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Compactness of isospectral conformal metrics on S^3

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Two Riemannian metrics g, g' on a compact manifold are said to be isospectral if their associated Laplacian operator on functions have identical spectrum. It is a well known problem to study the extent to which the spectrum determines the metric. In dimension two, Osgood, Phillips and Sarnak [OPS] studied this question and proved that the set of isospectral metrics on a compact surface form a compact family in the $\mathscr{C}^\infty$ topology. In that case there is available a criterion due to Wolpert [W] for compactness of the conformal structures in the Teichmuller space in terms of the determinant of the Laplacian. This reduces the problem to studying the isospectral conformal metrics on a fixed Riemann surface. It turns out that the determinant of the Laplacian played the key role for the compactness questions. In particular when the underlying surface is the two sphere, which is analytically the least transparent case, the compactness question reduces to an inequality of Onofri ([O], [OPS]) which is a sharp version of the Moser–Trudinger inequality on S^2.

We are interested in the situation in dimension 3. The well known solution of the Yamabe problem ([A], [S]) says that every conformal class of metrics on a compact Riemannian manifold contains a metric of constant scalar curvature. When (M^3, g_0) has constant negative scalar curvature, an isospectral set of metrics $g = u^4 g_0$ conformal to g_0 is compact in the $\mathscr{C}^\infty$ topology [BPY]. This result was proved directly using the heat invariants of the metric. The first step was to find a pointwise bound $0 < c_1 \leq u \leq c_2$ and a bound $\|u\|_{2,2} \leq c_3$ where c_i depend only on the heat invariants of g. The higher order derivative bounds required for $\mathscr{C}^\infty$ compactness is a consequence of this bound for u and the calculation for the coefficients for the terms involving the highest order derivatives of u in the asymptotic a_k of the heat kernel for g due to Gilkey ([G]).

In this paper we study the situation when M is the standard three sphere (S^3, g_0). As in the case of the two sphere, the conformal group G complicates the analysis.

DEFINITION. For a positive function u on S^3, and φ a conformal transforma-

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tion, define

$$u_\varphi = (u \circ \varphi) \cdot |d\varphi|^{1/2}$$

where $|d\varphi|$ is the linear stretch factor of $d\varphi$ measured with respect to the standard metric g_0. Thus $u_\varphi^4 g_0 = \varphi^*(u^4 g_0)$. We set

$$[u] = \{u_\varphi \mid \varphi \in G, \text{ the conformal group for } S^3\}.$$

The noncompactness of G shows that the class $[u]$ is noncompact in $H_1(S^3)$, although the metrics associated to $v \in [u]: \{g = v^4 g_0; v \in [u]\}$ are all isometric. We have the following result.

THEOREM 1. *For (S^3, g_0), if $\{g_i = u_i^4 g_0\}$ is a sequence of isospectral metrics, then there exists a subsequence g_i and conformal transformations φ_i such that $\varphi_i^* g_i$ converges in the $\mathscr{C}^\infty$ topology to a metric g which is also isospectral to $\{g_i\}$*

On $S^3 = \{x \in \mathbb{R}^4 \mid |x| = 1\}$ we have the standard metric $g_0 = \sum_{i=1}^4 dx_i^2$, with associated Laplacian Δ, scalar curvature $R_0 = 6$, volume form dv_0. For a conformal metric $g = u^4 g_0$ we have

$$dv = u^6 \, dv_0,$$

and its scalar curvature R is determined by the equation

$$8\Delta u + Ru^5 = R_0 u.$$

The trace of the heat kernel $\exp(-t\Delta_u)$, where Δ_u denotes the Laplacian associated to the metric $g = u^4 g_0$, has the well known expansion

$$\operatorname{Trace} \exp(-t\Delta_u) \sim (4\pi t)^{-n/2}(a_0 + a_1 t + a_2 t^2 + \cdots)$$

as $t \to 0$, where

$$a_0 = \int dv = \int u^6 \, dv_0.$$

$$a_1 = \int R \, dv = \int (8 |\nabla u|^2 + 6u^2) \, dv_0.$$

$$a_2 = \frac{1}{360} \int (3R^2 + 6 |\rho|^2) \, dv = \frac{1}{360} \int (3R^2 + 6 |\rho|^2 u^6) \, dv_0,$$

$|\rho|$ denoting the norm of the Ricci tensor of g, measured in the metric g. As the heat invariants a_k are determined by the spectrum of the Laplacian associated to

g, we get immediately the following bounds for isospectral conformal metrics $g_i = u_i^4 g_0$:

$$\int u^6 \, dv_0 = a_0,$$

$$\int (8\,|\nabla u|^2 + 6u^2)\, dv_0 = a_1.$$

$$\frac{1}{120}\int R^2 u^6 \, dv_0 = \int (8\Delta u - 6u)^2 u^{-4}\, dv_0 \leq a_2.$$

In fact we will prove the following preliminary version of Theorem 1:

THEOREM 1′. *On (S^3, g_0) if $g = u^4 g_0$ is a metric satisfying*

$$a_0(g) = \alpha_0 \tag{1}$$

$$a_1(g) \leq \alpha_1 \tag{2}$$

$$\int R^2 u^6 \, dv_0 \leq \alpha_2 \tag{3}$$

$$\Lambda \leq \lambda_1(g) \tag{4}$$

for positive constants α_0, α_1, α_2 and Λ, where $\lambda_1(g)$ is the first positive eigenvalue of the Laplacian, then there exist constants c_1, c_2 and c_3 (depending only on α_0, α_1, α_2 and λ), and a conformal transformation φ so that $v = u_\varphi$ satisfies:

$$0 < c_1 \leq v(x) \leq c_2, \tag{5}$$

$$\|v\|_{2,2} \leq c_3, \text{ where } \|v\|_{2,2}^2 = \int v^2 + |\nabla v|^2 + |\nabla\nabla v|^2. \tag{6}$$

Then Theorem 1 will be a direct consequence of Theorem 1′ and the following proposition proved in [BPY], which provides the required bounds for all the higher order derivatives of u.

PROPOSITION. *For a compact manifold (M^3, g_0) the set of conformal metrics $g = u^4 g_0$ with u satisfying* (5) (6) *and the conditions*

$$a_k(g) \leq a_k \text{ for } k = 3, 4, 5, \ldots$$

form a compact set in the $\mathscr{C}^\infty$ topology.

We break up the proof of Theorem 1′ into four lemmas.

LEMMA 1. *Assume u is a positive smooth function on S^n, for each $\varepsilon > 0$ sufficiently small, we can find*

$v \in [u]$ satisfying with $n^* = (n+2)/(n-2)$

$$\int_{S^n} v^{n^*+1+\varepsilon} x_j \, dv_0 = 0$$

for $j = 1, \ldots, n+1$, where x_j are the coordinate functions in $\mathbb{R}^{n+1}$.

Proof. Consider for each $\varphi \in G$, the function $(u_\varphi)^{n^*+1+\varepsilon}$ as a mass distribution on S^n and we define its center of mass to be

$$\text{C.M.}\,(\varphi) = \int (u_\varphi)^{n^*+1+\varepsilon} x \, dv_0 \Big/ \int (u_\varphi)^{n^*+1+\varepsilon} \, dv_0$$

It is clear that C.M. (φ) is continuous in φ. Using the stereographic projection $\pi : S^n \setminus \{Q\} \to \{y \in \mathbb{R}^{n+1} \mid \langle y, Q \rangle = 0\}$ as coordinates, we define the conformal transformation $\varphi_{Q,t}$ via its action on the y coordinates:

$$\varphi_{Q,t}(y) = ty.$$

The collection of conformal maps $\{\varphi_{Q,t} \mid Q \in S^n, t \geq 1\}$ is naturally identified with $B^{n+1} = \{x \in \mathbb{R}^{n+1} \mid |x| \leq 1\}$ by $\varphi_{Q,t} \to (t-1)t^{-1}Q$. Restricting the C.M. map to C.M.: $\{\varphi_{Q,t} \mid Q \in S^n, t \geq 1\} \approx B^{n+1} \to B^{n+1}$, we find easily that it extends continuously to ∂B to the identity map on ∂B. This implies that there exists $Q \in S^n$ and $t \geq 1$ so that C.M. $(\varphi_{Q,t}) = 0$. This is the claim of the lemma.

LEMMA 2. *Assume u is a positive function on S^3 which satisfies the hypothesis of Theorem* 1′, *there exists some $\varepsilon_0 > 0$ and a constant c depending only on the data $(\alpha_0, \alpha_1, \alpha_2, A)$ and some $v \in [u]$ with $\int v^{6+\varepsilon_0} \, dv_0 \leq c$.*

Remarks. The formulation of Lemma 2 is motivated by the proof of the Yamabe problem (c.f. [A], [S]) where a sequence u_k of solutions of the equation (for some constants μ_k which converges to constant μ and $\varepsilon_k \to 0$) satisfying the equation

$$c_n \Delta u_k + \mu_k u^{n^*-\varepsilon_k} = R_0 u_k$$

was shown to converge weakly to a nonnegative limit function u; then in order to show the weak limit is not identically zero, it was sufficient to find a similar bound for $\int u^{6+\varepsilon_0}\, dv_0$ as above. In that case the limit function u satisfies the Yamabe equation, so that the maximum principle shows $u>0$. In our situation, because there is no a priori limit for the scalar curvatures R_k in the equations

$$c_n \Delta u_k + R_k u_k^{n^*} = R_0 u_k.$$

we need to proceed differently to establish the following lemmas to complete our argument:

LEMMA 3. *Suppose $u \geq 0$ satisfies the hypothesis* (1), (2), (3) and (4) *of Theorem* 1 *and $\int u^{6+\varepsilon_0}\, dv_0 \leq c$, then there exists a positive c_1 (depending on α_0, α_1, α_2, Λ and c) so that $0<c_1 \leq u$ on S^3.*

LEMMA 4. *Suppose u is as in Lemma* 3, *then there exists a positive c_2* (*depending only on α_1, α_2, Λ and* c), *so that $u \leq c_2$ on S^3.*

Proof of Lemma 2. Multiplying the equation

$$8\Delta u + Ru^5 = 6u \text{ on } S^3 \tag{7}$$

by u^β (β to be chosen later) we have: (denote $\int_{S^3} dv_0$ as $\int dv_0$)

$$8\frac{4\beta}{(\beta+1)^2}\int |\nabla u^{(\beta+1)/2}|^2\, dv_0 + 6\int u^{\beta+1}\, dv_0 = \int Ru^4 u^{\beta+1}\, dv_0. \tag{8}$$

Let $w = u^{(\beta+1)/2}$ then we have

$$8\frac{4\beta}{(\beta+1)^2}\cdot\int |\nabla w|^2\, dv_0 + 6\int w^2\, dv_0 = \int Ru^4\, w^2\, dv_0. \tag{9}$$

From the Sobolev inequality for w:

$$Q\Big(\int w^6\, dv_0\Big)^{1/3} \leq 8\int |\nabla w|^2\, dv_0 + 6\int w^2 dv_0 \tag{10}$$

where $Q = Q(S^3) = 6\cdot(\int dv_0)^{2/3}$. We have for $\beta>0$, from (9) and (10):

$$Q\frac{4\beta}{(\beta+1)^2}\Big(\int w^6\, dv_0\Big)^{1/3} \leq \int Ru^4 w^2\, dv_0 + 6\Big(\frac{4\beta}{(\beta+1)^2} - 1\Big)\int w^2\, dv_0 \tag{11}$$

We proceed to estimate the term $I=\int Ru^4w^2\,dv_0$. Taking b large to be chosen later, on the region $|R|\geq b$ we have

$$b^2\int_{|R|\geq b} u^6\,dv_0 \leq \int_{|R|\geq b} R^2u^6\,dv_0 \leq A_2 \leq \alpha_2, \text{ where } A_2 = \mathrm{O}(a_2).$$

Thus

$$\begin{aligned}\int_{|R|\geq b} Ru^4w^2\,dv_0 &\leq \left(\int R^2u^6\,dv_0\right)^{1/2}\cdot\left(\int_{|R|\geq b} u^6\,dv_0\right)^{1/6}\left(\int w^6\,dv_0\right)^{1/3}\\ &\leq A_2^{1/2}\left(\frac{A_2}{b^2}\right)^{1/6}\cdot\left(\int w^6\,dv_0\right)^{1/3}.\end{aligned}\tag{12}$$

For the remaining part of I, write $\beta = 1+2\varepsilon$ and apply Lemma 1 to u with $\varepsilon>0$, we can find some $v=u_\varphi\in[u]$ so that $\int v^6v^\varepsilon x_j\,dv_0=0$ for $j=1,2,3,4$. We have v satisfies the equation

$$8\Delta v+(R\circ\varphi)v^5 = 6v \tag{7$'$}$$

and that the first eigenvalue $\lambda_1(v^4g_0)=\lambda_1(u^4g_0)\geq\Lambda$. We have as in (12)$'$ for $\tilde{w}=v^{(1+\beta)/2}$.

$$\int_{|R(v)|\geq b} R(v)v^4\,\tilde{w}^2\,dv_0 \leq \left(\frac{A_2}{b}\right)^{1/3}\left(\int \tilde{w}^6\,dv_0\right)^{1/3}. \tag{12$'$}$$

For $dv=v^6\,dv_0$ we have from the Raleigh–Ritz characterization for λ_1:

$$\int_{S^3}\psi^2\,dv \leq \left(\int_{S^3}\psi\,dv\right)^2+\frac{1}{\lambda_1}\int_{S^3}|\nabla_v\psi|^2\,dv \tag{13}$$

where $|\nabla_v\psi|^2\,dv=|\nabla\psi|^2v^2\,dv_0$.

Choose $\psi=v^\varepsilon x_j$ in (13), then by our assumption on v, we have

$$\begin{aligned}\int v^6v^{2\varepsilon}x_j^2\,dv_0 &\leq (1/\Lambda)\int|\nabla v^\varepsilon x_j|^2v^2\,dv_0\\ &\leq\frac{1}{\Lambda}\left[2\varepsilon^2\int|\nabla v|^2v^{2\varepsilon}x_j^2\,dv_0+2\int|\nabla x_j|^2v^{2+2\varepsilon}\,dv_0\right]\\ &\leq\frac{2\varepsilon^2}{\Lambda(1+\varepsilon)^2}\int|\nabla v^{1+\varepsilon}|^2x_j^2\,dv_0+L_j\end{aligned}\tag{14}$$

where $L_j=\int|\nabla x_j|^2v^{2+2\varepsilon}\,dv_0=\mathrm{O}(\int v^{2+2\varepsilon}\,dv_0)$.

Sum up (14) for $j = 1, 2, 3, 4$, we find

$$\int v^{6+2\varepsilon}\, dv_0 \le \frac{2\varepsilon^2}{\Lambda(1+\varepsilon)^2} \int |\nabla v^{1+\varepsilon}|^2\, dv_0 + L, \tag{14$'$}$$

with $L = O(\int v^{2+2\varepsilon}\, dv_0)$.

Since $v^4 g$ is isometric to $u^4 g_0$ estimates (1) (2) (3) and (9) (10) (11) (12)$'$ hold for v in place of u. Applying (8) for $\beta = 1 + 2\varepsilon$ and (14)$'$ we get

$$\int v^6 v^{2\varepsilon}\, dv_0 \le \frac{2\varepsilon^2}{\Lambda(1+\varepsilon)^2} \frac{(1+\varepsilon)^2}{8(1+2\varepsilon)} I + L. \tag{15}$$

Combining (15) with (12)$'$ and recall that $\tilde{w} = v^{1+\varepsilon}$, we find

$$\begin{aligned} I = \int R v^4 \tilde{w}^2 &\le \left(\frac{A_2}{b}\right)^{1/3} \left(\int \tilde{w}^6\, dv_0\right)^{1/3} + b \int v^4 \tilde{w}^2\, dv_0 \\ &\le \left(\frac{A_2}{b}\right)^{1/3} \left(\int \tilde{w}^6\, dv_0\right)^{1/3} + \frac{2b\varepsilon^2}{8 \cdot \Lambda} I + bL \end{aligned} \tag{16}$$

So that

$$\left(1 - \frac{2b\varepsilon^2}{8\Lambda}\right) I \le \left(\frac{A_2}{b}\right)^{1/3} \left(\int \tilde{w}^6\, dv_0\right)^{1/3} + bL. \tag{17}$$

Now choose b sufficiently large so that $\left(\frac{A_2}{b}\right)^{1/3} < \frac{1}{2}Q$, and then choose ε sufficiently small so that

$$\left(1 - \frac{2\beta\varepsilon^2}{8\Lambda}\right) > \frac{3}{4}, \qquad \frac{1+2\varepsilon}{(1+\varepsilon)^2} > \frac{3}{4}.$$

For this choice of b and ε, we have from (11) (16) and (17) that

$$\tfrac{3}{4} Q \left(\int \tilde{w}^6\right)^{1/3} \le I \le \tfrac{2}{3} Q \left(\int \tilde{w}^6\, dv_0\right)^{1/3} + \frac{4}{3} bL.$$

Recall $\tilde{w} = v^{1+\varepsilon}$, hence

$$\begin{aligned} \left(\int v^{6+6\varepsilon}\, dv_0\right)^{1/3} = \left(\int w^6\, dv_0\right)^{1/3} &< 16bL = 16b \int v^{2+2\varepsilon}\, dv_0 \\ &< b \left(\int v^6\, dv_0\right)^{(2+2\varepsilon)/6} \left(\int dv_0\right)^{(4-2\varepsilon)/6} = c < \infty. \end{aligned}$$

This proves lemma 2 with $\varepsilon_0 = 6\varepsilon$.

Proof of Lemma 3. Assume u satisfies the hypothesis of the Lemma, we will prove the following assertions:

(a) Denoting $E_\lambda=\{\xi\in S^3, u(\xi)\ge\lambda\}$, $|E_\lambda|=\int_{E_\lambda}dv_0$ then there exists $\lambda_0>0$ and $l_0>0$ so that $|E_{\lambda_0}|\ge l_0>0$.

(b) There exists $c'<\infty$ with $\int(\log u)^2\,dv_0\le c'$.

(c) There exists $c''<\infty$ with $-\log u\le c''$ on S^3.

It follows from (c) that $u\ge c_1>0$ for some fixed c_1.

To prove (a), we have

$$\int u^{6+\varepsilon}\,dv_0\le c,\ \text{with } \varepsilon\le\varepsilon_0,$$

and

$$a_0^2=\left(\int u^6\,dv_0\right)^2\le\left(\int u^{6-\varepsilon}\,dv_0\right)\left(\int u^{6+\varepsilon}\,dv_0\right)\le c\int u^{6-\varepsilon}\,dv_0.$$

Thus

$$\frac{a_0^2}{c}\le\int u^{6-\varepsilon}\,dv_0.$$

On the other hand, for all $\lambda>0$, we have

$$\int u^{6-\varepsilon}\,dv_0=\int_{E_\lambda}u^{6-\varepsilon}\,dv_0+\int_{E_\lambda^c}u^{6-\varepsilon}\,dv_0\le\left(\int_{E_\lambda}u^6\right)^{(6-\varepsilon)/6}|E_\lambda|^{\varepsilon/6}+\lambda^{6-\varepsilon}\,|E_\lambda^c|.$$

So for λ sufficiently small, say $\lambda^{6-\varepsilon}\mathrm{Vol}(S^3)<\frac{1}{2}\frac{a_0^2}{c}$ we have

$$\frac{1}{2}\frac{a_0^2}{c}\le a_0^{(6-\varepsilon)/6}|E_\lambda|^{\varepsilon/6}$$

thus

$$|E_\lambda|\ge(a_0/2c)^{6/\varepsilon}a_0=l_0.$$

To prove (b), choose λ_0, l_0 as in (a), and consider the Raleigh–Ritz characterization for $\lambda_1(D)$, D being the set $E_{\lambda_0}^c$, to find

$$\int_D\left|\log\left(\frac{u}{\lambda}\right)\right|^2dv_0\le\frac{1}{\lambda_1(D)}\int_D\left|\nabla\log\frac{u}{\lambda_0}\right|^2dv_0. \tag{18}$$

Since $|D| = \text{Volume } (S^3) - |E_\lambda| \le |S^3| - l_0$, the well known Faber–Krahn inequality ([C]), says that $\lambda_1(D) \ge C(l_0) > 0$. Thus we find

$$\begin{aligned}\int_{u\le\lambda_0}\left(\log\frac{u}{\lambda_0}\right)^2 dv_0 &\le \frac{1}{c(l_0)}\int\left|\frac{\nabla u}{u}\right|^2 dv_0\\ &\le \frac{1}{c(l_0)}\int\frac{\Delta u}{u}\,dv_0\\ &\le \frac{1}{c(l_0)}\frac{1}{8}\int(Ru^4-6)\,dv_0\\ &\le \frac{1}{c(l_0)}\frac{1}{8}\left[\left(\int R^2u^6\,dv_0\right)^{1/2}\left(\int u^6\,dv_0\right)^{1/6}\left(\int 1\,dv_0\right)^{1/3}+6\int dv_0\right]\\ &\le \frac{1}{c(l_0)8}\cdot[\alpha_0^{1/2}\alpha_2^{1/2}\,\text{Vol}\,(S^3)+6\,\text{Vol}\,(S^3)]. \end{aligned} \tag{19}$$

We have also

$$\int_{u\ge\lambda_0}\left(\log\frac{u}{\lambda_0}\right)^2 dv_0 \le \int_{u\ge\lambda_0}\left(\frac{u}{\lambda_0}\right)^2 dv_0 \le \frac{1}{\lambda_0^2}\int u^2\,dv_0 \le \frac{1}{\lambda_0^2}\alpha_0^{1/3}. \tag{20}$$

Combining (19) and (20) we have a bound for $\int(\log u)^2$ as claimed.

To prove (c). We use the integral identity:

$$-\psi(\xi)+\bar{\psi} = \int(\Delta\psi)(Q)G(\xi,Q)\,dv_0(Q) \tag{21}$$

where $G(\xi, Q)$ is the Green's function for Δ on S^3, and $\bar{\psi}$ is the average of ψ on S^3. We may add a suitable constant to G to make it positive. Apply this identity to $\psi = \log u$:

$$\begin{aligned}-\log u(\xi)+\overline{\log u} &= \int(u^{-1}\Delta u - u^{-2}|\nabla u|^2)(Q)G(\xi,Q)\,dv_0\\ &\le \int(u^{-1}\Delta u)(Q)G(\xi,Q)\,dv_0\\ &\le \frac{1}{8}\|Ru^4-6\|_p\,\|G(\xi,\cdot)\|_p\end{aligned}$$

for $\frac{1}{p}+\frac{1}{p'}=1$. Choose $p=\frac{3}{2}+\delta$, $\delta=\frac{\varepsilon}{16+2\varepsilon}$, we find $p'<3$ so that both $\|Ru^4-6\|_p$ and $\|G(\xi,\cdot)\|_{p'}$ are bounded. It now follows from (b) that $-\log u$ is bounded from above.

Proof of Lemma 4: Applying equation (21) to the function u we have

$$-u(p)+\bar{u}=\int \Delta u(Q)G(p,Q)\,dv_0(Q)$$

$$=\frac{1}{8}\int (6u-Ru^5)G(p,Q)\,dv_0(Q),$$

where $G(p,Q)\sim\frac{1}{d(p,Q)}+$ smooth function. We recall the following estimate ([A] p. 37): For $h(y)=\int_{\mathbf{R}^3}\frac{f(x)}{\|x-y\|}\,dx$ we have, when $\frac{1}{r}=\frac{1}{q}-\frac{2}{3}$, $r>1$

$$\|h\|_r\le c(q)\,\|f\|_q. \tag{22}$$

We will iterate this estimate with a sequence of suitably chosen r_j, q_j. Start with $q_0=\frac{2r_0}{4+r_0}$, $r_0=6+6\varepsilon$ where $6\varepsilon\le\varepsilon_0$, we have

$$\int (Ru^5)^{q_0}\le\left(\int R^2u^6\right)^{q_0/2}\left(\int u^{r_0}\right)^{1-(q_0/2)}$$

Thus by (22) we find a bound for $\|u\|_{r_1}$ with

$$r_1=\frac{6r_0}{12-r_0}>r_0.$$

Continuing with

$$r_2=\frac{6r_1}{12-r_1},\ q_1=\frac{2r_1}{r+r_1},\ \ldots,\ r_k=\frac{6r_{k-1}}{12-r_{k-1}},\ q_k=\frac{2r_k}{4+r_k}.$$

We find

$$r_{k+1}-r_k=\frac{r_k-6}{12-r_k}r_k\ge\varepsilon r_k>0 \quad \text{if} \quad 6<r_k<12.$$

Thus there will be a k_0 with $r_{k_0} > 12$ and $r_0 < r_1 < \cdots < r_{k_0-1} < 12 < r_{k_0}$ with

$$q_{k_0} = \frac{2r_{k_0}}{4 + r_{k_0}} > \frac{3}{2}.$$

So at the end of the iteration we find a bound for $\|u\|_{r_{k_0}}$, $\frac{3}{2} < q_{k_0} < 2$. This implies $u \in L^\infty$ from the Holder estimate:

$$\|u\|_\infty \le \|u\|_1 + \|Ru^5\|_{q_{k_0}} \|G\|_{q'} \qquad \text{where} \qquad \frac{1}{q'} + \frac{1}{q_{k_0}} = 1, \quad \text{with} \qquad q' < 3.$$

This finishes the proof of lemma 4.

End of the proof of Theorem 1′:

From Lemmas 3 and 4 we have

$$0 < c_1 \le u \le c_2.$$

From

$$\alpha_2 \ge a_2 \ge \int R^2 u^6 = \int \left(64 \frac{(\Delta u)^2}{u^4} - 96 \frac{\Delta u}{u^2} + \frac{36}{u^2}\right) dv_0$$

we conclude that

$$\int \frac{(\Delta u)^2}{u^4} \le \text{constant}.$$

This together with the uniform upper bound for u, yields a bound for $\int (\Delta u)^2$. Thus $\int (\Delta u)^2 + u^2\, dv_0$ is bounded, hence we have a bound for $\|u\|_{2,2}$.

Remark. In general for S^n with $n \ge 4$, Theorem 1′ continues to hold provided that we substitute condition (3) with the following

$$\int R^{(n/2)+\delta} u^{(2n/n-2)}\, dv_0 \le \alpha_2 \quad \text{for some } \delta > 0. \tag{3'}$$

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