

Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 64 (1989)

Artikel: Spanning homogeneous vector bundles.
Autor: Snow, Dennis M.
DOI: <https://doi.org/10.5169/seals-48954>

Nutzungsbedingungen

Die ETH-Bibliothek ist die Anbieterin der digitalisierten Zeitschriften auf E-Periodica. Sie besitzt keine Urheberrechte an den Zeitschriften und ist nicht verantwortlich für deren Inhalte. Die Rechte liegen in der Regel bei den Herausgebern beziehungsweise den externen Rechteinhabern. Das Veröffentlichen von Bildern in Print- und Online-Publikationen sowie auf Social Media-Kanälen oder Webseiten ist nur mit vorheriger Genehmigung der Rechteinhaber erlaubt. [Mehr erfahren](#)

Conditions d'utilisation

L'ETH Library est le fournisseur des revues numérisées. Elle ne détient aucun droit d'auteur sur les revues et n'est pas responsable de leur contenu. En règle générale, les droits sont détenus par les éditeurs ou les détenteurs de droits externes. La reproduction d'images dans des publications imprimées ou en ligne ainsi que sur des canaux de médias sociaux ou des sites web n'est autorisée qu'avec l'accord préalable des détenteurs des droits. [En savoir plus](#)

Terms of use

The ETH Library is the provider of the digitised journals. It does not own any copyrights to the journals and is not responsible for their content. The rights usually lie with the publishers or the external rights holders. Publishing images in print and online publications, as well as on social media channels or websites, is only permitted with the prior consent of the rights holders. [Find out more](#)

Download PDF: 05.04.2026

ETH-Bibliothek Zürich, E-Periodica, <https://www.e-periodica.ch>

Spanning homogeneous vector bundles

DENNIS M. SNOW¹

Let G be a semisimple complex Lie group, let P be a parabolic subgroup and let E be a rational P -module. In this note we give a simple criterion to determine whether a homogeneous vector bundle $\mathbf{E} = G \times_P E$ over the projective rational base G/P is spanned by global sections, or equivalently whether the evaluation map of the induced G -module, $E|_G \rightarrow E$, is surjective. This result complements earlier work [7] in which a formula for the ampleness of homogeneous vector bundles is derived, and generalizes results obtained in [5] for the case $\text{rank } G = 1$.

The criterion for spanning is as follows, see Corollary 2. Given a P -module E , we canonically associate to each simple root α a string of integers called the α -indices of E which are derived from the decomposition of E as a G_α -module. Then \mathbf{E} is spanned by global sections if and only if the α -indices are non-negative for all simple roots α . The criterion is actually phrased in slightly more general terms for Schubert varieties, see Theorem 2.

A condition on a vector bundle \mathbf{E} which is weaker than being spanned, but nevertheless quite useful, is to have some power of the tautological line bundle $\xi_{\mathbf{E}}$ over the projectivized bundle $\mathbf{P}(\mathbf{E})$ be spanned. A consequence of the above criterion for homogeneous vector bundles is that the condition of $\xi_{\mathbf{E}}^n$ being spanned is in fact equivalent to \mathbf{E} being spanned, see Theorem 3. This equivalence simplifies both the statement and proof of [7, Theorem 2.1].

1. Preliminaries

All algebraic groups and varieties are assumed to be defined over the complex numbers.

1.1. Desingularization of a Schubert variety. References for this paragraph are [1], [2], [6]. Let G be a semisimple complex Lie group, B a Borel subgroup generated by the *negative* roots of G , P a parabolic subgroup, and W the Weyl group of G . Let $w \in W$ have a reduced expression $s_{i_1} \dots s_{i_n}$ where s_j denotes the

¹ Partially supported by NSF grant DMS 8420315.

simple reflection associated to the simple root α_j . The Schubert variety in G/B associated to w , denoted by X_w , is defined to be the closure of BwB in G/B . Let P_i be the parabolic subgroup generated by the simple root α_i . A desingularization of X_w can be obtained as a quotient

$$Z_w = P_{i_1} \times \cdots \times P_{i_n}/B \times \cdots \times B$$

where the n -fold product $B \times \cdots \times B$ acts on $P_{i_1} \times \cdots \times P_{i_n}$ on the right via

$$\begin{aligned} (p_1, \dots, p_n) \cdot (b_1, \dots, b_n) \\ = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{n-1}^{-1} p_n b_n), \quad p_j \in P_{i_j}, b_j \in B. \end{aligned}$$

The desingularization map $\phi_w: Z_w \rightarrow X_w$ is induced by the multiplication $(p_1, \dots, p_n) \rightarrow p_1 \cdots p_n$. There is also the map $f_n: Z_w \rightarrow Z_{ws_n}$ induced from the projection $(p_1, \dots, p_n) \rightarrow (p_1, \dots, p_{n-1})$.

1.2. Homogeneous vector bundles. Let E be a P -module and $\mathbf{E} = G \times_P E$ the associated homogeneous vector bundle. Then \mathbf{E} is spanned by global sections if and only if the evaluation map of the induced G -module, $E|_G \rightarrow E$, is surjective. (Recall that $E|_G$ is defined to be the module of all P -equivariant algebraic maps $G \rightarrow E$, and the evaluation map sends a map $\nu: G \rightarrow E$ to $\nu(1)$.) Since $E|_G$ is the same G -module whether we induce from P or from B , see e.g. [3],

$$G \times_P E \text{ is spanned by global sections if and only if } G \times_B E \text{ is.} \tag{1}$$

For this reason, we usually let E stand for a B -module and $\mathbf{E} = G \times_B E$ for the associated homogeneous vector bundle over G/B . The restriction of \mathbf{E} to X_w is denoted by \mathbf{E}_w and the pull-back $\phi_w^* \mathbf{E}_w$ by $\tilde{\mathbf{E}}_w$. These bundles satisfy the following isomorphisms:

$$H^i(Z_w, \tilde{\mathbf{E}}_w) \cong H^i(X_w, \mathbf{E}_w), \quad i \geq 0, \tag{2}$$

$$f_{n*} \tilde{\mathbf{E}}_w \cong \tilde{\mathbf{H}}_{ws_{i_n}} \text{ where } H \text{ is the } B\text{-module } H^0(P_{i_n}/B, \mathbf{E}_{s_{i_n}}) = E|_{P_{i_n}}, \tag{3}$$

see [2, Theorem 3.1, Lemma 1.4]. Through these isomorphisms and standard Leray spectral sequences based on the tower of \mathbf{P}^1 -bundles $Z_w \rightarrow Z_{ws_n} \rightarrow \cdots \rightarrow Z_{s_n} \cong \mathbf{P}^1$ we also obtain:

$$H^0(X_w, \mathbf{E}_w) \cong E|_{P_{i_1} \cdots P_{i_n}}, \tag{4}$$

where $E|^{P_1 \cdots P_n}$ is the module obtained by successively restricting to B and inducing to P_j , $j = i_1, \dots, i_n$, see [4].

1.3. Rank one subgroups. Let G_α be the rank one simple subgroup of G generated by the positive root α , and let B_α be the intersection of G_α with B , $B_\alpha = T_\alpha U_{-\alpha}$, where T_α is a maximal torus of G_α and $U_{-\alpha}$ is the unipotent subgroup generated by $-\alpha$. Let E be a B -module. If we consider E as a $U_{-\alpha}$ -module, then it is well known that E extends to a G_α -module and has a unique (up to order of factors) decomposition into a direct sum of G_α -modules: $E = E_1 \oplus \cdots \oplus E_k$ where $E_i = m_{i,\alpha} \lambda_\alpha | G_\alpha$ is the G_α -module induced from a non-negative multiple of the fundamental dominant weight λ_α (considered either as a weight of G_α or of G), see [5], [8]. Note that $\dim E_i = m_{i,\alpha} + 1$. In particular, the 'zero' weight induces a one dimensional trivial module. Furthermore, each factor E_i is invariant under T_α with highest weight $t_{i,\alpha} \lambda_\alpha$, $1 \leq i \leq k$. Thus, as a B_α -module, $E_i = m_{i,\alpha} \lambda_\alpha |^{G_\alpha} \otimes n_{i,\alpha} \lambda_\alpha$, where $n_{i,\alpha} = t_{i,\alpha} - m_{i,\alpha}$, see [5].

DEFINITION. Let E be a B -module. For each positive root α , the α -indices of E are defined to be the string of integers $n_{i,\alpha}$, $1 \leq i \leq k$.

2. Criterion for spanning homogeneous vector bundles

The main results on spanning homogeneous vector bundles are consequences of the following lemma about B -modules induced to minimal parabolics.

LEMMA. Let E be a B -module, and let P_α be the minimal parabolic generated by a simple root α .

(1) The evaluation map $E|^{P_\alpha} \rightarrow E$ is surjective if and only if the α -indices of E are non-negative.

(2) Let α, β be two distinct simple roots. If the α -indices and the β -indices of E are non-negative, then they are also non-negative for the induced module $E|^{P_\alpha}$.

Proof. (1) The induced module $E|^{P_\alpha}$ is isomorphic to the space of sections of the homogeneous bundle $P_\alpha \times_B E = G_\alpha \times_{B_\alpha} E$, and thus $E|^{P_\alpha} = E|^{G_\alpha}$. As in 1.3, we write E as a B_α -module direct sum, $E = E_1 \oplus \cdots \oplus E_k$, with $E_i = m_{i,\alpha} \lambda_\alpha |^{G_\alpha} \otimes n_{i,\alpha} \lambda_\alpha$, $1 \leq i \leq k$. Since

$$E_i |^{G_\alpha} = m_{i,\alpha} \lambda_\alpha |^{G_\alpha} \otimes n_{i,\alpha} \lambda_\alpha |^{G_\alpha}$$

(see e.g. [3]), it is clear that $E|^{P_\alpha} \rightarrow E$ is surjective if and only if $n_{i,\alpha} \lambda_\alpha|^{G_\alpha} \neq 0$, i.e. $n_{i,\alpha} \geq 0$, $i = 1, \dots, k$.

(2) The α -indices of $E|^{P_\alpha}$ are obviously zero. To see why the β -indices remain non-negative in this induced module, let us determine explicitly the action of $b \in B_\beta$ on a B -equivariant morphism $s: P_\alpha \rightarrow E$ (i.e. $s \in E|^{P_\alpha}$). Let \mathfrak{u}_α be the Lie algebra of U_α , and let $u: \mathfrak{u}_\alpha \rightarrow U_\alpha$ be the exponential map which in this case is an algebraic isomorphism of groups. We can use $z \in \mathfrak{u}_\alpha \cong \mathbb{C}$ as a parameter for $\mathbb{P}^1 \cong P_\alpha/B$ via the correspondence $z \leftrightarrow u(z)B \in P_\alpha/B$. Express b as $b = \mu_\beta(t)w$ where w is in the root group $U_{-\beta}$ and $\mu_\beta: \mathbb{C}^* \rightarrow G$ is a one-parameter subgroup with image $T_\beta \subset G_\beta$ such that $\lambda_\beta(\mu_\beta(t)) = t$ for all $t \in \mathbb{C}^*$. Then the action of b on P_α/B is given by

$$bu(z)B = \mu_\beta(t)wu(z)B = \mu_\beta(t)u(z)\mu_\beta(t)^{-1}B = u(\alpha(\mu_\beta(t))z)B = u(t^{\langle \alpha, \beta \rangle} z)B,$$

since w and $u(z)$ always commute. Thus, in terms of the parameter z for \mathbb{P}^1 , the action is simply $z \rightarrow t^{\langle \alpha, \beta \rangle} z$.

Now let $E = E_1 \oplus \dots \oplus E_q$ be the decomposition of E as a B_β -module with $E_\nu = m_{\nu,\beta} \lambda_\beta|^{G_\beta} \otimes n_{\nu,\beta} \lambda_\beta$. Let ρ_ν denote the representation of G_β on $m_{\nu,\beta} \lambda_\beta|^{G_\beta}$. We may view $s \in E|^{P_\alpha}$ as a section of a direct sum of line bundles $\mathcal{O}(k_1) \oplus \dots \oplus \mathcal{O}(k_r)$ on \mathbb{P}^1 , where $r = \dim E$ and each k_j is one of the α -indices $n_{i,\alpha} \geq 0$, see [2], [5]. Therefore, we write $s = \sum_{1 \leq \nu \leq q} s_\nu$, $s_\nu = s_{\nu,1} e_{\nu,1} + \dots + s_{\nu,j(\nu)} e_{\nu,j(\nu)}$, where $e_{\nu,1}, \dots, e_{\nu,j(\nu)}$ is a basis for $E_\nu|^{P_\alpha}$, $\nu = 1, \dots, q$. We consider each component function $s_{\nu,v}$ to be a polynomial of degree $k(\nu, v)$ (i.e. one of the above k_j 's, depending on ν, v) in the parameter $z \in \mathbb{P}^1: s_{\nu,v}(z) = \sum_\eta c_{\nu,v}^\eta z^\eta$. Now the action of $b = \mu_\beta(t)w$ on s is given by: $(b.s)(z) = s(b^{-1}.z) = b.s(t^{-\langle \alpha, \beta \rangle} z)$. Note that on the left side of this equation b is acting in $E|^{P_\alpha}$ and on the right side the action is in E . Continuing to expand this expression further, we find

$$(b.s)(z) = \sum_{\nu=1}^q \sum_{v=1}^{j(\nu)} \sum_{\eta=0}^{k(\nu,v)} t^{n_{\nu,\beta} - \eta \langle \alpha, \beta \rangle} c_{\nu,v}^\eta z^\eta \rho_\nu(b) e_{\nu,v}$$

From this expression it is clear that the β -indices of $E|^{P_\alpha}$ are of the form $n_{\nu,\beta} - \eta \langle \alpha, \beta \rangle$, i.e. only non-negative multiples of $-\langle \alpha, \beta \rangle \geq 0$ added to the original β -indices of E . \square

As in section 1, let $w \in W$ and fix a reduced expression $w = s_{i_1} \dots s_{i_n}$. Let I be the set of simple roots corresponding to this sequence of reflections, $I = \{\alpha_j \mid j = i_1, \dots, i_n\}$. Let E be a B -module, $\mathbf{E} = G \times_B E$ the induced homogeneous vector bundle on G/B , X_w the Schubert variety associated to w in G/B , and \mathbf{E}_w the bundle \mathbf{E} restricted to X_w .

THEOREM. *The vector bundle \mathbf{E}_w is spanned by global sections if and only if the α -indices of E are non-negative for all simple roots $\alpha \in I$.*

Proof. First the necessity of the condition: If \mathbf{E}_w is spanned, then so is the bundle restricted to the G_α orbit $G_\alpha/B_\alpha \cong \mathbf{P}^1 \subset X_w \subset G/B$ for any $\alpha \in I$. Now the restricted bundle, $G_\alpha \times_{B_\alpha} E$, is spanned if and only if $E|^{P_\alpha} \rightarrow E$ is surjective. By the Lemma, this happens only when the α -indices of E are non-negative.

The sufficiency of the condition follows from the isomorphism 1.2(4) and repeated application of the previous Lemma. \square

An obvious consequence of the Theorem is the following:

COROLLARY. *A homogeneous vector bundle \mathbf{E} is spanned by global sections if and only if the α -indices of E are non-negative for all simple roots α .*

3. The tautological line bundle

Let \mathbf{E} be a vector bundle over a variety X . The projectivization of \mathbf{E} , denoted $\mathbf{P}(\mathbf{E})$, is the bundle over X defined as the space of 1-dimensional subspaces in the fibers of the dual bundle \mathbf{E}^* . Let $\xi_{\mathbf{E}}$ be the tautological line bundle over $\mathbf{P}(\mathbf{E})$ whose restriction to the fiber $\mathbf{P}(E)$ is $\mathcal{O}(1)$. There is a canonical isomorphism of sheaves $\pi_* \xi_{\mathbf{E}} \cong \mathbf{E}$ where $\pi: \mathbf{P}(\mathbf{E}) \rightarrow X$ is the bundle map. If the zero sections are removed, the two spaces are isomorphic: $\xi_{\mathbf{E}} \setminus \mathbf{P}(\mathbf{E}) \cong \mathbf{E} \setminus X$, and \mathbf{E} is spanned if and only if $\xi_{\mathbf{E}}$ is spanned. More generally, there is an isomorphism $\pi_* \xi_{\mathbf{E}}^n \cong S^n(\mathbf{E})$ where $S^n(\cdot)$ denotes the n th symmetric power. In this case, however, $\xi_{\mathbf{E}}^n$ being spanned does not necessarily imply that $S^n(\mathbf{E})$, or even \mathbf{E} , is spanned. As an application of the criterion in section 2, we prove that this implication does hold for homogeneous bundles:

THEOREM. *Let $\mathbf{E} = G \times_P E$ be a homogeneous vector bundle over a projective rational homogeneous space G/P . Then the following are equivalent:*

- (1) \mathbf{E} is spanned by global sections.
- (2) $\xi_{\mathbf{E}}$ is spanned by global sections.
- (3) $\xi_{\mathbf{E}}^n$ is spanned by global sections for some $n > 0$.
- (4) $S^n(\mathbf{E})$ is spanned by global sections for some $n > 0$.

Proof. The equivalence (1) \Leftrightarrow (2) is well-known and the implications (1) \Rightarrow (4) \Rightarrow (3) are obvious. Therefore it is sufficient to prove (3) \Rightarrow (2). Also, by 1.2(1), we may assume $P = B$.

Assume $\xi_{\mathbf{E}}$ is *not* spanned. Then by Theorem 2, there is a simple root α with a

negative α -index, $n_{i,\alpha} < 0$ for some integer i . Let $E_i = m_{i,\alpha}\lambda_\alpha|^{G_\alpha} \otimes n_{i,\alpha}\lambda_\alpha$ be the B_α -invariant submodule of E corresponding to this negative α -index. Let \mathbf{F} be the restriction of \mathbf{E}_i to the orbit under G_α of the identity coset: $G_\alpha/B_\alpha \subset G/B$, i.e. $\mathbf{F} = G_\alpha \times_{B_\alpha} E_i$. Let v be a weight vector in E_i of weight $(m_{i,\alpha} + n_{i,\alpha})\lambda_\alpha$ and let $p = 1 \times [v] \in \mathbf{P}(\mathbf{F}) = G_\alpha \times_{B_\alpha} \mathbf{P}(E_i)$, so that p is a B_α -fixed point in $\mathbf{P}(\mathbf{F})$ and $G_{\alpha \cdot p} \cong G_\alpha/B_\alpha$. If L denotes the restriction of $\xi_{\mathbf{F}}$ to $G_{\alpha \cdot p}$, then $L = G_\alpha \times_{B_\alpha} n_{i,\alpha}\lambda_\alpha$. Now, if $\xi_{\mathbf{E}}^n$ were spanned, then L^n would also be spanned, but this is impossible since $n_{i,\alpha} < 0$, so that no power of L has sections. \square

REFERENCES

- [1] ANDERSEN, H., *Vanishing theorems and induced representations*, J. Algebra 62, 86–100 (1980).
- [2] ANDERSEN, H., *Schubert varieties and Demazure's character formula*, Invent. math. 79, 611–618 (1985).
- [3] CLINE, E., PARSHALL, B. and SCOTT, L., *Induced modules and affine quotients*, Math. Ann. 230, 1–14 (1977).
- [4] CLINE, E., PARSHALL, B. and SCOTT, L., *Induced modules and extensions of representations*, Invent. math. 47, 41–51 (1978).
- [5] CLINE, E., PARSHALL, B. and SCOTT, L., *Induced modules and extensions of representations II*, J. London Math. Soc. (2) 20, 403–414 (1979).
- [6] DEMAZURE, M., *Désingularisation des variétés de Schubert généralisées*, Ann. Sc. École Norm. Sup. (4) 7, 53–88 (1974).
- [7] SNOW, D., *On the ampleness of homogeneous vector bundles*, Trans. Amer. Math. Soc. 294, 585–594 (1986).
- [8] SNOW, D., *Invariants of holomorphic affine flows*, Arch. Math. 49, 440–449 (1987).

*Department of Mathematics,
University of Notre Dame,
Notre Dame, Indiana 46556,
USA*

Received February 16, 1988