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Minimal measures

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Abstract. For a finite composition of exact, area preserving, positive twist diffeomorphisms of the infinite cylinder, we will show that an invariant measure whose support consists of minimal orbits strictly minimizes the average action.

§1. Introduction

In [6], we proved the existence of certain invariant sets for exact, area preserving twist diffeomorphisms of an annulus, which provides a kind of "weak solution" for the problem for which KAM theory provides a true solution, namely the problem of finding invariant circles. In [1], Aubry and Le Daeron independently developed a theory of minimal configurations. There is a corresponding theory of minimal orbits of twist diffeomorphisms, which slightly generalizes the author's theory in [6], and gives an alternative proof of the main result of [6].

The "Aubry–Mather sets", whose existence was proved in [1] and [6], carry unique invariant measures. In this paper, we characterize these measures by a minimality property. Our principal result is not at all surprising, given what is already known, and the proofs are basically an exercise in applying what is known, but they are not totally obvious, so we have written them down.

Our principal result concerns the average action $A(\bar{f}, \mu)$ of an $\bar{f}$-invariant measure μ, where $\bar{f}$ is a finite composition of exact, area preserving, positive twist diffeomorphisms of the infinite cylinder. The notion of average action is an obvious extension of the standard notion of the action of a periodic orbit. The definition of $A(\bar{f}, \mu) = A_{H,\eta}(\bar{f}, \mu)$ depends on a choice of a periodic Hamiltonian H such that $\bar{f}$ is the time one map of the associated flow, as well as a choice of a one form η on the infinite cylinder.

We show a couple of variants of standard results in our context. First, changing H changes $A_{H,\eta}(\bar{f}, \mu)$ only by a constant of integration. Second, changing η changes $A_{H,\eta}(\bar{f}, \mu)$ according to the formula

$$A_{H,\eta'}(\bar{f}, \mu) = A_{H,\eta}(\bar{f}, \mu) + [\eta' - \eta] \cdot \rho(f, \mu),$$

where $\rho(f, \mu)$ denotes the rotation number of μ and $[\eta' - \eta] = \int_\Gamma \eta' - \eta$, where Γ is any closed curve which winds once around the cylinder. Thus, $A(\bar{f}, \mu)$ is well defined modulo an affine function in the rotation number.

This leads to our definition of minimal measure: An $\bar{f}$-invariant measure μ will be said to be *minimal* if there exists $\lambda \in \mathbb{R}$ such that μ minimizes $A(\bar{f}, \mu) - \lambda\rho(\bar{f}, \mu)$ over all $\bar{f}$-invariant measures μ.

In terms of the Aubry theory of minimal orbits, the minimal measures (of $\bar{f}$) may be described as follows. Let $\omega \in \mathbb{R}$ and let RM_ω denote the set of recurrent minimal orbits of rotation number ω. Here, "minimal" is taken in Aubry's sense and recurrent in the usual sense of topological dynamics. Our principal result is that a measure is minimal if and only if its support lies in RM_ω, for some ω. Note that when ω is irrational, RM_ω is the "Aubry–Mather set" which, being a Denjoy minimal set (in the sense of topological dynamics), carries a unique invariant measure μ_ω. Thus, for $\omega \notin Q$, our principal result implies that, for suitable λ, the measure μ_ω minimizes $A(\bar{f}, \mu) - \lambda\rho(f, \mu)$ and this uniquely characterizes μ_ω.

Let $A(\omega) = A_{\bar{f}}(\omega) = A(\bar{f}, \mu_\omega)$, where μ_ω is any measure with support in RM_ω. Aubry proved in 1983 that $A(\omega)$ is strictly convex. As far as I know this proof has never been published, so I will present Aubry's proof in §4, since this result is needed to prove the results just referred to. Since A is convex, it has a sub-derivative at each ω. Conversely, if $\lambda \in \mathbb{R}$, then there is a unique ω at which λ is a sub-derivative of A. Uniqueness follows from the strict convexity, and existence is easily proved. Thus, our principal results may be restated, as follows: If λ is a sub-derivative of A at ω, then an $\bar{f}$-invariant probability measure μ on the infinite cylinder minimizes $A(\bar{f}, \mu) - \lambda\rho(f, \mu)$ if and only if $\operatorname{supp}\mu \subset RM_\omega$.

§2. Action

In this section, we review some standard notions (from e.g. Cartan [3]) in Hamiltonian dynamics. Let (M, Ω) denote a symplectic manifold, i.e. we suppose that M is connected and $2n$-dimensional and Ω is a symplectic 2-form on it, i.e. $d\Omega = 0$ and Ω^n $(= \Omega \wedge \cdots \wedge \Omega)$ vanishes nowhere. If H is a sufficiently smooth function on $M \times \mathbb{R}$ (called a Hamiltonian), then there is a time dependent vector field $X = X_H$, called the *symplectic gradient* of H, defined by

$$\Omega(Y, X_t) = Y \cdot H_t,$$

for every vector field Y on M. Here $H_t(x) = H(x, t)$, for $x \in M$, and $Y \cdot H_t$ denotes the directional derivative of H_t in the direction Y. We will suppose that X_H is *globally integrable*, in the sense that there is a (sufficiently smooth) mapping

$\phi: M \times \mathbb{R} \to M$ such that $\phi_0(x) = x$ and $d\phi_t(x)/dt = X_t(\phi_t(x))$, for all $x \in M$ and $t \in \mathbb{R}$. Here, $\phi_t(x) = \phi(x, t)$. By a *finite trajectory* Γ of H, we will mean a (sufficiently smooth) mapping $\Gamma: [t_0, t_1] \to M$ such that $d\Gamma(t)/dt = H(\Gamma(t), t)$, for $t_0 \le t \le t_1$. We will suppose that Ω is exact, i.e. there exists a 1-form η such that $d\eta = \Omega$. Following Hamilton, Poincaré, and Cartan (see [3]), we define the *action* of such a finite trajectory as

$$L_\eta(\Gamma) = \int_\Gamma \eta - H\,dt.$$

In what follows, we will be interested in the case when H is periodic of period one in t. We let $\bar{f}$ denote the time one map of the flow associated to H, i.e. $\bar{f} = \phi_1$. If Γ is a trajectory of the flow generated by H, then $(\ldots, \Gamma(i), \ldots)$ is an orbit of $\bar{f}$. In this way, we have a one–one correspondence between the orbits of $\bar{f}$ and the trajectories of the flow generated by H. If $\mathcal{O} = (\ldots, P_i, \ldots)$ is an orbit of $\bar{f}$, and Γ is the corresponding trajectory of the flow, we set (for $i < j \in \mathbb{Z}$)

$$L_\eta(\mathcal{O})(i, j) = L_\eta(\Gamma \mid [i, j]).$$

We still call this the *action* of the trajectory $(\Gamma(i), \ldots, \Gamma(j))$.

For $L_\eta(\mathcal{O})(i, j)$ to be invariant of $\bar{f}$, it would have to depend only on $\bar{f}$, i, and j, and be independent of H and η. However, this is not the case. Next, we analyze the dependence on H and η, in order to construct invariants of $\bar{f}$ out of $L_\eta(\mathcal{O})(i, j)$.

First, we consider changing H. Let H' be a second Hamiltonian such that $\bar{f}$ is still the time one map of the flow it generates. Let

$$C(x) = \int_{\Gamma'(x)} \eta - H'\,dt - \int_{\Gamma(x)} \eta - H\,dt,$$

where $\Gamma(x)$ is the trajectory of the flow generated by H between $t = 0$ and $t = 1$, starting at x and ending (of course) at $\bar{f}(x)$, and where $\Gamma'(x)$ is defined in the same way in terms of H'.

In fact, $C(x)$ is independent of x by an argument in Cartan [3]. Here is a version of this argument. Since M is connected, any two points in M may be connected by a smooth curve x_s, $0 \le s \le 1$. We let S denote the surface in $M \times \mathbb{R}$ swept out by $\tilde{\Gamma}(x_s)$, $0 \le s \le 1$, where $\tilde{\Gamma}(t) = (\Gamma(t), t)$. Since, for fixed s, the tangent to the curve $\Gamma(x_s)$ is in the kernel of $d(\eta - H\,dt) = \Omega - dH \wedge dt$ (by definition of the Hamiltonian flow), we have that the restriction of $d(\eta - H\,dt)$ to S vanishes. Hence, by Stokes's Theorem, the integral of $\eta - H\,dt$ around the boundary of S vanishes. We may apply the same argument to the surface S', swept out by $\Gamma'(x_s)$ and the 1-form $\eta - H'\,dt$; we find that the integral of $\eta - H'\,dt$ around the boundary of S' vanishes. Note that S and S' have the curves $\Lambda = \{x_s : 0 \le s \le 1\}$ in $M \times 0$ and $\bar{f}\Lambda$ in $M \times 1$ in common, and that the restriction of $\eta - H'\,dt$ to each of these curves is the same as the restriction of $\eta - H\,dt$. Therefore,

$$C(x_1) - C(x_0) = \int_{\partial S} \eta - H\,dt - \int_{\partial S'} \eta - H'\,dt = 0.$$

This shows that $C(x)$ is independent of x.

Let $C = C(x)$. Since the constant of integration C is independent of x, replacing H by H' changes $L_\eta(\mathcal{O})(i, j)$ by $C(j - i)$. Note that C is independent of $\mathcal{O}$, i, and j. It does depend, however, on η.

Thus, changing H changes $L_\eta(\mathcal{O})(i, j)$ only by a constant of integration. Changing η makes a more serious change. Clearly, $L_{\eta'}(\mathcal{O})(i, j) - L_\eta(\mathcal{O})(i, j) = \int_\Gamma \eta' - \eta$. If $\mathcal{O}$ is periodic of period $j - i$ then Γ is a closed trajectory and we have $L_{\eta'}(\mathcal{O})(i, j) - L_\eta(\mathcal{O})(i, j) = [\eta' - \eta] \cdot [\Gamma \mid [i, j]]$ where $[\eta' - \eta]$ denotes the de Rham cohomology class in $H^1(M, \mathbb{R})$ of $\eta' - \eta$ and $[\Gamma \mid [i, j]]$ denotes the homology class in $H_1(M, \mathbb{R})$ of $\Gamma \mid [i, j]$. Note that $\eta' - \eta$ is closed, since $d\eta' = d\eta = \Omega$.

Let $\tilde{M}$ denote the covering space of M whose fundamental group is the kernel of the Hurewiez homomorphism $\pi_1(M) \to H_1(M, \mathbb{R})$. Let $\tilde{\phi}$ be a lift of the flow to $\tilde{M}$, so $\tilde{\phi}_0 =$ identity, $\tilde{\phi}_t$ is a lift of ϕ_t to $\tilde{M}$, and $\tilde{\phi}_t$ depends continuously on all variable. Let $f = \tilde{\phi}_1$, i.e. f is the time one map of the flow on $\tilde{M}$ generated by H. Then f is a lift of $\bar{f}$ to $\tilde{M}$.

We define the *rotation number* of a periodic orbit $\mathcal{O}$ of $\bar{f}$ (with respect to a lift f of $\bar{f}$), by

$$\rho(f, \mathcal{O}) = [\Gamma \mid [i, j]]/(j - i),$$

if $\mathcal{O}$ is periodic of period $j - i$, where Γ is the trajectory of the flow generated by H corresponding to $\mathcal{O}$. Thus, $\rho(f, \mathcal{O}) \in H_1(M, \mathbb{R})$. Note that $\rho(f, \mathcal{O})$ depends only on the lift f of $\bar{f}$ to $\tilde{M}$, and not otherwise on H. In fact, we could alternatively define $\rho(f, \mathcal{O})$ by choosing a lift $\tilde{\mathcal{O}} = (\ldots, \tilde{P}_i, \ldots)$ of $\mathcal{O}$ and setting

$$\rho(f, \mathcal{O}) = [\Lambda]/(j - i),$$

where Λ is the image in M of a curve in $\tilde{M}$ connecting $\tilde{P}_i$ to $\tilde{P}_j$. It is an easy exercise in the theory of covering spaces to see that $[\Lambda] \in H_1(M, \mathbb{R})$ is independent of the curve chosen and that $[\Lambda] = [\Gamma \mid [i, j]]$.

This permits us to write the effect of changing η in the following form:

$$L_{\eta'}(\mathcal{O})(i, j) - L_\eta(\mathcal{O})(i, j) = (j - i)[\eta' - \eta] \cdot \rho(f, \mathcal{O}).$$

All this is standard. There is, furthermore, an obvious generalization to the case of invariant measures. Let μ be an $\bar{f}$-invariant probability (Borel) measure on M. We define the *average action,*

$$A(\bar{f}, \mu) = \int_M d\mu(x) \int_{\Gamma(x)} \eta - H\, dt,$$

if this integral exists. For example, if μ is the unique invariant measure supported by a periodic orbit $\mathcal{O}$ of period $j - i$, then $A(\bar{f}, \mu) = L_\eta(\mathcal{O})(i, j)/(j - i)$.

We will sometimes write this as $A_{H,\eta}(\bar{f}, \mu)$ or $A_\eta(\bar{f}, \mu)$ to make the dependence on H and η explicit. If we change H, the average action changes only by a constant of integration:

$$A_{H',\eta}(\bar{f}, \mu) - A_{H,\eta}(\bar{f}, \mu) = C,$$

where

$$C = \int_{\Gamma'(x)} \eta - H'\, dt - \int_{\Gamma(x)} \eta - H\, dt.$$

We proved above that C is independent of x, so, of course, it is independent of μ.

To analyze the dependence on η, we need the notion of the *rotation number* of an invariant measure μ. We consider a Borel function $U: \tilde{M} \to H_1(M, \mathbb{R})$, such that $U \circ T - U = [T]$ for any Deck transformation T, where $[T] \in H_1(M, \mathbb{R})$ denotes the homology class corresponding to T. (Recall that since the fundamental group of $\tilde{M}$ is the kernel of the Hurewicz homomorphism $\pi_1(M) \to H_1(M, \mathbb{R})$, the group of Deck transformations of $\tilde{M}$ over M is canonically isomorphic to the image of this homomorphism.) For simplicity, we will suppose that $H_1(M, \mathbb{R})$ is finite dimensional. We provide $H_1(M, \mathbb{R})$ with its usual topology as a finite dimensional vector space. Also, when we talk about "manifold", we *always* take as part of the definition the hypothesis that a manifold is Hausdorff and has a countable basis for its topology. A Borel function U as above is easily constructed: take any Borel fundamental domain of $\tilde{M}$ over M and choose U to be any Borel function on the fundamental domain. Then it has a unique extension to all of M satisfying $U \circ T - U = [T]$, for every Deck transformation T.

It is easily seen that $U \circ f - U$ is invariant under every Deck transformation. Hence, $U \circ f - U$ may be considered as a function on M; we define

$$\rho(f, \mu) = \int_M (U \circ f - U)\, d\mu \in H_1(M, \mathbb{R}),$$

as long as this integral exists, i.e. as long as $U \circ f - U$ is an L^1 function on M with respect to the measure μ. In fact, we will define $\rho(f, \mu)$ by the above formula, as long as there is one Borel function U satisfying $U \circ T - U = [T]$, for which the above integral exists. It is easily seen that $\rho(f, \mu)$ is independent of U satisfying these conditions, since μ is $\bar{f}$-invariant.

Moreover, if $\mathcal{O}$ is a periodic orbit and μ is the unique invariant measure supported in $\mathcal{O}$, then $\rho(f, \mu)$ is just the $\rho(f, \mathcal{O})$ defined above. The change in the average action when we change η is then given by

$$A_{H,\eta'}(\bar{f}, \mu) - A_{H,\eta}(\bar{f}, \mu) = [\eta' - \eta] \cdot \rho(f, \mu),$$

where f is the lift to $\tilde{M}$ of $\bar{f}$ which is the time one map of the flow on $\tilde{M}$ which is generated by the Hamiltonian H. This formula is valid as long as both sides are defined. It may be proved by observing that since $d\eta = d\eta'$, the lift of $\eta' - \eta$ to $\tilde{M}$ is exact, i.e. $\eta' - \eta = dU$ on $\tilde{M}$, where $U \circ T - U = [\eta' - \eta] \cdot [T]$, for any Deck transformation T, and that $\int_{\Gamma(x)} \eta' - \eta = U \circ f(x) - U(x)$, since $\partial\Gamma(x) = f(x) - x$.

Note that both sides are defined and therefore the formula above is valid when μ has compact support. From the above formulas for the change in the average action when we change H or η, we see that $\mu \to A_{H,\eta}(f, \mu)$ is changed by at most an affine function of $\rho(f, \mu) \in H_1(M, \mathbb{R})$.

§3. Minimal orbits

In this section, we recall the notion of minimal orbits, due to Aubry [1], as generalized by Bangert [2]. The notion of minimal orbits is defined by these authors only for a special class of mappings, although other authors have considered related set-ups [5], [11]. We need the following notation to describe the class of mappings which we will consider here. We let $\mathcal{T}^1_\beta$ denote the set of exact area preserving, orientation preserving, positive monotone twist mappings of the infinite cylinder $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$ which preserve the ends, twist each end infinitely, and have β as a uniform lower bound for the amount of twisting. A detailed definition of $\mathcal{T}^1_\beta$ is given in [8, §2], where it is called J_β. We let $\mathcal{P}^1_\beta$ denote the set of finite compositions of elements of $\mathcal{T}^1_\beta$. We let

$$\mathcal{T}^1 = \bigcup_{\beta>0} \mathcal{T}^1_\beta \quad \text{and} \quad \mathcal{P}^1 = \bigcup_{\beta>0} \mathcal{P}^1_\beta.$$

It is known that an exact area preserving C^1 diffeomorphism $\bar{f}$ of the infinite cylinder which preserves the ends is the time one map of the flow generated by a suitable Hamiltonian H. Of course, we must allow H to be time dependent for this to be true, but we may take H to be periodic of period one. In the language of differential topology, this amounts to the assertion that $\bar{f}$ is isotopic to the identity in the class of exact area preserving C^1 diffeomorphisms of the infinite cylinder. This result is well known to experts in differential topology. For a proof, see the fine print at the end of [4]. For example if $\bar{f} \in \mathcal{P}^1$, the average action $A_{H,\eta}(\bar{f}, \mu)$ is defined, where, as before, we choose a 1-form η such that $d\eta = \Omega$, the area form on the infinite cylinder. If $\bar{f} \in \mathcal{T}^1$, so it has a generating function h, the average action is given by

$$A_{H,\eta}(\bar{f}, \mu) = \int_M h(x, x')\, d\mu(x, y) + C,$$

in the case that $\eta = y\,dx$, where M is the infinite cylinder, f is the lift of $\bar{f}$ to the universal cover $\mathbb{R}^2$ of M which is the time one map of the flow generated by the Hamiltonian H, and $f(x, y) = (x', y')$. Note that since $f(x+1, y) = (x'+1, y')$ and $h(x+1, x'+1) = h(x, x')$, this integral is well defined as an integral over M, even though (x, y) and (x', y') denote points in the universal cover. Here, C is a constant of integration, which is independent of the measure μ.

To prove this formula, we relate the generating function h to the action, as follows. Let $\Lambda = \{(x_s, y_s) : 0 \le s \le 1\}$ be a smooth curve in the cylinder $M = (\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$. For each s, let $\Gamma_s = \Gamma(x_s, y_s) : [0, 1] \to M \times \mathbb{R}$ denote the curve used in the definition of the average action in the last section, i.e. $\Gamma_s(t) = (\phi_t(x_s, y_s), t)$, where ϕ is the flow generated by the Hamiltonian H. Let S denote the surface swept out by Γ_s as s varies between 0 and 1. By Stokes's theorem, we have

$$\int_{\partial S} \eta - H\,dt = \int_S \Omega - dH \wedge dt = 0,$$

since the tangent to Γ_s is in the kernel of $d(\eta - H\,dt)$ and therefore $\Omega - dH \wedge dt$ vanishes identically on S. We have $\partial S = \Gamma_0 + \Lambda' - \Gamma_1 - \Lambda$, where $\Lambda' = \bar{f}\Lambda$. Setting

$f(x_s, y_s) = (x'_s, y'_s)$, we have

$$h(x_1, x'_1) - h(x_0, x'_0) = \int_\Lambda y'\,dx' - y\,dx$$
$$= \int_{\Lambda'} \eta - \int_\Lambda \eta = \int_{\Gamma_1} \eta - H\,dt - \int_{\Gamma_0} \eta - H\,dt.$$

Here, the first equation is the definition of h, the second equation follows from the definitions and our choice $\eta = y\,dx$, and the third equation follows from the fact that dt vanishes identically on Λ and Λ', together with the previous equation.

We may express this equation more simply as

$$h(x, x') = \int_\Gamma \eta - H\,dt - C$$

for (x, y) in the cylinder, $f(x, y) = (x', y')$, and $\Gamma = \Gamma(x, y): [0, 1] \to M \times \mathbb{R}$ as above. Here, C is a constant of integration, independent of x and y. If μ is an $\bar{f}$ invariant probability measure, we may integrate the last formula with respect to μ, and obtain the formula for the average action we stated at the beginning of this section.

We may generalize this formula for the average action to the situation when $\bar{f} \in \mathscr{P}^1$, as follows. Let $\bar{f} = \bar{f}_k \cdots \bar{f}_1$ with $\bar{f}_i \in \mathscr{T}^1$. Let $h_1, \ldots, h_k$ be the corresponding generating functions. For $(x, y) \in \mathbb{R}^2$, let $f_1(x, y) = (x', y')$, $f_2 f_1(x, y) = (x'', y''), \ldots, f_k \cdots f_1(x, y) = (x^{(k)}, y^{(k)})$. Then

$$A_{H,\eta}(\bar{f}, \mu) = \int_M \sum_{i=0}^{k-1} h_{i+1}(x^{(i)}, x^{(i+1)})\,d\mu(x, y) + C,$$

for any $\bar{f}$-invariant measure μ, where $\eta = y\,dx$ and C is a constant, independent of μ.

In [8], we proved that for $\bar{f}_i \in \mathscr{T}^1$, the corresponding generating function h_i satisfies (H_1) and (H_2) of [8]. It follows that each h_i is uniformly bounded below. The above formula for $h(x, x')$ then shows that the average action $A_{H,\eta}(\bar{f}, \mu)$ is well defined, for any $\bar{f}$-invariant probability measure μ, although it may be $+\infty$.

So far in this section, we have considered the average action $A_{H,\eta}(\bar{f}, \mu)$ only for $\eta = y\,dx$. If the $\bar{f}$-invariant measure μ has compact support, we may apply the formula at the end of the last section for the dependence of the average action on H and η, and obtain a formula for $A_{H',\eta'}(\bar{f}, \mu)$ for any η'. In this case

$$[\eta' - \eta] = \int_\Gamma \eta' - \eta \in \mathbb{R} = H^1(M, \mathbb{R}),$$

where M is the cylinder and Γ is any smooth, closed curve, which goes once around M in the positive sense. Moreover, the rotation number is given by

$$\rho(f, \mu) = \int_M (x' - x)\, d\mu(x, y) \in \mathbb{R} = H_1(M, \mathbb{R}),$$

where $f(x, y) = (x', y')$, as usual.

In the general case, when μ does not necessarily have compact support, we have that if $A_{H,\eta}(\bar{f}, \mu) < +\infty$, then $\rho(f, \mu)$ is defined, in the sense that $x' - x \in L^1(M, \mu)$. This again follows from the last formula for $A_{H,\eta}(\bar{f}, \mu)$ and the formula before that for $h(x, x')$. For, consider $\bar{f} \in \mathcal{T}^1_\beta$ and its generating function h. We have that $y = -\partial_1 h(x, x')$ and $y' = \partial_2 h(x, x')$ go to $\pm\infty$ as $x' - x$ goes to $\pm\infty$, uniformly in x. Consequently, there exist constants C_1 and C_2 such that

$$|x' - x| < C_1 + C_2 h(x, x'),$$

for all $(x, x') \in \mathbb{R}^2$. The fact that $x' - x \in L^1(M, \mu)$ when $A_{H,\eta}(\bar{f}, \mu) < +\infty$ follows immediately. The other assertions may be proved as before.

For each $\bar{f} \in \mathcal{P}^1$ and each $\omega \in \mathbb{R}$, there is an $\bar{f}$-invariant probability measure μ_ω whose support consists entirely of minimal orbits of rotation number ω. If ω is a rational number p/q, we may obtain such a measure by choosing a minimal orbit, periodic of period (q, p), and taking μ_ω to be the unique $\bar{f}$-invariant probability measure supported by that orbit. If ω is irrational, the existence of μ_ω may be deduced from the theory developed in [6] or from the theory developed in [1], although we need to refer to [2] for proofs in the generality considered here. The measure μ_ω is the unique invariant measure with support in the "Aubry–Mather set" of rotation number ω. Note that in [8], we proved that the variational principal $h = h_1 * \cdots * h_k$ associated to f satisfies (H_1)–(H_4) of [8], where $h_1 * h_2$ is the "conjunction" of h_1 and h_2, defined by $h_1 * h_2(x, x') = \min_y (h_1(x, y) + h_2(y, x'))$. This permits us to apply the theory of [2].

We will denote $A(\bar{f}, \mu_\omega)$ by $A(\omega)$ or $A_{\bar{f}}(\omega)$. Note that $A(\omega)$ is well defined when ω is irrational because μ_ω is unique in that case. It is well defined when ω is rational because the action of all minimal orbits of the same period coincides. In what follows, it will be convenient to assume that $\eta = y\, dx$ and H is chosen so that the constant C of integration vanishes. With these conventions, we have

$$A(\omega) = \int_M h(x, x')\, d\mu_\omega(x, y),$$

where $f(x, y) = (x', y')$. This follows from the previous formula for the average action and the observation that

$$h(x, x^{(k)}) = \sum_{i=0}^{k-1} h_{i+1}(x^{(i)}, x^{(i+1)}),$$

for $(x, y) \in \operatorname{supp} \mu_\omega$. Note that for a general $\bar{f}$-invariant measure μ, this simplification is not possible; we have used the fact that any orbit in $\operatorname{supp} \mu_\omega$ is minimal.

§4. Strict convexity of $A_{\bar{f}}$

As Aubry pointed out to me in 1983, the function $A_{\bar{f}}(\omega) = A(\omega)$, defined in the last section, is a strictly convex function of ω, i.e. the line segment joining any two points of the graph of A lies above the graph of A. The proof which he told me (for the case $\bar{f} \in \mathcal{T}^1$) follows very simply from his theory. We present it here under the more general hypothesis that $\bar{f} \in \mathcal{P}^1$. The only fact about $\bar{f}$ which we use is the fact proved in [8] that its variational principal h satisfies the conditions (H_1)–(H_6).

We begin with the following approximate value for $h(x_i, \ldots, x_j)$ when $(x_i, \ldots, x_j)$ is a minimal segment of a configuration. This is more precise than what is needed for Aubry's proof, but may nonetheless be of interest. Since $\bar{f} \in \mathcal{P}^1 = \bigcup_{\beta>0} \mathcal{P}^1_\beta$, it is in $\mathcal{P}^1_\beta$ for some $\beta > 0$. Let $\theta = \cot \beta$. Then the variational principal h of $\bar{f}$ satisfies $(H_{6\theta})$, as was proved in [8]. We will show that

$$|h(x_i, \ldots, x_j) - (j-i)A((x_j - x_i)/(j-i))| \leq 8\theta.$$

The proof of this inequality follows very easily from properties of the conjunction developed in [8]. Let $H = h * \cdots * h$ denote the conjunction of h with itself $(j-i)$-times. Since $(x_i, \ldots, x_j)$ is minimal, we have $h(x_i, \ldots, x_j) = H(x_i, x_j)$. Let $\omega = (x_j - x_i)/(j-i)$. Then

$$(j-i)A(\omega) = \int_M H(\xi_i, \xi_j)\, d\mu_\omega(\xi, \eta),$$

where we set $f^i(\xi, \eta) = (\xi_i, \eta_i)$. This formula may be proved in the same way as the formula at the end of the last section for $A(\omega)$ was proved. From this formula, it follows that to prove the inequality, it is enough to approximate $H(x_i, x_j) - H(\xi_i, \xi_j)$, for all (ξ, η) in $\pi^{-1}(\operatorname{supp} \mu_\omega) \cap \mathcal{D}$ where $\pi : \mathbb{R}^2 \to (\mathbb{R}/\mathbb{Z}) \times$

$\mathbb{R} = M$ denotes the projection and $\mathscr{D}$ is a suitably chosen fundamental domain of the Deck transformation $(x, y) \to (x+1, y)$.

Let $n \in \mathbb{Z}$ be such that $|\omega(j-i) - n| \le \frac{1}{2}$. Let $(\xi^0, \eta^0) \in \pi^{-1}(\operatorname{supp} \mu_\omega)$ be such that $|\xi_i^0 - x_i| < 1$ and $|\xi_i^0 + n - x_j| < 1$ where ξ_i^0 is defined in terms of (ξ^0, η^0) in the same way as ξ_i was defined in terms of (ξ, η) above. Set $Y = -\partial_1 H(\xi_i^0, \xi_j^0) = \partial_2 H(\xi_{2i-j}^0, \xi_i^0)$. By [8, §5], we have that H satisfies (H_1)–(H_5) and $(H_{6\theta})$, since it is a conjunction of functions satisfying these conditions. By an obvious extension of [8, Lemma 6.1], we then have that

$$|Y + \partial_1 H(u\pm, u')| \le 2\theta, \qquad |Y - \partial_2 H(u, u'\pm)| \le 2\theta$$

when $|u - \xi_i^0| \le 1$ and $|u' - \xi_i^0 - n| \le 1$. Hence, by the mean value theorem,

$$|H(x_i, x_j) - H(\xi_i, \xi_j) - Y(\xi_i - x_i - \xi_j + x_j)| \le 8\theta,$$

whenever $|\xi_i^0 - \xi_i| \le 1$ and $|\xi_i^0 + n - \xi_j| \le 1$. We may obviously choose the fundamental domain $\mathscr{D}$ so that these conditions are satisfied whenever $(\xi, \eta) \in \pi^{-1}(\operatorname{supp} \mu_\omega) \cap \mathscr{D}$. But,

$$x_j - x_i = (j-i)\omega = \int_M (\xi_j - \xi_i)\, d\mu_\omega(\xi, \eta),$$

so the previous inequality implies

$$\left| H(x_i, x_j) - \int_M H(\xi_i, \xi_j)\, d\mu_\omega(\xi, \eta) \right| \le 8\theta.$$

By the last equation above, we then obtain $(j-i)A(\omega)$ as an approximate value of $h(x_i, \ldots, x_j) = H(x_i, x_j)$, with error bounded by 8θ.

Here is Aubry's proof. Consider real numbers $\omega_0 < \omega_1$ and let $\omega = (1-\lambda)\omega_0 + \lambda\omega_1$, where $0 < \lambda < 1$. Let u and v be minimal configurations of rotation numbers ω_0 and ω_1, resp. We consider a bi-infinite sequence of integers $\cdots < i_j < i_{j+1} < \cdots$ and a configuration w such that for even j, the segment $\{w_i : i_j \le i < i_{j+1}\}$ of w is the segment of a translate u^j of u, and for odd j, this segment is the segment of a translate v^j of v. We recall that to say that u^j as a translate of u means that there are integers a and b such that $u_i^j = u_{i-a} + b$, for all i. We suppose, in addition, that for j even and $i = i_j$, we have $|u_i^j - v_i^{j-1}| < 1$ and for j odd and $i = i_j$, we have $|v_i^j - u_i^{j-1}| < 1$. In fact, for any bi-infinite increasing sequence $\cdots < i_j < i_{j+1} < \cdots$ we may easily construct such a configuration w.

Now we consider such a sequence satisfying $i_{j+1}-i_j \to \infty$, $(i_{2j+1}-i_{2j})/(i_{2j}-i_{2j-1}) \to (1-\lambda)/\lambda$, and $(i_{j+1}-i_j)/|i_j| \to 0$, as $j \to \pm\infty$. There is no difficulty in finding such a sequence of i_j's. It is easily seen that ω is the rotation number of w and

$$A(\omega) \le \lim h(w_m, \ldots, w_n)/(n-m) = (1-\lambda)A(\omega_0) + \lambda A(\omega_1).$$

Here, the inequality is a consequence of the approximate value of $h(x_i, \ldots, x_j)$ obtained at the beginning of this section. This proves that A is convex.

To prove strict convexity, it is then enough to consider the case when ω_0 and ω_1 are rational and $\omega = (\omega_0 + \omega_1)/2$. Let Q be a common denominator of ω_0 and ω_1. We consider periodic minimal configurations u and v of rotation numbers ω_0 and ω_1, so $u_{i+Q} = u_i + P_0$ and $v_{i+Q} = v_i + P_i$, where $\omega_0 = P_0/Q$ and $\omega_1 = P_1/Q$. Since v has larger rotation number than u, their graphs cross, and since both are minimal, they cross exactly once. Choose an interval $[i, i+Q]$ in which these graphs cross. Then $v_i < u_i$, $u_{i+Q} < v_{i+Q} = v_i + P_1 < u_i + P_1 = u_{i+Q} + P_1 - P_0$, and $u_{i+2Q} + P_1 - P_0 = u_{i+Q} + P_1 < v_{i+Q} + P_1 = v_{i+2Q}$. Let $w_j = \max(u_j, v_j)$ for $i \le j \le i+Q$ and $w_j = \min(u_j + P_1 - P_0, v_j)$ for $i+Q \le j \le i+2Q$. Then $w_{i+2Q} = w_i + P_1 + P_0$. Define w_j for all j, by requiring $w_{j+2Q} = w_j + P_1 + P_0$. Then

$$\begin{aligned} A(\omega) &< (2Q)^{-1} h(w_1, \ldots, w_{2Q}) \\ &= (2Q)^{-1}[h(u \vee v_i, \ldots, u \vee v_{i+Q}) + h(u \wedge v_i, \ldots, u \wedge v_{i+Q})] \\ &\le (2Q)^{-1}[h(u_i, \ldots, u_{i+Q}) + h(v_i, \ldots, v_{i+Q})]. \end{aligned}$$

This completes Aubry's proof of the strict convexity of A.

§5. Minimal measures

An orbit is ω-(resp. α-) *recurrent* in the sense of topological dynamics if it comes back arbitrarily close to itself under forward (resp. backward) iteration. Let $\bar{f} \in \mathcal{P}^1$. By the theory of Aubry, as generalized by Bangert [2], a minimal orbit of $\bar{f}$ is α-recurrent if and only if it is ω-recurrent, and the set of all recurrent minimal orbits of rotation number ω is a closed subset of the cylinder, which we will denote $RM_\omega = RM_\omega(\bar{f})$. Moreover, if ω is irrational, then μ_ω is uniquely defined and $RM_\omega = \operatorname{supp} \mu_\omega$; if ω is rational, RM_ω is the union of all minimal periodic orbits of rotation number ω.

Recall that a number λ is said to be a *sub-derivative* of $A = A_{\bar{f}}$ at ω if the line in the plane of slope λ through $(\omega, A(\omega))$ lies below graph A; such a sub-derivative exists at each point, since A is convex, as we proved in the last section. Throughout this section, $\bar{f}$ will be a fixed element of $\mathcal{P}^1$. We will prove the main result of this paper:

PROPOSITION. *If λ is a sub-derivative of $A_{\bar{f}}$ at ω, then an $\bar{f}$-invariant probability measure μ minimizes $A(\bar{f}, \mu)-\lambda\rho(f, \mu)$ if and only if* $\operatorname{supp}\mu \subset RM_\omega$.

We will call an $\bar{f}$-invariant probability measure *minimal* if it minimizes $A(\bar{f}, \mu)-\lambda\rho(f, \mu)$ for some real number λ.

Note that if μ is an $\bar{f}$-invariant probability measure with support in RM_ω, then $A(\bar{f}, \mu)=A(\omega)$ and $\rho(f, \mu)=\omega$. In fact, if ω is irrational then $\mu=\mu_\omega$, and if ω is rational then μ is in the closed convex hull of invariant probability measures supported on periodic orbits of rotation number ω.

Consequently, to prove the proposition, it is enough to prove that if μ is an $\bar{f}$-invariant probability measure, then $A(\bar{f}, \mu)-\lambda\rho(f, \mu)\geq A(\omega)-\lambda\omega$, with equality if and only if $\operatorname{supp}\mu\subset RM_\omega$.

In fact, it is enough to consider ergodic measures. Recall that an $\bar{f}$-invariant probability measure μ is said to be *ergodic* if every $\bar{f}$-invariant Borel set has μ-measure 0 or 1. We assert that to prove the proposition, it is enough to show that if μ is an ergodic $\bar{f}$-invariant probability measures then $A(\bar{f}, \mu)-\lambda\rho(f, \mu)\geq A(\omega)-\lambda\omega$, with equality if and only if $\operatorname{supp}\mu\subset RM_\omega$.

This reduction to ergodic measures is a consequence of well known facts in functional analysis, together with our discussion of the average action in §2. We recall the relevant results.

The space of Borel probability measures on a compact metric space X is a compact, convex subset of the dual $C(X)^*$ of the Banach space $C(X)$ of continuous function on X with the sup norm, where $C(X)^*$ is provided with the weak topology defined by $C(X)$. If $T: X\rightarrow X$ is a continuous mapping, the set of T-invariant probability measures is compact and convex. A T-invariant probability measure μ is ergodic if and only if it is an extremal point of the set of all T-invariant probability measures. Since the set of T-invariant probability measures is compact and convex, it is the closed convex hull of the set of its extremal points, i.e. of the ergodic measures, by the Krein Milman theorem.

To apply these general results to our situation, we take for X the end compactification S^2 of the cylinder. This is the union of the cylinder and two points: the bottom end and the top end. It is homeomorphic to the two sphere. Obviously, $\bar{f}$ extends to a homeomorphism of S^2, which we continue to denote by $\bar{f}$. By the assumption we made at the outset, the extended homeomorphism $\bar{f}$ fixes each end. We extend the definition of $A(\bar{f}, \mu)$ to $\bar{f}$-invariant probability measures on S^2, by setting $A(\bar{f}, \mu)=+\infty$ if the μ-mass of either end is positive.

Moreover, $\mu \mapsto A(\bar{f}, \mu)$ is a lower semi-continuous function with values in $\mathbb{R} \cup \{\infty\}$, i.e. for every real number a, we have that $\{\mu : A(\bar{f}, \mu) > a\}$ is open in the weak topology on $\bar{f}$-invariant probability measures defined by $C(S^2)$. This follows from the formula in §3 for the average action in terms of the h_i's, and the fact that the h_i's are bounded below, since they are continuous and satisfy (H_1) and (H_2) of [2], as was verified in the example in [8, §3].

At this point, it is useful to recall that the definition of $A(\bar{f}, \mu)$ depends on the choice of coordinates (x, y) on the infinite cylinder, x being defined mod 1, and if we replace (x, y) by $(x, y - \lambda y)$ then $\eta = y\,dx$ is replaced by $\eta' = (y - \lambda)\,dx$ and $A(\bar{f}, \mu)$ is replaced by $A(\bar{f}, \mu) - \lambda\rho(f, \mu)$ by the formula we obtained above for the ambiguity of the average action. Consequently, we have that the mapping $\mu \mapsto A(\bar{f}, \mu) - \lambda\rho(f, \mu)$ is lower semi-continuous. Obviously, it is also affine.

Since the mapping $\mu \mapsto A(\bar{f}, \mu) - \lambda\rho(f, \mu)$ is lower semi-continuous and affine on the compact, convex set of all $\bar{f}$-invariant probability measures, we have that the set of μ which minimize $A(\bar{f}, \mu) - \lambda\rho(f, \mu)$ is also compact and convex, and its extremal points are extremal points of the set of all $\bar{f}$-invariant measures, i.e. they are ergodic measures.

Consequently, if $A(\bar{f}, \mu) - \lambda\rho(f, \mu) \geq A(\omega) - \lambda\omega$ for ergodic μ, this is also true for all μ; if equality implies $\operatorname{supp}\mu \subset RM_\omega$ for ergodic μ, we also have this implication for all μ. This completes our reduction to the case when μ is ergodic.

From now on, we let μ be a fixed ergodic $\bar{f}$-invariant probability measure on the cylinder. As we observed above, it is enough to prove that $A(\bar{f}, \mu) - \lambda\rho(f, \mu) \geq A(\omega) - \lambda\omega$ with equality if and only if $\operatorname{supp}\mu \subset RM_\omega$. We let (x, y) be a fixed generic point in the plane for μ and set $(x_i, y_i) = f^i(x, y)$. By a "generic point", we mean a point such that the Birkhoff sums $n^{-1}\sum_{i=0}^{n-1} u(x_i, y_i)$ and $n^{-1}\sum_{i=0}^{n-1} u(x_{-i}, y_{-i})$ converge $\int_M u\,d\mu$, for every continuous function u on the cylinder which is also in $L^1(M, \mu)$. The Birkhoff ergodic theorem shows that μ-almost every point is generic, since the space of functions we consider has a countable dense set with respect to the L^1-norm.

Since (x, y) is generic, we have

$$\lim_{\substack{i \to -\infty \\ j \to +\infty}} (x_j - x_i)/(j - i) = \rho(f, \mu)$$

and

$$\lim_{\substack{j \to -\infty \\ l \to \infty}} \sum_{i=j}^{l-1} \sum_{\alpha=0}^{k-1} h_{\alpha+1}(\xi_{ki+\alpha}, \xi_{ki+\alpha+1})/(l - j) = A(\bar{f}, \mu),$$

where we use the formula of §3 for the average action. It follows from the last equation that

$$\limsup_{\substack{j\to-\infty\\ l\to+\infty}} h(x_j, \ldots, x_l)/(l-j) \le A(\bar{f}, \mu),$$

in view of the definition of h as the conjunction of the h_α's.

For each positive integer N, we let $v_i^{(N)}$, $-N \le i \le N$, be a minimal segment of a configuration with $v_i^{(N)} = x_i$, $i = -N, N$. Then

$$\liminf_{N\to\infty} h(x_{-N}, \ldots, x_N)/2N \ge \lim_{N\to\infty} h(v_{-N}^{(N)}, \ldots, v_N^{(N)})/2N = A(\rho(f, \mu)),$$

by the estimate for $h(x_i, \ldots, x_j)$ when $(x_i, \ldots, x_j)$ is minimal, given at the beginning of §4.

The last two inequalities yield $A(\bar{f}, \mu) \ge A(\rho(f, \mu))$. Since λ is a subderivative of A at ω, we obtain $A(\bar{f}, \mu) - \lambda\rho(f, \mu) \ge A(\rho(f, \mu)) - \lambda\rho(f, \mu) \ge A(\omega) - \lambda\omega$. This is the first assertion we need to prove in order to prove the proposition; it remains to prove that $\operatorname{supp}\mu \subset RM_\omega$ when $A(\bar{f}, \mu) - \lambda\rho(f, \mu) = A(\omega) - \lambda\omega$.

In this case, the inequalities we just obtained become equations, i.e. $A(\bar{f}, \mu) - \lambda\rho(f, \mu) = A(\rho(f, \mu)) - \lambda\rho(f, \mu) = A(\omega) - \lambda\omega$. Since λ is a subderivative of A at ω, it follows that $\rho(f, \mu) = \omega$. Consequently, $A(\bar{f}, \mu) = A(\omega)$.

At this point, we have to recall the real valued function ψ_ω of a real variable which we introduced in [7]. It is a monotone non-decreasing function satisfying $\psi_\omega(t+1) = \psi_\omega(t) + 1$ which is continuous from the left and has the property that its derivative $d\psi_\omega(t)/dt$ (which may be thought of as a measure on $\mathbb{R}/\mathbb{Z}$ by monotonicity and periodicity) is the projection of μ_ω on $\mathbb{R}/\mathbb{Z}$. For identification purposes, we remark that ψ_ω is the generalized inverse of ϕ_ω in the sense that graph ψ_ω is the reflection of graph ϕ_ω about the $t = x$ axis. Here, ϕ_ω is the function of [6] which minimizes $F_\omega(\phi) = \int_0^1 h(\phi(t), \phi(t+\omega))\,dt$; in Aubry's terminology, it is a hull function of a minimal configuration of rotation number ω.

Note that since μ_ω is unique when ω is irrational, ψ_ω is unique up to addition of a constant. When ω is rational, there may be more than one choice of ψ_ω, just as there may be more than one choice of μ_ω.

When ω is irrational, μ_ω is non-atomic; consequently, ψ_ω is continuous. Moreover (when ω is irrational), $\mu_{\omega'} \to \mu_\omega$ weakly when $\omega' \to \omega$; consequently, $\psi_{\omega'} \to \psi_\omega$, uniformly. See [7].

We set

$$\Psi_\omega(x, y) = \psi_\omega(x') - \psi_\omega(x) - \omega,$$

where $f(x, y) = (x', y')$. Then Ψ_ω is a function on $\mathbb{R}^2$; but, $\Psi_\omega(x+1, y) = \Psi_\omega(x, y)$, so it may also be thought of as a function on the cylinder $(\mathbb{R}/\mathbb{Z}) \times \mathbb{R}$.

To prove that $\operatorname{supp} \mu \subset RM_\omega$ when $A(\bar{f}, \mu) - \lambda\rho(f, \mu) = A(\omega) - \lambda\omega$, we consider two cases, depending on whether ω is irrational or rational. First, we consider the case when ω is irrational. In this case, we will show that $\Psi_\omega = 0$ μ-almost everywhere, by contradiction. Let $v^{(N)}$ be defined in terms of $(\ldots, x_i, \ldots)$ as before. We then have the same inequalities as before. We will obtain a contradiction by showing that one of these inequalities may be replaced by a strict inequality, viz.,

$$\begin{aligned} A(\bar{f}, \mu) &\geq \limsup_{N\to\infty} h(x_{-N}, \ldots, x_N)/2N \\ &> \lim_{N\to\infty} h(v^{(N)}_{-N}, \ldots, v^{(N)}_N)/2N = A(\rho(f, \mu)). \end{aligned}$$

Since $\rho(f, \mu) = \omega$, this will contradict the fact that $A(\bar{f}, \mu) = A(\omega)$, which we proved above.

The only thing left to prove (in the case ω is irrational) is the strict inequality. For this, we use:

LEMMA. *Let v be a minimal segment of a configuration. There exists a real number ω' such that no ground configuration of rotation number ω' crosses v.*

Here, we use the terminology of Aubry: A ground configuration is a minimal configuration whose corresponding orbit in the cylinder is recurrent.

Proof of the lemma. Let U (resp. D) denote the set of $\omega' \in \mathbb{R}$ for which there is a ground configuration of rotation number ω' crossing v in the upwards (resp. downwards) direction. It is clear from the Aubry crossing lemma that if $\omega \in D$ and $\omega' \in U$ then $\omega < \omega'$. Moreover, D and U are open. This is a conseqeunce of the fact that if $\omega_i \to \omega$ as $i \to \infty$, then

$$RM_\omega \subset \liminf_{i\to\infty} RM_{\omega(i)}.$$

Here, lim inf is the standard notion, associated to convergence in the Hausdorff topology. See e.g. [10, §§8, 12]. The above inclusion is implicit in Aubry's theory.

Since D and U are open and $D < U$, there exists $\omega' \notin D \cup U$.

To prove the strict inequality above, we use the crossing equation

$$h(x \vee y_1, \ldots, x \vee y_j) + h(x \wedge y_i, \ldots, x \wedge y_j)$$
$$= h(x_i, \ldots, x_j) + h(y_i, \ldots, y_j) + \sum_R \mu_h(R).$$

Here, μ_h is the measure on the plane defined in [9, §3]. The sum is taken over all rectangles R of the form $[x_i, y_i] \times [y_{i+1}, x_{i+1}]$ with $x_i < y_i$ and $y_{i+1} < x_{i+1}$ or of the form $[y_i, x_i] \times [x_{i+1}, y_{i+1}]$ with $y_i < x_i$ and $x_{i+1} < y_{i+1}$.

We let $\omega_N \in \mathbb{R}$ be such that no ground configuration of rotation number ω_N crosses $v^{(N)}$. This exists by the lemma. We choose an increasing sequence u^α, $\alpha_0 \leq \alpha \leq \alpha_2$ of ground configurations of rotation number ω_N with the property that there is an index α_1 with $\alpha_0 < \alpha_1 < \alpha_2$ such that $u^{\alpha(1)}$ is below $v^{(N)}$ and $u^{\alpha(1)+1}$ is above $v^{(N)}$. We set $u_i^{\alpha(0)-1} = -\infty$ and $u_i^{\alpha(2)+1} = +\infty$, for all i. We let

$$w_i^\alpha = u_i^{\alpha-1} \vee x_i \wedge u_i^\alpha, \qquad -N \leq i \leq N,$$

for $\alpha_0 \leq \alpha \leq \alpha_1 + 1$. Applying the crossing equation $(\alpha_2 - \alpha_0 + 1)$-times, we obtain

$$h(x_{-N}, \ldots, x_N) + \sum_{\alpha(0)}^{\alpha(2)} h(u_{-N}^\alpha, \ldots, u_N^\alpha) = \sum_{\alpha(0)}^{\alpha(2)+1} h(w_{-N}^\alpha, \ldots, w_N^\alpha) + \sum_R \mu_h(R).$$

The final sum is taken over an appropriate collection of rectangles R. We will not need to determine this collection of rectangles explicitly, although this could be done, but only use certain properties of it, which will be discussed later. For $\alpha \leq \alpha_1$, we have $w_i^\alpha = u_i^\alpha$ for $i = -N, N$. For $\alpha = \alpha_1 + 1$, we have $w_i^\alpha = x_i = v_i^{(N)}$, for $i = -N, N$. For $\alpha > \alpha_1 + 1$, we have $w_i^\alpha = u_i^{\alpha-1}$, for $i = -N, N$. Since u^α and $v^{(N)}$ are minimal, we obtain

$$\begin{aligned} h(u_{-N}^\alpha, \ldots, u_N^\alpha) &\leq h(w_{-N}^\alpha, \ldots, w_N^\alpha), && \alpha \leq \alpha_1 \\ h(v_{-N}^{(N)}, \ldots, v_N^{(N)}) &\leq h(w_{-N}^\alpha, \ldots, w_N^\alpha), && \alpha = \alpha_1 + 1, \\ h(u_{-N}^{\alpha-1}, \ldots, u_N^{\alpha-1}) &\leq h(w_{-N}^\alpha, \ldots, w_N^\alpha), && \alpha > \alpha_1 + 1. \end{aligned}$$

From the previous equation, we then obtain

$$h(x_{-N}, \ldots, x_N) - h(v_{-N}^{(N)}, \ldots, v_N^{(N)}) \geq \sum_R \mu_h(R),$$

where the sum is taken over the same collection of rectangles as before.

Let Δ_i denote the contribution to $\sum_R \mu_h(R)$ of all crossings of graph x with some graph u^α between (i, x_i) and $(i+1, x_{i+1})$, so $\sum_R \mu_h(R) = \sum_{i=-N}^{N-1} \Delta_i$. If graph u^α crosses graph x between (i, x_i) and $(i+1, x_{i+1})$, we have $\mu_h([x_i, u_i^\alpha] \times [u_{i+1}^\alpha, x_{i+1}])$ or $\mu_h([u_i^\alpha, x_i] \times [x_{i+1}, u_{i+1}^\alpha])$ as a lower bound for Δ_i, since in applying the crossing equation to get the above inequality, we may introduce the u^α's in any order we wish. Furthermore, we may introduce as many u^α's as we wish, since introducing new u^α's can only increase $\sum_R \mu_h(R)$. Let $\delta_i^{(N)}$ denote the supremum of $\mu_h([x_i, u_i] \times [u_{i+1}, x_{i+1}])$ or $\mu_h([u_i, x_i] \times [x_{i+1}, u_{i+1}])$ over all ground configurations u of rotation number ω_N such that graph u crosses graph x between (i, x_i) and $(i+1, x_{i+1})$. We then have

$$h(x_{-N}, \dots, x_N) - h(v_{-N}^{(N)}, \dots, v_N^{(N)}) \geq \sum_{i=-N}^{N} \delta_i^{(N)},$$

by the previous inequality.

Our assumption that Ψ_ω does not vanish μ-almost everywhere implies that there exists a compactly supported continuous function σ on $\mathbb{R}$ with values in $[0, 1]$ and with $0 \notin \operatorname{supp} \sigma$ such that $\int_M \sigma \Psi_\omega \, d\mu > 0$. Let $K = |\omega| + \max |\operatorname{supp} \sigma| + 3$. By (H_1) and (H_5) of [8], there exists a positive constant ρ_0 such that $\mu_h \geq \rho_0 \, dx \, dx'$ on $\{|x' - x| \leq K\}$. If $\Psi_\omega(x_i, y_i) \in \operatorname{supp} \sigma$ and N is large enough, we have $\delta_i^{(N)} \geq \rho_0 (u_i - x_i)(x_{i+1} - u_{i+1})$. This is because $|u_{i+1} - u_i - \omega| \leq 1$, for N large enough, since $\omega_N \to \omega$ as $N \to \infty$ and ω is irrational, and because $|x_{i+1} - x_i - \omega| \leq \max |\operatorname{supp} \sigma| + 2$, since $\Psi_\omega(x_i, y_i) \in \operatorname{supp} \sigma$. Therefore, the relevant rectangle, $[x_i, u_i] \times [u_{i+1}, x_{i+1}]$ or $[u_i, x_i] \times [x_{i+1}, u_{i+1}]$, is in $\{|x' - x| \leq K\}$ and we obtain $\delta_i^{(N)} \geq \rho_0 (u_i - x_i)(x_{i+1} - u_{i+1})$, as asserted. Moreover, this inequality is valid for any ground configuration u such that graph u crosses graph x between (i, x_i) and $(i+1, x_{i+1})$.

In order to get a good lower bound for $\delta_i^{(N)}$, we choose u by setting $t = \psi_{\omega(N)}(x_i) - i\omega_N$, $t' = \psi_{\omega(N)}(x_{i+1}) - (i+1)\omega_N$, $\bar{t} = (t + t')/2$, and $u_j = \phi_{\omega(N)}(\bar{t} + j\omega_N)$. Let

$$\delta = \min \{\phi_\omega(t') - \phi_\omega(t) : t' - t \geq \min (|\operatorname{supp} \sigma|)/4\}.$$

Since ω is irrational, ϕ_ω is strictly increasing. Since $0 \notin \operatorname{supp} \sigma$, we then have $\delta > 0$. For N large enough, and $\Psi_\omega(x_i, y_i) \in \operatorname{supp} \sigma$, we have $|u_i - x_i|$, $|u_{i+1} - x_{i+1}| \geq \delta$. This is because $\Psi_{\omega(N)} \to \Psi_\omega$ uniformly, as $N \to \infty$. This convergence is valid because ω is irrational. It follows immediately from the fact that $\psi_{\omega(N)} \to \psi_\omega$, uniformly, which is a consequence of the uniqueness of the element minimizing ψ_ω, proved in [7].

Thus, we have $\delta_i^{(N)} \geq \rho_0 \delta^2 > 0$, when N is large enough (independently of i) and $\Psi_\omega(x_i, y_i) \in \operatorname{supp} \sigma$. The density in $\mathbb{Z}$ of the set of i for which $\Psi_\omega(x_i, y_i) \in \operatorname{supp} \sigma$ is $\geq \int_M \sigma \Psi_\omega \, d\mu > 0$, since (x, y) is μ-generic and $\sigma \leq 1$. It follows that

$$\liminf_{N\to\infty} [h(x_{-N}, \ldots, x_N) - h(v_{-N}^{(N)}, \ldots, v_N^{(N)})]/2N$$
$$\geq \liminf_{N\to\infty} \sum_{i=-N}^{N} \delta_i^{(N)}/2N \geq \rho_0 \delta^2 \int_M \sigma \Psi_\omega \, d\mu > 0.$$

Thus, we have obtained the strict inequality which was required to contradict our hypothesis that Ψ_ω does not vanish μ-almost everywhere.

For any orbit $(\ldots, (x_i, y_i), \ldots)$ on which Ψ_ω vanishes, we have that $\psi_\omega(x_i) - i\omega$ is a constant t, and hence $\phi_\omega(t + \omega i-) \leq x_i \leq \phi_\omega(t + \omega i+)$. It follows that such an orbit is forward and backward asymptotic to an orbit in $\operatorname{supp} \mu_\omega$. Since $\Psi_\omega = 0$, μ-almost everywhere, as we have just proved, it follows that $\operatorname{supp} \mu \subset \operatorname{supp} \mu_\omega = RM_\omega$. This completes the proof when ω is irrational.

Now we consider the case when ω is rational, say $\omega = p/q$. To show that $\operatorname{supp} \mu \subset RM_\omega$, we may reduce to the case $\omega = 0$ by replacing $\bar{f}$ by $\bar{f}^q$ and f by $f^q T^{-p}$, where $T(x, y) = (x + 1, y)$. We have $\rho(f^q T^{-p}, \mu) = q\rho(f, \mu) - p = 0$, $A(\bar{f}^q, \mu) = qA(\bar{f}, \mu) + C$, $A_{\bar{f}^q}(\omega) = qA_{\bar{f}}(\omega) + C$, and $RM_0(f^q T^{-p}) = RM_{p/q}(f)$, where the constant C of integration is the same in the two cases. Moreover, the assumption that μ is $\bar{f}$-invariant implies that it is also $\bar{f}^q$-invariant, and the assumption that $\bar{f}$ is in $\mathcal{P}^1$ implies that $\bar{f}^q$ is in $\mathcal{P}^1$. Thus to prove that $\operatorname{supp} \mu \subset RM_\omega(f)$ when ω is rational, it is enough to consider the case $\omega = 0$.

It will be convenient to assume $\min_x h(x, x) = 0$. Since h is determined only up to addition of a constant, we may assume this. Under this assumption, $A_{\bar{f}}(0) = 0$. Since μ is ergodic and has rotation number 0, we have

$$\lim_{i\to\pm\infty} x_i/i = 0, \qquad \lim_{N\to\infty} \sum_{i=-N}^{N} h(x_i, x_{i+1})/2N = 0.$$

By (3.4) of [9], we have

$$\sum_{i=-N}^{N-1} h(x_i, x_{i+1})/2N = \sum_{i=-N}^{N-1} h(x_i, x_i)/2N$$
$$+ \int_{x_{-N}}^{x_N} \partial_2 h(y, y+) \, dy/2N + \sum_{i=-N}^{N} \mu_h(\Delta_i)/2N,$$

where Δ_i is the triangle $\{(y, z) : x_i \leq y \leq z \leq x_{i+1}\}$ or $\{(y, z) : x_{i+1} \leq z \leq y \leq x_i\}$, according to whether x_i or x_{i+1} is greater. Since $\lim_{i\to\pm\infty} x_i/i = 0$, the limit of the second term on the right vanishes. The other two terms on the right are non-negative, so their limits vanish, too. Since the limit of the third term vanishes, the density in the integers of $\{i : |x_{i+1} - x_i| > \varepsilon\}$ vanishes, for every $\varepsilon > 0$. Since the limit of the first term vanishes, $\{i : h(x_i, x_i) > \varepsilon\}$ also has vanishing density, for every $\varepsilon > 0$. Since (x, y) is μ-generic, these facts imply that $\operatorname{supp} \mu \subset \{(x, y) : h(x, x) = 0 \text{ and } y = -\partial_1 h(x, x) = \partial_2 h(x, x)\} = RM_0$.

This completes the proof of the proposition stated at the beginning of this section.

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