**Zeitschrift:** Commentarii Mathematici Helvetici

Herausgeber: Schweizerische Mathematische Gesellschaft

**Band:** 64 (1989)

Artikel: On for dimensional s-cobordisms, II.

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DOI: https://doi.org/10.5169/seals-48950

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# On four-dimensional s-cobordisms, II

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In this note we complete the topological classification of s-cobordisms of 3-dimensional quaternionic space forms. Let  $Q_r$  be the quaternionic group of order  $2^{r+2}$  and let  $M_r = S^3/Q$  be the quotient of usual action of  $Q_r$  on  $S^3$ .

THEOREM. There exist precisely  $2^{2^r-r-1}$  distinct topological s-cobordisms of  $M_r$  to itself.

In [CS 1] (see also [CS 2]) it was shown that this number is a lower bound. Additional study using topological surgery, valid in this situation by the work of Freedman [F], showed that there are at most  $2^{2^r-r}$  distinct s-cobordisms. The further paper [KwS] on s-cobordisms of space forms also left this ambiguity unresolved. In [CS 3] we erroneously claimed that there are precisely  $2^{2^r-r}$  distinct s-cobordisms, and the new invariant was used to detect the topological non-triviality of an explicitly constructed smooth s-cobordism. In part [CS 3] used various exact sequences in Witt and L-theory, and the present note is the result of reconsideration of this material in light of [Ra] and the visible symmetric L-theory of Michael Weiss.

We will actually consider only the case r=1, and will prove that every s-cobordism of  $M=M_i$  to itself is homeomorphic to a product. From [KwS], the general result can easily be seen to follow from this case; alternatively, the argument to be given readily generalizes. (Similarly, the present methods also apply to the other space forms studied in [KwS].)

By [F], the topological surgery sequence  $(Q = Q_1)$ 

$$[\Sigma^{2}M_{+}; G/\text{TOP}] \xrightarrow{\theta} L_{s}(Q) \to \mathcal{G}(M \times I/\partial) \xrightarrow{\eta}$$
$$\to [\Sigma M_{+}; G/\text{TOP}] \to L_{4}(Q)$$

<sup>&</sup>lt;sup>1</sup> Partially supported by NSF grants.

for structures on  $M \times I$  relative the boundary is valid in this case. It is well known that all the elements in

$$[\Sigma M_+; G/\text{TOP}] \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$$

can be realized as the images under the normal invariant  $\eta$  of homotopy equivalences of  $M \times I$  to itself that are the identity on the boundary. It is not difficult to construct these directly and to compute their normal invariants from the obvious characteristic variety for  $M \times I$  rel boundary; this is also proven by a homotopy theoretic analysis in [KwS]. By [CS2], the image of  $\theta$  is a copy of  $\mathbb{Z}_2$  in

$$L_5(Q) = L_5^s(Q) = L_5^h(Q) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2.$$

Therefore there exists at most one non-trivial s-cobordism of M to itself. Now let h be the composite

$$M \times I \rightarrow (M \times I) \vee S^4 \rightarrow M \times I$$

where the first map is obtained by pinching off a cell in the interior and the second is the identity on  $M \times I$  and the non-trivial element of  $\pi_4(M) \cong \mathbb{Z}_2$  on  $S^4$ . It is also well known that  $\eta[h] = 0$ . (This is not essential, as we could compose with one of the self equivalences mentioned above.) Hence the theorem follows if it can be shown that the element  $[h] \in \mathcal{S}((M \times I)/\partial)$  is non-trivial. In fact, it will be shown that the surgery obstruction  $\xi \in L_5(Q)$  of a normal cobordism of h to the identity is not in the image of  $\theta$ .

Let  $VL^5(\mathbb{Z}Q)$  denote the visible symmetric L-group of Michael Weiss [We], but using chain complexes of stably free modules. The visible L-groups are refinement of the symmetric L-groups of Ranicki [Ra], have the same formal properties, and cobordism classes of finite Poincare complexes and degree one maps of them determine elements in the visible groups in the same way. Let

$$j_*: L_5(Q) \to VL^5(\mathbb{Z}Q)$$

be the natural map. The normal map  $(\Omega, b)$  of [CS 2],

$$\Omega: M \times T^2 \to M \times S^2$$

has surgery obstruction  $\sigma(\Omega, b)$  the non-trivial element in the image of  $\theta$ . The map  $\Omega$  obviously bounds a degree one map of manifolds

$$M \times S^1 \times D^2 \rightarrow M \times D^3$$
.

Hence the element  $\sigma^*(\Omega) = j_*(\sigma(\Omega, b)) \in VL^5(\mathbb{Z}Q)$  vanishes i.e.,  $j_* \circ \theta = 0$ . Hence it will suffice to show that

$$j_*(\xi) \neq 0$$
.

Let  $i:\pi\subset Q$  be the inclusion of the center. Recall the diagram [We] (compare [Ra], [WeI, II])

$$VL^{6}(\mathbb{Z}Q, \mathbb{Z}\pi) \longrightarrow V\hat{L}^{6}(\mathbb{Z}Q, \mathbb{Z}\pi) \xrightarrow{\Delta} L_{5}(Q, \pi)$$

$$\downarrow \qquad \qquad \downarrow \theta \qquad \qquad \downarrow$$

$$L_{5}(\pi) \longrightarrow VL^{5}(\mathbb{Z}\pi) \xrightarrow{\omega} V\hat{L}^{5}(\mathbb{Z}\pi) \longrightarrow L_{4}(\pi)$$

$$\downarrow \qquad \qquad \downarrow i_{*} \qquad \qquad \downarrow$$

$$L_{5}(Q) \xrightarrow{j_{*}} VL^{5}(\mathbb{Z}Q) \longrightarrow V\hat{L}^{5}(\mathbb{Z}Q)$$

with exact rows and columns. The main theorem of [We] yields compatible decompositions

$$V\hat{L}^6(\mathbb{Z}Q, \mathbb{Z}\pi) \cong \bigoplus_m H_{6-m}(Q, \pi; \hat{L}^m(\mathbb{Z}))$$

and

$$V\hat{L}^5(\mathbb{Z}G) \cong \bigoplus_m H_{5-m}(G; \hat{L}^m(\mathbb{Z})), \qquad G = Q \text{ or } \pi,$$

with

$$\hat{L}^{m}(\mathbb{Z}) = \begin{cases} \mathbb{Z}_{8} \\ \mathbb{Z}_{2} \\ 0 \\ \mathbb{Z}_{2} \end{cases} \text{ for } m \equiv \begin{cases} 0 \\ 1 \\ 2 \\ 3 \end{cases} \mod 4.$$

Let

$$f: M \times S^1 \to M \times S^1$$

be obtained from h by the obvious identification of boundary components. Performing the corresponding identifications on the boundary of a rel boundary normal cobordism of h to the identity and then gluing on copies of  $M \times D^2$  yields a normal map (g, b),

$$g:W\to M\times D^2$$

with  $\partial W = M \times S^1$ ,  $g \mid \partial W = f$ , and  $\sigma(g, b) = \xi$ . On the other hand, a simple homotopy-theoretic argument shows that f extends to a map

$$k: Y = (M \times D^2) \# E \rightarrow M \times D^2$$
,

where E is the nontrivial linear bundle over  $S^2$  with fiber  $S^3$ , with k trivial on the image of a cross-section of E and with  $k \mid S^3$  the covering projection of  $S^3$  to  $M \subset M \times D^2$ . By a standard cobordism-theoretic argument, using e.g. (17.6) of [C],  $W \cup_{\partial W} Y$  with the obvious map represents zero in the oriented cobordism of M. It follows readily that

$$g \cup k : W \cup_{\partial W} Y \rightarrow M \times D^2 \cup_{\partial} M \times D^2 = M \times S^2$$

is the boundary of a degree one map of an oriented manifold into  $M \times D^3$ . Hence the invariant  $\sigma^*(g \cup k) \in VL^5(\mathbb{Z}Q)$  vanishes, and so

$$j_*(\xi) = -j_*(\xi) = -\sigma^*(g) = \sigma^*(k).$$

Let  $P^3 = S^3/\pi$  be real projective 3-space and let

$$f_2: P^3 \times S^1 \to P^3 \times S^1$$

be defined, analogously to f, by twisting the top cell around the generator of  $\pi_3(P^3)$ , and define the extension

$$k_2: Y_2 = (P^3 \times D^2) \# E \to P^3 \times D^2$$

similarly to the above as well. Then we claim that

$$i_*(\sigma^*(k_2)) = \sigma^*(k),$$

 $i:\pi\to Q$  the inclusion as above. The crucial geometric fact that will be used in the proof of the claim is that for both k and  $k_2$ , the obvious generator of the kernel group  $K_3(k;\mathbb{Z}Q)$  or  $K_3(k_2;\mathbb{Z}\pi)$  is represented by an immersed sphere in  $Y\times\mathbb{R}$  or  $Y_2\times\mathbb{R}$  with the coefficient of the non-zero element  $g\in\pi$  in its self-intersection number equal to  $\pm 1$ . The generator of  $K_3(k;\mathbb{Z}Q)$  is represented by the sum of a generator of  $\pi_3(M)$  and a fiber of the summand E. Hence to see this fact, it suffices to check the same thing for the generators of  $\pi_3$  of  $M\times\mathbb{R}^3$  or  $P^3\times\mathbb{R}^3$ . This can be done using the immersions  $(z_i\in\mathbb{C},\ z_1\bar{z}_1+z_2\bar{z}_2=1)$ 

$$g(z_1, z_2) = (p(z_1, z_2), z_1, \text{Re}(z_2)),$$

p the appropriate covering projection  $(\mathbb{R}^3 = \mathbb{C} \times \mathbb{R})$  to compute the self-intersection number. The details are left to the reader.

The proof of the claim requires a more detailed review of some aspects of visible L-theory, taken from [We] and from some elaborations kindly provided to us in private communication by Weiss. A visible symmetric algebraic Poincare complex (VSAPC) of dimension n is a finite dimensional chain complex C of stably free finitely generated modules over a group ring  $\mathbb{Z}G$ , together with a class  $[\varphi]$  in

$$VQ^{n}(C) = H_{n}(P \otimes_{\mathbb{Z}G} \operatorname{Hom}_{\mathbb{Z}[\mathbb{Z}_{2}]}(W, C \otimes_{\mathbb{Z}} C)),$$

where P is a projective resolution of  $\mathbb{Z}$  over  $\mathbb{Z}G$ , and W is the usual resolution of  $\mathbb{Z}$  over  $\mathbb{Z}[\mathbb{Z}_2]$ . Further, the image of  $[\varphi]$  in the group  $Q^n(C)$  under the map induced by  $P \to \mathbb{Z}$  is assumed to be non-degenerate in the sense of [Ra]. The group  $VL^n(\mathbb{Z}G)$  is the algebraic bordism group of VSAPC's over  $\mathbb{Z}G$ . The obstruction theory for surgery to kill a class  $x \in H_k(C)$  in a VSAPC is formally the same as in [Ra] for symmetric algebraic Poincare complexes – the image of  $[\varphi]$  in  $VQ^n(\Sigma^{n-k}\mathbb{Z}G)$  under a chain map  $C \to \Sigma^{n-k}\mathbb{Z}G$  representing the Poincare dual of x, where the target is the chain complex consisting of  $\mathbb{Z}G$  concentrated in dimension n-k. Using the exact sequences 2.2 and 2.9 of [We] and the facts [Ra] that  $Q^n(\Sigma^{n-k}\mathbb{Z}G) = 0$  for k > n/2 and  $Q_n(\Sigma^{n-k}\mathbb{Z}G) = 0$  for k < n/2, one derives

$$VQ^{n}(\Sigma^{n-k}\mathbb{Z}G) = \begin{cases} \hat{H}^{k+1}(\mathbb{Z}_{2}; \mathbb{Z}G)/\hat{H}^{k+1}(\mathbb{Z}_{2}; \mathbb{Z}) & \text{for } k > n/2\\ \hat{H}^{k+1}(\mathbb{Z}_{2}; \mathbb{Z}) & \text{for } k < n/2 \end{cases}$$

Now consider  $(C(k), \varphi)$  representing  $\sigma^*(k)$ ; then  $H_i(C(k))$  vanishes for  $i \neq 2, 3$ , and for i = 2, 3,  $H_i(C(k)) = K_i(k)$  is free on generators  $x_i$ , say, with intersection number  $x_2 \cdot x_3 = 1$ . In this case the obstruction to killing  $x_2$  can be interpreted as the evaluation of  $w_2 Y$  on  $x_2$  and hence is the non-trivial element in  $VQ^5(\Sigma^3\mathbb{Z}Q) = \mathbb{Z}_2$ . Similarly, the obstruction to killing  $x_3$  can be interpreted as the coefficient mod 2 of g in the self intersection number of an immersion of  $S^3$  in  $Y \times \mathbb{R}$  representing  $x_3$  and so is also non-trivial in this case. Exactly the same discussion applies to  $C(k_2)$ ; in fact the inclusion induces isomorphisms  $VQ^n(\Sigma^{n-k}\mathbb{Z}\pi) \cong VQ^n(\Sigma^{n-k}\mathbb{Z}Q)$ . It follows that  $i_*(\sigma^*(k_2))$  is represented by a VSAPC  $(C, \psi)$  whose homology is free on generators  $y_2$  and  $y_3$  in dimensions two and three and is trivial elsewhere, with  $y_2 \cdot y_3 = 1$  and with the obstructions to killing  $y_i$  are non-trivial. Thus in the VSAPC  $(C(k), \varphi) \oplus -(C, \psi)$ , the homology class  $x_2 + y_2$  can be killed by algebraic surgery. The resulting VSAPC has homology free of rank one in dimensions two and three and zero elsewhere (by a standard argument), and the class  $x_3 - y_3$  survives to represent a generator of  $H_3$ 

of this complex. Hence this class can also be killed, and the result is a contractible complex; i.e. the above sum represents zero in  $VL^5(\mathbb{Z}Q)$ , the claim is proven, and hence

$$j_*(\xi) = i_*(\sigma^*(k_2)).$$

To analyze  $\sigma^*(k_2) = \beta$ , consider the Poincare complex

$$X = P^3 \times D^2 \cup_{f_2} P^3 \times D^2.$$

Since  $(f_2)^2$  is homotopic to the identity,  $k_2$  extends to a degree one map

$$g:(P^3\times S^2)\#E\to X$$

with  $\sigma^*(g) = \beta$ . Hence

$$\beta = \sigma^*(X) - \sigma^*((P^3 \times S^2) \# E) = \sigma^*(X),$$

as the domain of g is obviously an oriented boundary. As in [WeII, 7.1], there is a commutative square

$$\Omega_{5}^{P}(B\pi) \longrightarrow H_{5}(B\pi: \mathbb{MSG})$$

$$\downarrow \sigma^{*} \qquad \qquad \downarrow$$

$$VL^{5}(\mathbb{Z}\pi) \xrightarrow{\omega} H_{5}(B\pi; \hat{\mathbb{L}}^{*}) \cong V\hat{L}^{5}(\mathbb{Z}\pi).$$

(The above decompositions of  $V\hat{L}$  come from a decomposition of  $\hat{\mathbb{L}}^*$  into Eilenberg-Maclane spectra.) The top horizontal arrow is an (easy) map in the Levitt-Jones-Quinn exact sequence for oriented Poincare bordism, but in the notation used in [Ra 1]. Since the spectrum MSG has trivial homotopy in negative dimensions, this first of all implies that

$$\omega(\beta) \in \bigoplus_{m=0}^{5} H_{5-m}(B\pi; \hat{L}^{m}(\mathbb{Z})).$$

Further, let  $\kappa \in H^2(G/O; \mathbb{Z}_2)$  be the non-trivial element, and let  $\lambda \in H^3(BSG; \mathbb{Z}_2)$  be the image of  $\kappa$  under (all  $\mathbb{Z}_2$  coefficients)

$$H^2(G/O) \cong H^2(SG, SO) \cong H^3(BSG, BSO) \rightarrow H^3(BSG).$$

The image  $T\lambda$  of  $\lambda$  in  $H^3(MSG; \mathbb{Z}_2) = H^{k+3}(MSG_k; \mathbb{Z}_2)$ , k large, under the Thom

isomorphism detects the non-trivial element of

$$\pi_3(MSG) \cong \hat{L}^3(\mathbb{Z}) \cong L_2(e);$$

these isomorphisms are induced by maps in the sequences of Levitt and Ranicki. Hence, at least with respect to a suitable choice of the splitting of the spectrum  $\hat{\mathbb{L}}^*$ , the diagram

$$H_{5}(B\pi; \mathbb{MSG}) \longrightarrow H_{5}(B\pi; \Omega^{-3}\mathbb{K}(\mathbb{Z}_{2}))$$

$$\downarrow \qquad \qquad \qquad H_{2}(B\pi; \mathbb{Z}_{2})$$

$$H_{5}(B\pi; \hat{\mathbb{L}}^{*}) \longrightarrow H_{2}(B\pi; \hat{L}^{3}(\mathbb{Z})),$$

where the bottom arrow is given by the splitting above, commutes.

It follows from standard arguments and the above diagrams that the component of  $\omega(\sigma^*(X)) = \omega(\beta)$  in  $H_2(B\pi; \hat{L}^3(\mathbb{Z}))$  is given by the image in  $H_2(B\pi; \mathbb{Z}_2)$  of the class  $[X] \cap v^*\lambda \in H_2(X; \mathbb{Z}_2)$  under the obvious map, where

$$v: X \to BSG$$

classifies the Spivak normal fiber space of X. Clearly the restriction of v to the copies of  $P^3 \times D^2$  is trivial, and a map of trivial stable spherical fibrations covering  $f_2$  can be viewed as a "clutching function" for v. Further, its classifying map

$$P^3 \times S^1 \rightarrow SG$$

is a lift of the normal invariant

$$\eta(f_2): P^3 \times S^1 \to G/O.$$

It follows that  $v^*\lambda$  is the image of  $\eta(f_2)^*\kappa$  under the connecting homomorphism

$$\delta: H^2(P^3 \times S^1; \mathbb{Z}_2) \rightarrow H^3(X; \mathbb{Z}_2)$$

of the Mayer-Vietoris sequence. It follows that  $[X] \cap v^*\lambda$  is the image of

$$\lambda = [P^3 \times S^1] \cap \eta(f_2)^* \kappa \in H_2(P^3 \times S^1; \mathbb{Z}_2)$$

under the map induced by inclusion.

It is well-known that the normal invariant  $\eta(f_2)$  is non-trivial; one method of

proof is to show that the transverse inverse image under  $f_2$  of a framed  $S^1 \times S^2 \subset P^3 \times S^1$  has non-trivial Kervaire invariant. (A proof that  $\eta(f)$  is trivial can be based on the fact that similar constructions for f using tori in  $M \times S^1$  yield two copies of a 2-dimensional Kervaire manifold.) In fact, the above class  $\lambda$  is represented by the inclusion

$$P^2 \times \{pt\} \subset P^3 \times S^1$$

and hence has non-trivial image in  $H_2(B\pi; \mathbb{Z}_2)$ . Therefore the component of  $\omega(\beta)$  in  $H_2(B\pi; \hat{L}^3(\mathbb{Z}))$  is non-trivial.

It follows readily from the fact that  $\pi_4(P^3) \to \pi_4(P^4)$  is trivial that the canonical map  $X \to B\pi = P^{\infty}$  factors through  $P^4$  and hence [X] maps trivially in  $H_5(B\pi)$ . This then implies that the component of  $\omega(\beta)$  in  $H_5(B\pi; \hat{L}^{\theta})$  is trivial. Since  $i_*(\beta) = j_*(\xi)$ , we therefore know that  $\omega(\beta) = \partial(\gamma)$ ,

$$\gamma \in \bigoplus_{m=1}^{5} H_{6-m}(BQ, B\pi; \hat{L}^{m}(\mathbb{Z})).$$

Let  $\gamma_m$  be the component in  $H_{6-m}(BQ, B\pi; \hat{L}^m(\mathbb{Z}))$ . The diagram

$$\bigoplus_{m} H_{6-m}(BQ, B\pi; L^{m}(\mathbb{Z})) \longrightarrow \bigoplus_{m} H_{6-m}(BQ, B\pi; \hat{L}^{m}(\mathbb{Z}))$$

$$\downarrow \Delta$$

$$VL^{6}(\mathbb{Z}Q, \mathbb{Z}\pi) \longrightarrow VL^{6}(\mathbb{Z}Q, \mathbb{Z}\pi)$$

and the fact that  $L^m(\mathbb{Z}) \to \hat{L}^m(\mathbb{Z})$  is an isomorphism for m = 1, 2, 5 implies that  $\Delta(\gamma_m) = 0$  for m = 1, 2, 5. The facts that  $L^4(\mathbb{Z}) = \mathbb{Z}$  maps onto  $\hat{L}^4(\mathbb{Z}) = \mathbb{Z}_8$  and  $H_1(B\pi) \to H_1(BQ)$  is trivial imply that  $\gamma$  may be chosen so that  $\Delta(\gamma_4) = 0$  also.

Thus there is an least one  $\gamma$  with  $\partial(\gamma) = \omega(\beta)$  and  $\Delta(\gamma) = \Delta(\gamma_3)$ ; note that  $\gamma_3$  is the unique non-trivial element in  $H_3(BQ, B\pi; \hat{L}^3(\mathbb{Z}))$ . If  $\partial(\gamma) = \partial(\gamma')$ , then  $\gamma - \gamma'$  comes from  $H_6(BQ; \hat{\mathbb{L}}^*)$ . However, the composite of  $\Delta$  and the map from  $H_6(BQ; \hat{\mathbb{L}}^*)$  factors through the assembly map

$$A_*: H_6(BQ; \Omega^{-1}\mathbb{L}_*) = H_5(BQ; \mathbb{L}_*) \rightarrow L_5(Q).$$

Since the elements of  $H_{4i+1}(BQ)$  and  $H_{4i-1}(BQ; \mathbb{Z}_2)$  are represented by smooth oriented manifolds, everything in the image is the surgery obstruction of a closed manifold. Hence, by direct computation using representatives (see [CS2] or [Mi]) or by [H], this image is trivial at least when one maps to  $L_5^p(Q)$ . Therefore, for

any  $\gamma$  with  $\partial(\gamma) = \omega(\beta)$ , we have

$$\Delta^p(\gamma) = \Delta^p(\gamma_3) \in L_5^p(Q, \pi),$$

where  $\Delta^p$  is the composite of  $\Delta$  and the map to  $L_5^p(Q, \pi)$ .

We will now show that  $\Delta^p(\gamma_3) \neq 0$ . From the above, this implies that  $\beta$  does not come from  $VL^6(\mathbb{Z}Q, \mathbb{Z}\pi)$  and hence

$$i_{\star}\xi = i_{\star}\beta \neq 0$$

as was to be shown to prove the theorem. Let  $H \subset Q$  be a subgroup of index four, and consider the diagram  $(\hat{L}^3(\mathbb{Z}) = \mathbb{Z}_2)$ 

$$H_{2}(BQ; \mathbb{Z}_{2}) \longrightarrow L_{4}^{p}(Q, -)$$

$$\downarrow p^{!} \qquad \qquad \downarrow p^{!}$$

$$H_{3}(BQ, B\pi; \mathbb{Z}_{2}) \stackrel{\iota}{\longrightarrow} H_{3}(BQ, BH); \mathbb{Z}_{2}) \longrightarrow L_{5}^{p}(Q, H),$$

where p is the projection of the nontrivial line bundle over BQ corresponding to BH. As in [CS 2] for  $L^h$  or in [H],  $p^!$  on surgery groups is an isomorphism, and on homology groups  $p^!$  is an isomorphism by the Thom isomorphism theorem. The elements of  $H_2(BQ; \mathbb{Z}_2)$  are represented by Klein bottles, and a peeling argument similar to [CS 2] shows that  $(p^!)^{-1}\iota(\gamma_3)$  has non-trivial image in  $L_4^P(Q, -) \cong \mathbb{Z}_2$ . Alternatively it is well-known (e.g. [H]) that the nontrivial element in  $L_4^P(Q, -)$  is the surgery obstruction of a twisted product of a Klein bottle with the Arf invariant (the codimension two Arf invariant). The reader can easily check that the homology class represented by this Klein bottle maps under  $p^!$  to the image of  $\gamma_3$ ; this implies  $\Delta^P(\gamma_3) \neq 0$  and so completes the proof.

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Received August 25, 1987