let **E** be a vector bundle over a variety X. The projectivization of **E**, denoted P(E), is the bundle over X defined as the space of 1-dimensional subspaces in the fibers of the dual bundle E^* . Let ξ_E be the tautological line bundle over P(E) whose restriction to the fiber P(E) is O(1). There is a canonical isomorphism of sheaves $\pi_*\xi_E \cong E$ where $\pi:P(E) \to X$ is the bundle map. If the zero sections are removed, the two spaces are isomorphic: $\xi_E \setminus P(E) \cong E \setminus X$, and **E** is spanned if and only if ξ_E is spanned. More generally, there is an isomorphism $\pi_*\xi_E^n \cong S^n(E)$ where $S^n(\cdot)$ denotes the nth symmetric power. In this case, however, ξ_E^n being spanned does not necessarily imply that $S^n(E)$, or even **E**, is spanned. As an application of the criterion in section 2, we prove that this implication does hold for homogeneous bundles:

THEOREM. Let $\mathbf{E} = G \times_P E$ be a homogeneous vector bundle over a projective rational homogeneous space G/P. Then the following are equivalent:

- (1) E is spanned by global sections.
- (2) $\xi_{\mathbf{E}}$ is spanned by global sections.
- (3) $\xi_{\mathbb{E}}^n$ is spanned by global sections for some n > 0.
- (4) $S^n(\mathbf{E})$ is spanned by global sections for some n > 0.

Proof. The equivalence $(1) \Leftrightarrow (2)$ is well-known and the implications $(1) \Rightarrow (4) \Rightarrow (3)$ are obvious. Therefore it is sufficient to prove $(3) \Rightarrow (2)$. Also, by 1.2(1), we may assume P = B.

Assume $\xi_{\mathbf{E}}$ is not spanned. Then by Theorem 2, there is a simple root α with a

negative α -index, $n_{i,\alpha} < 0$ for some integer i. Let $E_i = m_{i,\alpha} \lambda_{\alpha} |_{G_{\alpha}} \otimes n_{i,\alpha} \lambda_{\alpha}$ be the B_{α} -invariant submodule of E corresponding to this negative α -index. Let F be the restriction of E_i to the orbit under G_{α} of the identity coset: $G_{\alpha}/B_{\alpha} \subset G/B$, i.e. $F = G_{\alpha} \times_{B_{\alpha}} E_i$. Let v be a weight vector in E_i of weight $(m_{i,\alpha} + n_{i,\alpha})\lambda_{\alpha}$ and let $p = 1 \times [v] \in P(F) = G_{\alpha} \times_{B_{\alpha}} P(E_i)$, so that p is a B_{α} -fixed point in P(F) and $G_{\alpha \cdot p} \cong G_{\alpha}/B_{\alpha}$. If E denotes the restriction of E to the impossible since E to the orbit under E to the orbit under E to E to

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