Zeitschrift: Commentarii Mathematici Helvetici
Herausgeber: Schweizerische Mathematische Gesellschaft
Band: 64 (1989)

Artikel: Spanning homogeneous vector bundles.
Autor: Snow, Dennis M.
DOI: https://doi.org/10.5169/seals-48954

Comment. Math. Helvetici 64 (1989) 395–400 0010–2571/89/030395–06$01.50 + 0.20/0

Spanning homogeneous vector bundles

DENNIS M. SNOW[1]

Let G be a semisimple complex Lie group, let P be a parabolic subgroup and let E be a rational P-module. In this note we give a simple criterion to determine whether a homogeneous vector bundle $\mathbf{E} = G \times_p E$ over the projective rational base G/P is spanned by global sections, or equivalently whether the evaluation map of the induced G-module, $E|^G \to E$, is surjective. This result complements earlier work [7] in which a formula for the ampleness of homogeneous vector bundles is derived, and generalizes results obtained in [5] for the case rank $G = 1$.

The criterion for spanning is as follows, see Corollary 2. Given a P-module E, we canonically associate to each simple root α a string of integers called the *α-indices of E* which are derived from the decomposition of E as a G_α-module. Then $\mathbf{E}$ is spanned by global sections if and only if the α-indices are non-negative for all simple roots α. The criterion is actually phrased in slightly more general terms for Schubert varieties, see Theorem 2.

A condition on a vector bundle $\mathbf{E}$ which is weaker than being spanned, but nevertheless quite useful, is to have some power of the tautological line bundle $\xi_{\mathbf{E}}$ over the projectivized bundle $\mathbf{P(E)}$ be spanned. A consequence of the above criterion for homogeneous vector bundles is that the condition of $\xi^n_{\mathbf{E}}$ being spanned is in fact equivalent to $\mathbf{E}$ being spanned, see Theorem 3. This equivalence simplifies both the statement and proof of [7, Theorem 2.1].

1. Preliminaries

All algebraic groups and varieties are assumed to be defined over the complex numbers.

1.1. Desingularization of a Schubert variety. References for this paragraph are [1], [2], [6]. Let G be a semisimple complex Lie group, B a Borel subgroup generated by the *negative* roots of G, P a parabolic subgroup, and W the Weyl group of G. Let $w \in W$ have a reduced expression $s_{i_1} \ldots s_{i_n}$ where s_j denotes the

[1] Partially supported by NSF grant DMS 8420315.

simple reflection associated to the simple root α_j. The Schubert variety in G/B associated to w, denoted by X_w, is defined to be the closure of BwB in G/B. Let P_i be the parabolic subgroup generated by the simple root α_i. A desingularization of X_w can be obtained as a quotient

$$Z_w = P_{i_1} \times \cdots \times P_{i_n}/B \times \cdots \times B$$

where the n-fold product $B \times \cdots \times B$ acts on $P_{i_1} \times \cdots \times P_{i_n}$ on the right via

$$(p_1, \ldots, p_n) \cdot (b_1, \ldots, b_n) \\ = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_{n-1}^{-1} p_n b_n), \qquad p_j \in P_{i_j}, b_j \in B.$$

The desingularization map $\phi_w : Z_w \to X_w$ is induced by the multiplication $(p_1, \ldots, p_n) \to p_1 \ldots p_n$. There is also the map $f_n : Z_w \to Z_{ws_n}$ induced from the projection $(p_1, \ldots, p_n) \to (p_1, \ldots, p_{n-1})$.

1.2. Homogeneous vector bundles. Let E be a P-module and $\mathbf{E} = G \times_P E$ the associated homogeneous vector bundle. Then $\mathbf{E}$ is spanned by global sections if and only if the evaluation map of the induced G-module, $E|^G \to E$, is surjective. (Recall that $E|^G$ is defined to be the module of all P-equivariant algebraic maps $G \to E$, and the evaluation map sends a map $v : G \to E$ to $v(1)$.) Since $E|^G$ is the same G-module whether we induce from P or from B, see e.g. [3],

$G \times_P E$ is spanned by global sections if and only if $G \times_B E$ is. (1)

For this reason, we usually let E stand for a B-module and $\mathbf{E} = G \times_B E$ for the associated homogeneous vector bundle over G/B. The restriction of $\mathbf{E}$ to X_w is denoted by $\mathbf{E}_w$ and the pull-back $\phi_w^* \mathbf{E}_w$ by $\tilde{\mathbf{E}}_w$. These bundles satisfy the following isomorphisms:

$$H^i(Z_w, \tilde{\mathbf{E}}_w) \cong H^i(X_w, \mathbf{E}_w), \qquad i \geq 0, \tag{2}$$

$$f_{n*} \tilde{\mathbf{E}}_w \cong \tilde{\mathbf{H}}_{ws_{i_n}} \text{ where } H \text{ is the } B\text{-module } H^0(P_{i_n}/B, \mathbf{E}_{s_{i_n}}) = E|^{P_{i_n}}, \tag{3}$$

see [2, Theorem 3.1, Lemma 1.4]. Through these isomorphisms and standard Leray spectral sequences based on the tower of $\mathbf{P}^1$-bundles $Z_w \to Z_{ws_n} \to \cdots \to Z_{s_n} \cong \mathbf{P}^1$ we also obtain:

$$H^0(X_w, \mathbf{E}_w) \cong E|^{P_{i_1} \cdots P_{i_n}}, \tag{4}$$

where $E|^{P_{i_1}\cdots P_{i_n}}$ is the module obtained by successively restricting to B and inducing to P_j, $j=i_1, \ldots, i_n$, see [4].

1.3. Rank one subgroups. Let G_α be the rank one simple subgroup of G generated by the positive root α, and let B_α be the intersection of G_α with B, $B_\alpha = T_\alpha U_{-\alpha}$, where T_α is a maximal torus of G_α and $U_{-\alpha}$ is the unipotent subgroup generated by $-\alpha$. Let E be a B-module. If we consider E as a $U_{-\alpha}$-module, then it is well known that E extends to a G_α-module and has a unique (up to order of factors) decomposition into a direct sum of G_α-modules: $E = E_1 \oplus \cdots \oplus E_k$ where $E_i = m_{i,\alpha}\lambda_\alpha|G_\alpha$ is the G_α-module induced from a non-negative multiple of the fundamental dominant weight λ_α (considered either as a weight of G_α or of G), see [5], [8]. Note that $\dim E_i = m_{i,\alpha}+1$. In particular, the 'zero' weight induces a one dimensional trivial module. Furthermore, each factor E_i is invariant under T_α with highest weight $t_{i,\alpha}\lambda_\alpha$, $1 \le i \le k$. Thus, as a B_α-module, $E_i = m_{i,\alpha}\lambda_\alpha|^{G_\alpha} \otimes n_{i,\alpha}\lambda_\alpha$, where $n_{i,\alpha} = t_{i,\alpha} - m_{i,\alpha}$, see [5].

DEFINITION. Let E be a B-module. For each positive root α, the *α-indices of E* are defined to be the string of integers $n_{i,\alpha}$, $1 \le i \le k$.

2. Criterion for spanning homogeneous vector bundles

The main results on spanning homogeneous vector bundles are consequences of the following lemma about B-modules induced to minimal parabolics.

LEMMA. *Let E be a B-module, and let P_α be the minimal parabolic generated by a simple root α.*

(1) *The evaluation map $E|^{P_\alpha} \to E$ is surjective if and only if the α-indices of E are non-negative.*

(2) *Let α, β be two distinct simple roots. If the α-indices and the β-indices of E are non-negative, then they are also non-negative for the induced module $E|^{P_\alpha}$.*

Proof. (1) The induced module $E|^{P_\alpha}$ is isomorphic to the space of sections of the homogeneous bundle $P_\alpha \times_B E = G_\alpha \times_{B_\alpha} E$, and thus $E|^{P_\alpha} = E|^{G_\alpha}$. As in 1.3, we write E as a B_α-module direct sum, $E = E_1 \oplus \cdots \oplus E_k$, with $E_i = m_{i,\alpha}\lambda_\alpha|^{G_\alpha} \otimes n_{i,\alpha}\lambda_\alpha$, $1 \le i \le k$. Since

$$E_i|^{G_\alpha} = m_{i,\alpha}\lambda_\alpha|^{G_\alpha} \otimes n_{i,\alpha}\lambda_\alpha|^{G_\alpha}$$

(see e.g. [3]), it is clear that $E|^{P_\alpha} \to E$ is surjective if and only if $n_{i,\alpha}\lambda_\alpha|^{G_\alpha} \neq 0$, i.e. $n_{i,\alpha} \geq 0$, $i = 1, \ldots, k$.

(2) The α-indices of $E|^{P_\alpha}$ are obviously zero. To see why the β-indices remain non-negative in this induced module, let us determine explicitly the action of $b \in B_\beta$ on a B-equivariant morphism $s: P_\alpha \to E$ (i.e. $s \in E|^{P_\alpha}$). Let $\mathbf{u}_\alpha$ be the Lie algebra of U_α, and let $u: \mathbf{u}_\alpha \to U_\alpha$ be the exponential map which in this case is an algebraic isomorphism of groups. We can use $z \in \mathbf{u}_\alpha \cong \mathbf{C}$ as a parameter for $\mathbf{P}^1 \cong P_\alpha/B$ via the correspondence $z \leftrightarrow u(z)B \in P_\alpha/B$. Express b as $b = \mu_\beta(t)w$ where w is in the root group $U_{-\beta}$ and $\mu_\beta: \mathbf{C}^* \to G$ is a one-parameter subgroup with image $T_\beta \subset G_\beta$ such that $\lambda_\beta(\mu_\beta(t)) = t$ for all $t \in \mathbf{C}^*$. Then the action of b on P_α/B is given by

$$bu(z)B = \mu_\beta(t)wu(z)B = \mu_\beta(t)u(z)\mu_\beta(t)^{-1}B = u(\alpha(\mu_\beta(t))z)B = u(t^{\langle\alpha,\beta\rangle}z)B,$$

since w and $u(z)$ always commute. Thus, in terms of the parameter z for $\mathbf{P}^1$, the action is simply $z \to t^{\langle\alpha,\beta\rangle}z$.

Now let $E = E_1 \oplus \cdots \oplus E_q$ be the decomposition of E as a B_β-module with $E_\nu = m_{\nu,\beta}\lambda_\beta|^{G_\beta} \otimes n_{\nu,\beta}\lambda_\beta$. Let ρ_ν denote the representation of G_β on $m_{\nu,\beta}\lambda_\beta|^{G_\beta}$. We may view $s \in E|^{P_\alpha}$ as a section of a direct sum of line bundles $\mathcal{O}(k_1) \oplus \cdots \oplus \mathcal{O}(k_r)$ on $\mathbf{P}^1$, where $r = \dim E$ and each k_j is one of the α-indices $n_{i,\alpha} \geq 0$, see [2], [5]. Therefore, we write $s = \sum_{1 \leq \nu \leq q} s_\nu$, $s_\nu = s_{\nu,1}e_{\nu,1} + \cdots + s_{\nu,j(\nu)}e_{\nu,j(\nu)}$, where $e_{\nu,1}, \ldots, e_{\nu,j(\nu)}$ is a basis for $E_\nu|^{P_\alpha}$, $\nu = 1, \ldots, q$. We consider each component function $s_{\nu,\upsilon}$ to be a polynomial of degree $k(\nu, \upsilon)$ (i.e. one of the above k_j's, depending on ν, υ) in the parameter $z \in \mathbf{P}^1 : s_{\nu,\upsilon}(z) = \sum_\eta c^\eta_{\nu,\upsilon} z^\eta$. Now the action of $b = \mu_\beta(t)w$ on s is given by: $(b.s)(z) = s(b^{-1}.z) = b.s(t^{-\langle\alpha,\beta\rangle}z)$. Note that on the left side of this equation b is acting in $E|^{P_\alpha}$ and on the right side the action is in E. Continuing to expand this expression further, we find

$$(b.s)(z) = \sum_{\nu=1}^{q} \sum_{\upsilon=1}^{j(\nu)} \sum_{\eta=0}^{k(\nu,\upsilon)} t^{n_{\nu,\beta} - \eta\langle\alpha,\beta\rangle} c^\eta_{\nu,\upsilon} z^\eta \rho_\nu(b) e_{\nu,\upsilon}$$

From this expression it is clear that the β-indices of $E|^{P_\alpha}$ are of the form $n_{\nu,\beta} - \eta\langle\alpha, \beta\rangle$, i.e. only non-negative multiples of $-\langle\alpha, \beta\rangle \geq 0$ added to the original β-indices of E. $\square$

As in section 1, let $w \in W$ and fix a reduced expression $w = s_{i_1} \ldots s_{i_n}$. Let I be the set of simple roots corresponding to this sequence of reflections, $I = \{\alpha_j \mid j = i_1, \ldots, i_n\}$. Let E be a B-module, $\mathbf{E} = G \times_B E$ the induced homogeneous vector bundle on G/B, X_w the Schubert variety associated to w in G/B, and $\mathbf{E}_w$ the bundle $\mathbf{E}$ restricted to X_w.

THEOREM. *The vector bundle* $\mathbf{E}_w$ *is spanned by global sections if and only if the* α*-indices of* E *are non-negative for all simple roots* $\alpha \in I$.

Proof. First the necessity of the condition: If $\mathbf{E}_w$ is spanned, then so is the bundle restricted to the G_α orbit $G_\alpha/B_\alpha \cong \mathbf{P}^1 \subset X_w \subset G/B$ for any $\alpha \in I$. Now the restricted bundle, $G_\alpha \times_{B_\alpha} E$, is spanned if and only if $E|^{P_\alpha} \to E$ is surjective. By the Lemma, this happens only when the α-indices of E are non-negative.

The sufficiency of the condition follows from the isomorphism 1.2(4) and repeated application of the previous Lemma. □

An obvious consequence of the Theorem is the following:

COROLLARY. *A homogeneous vector bundle* $\mathbf{E}$ *is spanned by global sections if and only if the* α*-indices of* E *are non-negative for all simple roots* α.

3. The tautological line bundle

Let $\mathbf{E}$ be a vector bundle over a variety X. The projectivization of $\mathbf{E}$, denoted $\mathbf{P}(\mathbf{E})$, is the bundle over X defined as the space of 1-dimensional subspaces in the fibers of the dual bundle $\mathbf{E}^*$. Let $\xi_{\mathbf{E}}$ be the tautological line bundle over $\mathbf{P}(\mathbf{E})$ whose restriction to the fiber $\mathbf{P}(E)$ is $\mathcal{O}(1)$. There is a canonical isomorphism of sheaves $\pi_*\xi_{\mathbf{E}} \cong \mathbf{E}$ where $\pi: \mathbf{P}(\mathbf{E}) \to X$ is the bundle map. If the zero sections are removed, the two spaces are isomorphic: $\xi_{\mathbf{E}} \backslash \mathbf{P}(\mathbf{E}) \cong \mathbf{E} \backslash X$, and $\mathbf{E}$ is spanned if and only if $\xi_{\mathbf{E}}$ is spanned. More generally, there is an isomorphism $\pi_*\xi_{\mathbf{E}}^n \cong S^n(\mathbf{E})$ where $S^n(\cdot)$ denotes the nth symmetric power. In this case, however, $\xi_{\mathbf{E}}^n$ being spanned does not necessarily imply that $S^n(\mathbf{E})$, or even $\mathbf{E}$, is spanned. As an application of the criterion in section 2, we prove that this implication does hold for homogeneous bundles:

THEOREM. *Let* $\mathbf{E} = G \times_P E$ *be a homogeneous vector bundle over a projective rational homogeneous space* G/P. *Then the following are equivalent*:

(1) $\mathbf{E}$ *is spanned by global sections.*
(2) $\xi_{\mathbf{E}}$ *is spanned by global sections.*
(3) $\xi_{\mathbf{E}}^n$ *is spanned by global sections for some* $n > 0$.
(4) $S^n(\mathbf{E})$ *is spanned by global sections for some* $n > 0$.

Proof. The equivalence $(1) \Leftrightarrow (2)$ is well-known and the implications $(1) \Rightarrow (4) \Rightarrow (3)$ are obvious. Therefore it is sufficient to prove $(3) \Rightarrow (2)$. Also, by 1.2(1), we may assume $P = B$.

Assume $\xi_{\mathbf{E}}$ is *not* spanned. Then by Theorem 2, there is a simple root α with a negative α-index, $n_{i,\alpha} < 0$ for some integer i. Let $E_i = m_{i,\alpha}\lambda_\alpha|^{G_\alpha} \otimes n_{i,\alpha}\lambda_\alpha$ be the B_α-invariant submodule of E corresponding to this negative α-index. Let $\mathbf{F}$ be the restriction of $\mathbf{E}_i$ to the orbit under G_α of the identity coset: $G_\alpha/B_\alpha \subset G/B$, i.e. $\mathbf{F} = G_\alpha \times_{B_\alpha} E_i$. Let v be a weight vector in E_i of weight $(m_{i,\alpha} + n_{i,\alpha})\lambda_\alpha$ and let $p = 1 \times [v] \in \mathbf{P}(\mathbf{F}) = G_\alpha \times_{B_\alpha} \mathbf{P}(E_i)$, so that p is a B_α-fixed point in $\mathbf{P}(\mathbf{F})$ and $G_{\alpha \cdot p} \cong G_\alpha/B_\alpha$. If L denotes the restriction of $\xi_{\mathbf{F}}$ to $G_{\alpha \cdot p}$, then $L = G_\alpha \times_{B_\alpha} n_{i,\alpha}\lambda_\alpha$. Now, if $\xi_{\mathbf{E}}^n$ were spanned, then L^n would also be spanned, but this is impossible since $n_{i,\alpha} < 0$, so that no power of L has sections. $\square$

REFERENCES

[1] ANDERSEN, H., *Vanishing theorems and induced representations,* J. Algebra *62,* 86–100 (1980).

[2] ANDERSEN, H., *Schubert varieties and Demazure's character formula,* Invent. math. *79,* 611–618 (1985).

[3] CLINE, E., PARSHALL, B. and SCOTT, L., *Induced modules and affine quotients,* Math. Ann. *230,* 1–14 (1977).

[4] CLINE, E., PARSHALL, B. and SCOTT, L., *Induced modules and extensions of representations,* Invent. math. *47,* 41–51 (1978).

[5] CLINE, E., PARSHALL, B. and SCOTT, L., *Induced modules and extensions of representations II,* J. London Math. Soc. (2) *20,* 403–414 (1979).

[6] DEMAZURE, M., *Désingularisation des variétés de Schubert généralisées,* Ann. Sc. École Norm. Sup. (4) *7,* 53–88 (1974)

[7] SNOW, D., *On the ampleness of homogeneous vector bundles,* Trans. Amer. Math. Soc. *294,* 585–594 (1986).

[8] SNOW, D., *Invariants of holomorphic affine flows,* Arch. Math. *49,* 440–449 (1987).

Department of Mathematics,
University of Notre Dame,
Notre Dame, Indiana 46556,
USA

Received February 16, 1988